African Diaspora Journal of Mathematics
Volume 16, Number 1, pp. 28–36 (2013)

ISSN 1539-854X
www.math-res-pub.org/adjm

Brackets in the Free Loop Space Homology of Some Homogeneous Spaces.

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Abstract

Let X be a simply connected homogeneous space of which $\pi_*(X)\otimes\mathbb{Q}$ is finite dimensional. We consider the homology of the free loop space $\mathrm{map}(S^1,X)$ with the bracket defined by Chas and Sullivan. We show that the Lie algebra $s\mathbb{H}_*(\mathrm{map}(S^1,X),\mathbb{Q})$ is not nilpotent.

AMS Subject Classification: Primary 55P62; Secondary 55M35.

Keywords: Derivations, Hochschild cohomology, Free loop space homology.

1 Introduction

In this paper we study the Lie bracket in the homology of the free loop space of a homogeneous space. We make extensive use of the theory of Sullivan algebras of which details can be found in [2, 12, 13].

Let (A,d) be a commutative cochain algebra over a field $\mathbb{k}$. A derivation θ of degree i is a linear mapping $A^n \to A^{n-i}$ such that $\theta(ab) = \theta(a)b + (-1)^{i|a|}a\theta(b)$. Denote by $\mathrm{Der}_i A$ the vector space of all derivations of degree i and let $\mathrm{Der}A = \oplus_{i\in\mathbb{Z}} \mathrm{Der}_i A$. With the commutator bracket $\mathrm{Der}A$ becomes a graded Lie algebra. There is a differential $\delta : \mathrm{Der}_i A \to \mathrm{Der}_{i-1} A$ defined by $\delta\theta = [d,\theta]$. Hence $(\mathrm{Der}A,\delta)$ is a differential graded Lie algebra. Using the grading convention $A^n = A_{-n}$, we may view a derivation of degree i as increasing the lower degree by i.

Moreover $\mathrm{Der}A$ is a differential graded A-module with the action $(a\theta)(x) = a\theta(x)$. If $A = (\wedge V,d)$ is a Sullivan algebra of which V is finite dimensional, we show that $\mathrm{Der}A \cong A\otimes V^\#$, where $V^\#$ is the graded dual of V (Lemma 4.1). With the above grading convention $V^\# = \oplus_{i\geq 1}(V^\#)_i$ is positively graded.

On $s^{-1}\mathrm{Der}A$, we define a bracket of degree 1 by $\{\alpha,\beta\} = s^{-1}[s\alpha, s\beta]$ and a differential $\delta'(\alpha) = -s^{-1}\delta(s\alpha) = -\{d',\alpha\}$, where $d' = s^{-1}d$ is of degree -2.
Let $\bar{A}$ be the kernel of the augmentation $\epsilon : A \to \mathbb{k}$. We denote by $C^*(A;A) = \mathrm{Hom}(T(s\bar{A}),A)$

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(resp. $HH^*(A;A)$) the Hochschild complex (resp. cohomology) of the cochain algebra A with coefficients in A [9]. Moreover

$$HH^*(A;A) \cong \mathrm{Ext}_{A\otimes A}(A,A),$$

where A is considered as an $A\otimes A$-module by the action $(a\otimes b)c = abc$. Therefore, in order to compute the Hochschild cohomology of a commutative differential graded algebra A, it is sufficient to find a free resolution of A as an $A\otimes A$-module. In particular, for the minimal Sullivan algebra $(\wedge V,d)$, one can consider a relative Sullivan model of the multiplication $m : \wedge V\otimes\wedge V \to \wedge V$. Such a model is given by

$$\begin{array}{ccc} (\wedge V,d)\otimes(\wedge V,d) & \xrightarrow{m} & (\wedge V,d) \\ \downarrow & & \| \\ (\wedge V\otimes\wedge V\otimes\wedge\bar{V},D) & \xrightarrow{\simeq} & (\wedge V,d), \end{array}$$

where $\bar{V}^n = V^{n+1}$, $D(1\otimes 1\otimes\bar{v}) = v\otimes 1\otimes 1 - 1\otimes v\otimes 1 + \alpha$ with $\alpha\in(\wedge V\otimes\wedge V)^{>0}\otimes\bar{V}$ [2]. Therefore

$$HH(\wedge V;\wedge V) \cong H_*(\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge\bar{V},\wedge V),D)$$

Define

$$\psi : (s^{-1}\,\mathrm{Der}\wedge V,\delta') \to (\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge\bar{V},\wedge V),D)$$

by

$$\psi(s^{-1}\theta)(\bar{v}) = (-1)^{|\theta|}\theta(v), \quad \psi(s^{-1}\theta)(\wedge^{\geq 2}\bar{V}) = \psi(s^{-1}\theta)(1\otimes 1\otimes 1) = 0. \tag{1.1}$$

Then $\psi(s^{-1}\theta)$ is extended to $\wedge V\otimes\wedge V\otimes\wedge\bar{V}$ as a morphism of $\wedge V\otimes\wedge V$-modules. Moreover ψ commutes with differentials.

Our main result states.

Theorem 1.1. *The inclusion*

$$\psi : (s^{-1}\,\mathrm{Der}\wedge V,\delta') \hookrightarrow (\mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge\bar{V},\wedge V),D)$$

induces an injective graded Lie algebra morphism

$$H_*(s^{-1}\,\mathrm{Der}\wedge V,\delta') \hookrightarrow HH(\wedge V;\wedge V).$$

We do not know if the result holds for any graded commutative differential algebra (A,d) as stated [8, Theorem 1] as some gaps in the proof were later found.

Let X be a closed oriented manifold of dimension m and $LX = \mathrm{map}(S^1,X)$ the space of free loops on X. The loop homology of X is the homology of LX with a shift of degrees, that is, $\mathbb{H}_*(LX) = H_{*+m}(LX)$. In [1], Chas and Sullivan define a product, called loop product and a Lie bracket (called loop bracket) on $\mathbb{H}_*(LX)$ turning $\mathbb{H}_*(LX)$ into a Gerstenhaber algebra. We use the above result to show that the free loop space homology of a homogeneous space contains Gerstenhaber brackets of arbitrary length.

Theorem 1.2. *Let X be a 1-connected homogeneous space of which $\pi_*(X)\otimes\mathbb{Q}$ is finite dimensional, then the graded Lie algebra $s\mathbb{H}_*(LX,\mathbb{Q})$ is not nilpotent.*

2 Hochschild cohomology

We define here the Hochschild cohomology through the bar construction of an augmented differential graded algebra (A,d), not necessarily commutative. The bar construction $\mathbb{B}(A;A;A)$ provides a free resolution of A as an $A\otimes A^{op}$-module. It is defined by

$$\mathbb{B}_k(A;A;A) = A\otimes T^k(s\bar{A})\otimes A.$$

An element $a[a_1|a_2|\cdots a_k]b \in A\otimes T^k(s\bar{A})\otimes A$ is of degree $|a|+|b|+\sum_{i=1}^k |sa_i|$. The differential $d = d_0 + d_1$ is defined as follows (see for instance [3]).

$$d_0 : \mathbb{B}_k(A;A;A) \to \mathbb{B}_k(A;A;A), \quad d_1 : \mathbb{B}_k(A;A;A) \to \mathbb{B}_{k-1}(A;A;A),$$

$$\begin{aligned} d_0(a[a_1|a_2|\cdots a_k]b) &= (da)[a_1|a_2|\cdots a_k]b - \textstyle\sum_{i=1}^k (-1)^{\epsilon(i)} a[a_1|\cdots|da_i|\cdots|a_k]b \\ &\quad +(-1)^{\epsilon(k+1)} a[a_1|a_2|\cdots a_k](db), \end{aligned}$$

$$\begin{aligned} d_1(a[a_1|a_2|\cdots a_k]b) = \; & (aa_1)[a_2|\cdots a_k]b - \textstyle\sum_{i=2}^k (-1)^{\epsilon(i)} a[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]b \\ & -(-1)^{\epsilon(k)} a[a_1|a_2|\cdots a_{k-1}](a_k b), \end{aligned}$$

where $\epsilon(i) = |a| + \sum_{j=1}^{i-1} |sa_j|$. Therefore the Hochschild cochain complex is given by

$$(C^*(A;A),D) = \mathrm{Hom}_{A\otimes A^{op}}(\mathbb{B}(A;A;A),A) \cong (\mathrm{Hom}(T(s\bar{A}),A), D_0 + D_1),$$

where the differential is expressed as follows [7].

$$\begin{aligned} (D_0 f)([a_1|a_2|\ldots|a_k]) = \; & d(f([a_1|a_2|\ldots|a_k])) \\ & + \textstyle\sum_{i=1}^k (-1)^{\bar{\epsilon}(i)} f([a_1|\ldots|da_i|\ldots|a_k]) \end{aligned}$$

and

$$\begin{aligned} (D_1 f)([a_1|a_2|\ldots|a_k]) = \; & -(-1)^{|sa_1||f|} a_1 f([a_2|\ldots|a_k]) \\ & +(-1)^{\bar{\epsilon}(k)} f([a_1|\ldots|a_{k-1}])a_k \\ & + \textstyle\sum_{i=2}^k (-1)^{\bar{\epsilon}(i)} f([a_1|\ldots|a_{i-1}a_i|\ldots|a_k]), \end{aligned}$$

where $\bar{\epsilon}(i) = |f| + |sa_1| + \cdots + |sa_{i-1}|$.

Moreover, there is a bracket on $C^*(A;A)$, inducing a Gerstenhaber algebra structure on $HH^*(A,A)$ [9]. The Lie bracket is defined by the formula

$$\{f,g\} = f\bar{\circ}g - (-1)^{(|f|+1)(|g|+1)} g\bar{\circ}f, \tag{2.1}$$

where

$$(f\bar{\circ}g)([a_1|a_2|\ldots|a_k]) = \sum_{0\le i\le j\le k} (-1)^{\eta(i)} f([a_1|\ldots|a_i|g([a_{i+1}|\ldots|a_j])|a_{j+1}\ldots|a_k]),$$

and $\eta(i) = |g|(|sa_1| + \cdots |sa_i|)$. If $f \in C^p(A;A)$ and $g \in C^q(A;A)$, then $\{f,g\} \in C^{p+q-1}(A;A)$. As $C^1(A;A)$ is closed under this bracket, $sHH^1(A;A)$ is a sub Lie algebra of $sHH^*(A;A)$. The differential $d : A \to A$ corresponds to an element $\tilde{d} \in C^1(A;A)$ of total degree -2 defined by $\tilde{d}([a]) = -da$. It is easily verified that $D_0 f = -\{\tilde{d}, f\}$. Moreover, if $\mu \in C^2(A;A)$ is defined by $\mu([a|b]) = ab$, then $D_1 f = -\{\mu, f\}$ [11].

Define

$$F_1C^1(A;A) = \{f \in C^*(A;A) | f(T^{>1}(sA)) = 0\}.$$

Consider the composition mapping

$$\varphi : s^{-1}\operatorname{Der} A \hookrightarrow F_1C^1(A;A) \xrightarrow{p} C^1(A;A) \subset C^*(A;A),$$

where p is the canonical projection.

Lemma 2.1. *The inclusion $\varphi : s^{-1}\operatorname{Der} A \to C^*(A;A)$ respects the brackets.*

Proof. Note that if $\theta \in \operatorname{Der} A$, then $(\varphi(s^{-1}\theta))([a]) = (-1)^{|\theta|}\theta(a)$. Given $\theta_1, \theta_2 \in \operatorname{Der} A$, it is easily checked that

$$\varphi(\{s^{-1}\theta_1, s^{-1}\theta_2\})([a]) = \{\varphi(s^{-1}\theta_1), \varphi(s^{-1}\theta_2)\}([a]). \quad \square$$

Lemma 2.2. *The inclusion $\varphi : (s^{-1}\operatorname{Der} A, \delta') \to (C^*(A;A), D_0 + D_1)$ commutes with differentials.*

Proof. As $\delta'(\theta) = -\{d', \theta\}$, $D_0 f = -\{\tilde{d}, f\} = -\{\varphi(d'), f\}$, therefore

$$\varphi(\{-d', \theta\}) = -\{\varphi(d'), \varphi(\theta)\} = -\{\tilde{d}, \varphi(\theta)\} = D_0(\varphi(\theta)) = (D_0 + D_1)(\varphi(\theta)),$$

as $D_1(\varphi(\theta)) = 0$, since $s\theta$ is a derivation. $\square$

3 Proof of Theorem 1.1

We recall that

$$\psi : (s^{-1}\operatorname{Der} \wedge V, \delta') \to (\operatorname{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \wedge V), D)$$

is defined

$$\psi(s^{-1}\theta)(\bar{v}) = (-1)^{|\theta|}\theta(v), \quad \psi(s^{-1}\theta)(\wedge^{\geq 2}\bar{V}) = \psi(s^{-1}\theta)(1 \otimes 1 \otimes 1) = 0.$$

Clearly ψ is injective and its range is isomorphic to $\operatorname{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \bar{V}, \wedge V)$. To show that ψ commutes with differentials, we first observe that

$$(\operatorname{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \wedge V), D) \cong (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, \wedge V), D'),$$

where the differential on $\wedge V \otimes \wedge \bar{V}$ is defined by $d\bar{v} = v - s(dv)$ and s is the derivation of $\wedge V \otimes \wedge \bar{V}$ which satisfies $s(v) = \bar{v}$ and $s(\bar{v}) = 0$ [2]. Hence we can view ψ as a map

$$\psi : (s^{-1}\operatorname{Der} \wedge V, \delta') \to (\operatorname{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, \wedge V), D').$$

Therefore

$$\begin{aligned} \psi(\delta'(s^{-1}\theta))(\bar{v}) &= \psi(-s^{-1}[d, \theta])(\bar{v}) = (-1)^{|\theta|}[d, \theta](v) \\ &= (-1)^{|\theta|}(d\theta(v) - (-1)^{|\theta|}\theta(dv)). \end{aligned}$$

Moreover

$$\begin{aligned}(D'(\psi(s^{-1}\theta))(\bar v) &= d(\psi(s^{-1}\theta)(\bar v) - (-1)^{|\theta|+1}(\psi(s^{-1}\theta))(d\bar v)\\ &= (-1)^{|\theta|}d\theta(v) - (-1)^{|\theta|+1}\psi(s^{-1}\theta)(v\otimes 1 - sdv)\\ &= (-1)^{|\theta|}d\theta(v) - (-1)^{|\theta|}\psi(s^{-1}\theta)(sdv)\\ &= (-1)^{|\theta|}(d\theta(v) - (-1)^{|\theta|}\theta(dv)).\end{aligned}$$

Hence ψ commutes with differentials.

Moreover $(\wedge V\otimes\wedge^n\bar V,d)$ is a sub complex of $(\wedge V\otimes\wedge\bar V,d)$. Hence there is a decomposition (see also [4])

$$H_*(\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge^n\bar V,\wedge V),D') = \oplus_{n\geq 0}H_*(\mathrm{Hom}_{\wedge V}(\wedge V\otimes\wedge^n\bar V,\wedge V),D').$$

Therefore ψ restricts to a differential isomorphism

$$(s^{-1}\,\mathrm{Der}\wedge V,\delta')\xrightarrow{\cong}(\mathrm{Hom}_{\wedge V}(\wedge V\otimes\bar V,\wedge V),D').$$

Hence

$$H_*(\psi): H_*(s^{-1}\,\mathrm{Der}\wedge V,\delta')\to HH(\wedge V;\wedge V)$$

is injective.

As $(\wedge V\otimes\wedge V\otimes\wedge\bar V,D)$ and $\mathbb{B}(\wedge V;\wedge V;\wedge V)$ are free resolutions of $\wedge V$ as $\wedge V\otimes\wedge V$-modules, then there is a quasi-isomorphism

$$(\wedge V\otimes\wedge V\otimes\wedge\bar V,D)\to\mathbb{B}(\wedge V;\wedge V;\wedge V).$$

An explicit quasi-isomorphism $\jmath:(\wedge V\otimes\wedge V\otimes\wedge\bar V,D)\to\mathbb{B}(\wedge V;\wedge V;\wedge V)$ is defined as follows. If $dv=0$ then $\jmath(\bar v)=1\otimes[v]\otimes 1$. Otherwise $\jmath(\bar v)=1\otimes[v]\otimes 1+\alpha$, $\alpha\in 1\otimes T^{\geq 2}(s(\wedge^+V))\otimes 1$. One extends $\jmath$ to $\wedge^{\geq 2}\bar V$ by

$$\jmath(\bar v_1\wedge\ldots\wedge\bar v_n)=\frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma)[\jmath(v_{\sigma(1)})|\ldots|\jmath(v_{\sigma(n)})],$$

where $v_i\in V$. As the following diagram commutes,

$$\begin{array}{ccc}(\wedge V\otimes\wedge V\otimes\wedge\bar V,D) & \xrightarrow{\ \jmath\ } & \mathbb{B}(\wedge V;\wedge V;\wedge V)\\ \downarrow\simeq & & \simeq\downarrow\\ (\wedge V,d) & = & (\wedge V,d)\end{array}$$

we deduce that $\jmath$ is quasi-isomorphism.

We consider the following commutative diagram.

$$\begin{array}{ccc}(s^{-1}\,\mathrm{Der}\wedge V,\delta') & \xrightarrow{\ \psi\ } & \mathrm{Hom}_{\wedge V\otimes\wedge V}(\wedge V\otimes\wedge V\otimes\wedge\bar V,\wedge V)\\ \downarrow\varphi & & \simeq\uparrow\mathrm{Hom}(j)\\ C^*(\wedge V;\wedge V) & = & C^*(\wedge V;\wedge V)\end{array}$$

As $H_*(\psi)$ is injective and $H_*(\mathrm{Hom}(j))$ is an isomorphism, we conclude that $H_*(\varphi)$ is injective.

4 Spectral sequence for an n-stage Postnikov tower

We first show the following Lemma.

Lemma 4.1. *Let $\{v_1, v_2, \dots, v_n\}$ be a homogeneous linear basis of V and, for $1 \le i \le n$, let θ_i be the derivation of $\wedge V$ uniquely determined by*

$$\theta_i(v_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The graded $\wedge V$-module $\mathrm{Der}\wedge V$ *is freely generated by the derivations θ_i $(1 \le i \le n)$.*

Proof. We denote by $V^\#$ the graded dual of V. By restriction to V, we have isomorphisms of graded $\wedge V$-modules

$$\mathrm{Der}\wedge V \cong \mathrm{Hom}(V, \wedge V) \cong (\wedge V) \otimes V^\#. \quad \square$$

If X is an n-stage Postnikov tower, then X admits a Sullivan algebra of the form $(\wedge(V_1 \oplus \cdots \oplus V_n), d)$, where $dV_1 = 0$ and $dV_i \subset \wedge(V_1 \oplus \cdots \oplus V_{i-1})$. We will assume that each V_i is finite dimensional. Define a filtration on the Lie algebra of derivations $L = \mathrm{Der}\wedge(V_1 \oplus \cdots V_n)$ as follows.

$$F_pL = \{\theta \in \mathrm{Der}\wedge V : \theta(V_1 \oplus \cdots \oplus V_{n-p-1}) = 0\}.$$

We get a filtration $0 \subset F_0L \subset F_1L \subset \cdots \subset F_{n-1}L = L$. Moreover $F_0L = (\wedge V) \otimes Z^0$ where $Z^0 = V_n^\#$. In general, $F_kL/F_{k-1}L = (\wedge V) \otimes Z^k$ where $Z^k = V_{n-k}^\#$ and $\delta Z^k \subset (\wedge V) \otimes (Z^0 \oplus \cdots \oplus Z^{k-1})$. This defines a semifree filtration of L, hence (L, δ) is a semifree differential module over $(\wedge V, d)$.

It comes from the definition that $[F_pL, F_qL] \subset F_rL$, where $r = \max\{p, q\}$. Hence $[F_pL, F_qL] \subset F_{p+q}L$. The filtration induces a spectral sequence of differential graded Lie algebras such that $E^0_{k,*} = F_kL/F_{k-1}L \cong A \otimes Z^{k,*}$ and $d_0 = d \otimes 1$. Hence $E^1_{k,*} \cong H(A) \otimes Z^k$. The E^1-term, together with differentials, yields

$$\begin{array}{ccccccc} E^1_{n-1,*} & \xrightarrow{d_1} & E^1_{n-2,*} & \xrightarrow{d_1} \cdots \xrightarrow{d_1} & E^1_{0,*} \\ \| & & \| & & \| \\ H(A) \otimes Z^{n-1}_* & \xrightarrow{d_1} & H(A) \otimes Z^{n-2}_* & \xrightarrow{d_1} \cdots \xrightarrow{d_1} & H(A) \otimes Z^0_*. \end{array}$$

In particular if X is a homogeneous space, then its minimal Sullivan model is of the form $(\wedge V, d) = (\wedge(V_1 \oplus V_2), d)$ with $dV_1 = 0$ and $dV_2 \subset \wedge V_1$, then the above spectral sequence collapses at E^2-level.

5 Computations for homogeneous spaces

Let X be a closed oriented manifold of dimension m. The loop homology $\mathbb{H}_*(LX) = H_{*+m}(LX)$ is endowed with a loop product and a loop bracket turning it into a graded Gerstenhaber algebra [1]. When coefficients are taken in a field there is an isomorphism of graded vector spaces [10]

$$HH_*(C^*X; C^*X) \cong H^*(LX)$$

which dualizes in

$$HH^*(C^*X;C_*X)\cong H_*(LX).$$

If $\mathbb{k}$ is of characteristic 0 and X is simply connected, there is an isomorphism of Gerstenhaber algebras [6, 7, 5]

$$\mathbb{H}_*(LX)\cong HH^*(C^*X;C^*X).$$

Moreover if X is simply connected and $A=(\wedge V,d)$ is a Sullivan model of X, one has an isomorphism of Gerstenhaber algebras [3, Proposition 3.3]

$$HH^*(A;A)\cong HH^*(C^*X;C^*X).$$

Therefore $H_*(s^{-1}\operatorname{Der}\wedge V,\delta')$ is a sub Lie algebra of $\mathbb{H}_*(LX)$. We note that if $\theta,\theta'\in\operatorname{Der}\wedge V$, where $|\theta|=k$ and $a\in(\wedge V)^i$, then $a\theta\in(\operatorname{Der}\wedge V)_{k-i}$. Moreover

$$\begin{aligned}[\theta,a\theta'](x) &= \theta(a\theta'(x))+(-1)^{|\theta||a\theta'|}(a\theta')(\theta(x))\\ &= \theta(a)\theta'(x)+(-1)^{|\theta||a|}a(\theta\theta')(x)+(-1)^{|\theta||a\theta'|}a(\theta'\theta)(x)\\ &= \theta(a)\theta'(x)+(-1)^{|\theta||a|}a[\theta,\theta'](x).\end{aligned}$$

Hence

$$[\theta,a\theta']=\theta(a)\theta'+(-1)^{|\theta||a|}a[\theta,\theta']. \tag{5.1}$$

We can now compute brackets in the E^2-term of the spectral sequence of $s^{-1}\operatorname{Der}\wedge V$, when $(\wedge V,d)$ is the minimal Sullivan model of a homogeneous space. We simply denote by d the differential d_1 of the E^1-term of the spectral sequence.

Example 5.1. Consider $X=\mathbb{C}P(n)$ of which the minimal Sullivan model is $(\wedge(x,y),d)$, $|x|=2,|y|=2n+1$, $dx=0$, $dy=x^{n+1}$. The E^1-term is given by $(\wedge x/(x^{n+1})\otimes\mathbb{Q}<z_1,z_{2n}>,d)$, where $z_1=s^{-1}x^{\#}$ and $z_{2n}=s^{-1}y^{\#}$. The differential is given by $dz_{2n}=0$, $dz_1=(n+1)x^nz_{2n}$. Here subscripts indicate degrees. Non zero homology classes are $\{x^jz_{2n}, x^iz_1,\quad 0\le j\le n-1,\quad 1\le i\le n\}$. In particular $\{xz_1,x^jz_{2n}\}=jx^jz_{2n}$, hence $\operatorname{ad}^k(xz_1)\neq 0$, for $k\ge 1$.

Example 5.2. We consider the minimal Sullivan model of $X=Sp(5)/SU(5)$ which is given by $(\wedge(x_6,x_{10},y_{11},y_{15},y_{19},d)$ with $dx_i=0$, $dy_{11}=x_6^2$, $dy_{15}=x_6x_{10}$, $dy_{19}=x_{10}^2$, where subscripts indicate degrees. The rational cohomology $H^*(X,\mathbb{Q})$ is given by classes of $\{1,x_6,x_{10},x_6y_{15}-x_{10}y_{11},x_{10}y_{15}-x_6y_{19},x_6(x_{10}y_{15}-x_6y_{19})\}$. Hence the E^1-term is $(H^*(X,\mathbb{Q})\otimes Z,d)$, where Z is spanned by $\{z_{10},z_{14},z_{18},w_5,w_9\}$, $z_i=s^{-1}y^{\#}_{i+1}$, $w_i=s^{-1}x^{\#}_{i+1}$ and $dz_i=0,dw_5=2x_6z_{10}+x_{10}z_{14}$, $dw_9=x_6z_{14}+2x_{10}z_{18}$. It is easily checked that $x_6w_5,x_6z_i^k,x_6w_9,x_{10}z_i^k$ are non zero homology classes. Moreover $\{x_6w_5,x_6z_i^k\}=x_6z_i^k$, $\{x_{10}w_9,x_{10}z_i^k\}=x_{10}z_i^k$. Hence for $\alpha=x_6w_5$, $\operatorname{ad}^k\alpha\neq 0$, $k\ge 1$. It is the same for $\beta=x_{10}w_9$.

We have the more general result.

Theorem 5.3. *Let X be a homogeneous space of which the minimal Sullivan model is $(A,d)=(\wedge(x_1,\dots,x_n,y_1,\dots,y_m),d)$, where $|x_i|$ is even, $|y_i|$ is odd and $dx_i=0$, $f_i=dy_i\in\wedge(x_1,\dots,x_n)$. Then the graded Lie algebra $s\mathbb{H}_*(LX,\mathbb{Q})$ is not nilpotent.*

Proof. It is sufficient to show that $H_*(s^{-1}\operatorname{Der}A,\delta')\subset HH_*(A;A)$ is not nilpotent. Like in the previous examples, we consider the spectral sequence for $s^{-1}\operatorname{Der}A$. The E^1-term is given by

$$(H^*(A,d)\otimes\mathbb{Q}<z_1,\dots,z_m,w_1,\dots,w_n>,d),$$

where $z_j = s^{-1}y_j^{\#}$, $w_i = s^{-1}x_i^{\#}$, $dz_j = 0$ and $dw_i = \sum_j \frac{\partial f_j}{\partial x_i} z_j$. We are looking for coefficients $q_i \in \mathbb{Q}$ such that $\alpha = \sum_i q_i x_i w_i$ is a d-cocycle.

$$\begin{aligned} d(\textstyle\sum_i q_i x_i w_i) &= \textstyle\sum_i \sum_j q_i x_i \frac{\partial f_j}{\partial x_i} z_j \\ &= \textstyle\sum_j (\sum_i q_i x_i \frac{\partial f_j}{\partial x_i}) z_j. \end{aligned}$$

In particular $d\alpha = 0$ if $\sum_i q_i x_i \frac{\partial f_j}{\partial x_i} = c_j f_j$, for $j = 1, 2, \dots, m$ and the c_j's are rational numbers. It is the case if one takes $q_i = |x_i|$ and $c_j = |f_j|$. This is the Euler Theorem for homogeneous functions in the graded case.

If we denote by Z^0 and Z^1 the respective spans of $\{z_j\}$ and $\{w_i\}$ and $H = H^*(X, \mathbb{Q})$, then $dZ^0 = 0$ and $dZ^1 \subset H \otimes Z^0$. As $\alpha \in H \otimes Z^1$, then α cannot be a d-boundary. Moreover $\{\alpha, x_i z_i\} = |x_i| x_i z_i$, hence $s\mathbb{H}_*(LX, \mathbb{Q})$ is not nilpotent. □

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