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A NOTE ON THE SPINNED POISSON DISTRIBUTION

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Abstract

Recently a new probability distribution, the *spinned Poisson distribution* (SPD), was introduced to model data collected in various health management situations, such as the spread of an infectious disease when infected cases are removed from the observed population. As the name suggests, the SPD is a generalization of the well-known Poisson distribution. In this paper, simple data sets are used to show that there is no guarantee that the equations for the maximum likelihood estimates of the parameters associated with the SPD have a non-trivial solution. It follows that even for a simple data set where a non-trivial SPD may seem appropriate, the fitted SPD may be just a Poisson distribution.

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1 Introduction

In a recent paper, Shanmugam [3] introduced a new probability distribution, the *spinned Poisson distribution* (SPD), to model the spread of an infectious disease when infected cases are removed (e.g. by health management authorities) during the data collection process. For parameters $\rho \geq 0$ and $\theta > 0$, the SPD is defined by

$$p(y|\rho,\theta) = \Pr[Y = y] = \frac{1+\rho y}{1+\rho\theta}\,\frac{e^{-\theta}\theta^{y}}{y!} \qquad (y = 0,1,2,\ldots). \tag{1.1}$$

Observe that $p(y|0,\theta) = e^{-\theta}\theta^{y}/y!$, so the SPD is a generalization of the well-known Poisson distribution. Also, if $\alpha = 1/(1+\rho\theta)$, then

$$p(y|\rho,\theta) = \frac{1}{1+\rho\theta}\,\frac{e^{-\theta}\theta^{y}}{y!} + \frac{\rho\theta}{1+\rho\theta}\,\frac{e^{-\theta}\theta^{y-1}}{(y-1)!} = \alpha\, p(y|0,\theta) + (1-\alpha)\,p(y-1|0,\theta), \tag{1.2}$$

so the spinned Poisson probability $\Pr[Y = y]$ can be regarded as a convex combination of the Poisson probabilities $\Pr[Y = y]$ and $\Pr[Y = y-1]$.

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Given a random sample $y_1, y_2, \ldots, y_n$ with mean $\bar{y}$ from the SPD (1.1), the parameters ρ and θ can be estimated by using the maximum likelihood estimates (MLEs) $\hat{\rho}$ and $\hat{\theta}$. These are the values of ρ and θ that maximize the likelihood function

$$L = \prod_{i=1}^{n} p(y_i|\rho,\theta) = \frac{e^{-n\theta}\theta^{n\bar{y}}}{(1+\rho\theta)^n} \prod_{i=1}^{n} \frac{1+\rho y_i}{y_i!} \tag{1.3}$$

(or, equivalently, maximize the logarithm of L) for $\rho \geq 0$ and $\theta > 0$. To avoid trivial cases, assume the y_i are not all equal. Also assume (after relabelling the data if necessary) that $y_1, \ldots, y_k$ are positive and $y_{k+1} = \ldots = y_n = 0$ (where $1 \leq k \leq n$).

To find $\hat{\rho}$ and $\hat{\theta}$, firstly consider the values of L on the boundary of the parameter space. Here it is helpful (especially for the case $\rho \to \infty$) to use (1.2) to write L in terms of α and θ:

$$L = L(\alpha,\theta) = e^{-n\theta}\theta^{n(\bar{y}-1)} \prod_{i=1}^{n} \frac{\alpha\theta + (1-\alpha)y_i}{y_i!} = e^{-n\theta}\theta^{n\bar{y}-k}\alpha^{n-k} \prod_{i=1}^{k} \frac{\alpha\theta + (1-\alpha)y_i}{y_i!}. \tag{1.4}$$

Now look at L on each component of the boundary of the region $0 \leq \alpha \leq 1$, $\theta \geq 0$.

(i). If $\alpha = 0$ (which corresponds to $\rho = \infty$), L is 0 if any of the y_i are 0. Otherwise (i.e. if $k = n$), L is maximized if $\theta = \bar{y} - 1$.

(ii). If $\alpha = 1$ (i.e. $\rho = 0$), L is maximized if $\theta = \bar{y}$.

(iii). When $\theta = 0$, $L = 0$ unless $\bar{y} = k/n$ (i.e. $y_1 = \ldots = y_k = 1$), in which case $L = \alpha^{n-k}(1-\alpha)^k$ is maximized if $\alpha = 1 - k/n = 1 - \bar{y}$.

(iv). As $\theta \to \infty$, $L \to 0$ uniformly in α.

Next consider possible local maxima of L in the interior $0 < \rho, \theta < \infty$ of the parameter space. These occur when $\partial \ln L/\partial\rho = \partial \ln L/\partial\theta = 0$, which by (1.3) leads to the equations (as stated in equivalent form by Shanmugam)

$$\bar{y} = \theta + \frac{\rho\theta}{1+\rho\theta} \tag{1.5}$$

and

$$\sum_{i=1}^{n} \frac{1}{1+\rho y_i} = \frac{n}{1+\rho\theta}. \tag{1.6}$$

As pointed out in [3], it can be tedious to find exact solutions to these equations. However, an elementary observation from (1.5) that will be useful later is:

$$\text{Any solution to (1.5) and (1.6) in } 0 < \rho, \theta < \infty \text{ must satisfy } \theta < \bar{y}. \tag{1.7}$$

The principal aim of this note is to demonstrate that even for simple data sets, there is no guarantee that the system (1.5) and (1.6) has a solution for $0 < \rho, \theta < \infty$. This aim will be achieved in Sections 2 and 3, where it is shown that the fitted SPD based on MLEs $\hat{\rho}$ and $\hat{\theta}$ may be simply one of the three 'boundary distributions' $p(y-1|0,\bar{y}-1)$, $p(y|0,\bar{y})$ and $(1-\bar{y})p(y) + \bar{y}p(y-1)$, corresponding to cases (i)–(iii) above, where $p(y)$ is the limiting Poisson distribution as $\theta \to 0$,

$$p(y) = \Pr[Y = y] = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2 Some Simple Examples

Typically the Poisson distribution is used to model count data such as the numbers of new infections of a particular disease occurring in a population in a fixed time interval, although Shanmugam illustrated the SPD with counts of the time intervals between successive infections being detected. The following two examples illustrate the modelling of some (hypothetical) elementary sample count data by a SPD.

Example 1: Suppose a sample contains n observations, with a count of $r \geq 1$ being recorded on k of these occasions (where $1 \leq k \leq n-1$) and a count of 0 recorded on the remaining $n-k$ occasions (so $\bar{y} = kr/n$). Then (1.6) becomes

$$(n-k) + \frac{k}{1+\rho r} = \frac{n}{1+\rho\theta},$$

which yields $\rho = 0$ or $\rho\theta = (kr - n\theta)/[r(n-k)]$. Substituting the result for $\rho\theta$ into (1.5) gives $\theta = kr/n$ (so $\rho = 0$) or $\theta = r-1$. Thus if $r = 1$, there is no solution to the system (1.5) and (1.6) for $0 < \rho, \theta < \infty$, while for $r > 1$, the condition $\theta = r-1$ gives $\rho = \rho^*$ where $\rho^* = (kr - nr + n)/[r(r-1)(n-k)]$ (which is positive for $r < n/(n-k)$) and $\alpha = r - rk/n$.

Case (a): If $r = 1$ it follows from the above comments and the remarks in Section 1 about the behaviour of L on the boundary of the parameter space that the fitted SPD is whichever of $p(y|0,\bar{y})$ and $(1-\bar{y})p(y) + \bar{y}p(y-1)$ has the greater likelihood. Now, the associated likelihood values are $L_1 = e^{-k}(k/n)^k$ and $L_2 = (1-k/n)^{n-k}(k/n)^k$, respectively, with ratio of

$$\frac{L_1}{L_2} = f_1(k) = e^{-k}(1-k/n)^{k-n}.$$

It is readily checked that $g_1(k) = \ln f_1(k)$ satisfies $g_1(0) = 0$ and $g_1'(k) = \ln(1-k/n) < 0$ for $0 < k < n$, so $g_1(k) < 0$ and $f_1(k) < 1$ for $1 \leq k \leq n-1$, which means $L_1 < L_2$.

Hence for $r = 1$ in this example, the fitted SPD is $(1-\bar{y})p(y) + \bar{y}p(y-1)$.

Case (b): If $r > 1$ it follows from the comments in Section 1 that the fitted SPD is just the Poisson distribution $p(y|0, kr/n)$ except perhaps if $r < n/(n-k)$. For $r < n/(n-k)$, use (1.4) to calculate the ratio of likelihood values

$$\frac{L(r - rk/n, r-1)}{L(1, kr/n)} = \frac{(r-1)^{k(r-1)}(n-k)^{n-k}n^{kr-n}}{r^{kr-n}k^{k(r-1)}}\, e^{kr-n(r-1)}.$$

If $x = kr - n(r-1)$, so $r = (n-x)/(n-k)$, this can be written as

$$\frac{L(r - rk/n, r-1)}{L(1, kr/n)} = f_2(x) = \left(\frac{n(k-x)}{k(n-x)}\right)^{\frac{k(k-x)}{n-k}} \left(\frac{n-x}{n}\right)^{n-k} e^x,$$

where $0 < x < k < n$. If $g_2(x) = \ln f_2(x)$, it is straightforward to show $g_2(0) = g_2'(0) = 0$ and $g_2''(x) = x(n-k)/[(k-x)(n-x)^2]$. Thus $g_2''(x)$, $g_2'(x)$ and $g_2(x)$ are all positive for $0 < x < k$, which means $f_2(x) > 1$, and so $L(r - rk/n, r-1) > L(1, kr/n)$.

Hence for $r > 1$ in this example, the fitted SPD is $p(y|\rho^*, r-1)$ if $r < n/(n-k)$, and otherwise is the Poisson distribution $p(y|0, kr/n)$. □

Example 2: Suppose the sample count data contains n observations, with 1 recorded on $a > 0$ occasions, 2 recorded on $b > 0$ occasions, and 0 recorded on the remaining $n-a-b$ occasions (so $\bar{y} = (a+2b)/n$).

For this data, (1.6) becomes

$$(n-a-b) + \frac{a}{1+\rho} + \frac{b}{1+2\rho} = \frac{n}{1+\rho\theta},$$

which after rearrangement gives either $\rho = 0$ or

$$2\theta(n-a-b)\rho^2 + [(3n-a-2b)\theta - 2a - 2b]\rho + (n\theta - a - 2b) = 0. \tag{2.1}$$

Also (1.5) gives

$$n\theta - a - 2b = \rho\theta(a+2b-n-n\theta), \tag{2.2}$$

which when substituted into (2.1) yields either $\rho = 0$ or

$$2(n-a-b)\rho\theta = n\theta^2 - 2n\theta + 2a + 2b. \tag{2.3}$$

Now, if $a+b = n$, (2.3) has no real solution for θ, which means the the MLEs correspond to the greater of the likelihood values $L(1, 1+b/n)$ and $L(0, b/n)$. By (1.4), the ratio of these likelihood values is

$$f_3(b) = \frac{L(1,1+b/n)}{L(0,b/n)} = e^{-n}\left(1+\frac{b}{n}\right)^{n+b}\left(\frac{n}{2b}\right)^b.$$

If $g_3(b) = \ln f_3(b)$, it is straightforward to show $g_3(n) = n\ln(2/e) < 0$ and $g_3'(b) = \ln(n+b) - \ln(2b) \geq 0$ for $1 \leq b \leq n$. Thus $g_3(b) < 0$ for $1 \leq b \leq n-1$, which means $f_3(b) < 1$, and so $L(1, 1+b/n) < L(0, b/n)$. Hence if $a+b = n$, the fitted SPD is $p(y-1|0, b/n)$.

If all of a, b and $n-a-b$ are positive, then substituting the result for $\rho\theta$ from (2.3) into (2.2) gives

$$n\theta^3 - (n+a+2b)\theta^2 + 2(a+2b)\theta - 2b = 0. \tag{2.4}$$

If, for example, $a = 0.3n$ and $b = 0.2n$ (so $\bar{y} = 0.7$), the unique real solution to (2.4) is $\theta = 0.5$, so $\rho = 0.5$ and $\alpha = 0.8$. Straightforward numerical calculation using (1.4) shows that $L(0.8, 0.5)$ exceeds the maximum boundary value $L(1, 0.7)$, so the fitted SPD is $p(y|0.5, 0.5)$. On the other hand, if $a = 0.2n$ and $b = 0.2n$, then the only real solution to (2.4) is $\theta \simeq 0.702319 > \bar{y}$. By (1.7), L has no local maximum in the interior of the parameter space, and so in this case the fitted SPD is the Poisson distribution $p(y|0, 0.6)$. □

Observe that Examples 1 and 2 provide explicit simple data sets for each of the four possible locations in the parameter space (i.e. at an interior point, or on one of the boundary components $\alpha = 0$, $\alpha = 1$ and $\theta = 0$) for the parameters that correspond to the fitted SPD.

3 Fitting an SPD to Infection Count Data

In his paper [3] in which the SPD was introduced, Shanmugam modelled a classic data set from a smallpox outbreak in a closed community of about 120 members of the Faith Tabernacle religious group in Abakaliki, Nigeria in 1967. This data has been reported by

Bailey [1] and O'Neill and Becker [2], amongst others. However, rather than take Shanmugam's approach and use the SPD to model counts of the time intervals between successive infections being detected, we will fit an SPD to counts of the numbers of new smallpox infections per day. Often a Poisson distribution is fitted to data of this type, but for the Abakaliki data, "the usual Poisson distribution is inappropriate to use because of the impact of removing infected cases [by isolation at the Infectious Disease Hospital] on the incidence rate and the chance of observing a new case" ([3, p. 299]). Based on the inter-removal times reported in [2, p. 101], the data for a period of 77 days were:

No of new infections	0	1	2	3
No of days	54	17	5	1

For this data the maximum likelihood equations (1.5) and (1.6) are

$$\frac{30}{77} = \theta + \frac{\rho\theta}{1+\rho\theta} \tag{3.1}$$

and

$$54 + \frac{17}{1+\rho} + \frac{5}{1+2\rho} + \frac{1}{1+3\rho} = \frac{77}{1+\rho\theta}. \tag{3.2}$$

In fact, there are no solutions to the system (3.1) and (3.2) for $0 < \rho, \theta < \infty$ that give a higher value of the likelihood function than at $\rho = 0$ and $\theta = 30/77$. This can be demonstrated in several ways, but the following approach has the advantage that it uses the results of Example 2, rather than working directly with the more complicated (3.1) and (3.2), and can be adapted readily to certain other data sets.

By (1.4) the likelihood function for the Abakaliki data is

$$L(\alpha,\theta) = e^{-77\theta}\theta^7\alpha^{54}\left(\alpha\theta+(1-\alpha)\right)^{17}\left(\frac{\alpha\theta+2(1-\alpha)}{2}\right)^5\left(\frac{\alpha\theta+3(1-\alpha)}{3!}\right). \tag{3.3}$$

Since $\alpha\theta(\alpha\theta+3(1-\alpha)) \le (\alpha\theta+(1-\alpha))(\alpha\theta+2(1-\alpha))$, then

$$L(\alpha,\theta) \le \frac{1}{3}e^{-77\theta}\theta^6\alpha^{53}\left(\alpha\theta+(1-\alpha)\right)^{18}\left(\frac{\alpha\theta+2(1-\alpha)}{2}\right)^6 = \frac{1}{3}L^*(\alpha,\theta), \tag{3.4}$$

where $L^*(\alpha,\theta)$ is the likelihood function for the "modified Abakaliki data":

No of new infections	0	1	2
No of days	53	18	6

To fit an SPD to this modified data, consider its MLE equation (2.4) which is $77\theta^3 - 107\theta^2 + 60\theta - 12 = 0$. The only real solution to this equation is $\theta \simeq 0.417249 > \bar{y}$. By (1.7), the likelihood function for the modified data has no local maximum in the interior of the parameter space, and so is maximized when $\alpha = 1$ and $\theta = 30/77$. Hence if $0 \le \alpha \le 1$ and $\theta \ge 0$, then by (3.4) and (3.3),

$$L(\alpha,\theta) \le \frac{1}{3}L^*(\alpha,\theta) \le \frac{1}{3}L^*(1,30/77) = L(1,30/77).$$

Thus the likelihood function for the original Abakaliki data is maximized when $\alpha = 1$ and $\theta = 30/77$, so the fitted SPD for this data is the Poisson distribution $p(y|0, 30/77)$.

This example in Section 3 shows that for a real-life data set, it is still possible that the fitted SPD is just a Poisson distrbution, even if initial considerations suggest that the additional generalizations afforded by the SPD model will yield a better fit to the data than the Poisson distribution.

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References

[1] N. T. J. Bailey, *The Mathematical Theory of Infectious Diseases and Its Applications*, 2nd edn, Griffin, London 1975.

[2] P. D. O'Neill and N. G. Becker, Inference for an epidemic when susceptibility varies. *Biostatistics* **2** (2001), 99-108.

[3] R. Shanmugam, Spinned Poisson distribution with health management application. *Health Care Management Science* **14** (2011), 299-306.