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Abstract

In this paper, we obtain a collection of existence results of almost automorphic mild solutions to several stochastic functional differential equations in a real separable Hilbert space. The main technique is based upon some appropriate composition theorems combined with the fixed point method. Moreover, an example is also given to justify the practical usefulness of the established general theorems.

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1 Introduction

In this paper, we are mainly concerned with the existence of square-mean almost automorphic mild solutions to the following stochastic functional differential equations with a constant delay

$$dx(t) = Ax(t)dt + f(t, x(t-r))\,dt + g(t, x(t-r))\,dW(t),\ t \in \mathbb{R}, \tag{1.1}$$

$$dx(t) = Ax(t)dt + f(t, B_1 x(t), x(t-r))\,dt + g(t, B_2 x(t), x(t-r))\,dW(t),\ t \in \mathbb{R}, \tag{1.2}$$

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where $r \geq 0$ is a fixed constant, and A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(\mathbb{P},\mathbb{H})$, and $B_i, i = 1,2$, are bounded linear operators, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathscr{F},\mathbb{P},\mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u)-W(v); u,v \leq t\}$. Here f and g are appropriate functions to be specified later.

The concept of almost automorphy is an important generalization of the classical almost periodicity. It was introduced by S. Bochner [8, 9], for more details about this topics we refer the reader to [19, 21, 23]. In recent years, the existence of almost periodic and almost automorphic solutions on different kinds of deterministic differential equations have been considerably investigated in lots of publications [1, 2, 3, 4, 7, 12, 15, 17, 18, 22, 29, 30, 31] because of their significance and applications in physics, mechanics and mathematical biology.

Since noise or stochastic perturbation is unavoidable in real world, it is of great importance to consider the stochastic effects in the investigation of differential systems [24, 25, 26, 27, 28]. Recently, the existence of almost periodic or pseudo almost periodic solutions to some stochastic differential equations have been considered such as [5, 11, 10] and references therein. Especially, Bezandry and Diagana systematically studied the fundamental properties of almost periodic stochastic processes and investigated almost periodic solutions to different kinds of stochastic differential equations in a recent monograph [6]. In Ref. [20], Fu and Liu introduced a new concept of square-mean almost automorphic stochastic processes, and established some basic properties including a composition theorem, and then they investigated the existence and uniqueness of square-mean almost automorphic mild solutions to some linear and nonlinear stochastic differential equations. Chang et al. [13, 14, 16, 32] extended further some basic properties of square-mean almost automorphic process, from another perspective. Specifically, in [13], a new concept of S^2-almost automorphy for stochastic processes including a composition theorem was studied. And more recently, in [16], Chen and Lin presented the concept of square-mean pseudo almost automorphic process and investigated the existence and uniqueness of square-mean pseudo almost automorphic solutions for some stochastic differential equations, and Zhao et al. introduced the notation of square-mean asymptotically almost automorphy for stochastic processes with some properties of such stochastic processes in [32].

Motivated by the above mentioned works [11, 13, 14, 20], we investigate in this paper the existence of square-mean almost automorphic mild solutions to the problems (1.1)-(1.2). Our main results are established by means of the fixed point method. The obtained results can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows. In section 2, we recall some basic definitions and notations. We also present and prove some preliminary facts which will be used throughout this paper. In section 3, we prove the existence of square-mean almost automorphic mild solutions to (1.1)-(1.2). An example is given in Section 4 to illustrate the results obtained.

2 Preliminaries and a new composition theorem

In this section, we introduce some basic definitions, notations, lemmas and technical results which will be used in the sequel. For more details on this section, we refer the reader to

[13, 14, 20].

Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ are two real separable Hilbert spaces. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. The notation $L^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty.$$

For $x \in L^2(\mathbb{P}, \mathbb{H})$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 d\mathbb{P}\right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. The notations $C\big(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\big)$ and $BC\big(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\big)$ stand for the collection of all continuous stochastic processes from $\mathbb{R}$ into $L^2(\mathbb{P}, \mathbb{H})$ and the space of all bounded continuous stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, respectively. It is then easy to check that $BC\big(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\big)$ is a Banach space when it is endowed with the norm $\|x\|_{BC(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))} := \sup_{t\in\mathbb{R}} \|x(t)\|_2$. We let $\mathcal{L}(\mathbb{K}, \mathbb{H})$ denote the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{\mathcal{L}(\mathbb{K},\mathbb{H})}$; in particular, this is simply denoted by $\mathcal{L}(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$. In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$.

Definition 2.1. ([20]) A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be stochastically continuous if

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Definition 2.2. ([13, 20]) A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AA\big(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\big)$.

Definition 2.3. ([13, 20]) A function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P}, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that for some function $\widetilde{f}$

$$\lim_{n\to\infty} E\|f(t+s_n, x) - \widetilde{f}(t, x)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f}(t-s_n, x) - f(t, x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in L^2(\mathbb{P}, \mathbb{H})$.

The next definition is a little different from the definition 2.3. We will use the new definition of square-mean almost automorphic functions to study the existence of square-mean almost automorphic mild solutions of Eqs. (1.1)-(1.2) when the perturbation f and g are not Lipschitz continuous.

Definition 2.4. ([13, 20]) A function $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $(t,x) \to f(t,x)$, which is jointly continuous, is said to be square-mean almost automorphic if $f(t,x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of $L^2(\mathbb{P},\mathbb{H})$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\widetilde{f} : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ such that

$$\lim_{n\to\infty} E\|f(t+s_n,x)-\widetilde{f}(t,x)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{f}(t-s_n,x)-f(t,x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in K$.

Lemma 2.5. *([20]) If x, x_1 and x_2 are all square-mean almost automorphic stochastic processes, then the following hold true:*

(i) $x_1 + x_2$ is square-mean almost automorphic.

(ii) λx is square-mean almost automorphic for every scalar λ.

(iii) There exists a constant $M > 0$ such that $\sup_{t\in\mathbb{R}} \|x(t)\|_2 \leq M$. That is, x is bounded in $L^2(\mathbb{P},\mathbb{H})$.

Lemma 2.6. *([20]) $\Big(AA\big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\big),\|\cdot\|_\infty\Big)$ is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA\big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\big)$.

Lemma 2.7. *[14] Let $f \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$. Then we have*

(I) $h(t) := f(-t) \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$.

(II) $f_a(t) := f(t+a) \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$, $a \in \mathbb{R}$ being fixed.

Lemma 2.8. *[14] Let $B \in \mathcal{L}(L^2(\mathbb{P},\mathbb{H}))$ and assume that $f \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$. Then $Bf \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$.*

Lemma 2.9. *[14] Let $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $(t,x) \to f(t,x)$ be square-mean almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$, that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x,y \in K$ and $E\|x-y\|^2 < \delta$ imply that $E\|f(t,x)-f(t,y)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ given by $F(\cdot) := f(\cdot,x(\cdot))$ is square-mean almost automorphic.*

Now, we introduce a few preliminary and important results.

Theorem 2.10. *Let $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $(t,x,y) \to f(t,x,y)$ be square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x,y) \in K$, where K is any bounded subset of $L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$, and assume that $f(t,\cdot,\cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$. Then for any square-mean almost automorphic stochastic process $\varphi : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$, the stochastic process $\Psi : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ given by $\Psi(t) := f(t,\varphi(t),\varphi(t-r))$ is square-mean almost automorphic.*

Proof. Combining Lemma 2.5 with Lemma 2.7, we can establish this result by using the same argument as in [14, Theorem 2.1], and we omit the details here. □

Theorem 2.11. *Let $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $(t,x) \to f(t,x)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P},\mathbb{H})$, and assume that there exists a bounded continuous function $L : \mathbb{R} \to \mathbb{R}^+$ such that*

$$E\|f(t,x) - f(t,y)\|^2 \le L(t)E\|x - y\|^2$$

for all $x,y \in L^2(\mathbb{P},\mathbb{H})$ and for each $t \in \mathbb{R}$. Then for any $x \in AA(\mathbb{R};L^2(\mathbb{P},\mathbb{H}))$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ given by $F(\cdot) := f(\cdot,x(\cdot))$ is square-mean almost automorphic.

Proof. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $x \in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, there exit a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ such that

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \ \text{ and } \ \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0 \tag{2.1}$$

hold for each $t \in \mathbb{R}$. On the other hand, since $(t,x) \to f(t,x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P},\mathbb{H})$, we can extract a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ (for convenience, we also denote it by $\{s_n\}_{n\in\mathbb{N}}$) and a function $\widetilde{f}$ such that

$$\lim_{n\to\infty} E\|f(t+s_n,x) - \widetilde{f}(t,x)\|^2 = 0 \ \text{ and } \ \lim_{n\to\infty} E\|\widetilde{f}(t-s_n,x) - f(t,x)\|^2 = 0 \tag{2.2}$$

for each $t \in \mathbb{R}$ and each $x \in L^2(\mathbb{P},\mathbb{H})$.

Now, let us consider the function $\widetilde{F} : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ defined by $\widetilde{F}(t) := \widetilde{f}(t,y(t))$, $t \in \mathbb{R}$.

Note that

$$F(t+s_n) - \widetilde{F}(t) = f(t+s_n,x(t+s_n)) - f(t+s_n,y(t)) + f(t+s_n,y(t)) - \widetilde{f}(t,y(t)).$$

Then, we have

$$\begin{aligned} & E\|F(t+s_n) - \widetilde{F}(t)\|^2 \\ \le\ & 2E\|f(t+s_n,x(t+s_n)) - f(t+s_n,y(t))\|^2 + 2E\|f(t+s_n,y(t)) - \widetilde{f}(t,y(t))\|^2 \\ \le\ & 2L(t+s_n)E\|x(t+s_n) - y(t)\|^2 + 2E\|f(t+s_n,y(t)) - \widetilde{f}(t,y(t))\|^2. \end{aligned}$$

By (2.1) and the bounded continuity of $L(t)$, we have

$$\lim_{n\to\infty} L(t+s_n)E\|x(t+s_n) - y(t)\|^2 = 0. \tag{2.3}$$

Moreover, by (2.2), we get that

$$\lim_{n\to\infty} E\|f(t+s_n,y(t)) - \widetilde{f}(t,y(t))\|^2 = 0. \tag{2.4}$$

Therefore, we can deduce from (2.3), (2.4) and the above inequality that

$$\lim_{n\to\infty} E\|F(t+s_n) - \widetilde{F}(t)\|^2 = 0 \ \text{ for each } \ t \in \mathbb{R}.$$

Similarly, using the same argument as above, we obtain $\lim_{n\to\infty} E\|\widetilde{F}(t-s_n) - F(t)\|^2 = 0$ for each $t \in \mathbb{R}$. That is, $F(t)$ is square-mean almost automorphic. The proof is now complete. □

Theorem 2.12. *Suppose that $f:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $(t,x,y)\to f(t,x,y)$ be square-mean almost automorphic in $t\in\mathbb{R}$ for each $(x,y)\in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$, and that there exists a bounded continuous function $\overline{L}:\mathbb{R}\to\mathbb{R}^+$ such that*

$$E\|f(t,x,y)-f(t,\overline{x},\overline{y})\|^2\le\overline{L}(t)\Big(E\|x-\overline{x}\|^2+E\|y-\overline{y}\|^2\Big),$$

for all $t\in\mathbb{R}$ and each $(x,y),(\overline{x},\overline{y})\in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$. Then for any square-mean almost automorphic stochastic process $\phi:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$, the stochastic process $\Phi:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ given by $\Phi(t):=f(t,\phi(t),\phi(t-r))$ is square-mean almost automorphic.

Proof. The proof is similar to the proof of Theorem 2.11. For the reader's convenience, we offer the proof here. Since $\phi:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ is square-mean almost automorphic, it follows from Lemma 2.7 that the function $t\to\phi(t-r)\in AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. By square-mean almost automorphy of ϕ, we can extract a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ and a stochastic process $\overline{\phi}$ such that for each $t\in\mathbb{R}$,

$$\lim_{n\to\infty}E\|\phi(t+s_n)-\overline{\phi}(t)\|^2=0,\ \ \lim_{n\to\infty}E\|\overline{\phi}(t-s_n)-\phi(t)\|^2=0, \tag{2.5}$$

and

$$\lim_{n\to\infty}E\|\phi(t+s_n-r)-\overline{\phi}(t-r)\|^2=0,\ \ \lim_{n\to\infty}E\|\overline{\phi}(t-s_n-r)-\phi(t-r)\|^2=0. \tag{2.6}$$

On the other hand, since $(t,x,y)\to f(t,x,y)$ is square-mean almost automorphic in $t\in\mathbb{R}$ for each $(x,y)\in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$, there exist a subsequence $\{\underset{\sim}{s_n}\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ (for convenience, we also denote it by $\{s_n\}_{n\in\mathbb{N}}$) and a stochastic process $\widetilde{f}$ such that

$$\lim_{n\to\infty}E\|f(t+s_n,x,y)-\widetilde{f}(t,x,y)\|^2=0\ \text{ and }\ \lim_{n\to\infty}E\|\widetilde{f}(t-s_n,x,y)-f(t,x,y)\|^2=0 \tag{2.7}$$

for each $t\in\mathbb{R}$ and each $(x,y)\in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$.

Now, let us consider the function $\widetilde{\Phi}:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ defined by $\widetilde{\Phi}(t):=\widetilde{f}\Big(t,\overline{\phi}(t),\overline{\phi}(t-r)\Big)$, $t\in\mathbb{R}$.

We can see that

$$\begin{aligned}\Phi(t+s_n)-\widetilde{\Phi}(t) &= f(t+s_n,\phi(t+s_n),\phi(t+s_n-r))-\widetilde{f}\Big(t,\overline{\phi}(t),\overline{\phi}(t-r)\Big)\\ &= f(t+s_n,\phi(t+s_n),\phi(t+s_n-r))-f\Big(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\Big)\\ &\quad +f\Big(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\Big)-\widetilde{f}\Big(t,\overline{\phi}(t),\overline{\phi}(t-r)\Big).\end{aligned}$$

Then, we have

$$\begin{aligned}E\|\Phi(t+s_n)-\widetilde{\Phi}(t)\|^2 &\le 2E\|f(t+s_n,\phi(t+s_n),\phi(t+s_n-r))-f\Big(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\Big)\|^2\\ &\quad +2E\|f\Big(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\Big)-\widetilde{f}\Big(t,\overline{\phi}(t),\overline{\phi}(t-r)\Big)\|^2\\ &\le 2\overline{L}(t+s_n)\Big(E\|\phi(t+s_n)-\overline{\phi}(t)\|^2+E\|\phi(t+s_n-r)-\overline{\phi}(t-r)\|^2\Big)\\ &\quad +2E\|f\Big(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\Big)-\widetilde{f}\Big(t,\overline{\phi}(t),\overline{\phi}(t-r)\Big)\|^2.\end{aligned}$$

By (2.5)-(2.6) and the bounded continuity of $\overline{L}(t)$, we know that

$$\lim_{n\to\infty}\overline{L}(t+s_n)\Big(E\|\phi(t+s_n)-\overline{\phi}(t)\|^2+E\|\phi(t+s_n-r)-\overline{\phi}(t-r)\|^2\Big)=0. \tag{2.8}$$

Moreover, by (2.7), we get that

$$\lim_{n\to\infty} E\|f\left(t+s_n,\overline{\phi}(t),\overline{\phi}(t-r)\right)-\widetilde{f}\left(t,\overline{\phi}(t),\overline{\phi}(t-r)\right)\|^2=0,$$

which combines with (2.8) yields that

$$\lim_{n\to\infty} E\|\Phi(t+s_n)-\widetilde{\Phi}(t)\|^2=0 \text{ for each } t\in\mathbb{R}.$$

Similarly, we can prove that

$$\lim_{n\to\infty} E\|\widetilde{\Phi}(t-s_n)-\Phi(t)\|^2=0 \text{ for each } t\in\mathbb{R}.$$

Hence $\Phi(t):=f(t,\phi(t),\phi(t-r))\in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, which ends the proof. □

Definition 2.13. An $\mathscr{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of problem (1.1) on $\mathbb{R}$ if it satisfies the corresponding stochastic integral equation

$$x(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s,x(s-r))\,ds+\int_a^t T(t-s)g(s,x(s-r))\,dW(s)$$

for all $t\geq a$ and for each $a\in\mathbb{R}$.

Definition 2.14. An $\mathscr{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of problem (1.2) on $\mathbb{R}$ if it satisfies the corresponding stochastic integral equation

$$x(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s,B_1x(s),x(s-r))\,ds+\int_a^t T(t-s)g(s,B_2x(s),x(s-r))\,dW(s)$$

for all $t\geq a$ and $a\in\mathbb{R}$.

3 Main results

In this section, we investigate the existence of a square-mean almost automorphic mild solutions for the problems (1.1)-(1.2). We first list the following basic assumptions.
(H1) The operator $A:D(A)\subset L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$ is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(\mathbb{P},\mathbb{H})$; that is, there exist constants $M>0$, $\delta>0$ such that $\|T(t)\|\leq Me^{-\delta t}$ for all $t\geq 0$.
(H2) The function $f\in AA\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H})\right)$ and there exists a bounded continuous function $L_f:\mathbb{R}\to\mathbb{R}^+$ such that

$$E\|f(t,x)-f(t,y)\|^2\leq L_f(t)E\|x-y\|^2$$

for all $t\in\mathbb{R}$ and $x,y\in L^2(\mathbb{P},\mathbb{H})$.
(H3) The function $g\in AA\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H});L^2(\mathbb{P},\mathbb{H})\right)$ and there exists a bounded continuous function $L_g:\mathbb{R}\to\mathbb{R}^+$ such that

$$E\|g(t,x)-g(t,y)\|^2\leq L_g(t)E\|x-y\|^2$$

for all $t\in\mathbb{R}$ and $x,y\in L^2(\mathbb{P},\mathbb{H})$.

Now, we are ready to state our first main result.

Theorem 3.1. *Assume the conditions (H1)-(H3) hold. Then Eq. (1.1) has a unique square-mean almost automorphic mild solution on* $\mathbb{R}$ *whenever* $\sqrt{L_0} < 1$, *where*

$$L_0 = M^2\left[\frac{2}{\delta^2}\sup_{t\in\mathbb{R}} L_f(t) + \frac{1}{\delta}\sup_{t\in\mathbb{R}} L_g(t)\right]. \tag{3.1}$$

Proof. We define the operator $\Lambda : AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right) \to AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ by

$$\Lambda x(t) \;=\; \int_{-\infty}^{t} T(t-s)f(s,x(s-r))\,ds + \int_{-\infty}^{t} T(t-s)g(s,x(s-r))\,dW(s),\ t\in\mathbb{R}.$$

First, let us check that $\Lambda\left(AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)\right) \subset AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$. Take $x \in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, by Lemma 2.7 and Theorem 2.11, we infer that both $F(\cdot) = f(\cdot,x(\cdot-r))$ and $G(\cdot) = g(\cdot,x(\cdot-r)) \in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$. Then, the similar reasoning as in the proof of [15, Theorem 3.1] proves that $\Lambda x(\cdot) \in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$. Thus, Λ maps $AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ into itself.

Next, we prove that Λ is a strict contraction mapping on $AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$. To this end, for each $t\in\mathbb{R}$, $x,y \in AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, we have

$$\begin{aligned}
E\|\Lambda x(t)-\Lambda y(t)\|^2 \;&=\; E\Big\|\int_{-\infty}^{t} T(t-s)[f(s,x(s-r))-f(s,y(s-r))]ds \\
&\quad + \int_{-\infty}^{t} T(t-s)[g(s,x(s-r))-g(s,y(s-r))]dW(s)\Big\|^2 \\
&\le 2E\Big\|\int_{-\infty}^{t} T(t-s)[f(s,x(s-r))-f(s,y(s-r))]ds\Big\|^2 \\
&\quad +2E\Big\|\int_{-\infty}^{t} T(t-s)[g(s,x(s-r))-g(s,y(s-r))]dW(s)\Big\|^2.
\end{aligned}$$

We first evaluate the first term of the right-hand side as follows:

$$\begin{aligned}
&2E\Big\|\int_{-\infty}^{t} T(t-s)[f(s,x(s-r))-f(s,y(s-r))]ds\Big\|^2 \\
\le\;& 2M^2E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s,x(s-r))-f(s,y(s-r))\|ds\right)^2 \\
\le\;& 2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}E\|f(s,x(s-r))-f(s,y(s-r))\|^2ds\right) \\
\le\;& 2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}L_f(s)E\|x(s-r)-y(s-r)\|^2ds\right) \\
\le\;& 2M^2\sup_{t\in\mathbb{R}} L_f(t)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}} E\|x(t-r)-y(t-r)\|^2 \\
\le\;& \frac{2M^2}{\delta^2}\sup_{t\in\mathbb{R}} L_f(t)\sup_{t\in\mathbb{R}} E\|x(t-r)-y(t-r)\|^2,
\end{aligned}$$

by using the Cauchy-Schwarz inequality.

As to the second term, by the Ito integral, we get

$$
\begin{aligned}
& 2E\left\|\int_{-\infty}^{t} T(t-s)[g(s,x(s-r))-g(s,y(s-r))]dW(s)\right\|^2 \\
\leq\ & 2E\left(\int_{-\infty}^{t} \|T(t-s)[g(s,x(s-r))-g(s,y(s-r))]\|^2 ds\right) \\
\leq\ & 2M^2\int_{-\infty}^{t} e^{-2\delta(t-s)}E\|g(s,x(s-r))-g(s,y(s-r))\|^2 ds \\
\leq\ & 2M^2\int_{-\infty}^{t} e^{-2\delta(t-s)}L_g(s)E\|x(s-r)-y(s-r)\|^2 ds \\
\leq\ & 2M^2\sup_{t\in\mathbb{R}} L_g(t)\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}} E\|x(t-r)-y(t-r)\|^2 \\
\leq\ & \frac{M^2}{\delta}\sup_{t\in\mathbb{R}} L_g(t)\sup_{t\in\mathbb{R}} E\|x(t-r)-y(t-r)\|^2 .
\end{aligned}
$$

Thus, by combining the above inequality together, we obtain that, for each $t\in\mathbb{R}$,

$$
E\|\Lambda x(t)-\Lambda y(t)\|^2 \leq M^2\left[\frac{2}{\delta^2}\sup_{t\in\mathbb{R}} L_f(t)+\frac{1}{\delta}\sup_{t\in\mathbb{R}} L_g(t)\right]\sup_{t\in\mathbb{R}} E\|x(t-r)-y(t-r)\|^2,
$$

that is,

$$
\|\Lambda x(t)-\Lambda y(t)\|_2^2 \leq L_0\sup_{t\in\mathbb{R}}\|x(t-r)-y(t-r)\|_2^2. \tag{3.2}
$$

Note that

$$
\sup_{t\in\mathbb{R}}\|x(t-r)-y(t-r)\|_2^2 \leq \left(\sup_{t\in\mathbb{R}}\|x(t-r)-y(t-r)\|_2\right)^2, \tag{3.3}
$$

and (3.2) together with (3.3) gives, for each $t\in\mathbb{R}$,

$$
\|\Lambda x(t)-\Lambda y(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty .
$$

Hence, we obtain

$$
\|\Lambda x-\Lambda y\|_\infty = \sup_{t\in\mathbb{R}}\|\Lambda x(t)-\Lambda y(t)\|_2 \leq \sqrt{L_0}\|x-y\|_\infty ,
$$

which implies that Λ is a contraction by (3.1). So by the Banach contraction principle, we conclude that there exists a unique fixed point $x(\cdot)$ for Λ in $AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, such that $\Lambda x = x$. After these, using the same lines as in the proof of [15, Theorem 3.1], we conclude that $x(t)$ is a unique square-mean almost automorphic mild solution of Eq. (1.1), which completes the proof. □

We next study the existence of square-mean almost automorphic mild solutions of Eq. (1.1) when the functions f and g are not Lipschitz continuous. We need to assume that f and g satisfy appropriate compactness conditions. To establish the result, we begin by introducing the following assumptions.
(H4) The semigroup $\{T(t)\}_{t\geq 0}$ is compact for $t>0$.
(H5) The function $f:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$ satisfies the following conditions:

(i) f is square-mean almost automorphic and $f(t,\cdot)$ is uniformly continuous in every bounded subset $K \subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$.

(ii) There exist an integrable function $m_f : \mathbb{R} \to [0,\infty)$ and a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that

$$E\|f(t,\varphi)\|^2 \le m_f(t)W_f(E\|\varphi\|^2), \quad \text{for all} \quad (t,\varphi) \in \mathbb{R}\times K.$$

(iii) Let $\{x_n\} \subset AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{f(\cdot,x_n(\cdot-r))\}$ is relatively compact in $BC\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$.

(H6) The function $g : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ satisfies the following conditions:

(i) g is square-mean almost automorphic and $g(t,\cdot)$ is uniformly continuous in every bounded subset $K' \subset L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$.

(ii) There exist an integrable function $m_g : \mathbb{R} \to [0,\infty)$ and a continuous nondecreasing function $W_g : [0,\infty) \to (0,\infty)$ such that

$$E\|g(t,\varphi)\|^2 \le m_g(t)W_g(E\|\varphi\|^2), \quad \text{for all} \quad (t,\varphi) \in \mathbb{R}\times K'.$$

(iii) Let $\{x_n\} \subset AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{g(\cdot,x_n(\cdot-r))\}$ is relatively compact in $BC\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$.

Theorem 3.2. *Let (H1), (H4), (H5) and (H6) be satisfied. Then Eq. (1.1) admits at least one square-mean almost automorphic mild solution on* $\mathbb{R}$ *provided that*

$$\overline{L_f} = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} m_f(s)ds < \infty, \quad \overline{L_g} = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\delta(t-s)} m_g(s)ds < \infty$$

and

$$\frac{2M^2\overline{L_f}}{\delta} \liminf_{r\to\infty} \frac{W_f(r)}{r} + 2M^2\overline{L_g} \liminf_{r\to\infty} \frac{W_g(r)}{r} < 1.$$

Proof. Let the operator Λ be defined the same as in Theorem 3.1. Now, from Lemmas 2.7 and 2.9, it is clear that both $F(\cdot) = f(\cdot,x(\cdot-r))$ and $G(\cdot) = g(\cdot,x(\cdot-r)) \in AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$ whenever $x \in AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$. Moreover, by Lemmas 3.1 and 3.2 in [14], we deduce that Λ is continuous and maps $AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$ into itself. So Λ is well defined and continuous.

Next, we shall use the Schauder fixed point theorem to prove that Λ has a fixed point. For the sake of convenience, we break the proof into several steps.

Step 1. Let $B_r = \{x \in AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big) : \|x\|_\infty \le r\}$ for each $r > 0$. We prove that there exists a number r such that $\Lambda(B_r) \subseteq B_r$.

Step 2. This step consists of showing that the operator Λ is completely continuous on B_r. It suffices to prove that the following statements are true.

(i) $V(t) = \{\Lambda x(t) : x \in B_r\}$ is relatively compact in $L^2(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$.

(ii) $\{\Lambda x : x \in B_r\} \subset AA\Big(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\Big)$ is a family of equicontinuous functions.

It follows from Step 1 and Step 2 in [14, Theorem 3.1] that the above assertions hold. After these, we follow the same reasoning as in the proof of Theorem 3.1 in [14]. We denote the closed convex hull of ΛB_r by $\overline{conv}\Lambda B_r$. Since $\Lambda B_r \subset B_r$ and B_r is closed convex, $\overline{conv}\Lambda B_r \subset B_r$. Thus, $\Lambda(\overline{conv}\Lambda B_r) \subset \Lambda B_r \subset \overline{conv}\Lambda B_r$. This implies that Λ is a continuous mapping from $\overline{conv}\Lambda B_r$ to $\overline{conv}\Lambda B_r$. It is easy to verify that $\overline{conv}\Lambda B_r$ has the properties (i) and (ii). More explicitly, $x(t) : x \in \overline{conv}\Lambda B_r$ is relatively compact in $L^2(\mathbb{P},\mathbb{H})$ for each $t \in \mathbb{R}$, and $\overline{conv}\Lambda B_r \subset BC\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ is uniformly bounded and equicontinuous. By the Ascoli-Arzelà theorem, the restriction of $\overline{conv}\Lambda B_r$ to every compact subset K'' of $\mathbb{R}$, namely $\{x(t) : x \in \overline{conv}\Lambda B_r\}_{x\in K''}$ is relatively compact in $C\Big(K''; L^2(\mathbb{P},\mathbb{H})\Big)$. Thus, by the conditions (H5)(iii) and (H6)(iii) we deduce that $\Lambda : \overline{conv}\Lambda B_r \to \overline{conv}\Lambda B_r$ is a compact operator. So by Schauder's fixed point theorem, we conclude that there is a fixed point $x(\cdot)$ for Λ in $\overline{conv}\Lambda B_r$. That is Eq. (1.1) has at least one square-mean almost automorphic mild solutions $x \in B_r$. This completes the proof. □

In order to investigate the solution to (1.2), we need the following assumptions:

(H7) The operators $B_i : L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ for $i = 1,2$, are bounded linear operators and $\varpi := \max_{i=1,2}\Big\{\|B_i\|_{\mathcal{L}(L^2(\mathbb{P},\mathbb{H}))}\Big\}$.

(H8) The function $f \in AA\Big(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\Big)$ and there exists a bounded continuous function $\widetilde{L_f} : \mathbb{R} \to \mathbb{R}^+$ such that

$$E\|f(t,\varphi,\psi) - f(t,\overline{\varphi},\overline{\psi})\|^2 \le \widetilde{L_f}(t)\Big(E\|\varphi-\overline{\varphi}\|^2 + E\|\psi-\overline{\psi}\|^2\Big)$$

for all $t \in \mathbb{R}$ and $(\varphi,\psi),(\overline{\varphi},\overline{\psi}) \in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$.

(H9) The function $g \in AA\Big(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\Big)$ and there exists a bounded continuous function $\widetilde{L_g} : \mathbb{R} \to \mathbb{R}^+$ such that

$$E\|g(t,\varphi,\psi) - g(t,\overline{\varphi},\overline{\psi})\|^2 \le \widetilde{L_g}(t)\Big(E\|\varphi-\overline{\varphi}\|^2 + E\|\psi-\overline{\psi}\|^2\Big)$$

for all $t \in \mathbb{R}$ and $(\varphi,\psi),(\overline{\varphi},\overline{\psi}) \in L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$.

Now, we present another main result.

Theorem 3.3. *Let (H1), (H7), (H8) and (H9) be satisfied. Then the problem (1.2) has a unique square-mean almost automorphic mild solution on* $\mathbb{R}$ *whenever* $\sqrt{L_1(\varpi^2+1)} < 1$, *where*

$$L_1 = M^2\left[\frac{2}{\delta^2}\sup_{t\in\mathbb{R}}\widetilde{L_f}(t) + \frac{1}{\delta}\sup_{t\in\mathbb{R}}\widetilde{L_g}(t)\right]. \tag{3.4}$$

Proof. Let $\Lambda : AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big) \to AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ be the operator defined by

$$\begin{aligned}\Lambda x(t) &= \int_{-\infty}^{t} T(t-s)f(s,B_1x(s),x(s-r))\,ds + \int_{-\infty}^{t} T(t-s)g(s,B_2x(s),x(s-r))\,dW(s)\\ &= \Lambda_1x(t) + \Lambda_2x(t),\end{aligned}$$

where

$$\Lambda_1x(t) = \int_{-\infty}^{t} T(t-s)f(s,B_1x(s),x(s-r))\,ds,$$

and

$$\Lambda_2x(t) = \int_{-\infty}^{t} T(t-s)g(s,B_2x(s),x(s-r))\,dW(s)$$

for each $t \in \mathbb{R}$.

Now, let us prove that Λx is well defined. First, we will show that $\Lambda_1 x(\cdot) \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$. Indeed, let $x \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$, then $s \to B_i x(s)$ is in $AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ as $B_i \in \mathcal{L}\Big(L^2(\mathbb{P},\mathbb{H})\Big)$, $i = 1,2$ by Lemma 2.8. And hence, by Lemma 2.7, one can easily see that $s \to x(s-r) \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$. In view of (H8)-(H9) and Theorem 2.12, the function $s \to f(s, B_1 x(s), x(s-r))$ is in $AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$. Let $\{s_n'\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $F(\cdot) = f(\cdot, B_1 x(\cdot), x(\cdot - r)) \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s_n'\}_{n\in\mathbb{N}}$ and a stochastic process $\widetilde{F}$ such that

$$\lim_{n\to\infty} E\|F(t+s_n) - \widetilde{F}(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{F}(t-s_n) - F(t)\|^2 = 0 \tag{3.5}$$

hold for each $t \in \mathbb{R}$. Moreover, if we let $(\widetilde{\Lambda_1} x)(t) = \int_{-\infty}^{t} T(t-s)\widetilde{F}(s)ds$, by using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& E\|\Lambda_1 x(t+s_n) - \widetilde{\Lambda_1} x(t)\|^2 \\
= \;& E\left\| \int_{-\infty}^{t+s_n} T(t+s_n-s) f(s, B_1 x(s), x(s-r))\, ds - \int_{-\infty}^{t} T(t-s)\widetilde{F}(s)ds \right\|^2 \\
= \;& E\left\| \int_{-\infty}^{t} T(t-s) f(s+s_n, B_1 x(s+s_n), x(s+s_n-r))\, ds - \int_{-\infty}^{t} T(t-s)\widetilde{F}(s)ds \right\|^2 \\
\le \;& E\left(\int_{-\infty}^{t} \|T(t-s)[F(s+s_n) - \widetilde{F}(s)]\| ds \right)^2 \\
\le \;& M^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)} ds \right) \left(\int_{-\infty}^{t} e^{-\delta(t-s)} E\|F(s+s_n) - \widetilde{F}(s)\|^2 ds \right) \\
\le \;& M^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)} ds \right)^2 \sup_{t\in\mathbb{R}} E\|F(t+s_n) - \widetilde{F}(t)\|^2 \\
\le \;& \frac{M^2}{\delta^2} \sup_{t\in\mathbb{R}} E\|F(t+s_n) - \widetilde{F}(t)\|^2.
\end{aligned}$$

Thus, by (3.5), we immediately obtain that

$$\lim_{n\to\infty} E\|\Lambda_1 x(t+s_n) - \widetilde{\Lambda_1} x(t)\|^2 = 0,$$

for each $t \in \mathbb{R}$. And we can show in a similar way that

$$\lim_{n\to\infty} E\|\widetilde{\Lambda_1} x(t-s_n) - \Lambda_1 x(t)\|^2 = 0,$$

for each $t \in \mathbb{R}$. Thus we conclude that $\Lambda_1 x(t) \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$.

Next, we show that $\Lambda_2 x(\cdot)$ is square-mean almost automorphic whenever x is square-mean almost automorphic. Let $x \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$, then $s \to B_2 x(s)$ is in $AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ as $B_2 \in \mathcal{L}\Big(L^2(\mathbb{P},\mathbb{H})\Big)$. And hence, by Lemma 2.7 and Theorem 2.12, one can easily see that $s \to g(s, B_2 x(s), x(s-r))$ is in $AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$. Since $G(\cdot) = g(\cdot, B_2 x(\cdot), x(\cdot - r)) \in AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$, then for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}} \subset \{s_n'\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{G}$

$$\lim_{n\to\infty} E\|G(t+s_n) - \widetilde{G}(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\|\widetilde{G}(t-s_n) - G(t)\|^2 = 0 \tag{3.6}$$

hold for each $t \in \mathbb{R}$. The next step consists of showing that $\Lambda_2 x(t) \in AA\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. Let $\widetilde{W}(\sigma) := W(\sigma + s_n) - W(s_n)$ for each $\sigma \in \mathbb{R}$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $\widetilde{\Lambda_2}x(t) = \int_{-\infty}^{t} T(t-s)\widetilde{G}(s)d\mathbb{W}(s)$, then by making a change of variables $\sigma = s - s_n$ to get

$$
\begin{aligned}
& E\|\Lambda_2 x(t+s_n) - \widetilde{\Lambda_2}x(t)\|^2 \\
= \ & E\left\| \int_{-\infty}^{t+s_n} T(t+s_n-s)g\left(s, B_2x(s), x(s-r)\right)dW(s) - \int_{-\infty}^{t} T(t-s)\widetilde{G}(s)dW(s)\right\|^2 \\
= \ & E\left\| \int_{-\infty}^{t} T(t-\sigma)[G(\sigma+s_n) - \widetilde{G}(\sigma)]d\widetilde{W}(\sigma)\right\|^2.
\end{aligned}
$$

Thus using the Ito's isometry property of stochastic integral, we obtain that

$$
\begin{aligned}
& E\|\Lambda_2 x(t+s_n) - \widetilde{\Lambda_2}x(t)\|^2 \\
\leq \ & E\left(\int_{-\infty}^{t} \|T(t-\sigma)[G(\sigma+s_n) - \widetilde{G}(\sigma)]\|^2 d\sigma \right) \\
\leq \ & M^2 \int_{-\infty}^{t} e^{-2\delta(t-\sigma)} E\|G(\sigma+s_n) - \widetilde{G}(\sigma)\|^2 d\sigma \\
\leq \ & \frac{M^2}{2\delta} \sup_{t\in\mathbb{R}} E\|G(t+s_n) - \widetilde{G}(t)\|^2.
\end{aligned}
$$

Thus, by (3.6), it leads to

$$
\lim_{n\to\infty} E\|\Lambda_2 x(t+s_n) - \widetilde{\Lambda_2}x(t)\|^2 = 0,
$$

for each $t \in \mathbb{R}$. Arguing in a similar way, we can obtain

$$
\lim_{n\to\infty} E\|\widetilde{\Lambda_2}x(t-s_n) - \Lambda_2 x(t)\|^2 = 0,
$$

for each $t \in \mathbb{R}$. Consequently, $\Lambda_2 x(t) \in AA\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. In view of the above, it is clear that Λ maps $AA\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ into itself.

Finally, we prove that Λ is a contraction mapping on $AA\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. To this end, for each $t \in \mathbb{R}$, $x, y \in AA\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$, we see that

$$
\begin{aligned}
& E\|\Lambda x(t) - \Lambda y(t)\|^2 \\
= \ & E\bigg\| \int_{-\infty}^{t} T(t-s)[f\left(s, B_1x(s), x(s-r)\right) - f\left(s, B_1y(s), y(s-r)\right)]ds \\
& + \int_{-\infty}^{t} T(t-s)[g\left(s, B_2x(s), x(s-r)\right) - g\left(s, B_2y(s), y(s-r)\right)]dW(s)\bigg\|^2 \\
\leq \ & 2E\left\| \int_{-\infty}^{t} T(t-s)[f\left(s, B_1x(s), x(s-r)\right) - f\left(s, B_1y(s), y(s-r)\right)]ds\right\|^2 \\
& + 2E\left\| \int_{-\infty}^{t} T(t-s)[g\left(s, B_2x(s), x(s-r)\right) - g\left(s, B_2y(s), y(s-r)\right)]dW(s)\right\|^2.
\end{aligned}
$$

We first evaluate the first term of the right-hand side as follows:

$$\begin{aligned}
&2E\left\|\int_{-\infty}^{t} T(t-s)[f(s,B_1x(s),x(s-r))-f(s,B_1y(s),y(s-r))]ds\right\|^2\\
\leq\;& 2M^2E\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s,B_1x(s),x(s-r))-f(s,B_1y(s),y(s-r))\|ds\right)^2\\
\leq\;& 2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}E\|f(s,B_1x(s),x(s-r))-f(s,B_1y(s),y(s-r))\|^2ds\right)\\
\leq\;& 2M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\widetilde{L_f}(s)\Big(E\|B_1x(s)-B_1y(s)\|^2+E\|x(s-r)-y(s-r)\|^2\Big)ds\right)\\
\leq\;& 2M^2\sup_{t\in\mathbb{R}}\widetilde{L_f}(t)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)^2\sup_{t\in\mathbb{R}}\Big(E\|B_1x(t)-B_1y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big)\\
\leq\;& \frac{2M^2}{\delta^2}\sup_{t\in\mathbb{R}}\widetilde{L_f}(t)\sup_{t\in\mathbb{R}}\Big(\varpi^2E\|x(t)-y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big),
\end{aligned}$$

by using the Cauchy-Schwarz inequality.

As to the second term, by the Ito integral, we get

$$\begin{aligned}
&2E\left\|\int_{-\infty}^{t} T(t-s)[g(s,B_2x(s),x(s-r))-g(s,B_2y(s),y(s-r))]dW(s)\right\|^2\\
\leq\;& 2E\left(\int_{-\infty}^{t} \|T(t-s)[g(s,B_2x(s),x(s-r))-g(s,B_2y(s),y(s-r))]\|^2ds\right)\\
\leq\;& 2M^2\int_{-\infty}^{t} e^{-2\delta(t-s)}E\|g(s,B_2x(s),x(s-r))-g(s,B_2y(s),y(s-r))\|^2ds\\
\leq\;& 2M^2\int_{-\infty}^{t} e^{-2\delta(t-s)}\widetilde{L_g}(s)\Big(E\|B_2x(s)-B_2y(s)\|^2+E\|x(s-r)-y(s-r)\|^2\Big)ds\\
\leq\;& 2M^2\sup_{t\in\mathbb{R}}\widetilde{L_g}(t)\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\right)\sup_{t\in\mathbb{R}}\Big(\varpi^2E\|x(t)-y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big)\\
\leq\;& \frac{M^2}{\delta}\sup_{t\in\mathbb{R}}\widetilde{L_g}(t)\sup_{t\in\mathbb{R}}\Big(\varpi^2E\|x(t)-y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big).
\end{aligned}$$

Thus, by combining the above inequality together, we obtain that, for each $t\in\mathbb{R}$,

$$E\|\Lambda x(t)-\Lambda y(t)\|^2\leq M^2\left[\frac{2}{\delta^2}\sup_{t\in\mathbb{R}}\widetilde{L_f}(t)+\frac{1}{\delta}\sup_{t\in\mathbb{R}}\widetilde{L_g}(t)\right]\sup_{t\in\mathbb{R}}\Big(\varpi^2E\|x(t)-y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big),$$

that is,

$$\|\Lambda x(t)-\Lambda y(t)\|_2^2\leq L_1\sup_{t\in\mathbb{R}}\Big(\varpi^2E\|x(t)-y(t)\|^2+E\|x(t-r)-y(t-r)\|^2\Big). \tag{3.7}$$

Note that

$$\sup_{t\in\mathbb{R}}\|x(t)-y(t)\|_2^2\leq\left(\sup_{t\in\mathbb{R}}\|x(t)-y(t)\|_2\right)^2, \tag{3.8}$$

and (3.7) together with (3.8) gives, for each $t \in \mathbb{R}$,

$$\|\Lambda x(t) - \Lambda y(t)\|_2 \leq \sqrt{L_1(\varpi^2+1)}\|x-y\|_\infty.$$

Hence, we obtain

$$\|\Lambda x - \Lambda y\|_\infty = \sup_{t\in\mathbb{R}} \|\Lambda x(t) - \Lambda y(t)\|_2 \leq \sqrt{L_1(\varpi^2+1)}\|x-y\|_\infty,$$

which implies that Λ is a contraction by (3.4). So by the Banach contraction principle, we conclude that there exists a unique fixed point $x(\cdot)$ for Λ in $AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$, such that $\Lambda x = x$. Moreover, using the same argument as in [15, Theorem 3.1], we can see that $x(t)$ is the unique square-mean almost automorphic mild solution to Eq. (1.2). The proof is complete. □

(H10) The function $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ satisfies the following conditions:

(i) f is square-mean almost automorphic and $f(t,\cdot,\cdot)$ is uniformly continuous in every bounded subset $K \subset L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$.

(ii) There exist an integrable function $\overline{m_f} : \mathbb{R} \to [0,\infty)$ and a continuous nondecreasing function $\overline{W_f} : [0,\infty) \to (0,\infty)$ such that

$$E\|f(t,\varphi,\psi)\|^2 \leq \overline{m_f}(t)\overline{W_f}(E\|\varphi\|^2 + E\|\psi\|^2), \quad \text{for all} \quad (t,\varphi,\psi) \in \mathbb{R}\times K.$$

(iii) Let $\{x_n\} \subset AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{f(\cdot, B_1x_n(\cdot), x_n(\cdot - r))\}$ is relatively compact in $BC\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$.

(H11) The function $g : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ satisfies the following conditions:

(i) g is square-mean almost automorphic and $g(t,\cdot,\cdot)$ is uniformly continuous in every bounded subset $K' \subset L^2(\mathbb{P},\mathbb{H})\times L^2(\mathbb{P},\mathbb{H})$ uniformly for $t \in \mathbb{R}$.

(ii) There exist an integrable function $\overline{m_g} : \mathbb{R} \to [0,\infty)$ and a continuous nondecreasing function $\overline{W_g} : [0,\infty) \to (0,\infty)$ such that

$$E\|g(t,\varphi,\psi)\|^2 \leq \overline{m_g}(t)\overline{W_g}(E\|\varphi\|^2 + E\|\psi\|^2), \quad \text{for all} \quad (t,\varphi,\psi) \in \mathbb{R}\times K'.$$

(iii) Let $\{x_n\} \subset AA\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{g(\cdot, B_2x_n(\cdot), x_n(\cdot - r))\}$ is relatively compact in $BC\Big(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\Big)$.

Theorem 3.4. *Assume that conditions (H1), (H4), (H7) and (H10)-(H11) hold. Then Eq. (1.2) at least has a square-mean almost automorphic mild solution on* $\mathbb{R}$ *provided that*

$$\widehat{L_f} = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)}\overline{m_f}(s)ds < \infty, \quad \widehat{L_g} = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\delta(t-s)}\overline{m_g}(s)ds < \infty$$

and

$$\frac{2M^2\widehat{L_f}}{\delta} \liminf_{r\to\infty} \frac{\overline{W_f}[(\varpi^2+1)r]}{r} + 2M^2\widehat{L_g} \liminf_{r\to\infty} \frac{\overline{W_g}[(\varpi^2+1)r]}{r} < 1.$$

Proof. Let the operator Λ be defined the same as in Theorem 3.3. Then, from Lemmas 2.7-2.8 and Theorem 2.10, we can see that both $F(\cdot) = f(\cdot, B_1x(\cdot), x(\cdot - r))$ and $G(\cdot) = g(\cdot, B_2x(\cdot), x(\cdot - r)) \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ whenever $x \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$. Moreover, by using the same arguments of Lemmas 3.1 and 3.2 in [14], one can easily see that Λ is continuous and maps $AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ into itself. So Λ is well defined and continuous.

Next, by an argument similar to the proof of Theorem 3.1 in [14], we can obtain that the following statements are true.

I. Let $B_r = \{x \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right) : \|x\|_\infty \le r\}$ for each $r > 0$. Then there exists a number r such that $\Lambda(B_r) \subseteq B_r$.

II. The operator Λ is completely continuous on B_r, that is, the following statements hold.

(i) $V(t) = \{\Lambda x(t) : x \in B_r\}$ is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$ for each $t \in \mathbb{R}$.

(ii) $\{\Lambda x : x \in B_r\} \subset AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ is a family of equicontinuous functions.

Now we denote the closed convex hull of ΛB_r by $\overline{conv}\Lambda B_r$. Since $\Lambda B_r \subset B_r$ and B_r is closed convex, $\overline{conv}\Lambda B_r \subset B_r$. Thus, $\Lambda(\overline{conv}\Lambda B_r) \subset \Lambda B_r \subset \overline{conv}\Lambda B_r$. This implies that Λ is a continuous mapping from $\overline{conv}\Lambda B_r$ to $\overline{conv}\Lambda B_r$. It is easy to verify that $\overline{conv}\Lambda B_r$ has the properties (i) and (ii). More explicitly, $x(t) : x \in \overline{conv}\Lambda B_r$ is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$ for each $t \in \mathbb{R}$, and $\overline{conv}\Lambda B_r \subset BC\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ is uniformly bounded and equicontinuous. By the Ascoli-Arzelà theorem, the restriction of $\overline{conv}\Lambda B_r$ to every compact subset K'' of $\mathbb{R}$, namely $\{x(t) : x \in \overline{conv}\Lambda B_r\}_{x \in K''}$ is relatively compact in $C\left(K''; L^2(\mathbb{P}, \mathbb{H})\right)$. Thus, by the conditions (H10)(iii) and (H11)(iii) we deduce that $\Lambda : \overline{conv}\Lambda B_r \to \overline{conv}\Lambda B_r$ is a compact operator. So by Schauder's fixed point theorem, we conclude that there is a fixed point $x(\cdot)$ for Λ in $\overline{conv}\Lambda B_r$. That is Eq. (1.2) has at least one square-mean almost automorphic mild solutions $x \in B_r$. This completes the proof. □

4 Applications

In this section, we provide an example to illustrate the practical usefulness of our main results established in the preceding section. We consider the following one-dimensional and semilinear stochastic partial functional differential equations with the Dirichlet boundary conditions:

$$\begin{aligned} du(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x)dt + \sin\frac{1}{2+\cos t + \cos\sqrt{2}t}[\gamma u(t,x) + \sin(u(t,x(t-\tau)))]dt \\ &\quad + \sin\frac{1}{2+\cos t+\cos\sqrt{3}t}[\eta u(t,x) + \sin(u(t,x(t-\tau)))]dW(t), \quad (4.1) \\ u(t,0) &= u(t,1) = 0, \quad t \in \mathbb{R}, \quad (4.2) \end{aligned}$$

where $(t,x) \in \mathbb{R} \times [0,1], \tau \ge 0$ is a fixed constant, $\gamma, \beta > 0$, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$.

Set $\mathbb{X} = L^2\left(\mathbb{P}, L^2[0,1]\right)$ and define

$$D(A) := \{u \in C^1[0,1] : u''(\xi) \in \mathbb{X}, u'(\xi) \in \mathbb{X} \text{ is absolutely continuous on } [0,1], u(0) = u(1) = 0\},$$

$$Au = \Delta u = u'', \quad \forall u \in D(A).$$

Then, A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$ given by

$$(T(t)x)(\xi) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x, e_n\rangle_{L^2} e_n(\xi),$$

where $e_n(\xi) = \sqrt{2}\sin(n\pi\xi)$ for $n = 1, 2, \cdots$. Moreover, $\|T(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$.

It is easy to see that $\{T(t)\}_{t\geq 0}$ satisfies (H1) with $M = 1$ and $\delta = \pi^2$. Let $B_1 = \gamma I_d$ and $B_2 = \eta I_d$. Under the previous conditions, we can define the functions $f, g : \mathbb{R}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$ by

$$f(t, B_1\varphi, \varphi_t)(x) = \sin\frac{1}{2+\cos t+\cos\sqrt{2}t}[\gamma\varphi(x) + \sin(\varphi(x_t))],$$

$$g(t, B_2\varphi, \varphi_t)(x) = \sin\frac{1}{2+\cos t+\cos\sqrt{3}t}[\eta\varphi(x) + \sin(\varphi(x_t))],$$

which permits to transform the system (4.1)-(4.2) into the abstract system (1.2). Clearly, both f and $g \in AA(\mathbb{R}\times\mathbb{X}\times\mathbb{X}, \mathbb{X})$, and satisfy the Lipschitz conditions with $L_f = 2$ and $L_g = 4$, respectively. Hence, choosing γ and η such that

$$\sqrt{\frac{4}{\pi^4}(\gamma^2+1) + \frac{4}{\pi^2}(\eta^2+1)} < 1$$

assumption of Theorem 3.2 is satisfied and the problem (4.1)-(4.2) has a unique mild solution in $AA(\mathbb{X})$.

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