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Third Order BVPs with ϕ-Laplacian Operators on $[0,+\infty)$

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Abstract

This work deals with the existence of multiple positive solutions for a third order boundary value problem with a ϕ-Laplacian operator on the half-line. The existence results are obtained both for the regular and the singular cases using the fixed point index theory on a suitable cone of a Banach space. The singularity is treated by an approximation technique and sequential arguments. Examples of applications are included to illustrate the existence results.

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1 Introduction

This paper is devoted to the existence of positive solutions to the following boundary value problem (bvp for short) posed on the positive half-line:

$$\begin{cases} (\phi(-x''))'(t)+m(t)f(t,x(t),x'(t))=0, \quad t\in I, \\ \\ \alpha x(0)-\beta x'(0)=x'(+\infty)=x''(+\infty)=0 \end{cases} \tag{1.1}$$

where $\alpha,\beta>0$ are positive constants, $I:=(0,+\infty)$ denotes the set of positive real numbers, and $\mathbb{R}^+:=[0,+\infty)$. The functions $m: I\longrightarrow I$ and the nonlinearity $f:\mathbb{R}^+\times I\times\mathbb{R}^+\longrightarrow\mathbb{R}^+$ are

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continuous. The map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ stands for a continuous, increasing homeomorphism with $\phi(0) = 0$ (for instance the p-Laplacian $\varphi_p(s) = |s|^{p-1}s,\ p > 1$.)

Boundary value problems on the half line originate from many applications in physics such as the modeling of the unsteady flow of a gas through a semi-infinite porous media, in determining the electrical potential in an isolated neutral atom, in the propagation of flames in combustion theory or in plasma physics (see, e.g., [1] and references therein). From the mathematical point of view, second-order bvps on the half-line have received a great deal of attention and interest in the recent literature (see [5]-[9], [13]) over the past few years. However only few papers have considered the existence of positive solutions for higher order differential equations in finite intervals of the real line (see, e.g., [10], [12], [14]-[15]). The goal of this paper is to fill the gap in this area by considering the third-order bvp (1.1) on the positive half-line. We shall discuss the questions of existence and multiplicity of positive solutions in the regular and the singular cases, i.e., even when f may present a singularity with respect to the second argument. The difficulties are that the problem is posed on an unbounded interval and the nonlinearity also depends on the first derivative; we will use the fixed point index theory in a suitable cone of a weighted Banach space to prove existence of single and twin positive solutions.

This paper has mainly four sections. In Section 2, we define a special cone and we prove some lemmas which are needed in this work as well as some auxiliary results; a fixed point formulation is given. In Section 3, we suppose that f has no singularities and then we prove the existence of at least one and then two positive solutions using the fixed point index theory in an appropriate cone. Extension to the case when the nonlinearity $f = f(t,x,y)$ is singular at $x = 0$ is presented in Section 4. The singularity is treated by means of regularization, approximation, and on compactness arguments. Two examples of application are given in Section 2 and 3 to illustrate the results obtained.

A function x is said to be a positive solution of problem (1.1) if $x \in C^2(\mathbb{R}^+,\mathbb{R})$, $\phi(-x'') \in C^1(I,\mathbb{R})$, x satisfies (1.1) with $x(t) > 0$, for all positive t.

2 Preliminaries

First, recall that a mapping $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets. A nonempty subset $\mathcal{P}$ of a Banach space E is called a cone if it is convex, closed, and satisfies

$$[0,\infty).\mathcal{P} \subset \mathcal{P} \text{ and } \mathcal{P} \cap (-\mathcal{P}) = \{0\}.$$

Definition 2.1. The operator ϕ is said to be:

(a) sub-multiplicative if $\phi(xy) \leq \phi(x)\phi(y),\ \forall\, x,y \in [0,+\infty)$,

(b) super-multiplicative if $\phi(xy) \geq \phi(x)\phi(y),\ \forall\, x,y \in [0,+\infty)$,

(c) multiplicative if $\phi(xy) = \phi(x)\phi(y),\ \forall\, x,y \in [0,+\infty)$.

Notice that if ϕ is sub-multiplicative, then ϕ^{-1} is super-multiplicative and that the p-Laplacian is multiplicative. The following lemmas will be used to prove our main existence results. More details on the theory and the computation of the fixed point index on cones in Banach spaces may be found in [1, 2, 4, 11].

Lemma 2.2. *Let Ω be a bounded open subset in a real Banach space E, $\mathcal{P}$ a cone of E, and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $\lambda Ax \neq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$, $\forall\, \lambda \in$* $(0,1]$. *Then* $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.

Lemma 2.3. *Let Ω be a bounded open subset in a real Banach space E, $\mathcal{P}$ a cone of E, and $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose that $Ax \nleq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$. Then* $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.

Define the functional space

$$C_l([0,\infty),\mathbb{R}) = \{x \in C([0,\infty),\mathbb{R}) \mid \lim_{t\to+\infty} x(t) \text{ exists}\}.$$

For $x \in C_l([0,\infty),\mathbb{R})$, let $\|x\|_l = \sup\limits_{t\in\mathbb{R}^+} |x(t)|\cdot$ This makes C_l a Banach space. However, the basic space to study Problem (1.1) is

$$E = \{x \in C^1([0,\infty),\mathbb{R}) \mid \lim_{t\to+\infty} \frac{x(t)}{1+t} \text{ exists and } \lim_{t\to+\infty} x'(t) = 0\}.$$

It is clear that $(E, \|\cdot\|)$ is a Banach space with norm $\|x\| = \max\{\|x\|_1, \|x\|_2\}$ where $\|x\|_1 = \sup\limits_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$ and $\|x\|_2 = \sup\limits_{t\in\mathbb{R}^+} |x'(t)|$. The following result is a useful compactness criterion.

Lemma 2.4. *[3, p.62] Let $M \subseteq C_l(\mathbb{R}^+,\mathbb{R})$. Then M is relatively compact in $C_l(\mathbb{R}^+,\mathbb{R})$ if the following three conditions hold:*

1. *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$.*
2. *The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e., equicontinuous on every compact interval of $\mathbb{R}^+$.*
3. *The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

A similar result is then derived in the space E:

Lemma 2.5. *Let $M \subseteq E$. Then M is relatively compact in E if the following conditions hold:*

1. *M is bounded in E,*
2. *the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z \mid z(t) = x'(t), x \in M\}$ are locally equicontinuous on* $[0,+\infty)$,
3. *the functions belonging to $\{u \mid u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z \mid z(t) = x'(t), x \in M\}$ are equiconvergent at $+\infty$.*

Lemma 2.6. *[7] Let $x \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a positive concave function. Then x is nondecreasing on* $[0,+\infty)$.

Since ϕ is an increasing homeomorphism, it is easy to prove

Lemma 2.7. *If x is a solution of Problem (1.1), then x is positive, monotone increasing, and concave on* $[0,+\infty)$.

Define the positive cone

$$\begin{aligned}\mathcal{P} = \{x \in E \mid \quad & x \text{ is concave }, x(t) \geq 0,\\ & x(0) \geq \tfrac{\beta}{\alpha+\beta}\|x\|_2, \text{ and } \alpha x(0) = \beta x'(0)\}.\end{aligned}$$

Notice that $x \in \mathcal{P}$ is nondecreasing and by L'Hopital's rule $\lim\limits_{t\to+\infty} \frac{x(t)}{1+t} = 0$. Now in a series of lemmas, we will study the main properties of this cone.

Lemma 2.8. *[7] Let $x \in \mathcal{P}$.*
(1) If $\theta \in (1,+\infty)$, then

$$x(t) \geq \frac{1}{\theta}\|x\|_1, \quad \forall\, t \in [1/\theta, \theta].$$

(2) If

$$\rho(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in (1,+\infty), \end{cases} \tag{2.1}$$

then

$$x(t) \geq \rho(t)\|x\|_1, \quad \forall\, t \geq 0.$$

Lemma 2.9. *[9] Let $x \in \mathcal{P}$ and $M = \max\{\frac{\beta}{\alpha}, 1\}$. Then*

(1) $\|x\|_1 \leq M\|x\|_2$ and thus $\|x\| \leq M\|x\|_2$.

(2)

$$x(t) \geq \rho(t)\frac{\beta}{\alpha+\beta}\|x\|, \ \forall\, t \geq 0.$$

Lemma 2.10. *Let $\delta \in C(\mathbb{R}^+,\mathbb{R}^+) \cap L^1((0,\infty),\mathbb{R}^+)$ and*

$$x(t) = \int_0^{+\infty} G(t,s)\delta(s)ds,$$

where

$$G(t,s) = \begin{cases} s + \frac{\beta}{\alpha}, & 0 \leq s \leq t < +\infty, \\ t + \frac{\beta}{\alpha}, & 0 \leq t \leq s < +\infty. \end{cases}$$

Then

$$\begin{cases} x''(t) + \delta(t) = 0, \quad t > 0, \\ \alpha x(0) - \beta x'(0) = \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases} \tag{2.2}$$

Lemma 2.11. *Assume that $\delta \in C(\mathbb{R}^+,\mathbb{R}^+)$ is such that*

$$\int_0^{+\infty} \delta(s)ds < +\infty \ \text{ and } \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds < +\infty$$

and let

$$x(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds. \tag{2.3}$$

Then x is a solution of

$$\begin{cases} (\phi(-x''(t)))' + \delta(t) = 0, \quad t > 0, \\ \alpha x(0) - \beta x'(0) = x'(+\infty) = x''(+\infty) = 0. \end{cases} \tag{2.4}$$

Moreover $x \in \mathcal{P}$.

Proof. Differentiating (2.3) yields

$$x'(t) = \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds$$

and

$$x''(t) = -\phi^{-1}\left(\int_t^{+\infty} \delta(\tau)d\tau\right).$$

Then $(\phi(-x''(t)))' + \delta(t) = 0$ and routine calculation ensures that x satisfies the boundary conditions in (2.4). Finally x is nondecreasing,

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds = 0,$$

and

$$\lim_{t\to+\infty} \frac{x(t)}{1+t} = \begin{cases} 0, & \text{if } \lim\limits_{t\to+\infty} x(t) < \infty, \\ \lim\limits_{t\to+\infty} x'(t) = 0, & \text{if } \lim\limits_{t\to+\infty} x(t) = \infty, \end{cases}$$

which means that $x \in E$. Since for all $t \geq 0$, $x(t) \geq 0$, $x'(t) \geq 0$, $x''(t) \leq 0$, and

$$x(0) = \frac{\beta}{\alpha}\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds = \frac{\beta}{\alpha}x'(0) = \frac{\beta}{\alpha}\|x\|_2 \geq \frac{\beta}{\alpha+\beta}\|x\|_2.$$

Thus we have proved that $x \in \mathcal{P}$, as claimed.

3 The regular case

In this section, we suppose that $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and there exists $t_0 > 0$ such that $f(t_0, 0, 0) \not\equiv 0$ so that the trivial solution is ruled out. Let $\widetilde{\rho}(t) = \frac{\rho(t)}{1+t}$, $g(t,x,y) = f(t,(1+t)x,y)$, and for all $r > 0$ and $t \geq 0$ define the function $g_r(t) = \sup\{g(t,x,y), 0 \leq x \leq r, 0 \leq y \leq r\}$. Consider the assumptions:

$(\mathcal{H}_1)$ For each $r > 0$,

$$\int_0^{+\infty} m(\tau)g_r(\tau)d\tau < +\infty \quad \text{and} \quad \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)g_r(\tau)d\tau\right)ds < +\infty.$$

$(\mathcal{H}_2)$ There exists

$$R > M\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)g_R(\tau)d\tau\right)ds. \tag{3.1}$$

Now define an operator A on $\mathcal{P}$ by

$$Ax(t) = \int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds.$$

Lemma 3.1. *Suppose that $(\mathcal{H}_1)$ holds. Then the operator A sends $\mathcal{P}$ into $\mathcal{P}$ and A is completely continuous.*

Proof. Since $f(t,x(t),x'(t)) = g(t,\frac{x(t)}{1+t},x'(t)) \leq g_{\|x\|}(t)$, then Lemma 2.11 and the condition $(\mathcal{H}_1)$ guarantee that $A(\mathcal{P}) \subset \mathcal{P}$. It remains to show that A is completely continuous.

Step 1: A is continuous. Let $\{x_n\}_{n\geq 0} \subseteq \mathcal{P}$ be some sequence converging to some limit x_0. Then there exists $r > 0$ such that $\|x_n\| \leq r$, $\forall\, n \geq 0$. By $(\mathcal{H}_1)$, we have

$$\begin{aligned}
&\|Ax_n - Ax_0\|_2\\
=\;& \sup_{t\in\mathbb{R}^+} |(Ax_n(t))' - (Ax_0(t))'|\\
\leq\;& \sup_{t\in\mathbb{R}^+} \int_t^{+\infty} \Big|\phi^{-1}\left(\int_s^{+\infty} m(\tau)f(\tau,x_n(\tau),x_n'(\tau))d\tau\right)\\
&-\phi^{-1}(\int_s^{+\infty} m(\tau)f(\tau,x_0(\tau),x_0'(\tau))d\tau)\Big|ds\\
\leq\;& \int_0^{+\infty} \Big|\phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau,\tfrac{x_n(\tau)}{1+\tau},x_n'(\tau))d\tau\right)\\
&-\phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau,\tfrac{x_0(\tau)}{1+\tau},x_0'(\tau))d\tau\right)\Big|ds.
\end{aligned}$$

Also

$$\left\{\begin{array}{l}
|m(\tau)g(\tau,\frac{x_n(\tau)}{1+\tau},x_n'(\tau)) - m(\tau)g(\tau,\frac{x_0(\tau)}{1+\tau},x_0'(\tau))| \leq 2m(\tau)g_r(\tau),\\
\text{and}\\
\Big|\phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau,\frac{x_n(\tau)}{1+\tau},x_n'(\tau))d\tau\right)\\
-\phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau,\frac{x_0(\tau)}{1+\tau},x_0'(\tau))d\tau\right)\Big| \leq 2\phi^{-1}\left(\int_s^{+\infty} m(\tau)g_r(\tau)d\tau\right).
\end{array}\right.$$

Then the condition $(\mathcal{H}_1)$, the continuity of g, m, and ϕ^{-1}, and the Lebesgue dominated convergence theorem imply that $\|Ax_n - Ax_0\|_2 \to 0$, as $n \to +\infty$. Finally Lemma 2.9 implies that $\|Ax_n - Ax_0\|$ tends to 0, as $n \to +\infty$.

Step 2: Let D be a bounded set. We prove that $A(D)$ is relatively compact. Indeed there exists $r > 0$ such that $\|x\| \leq r, \forall\, x \in D$. We shall proceed in three steps:

(a) A(D) is uniformly bounded. For $x \in D$

$$\begin{aligned}
\|Ax\| &\leq M\|Ax\|_2\\
&\leq M\sup_{t\in\mathbb{R}^+} |(Ax)'(t)|\\
&\leq M\sup_{t\in\mathbb{R}^+} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&\leq M\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau,\tfrac{x(\tau)}{1+\tau},x'(\tau))d\tau\right)\\
&\leq M\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)g_r(\tau)d\tau\right)ds < \infty,
\end{aligned}$$

proving that $A(D)$ is bounded.

(b) For all $T > 0$ and $t, t' \in [0, T]$ $(t > t')$, we have

$$\begin{aligned}
&\left|\tfrac{Ax(t)}{1+t} - \tfrac{Ax(t')}{1+t'}\right| \\
\leq\ & \int_0^{+\infty} \left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right| \phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x(\tau), x'(\tau)) d\tau\right) ds \\
\leq\ & \int_0^{+\infty} \left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right| \phi^{-1}\left(\int_s^{+\infty} m(\tau) g(\tau, \tfrac{x(\tau)}{1+\tau}, x'(\tau)) d\tau\right) ds \\
\leq\ & \int_0^{T} \left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right| \phi^{-1}\left(\int_s^{+\infty} m(\tau) g_r(\tau) d\tau\right) ds \\
&+ \left|\tfrac{t+\frac{\beta}{\alpha}}{1+t} - \tfrac{t'+\frac{\beta}{\alpha}}{1+t'}\right| \int_T^{\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g_r(\tau) d\tau\right) ds.
\end{aligned}$$

Similarly

$$\begin{aligned}
&|((Ax)'(t)) - ((Ax)'(t'))| \\
=\ & \Big|\int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x(\tau), x'(\tau)) d\tau\right) ds \\
&- \int_{t'}^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x(\tau), x'(\tau)) d\tau\right) ds\Big| \\
\leq\ & \int_{t'}^{t} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g(\tau, \tfrac{x(\tau)}{1+\tau}, x'(\tau)) d\tau\right) ds \\
\leq\ & \int_{t'}^{t} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g_r(\tau) d\tau\right) ds.
\end{aligned}$$

By condition $(\mathcal{H}_1)$, for every $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}| < \varepsilon$ and $|(Ax)'(t) - (Ax)'(t')| < \varepsilon$, for all $t, t' \in [0, T]$ such that $|t - t'| < \delta$.

(c) For $x \in D$, $\lim\limits_{t\to+\infty} \frac{Ax(t)}{1+t} = \lim\limits_{t\to+\infty} (Ax)'(t) = 0$ follows from L'Hopital's rule. As a consequence

$$\begin{aligned}
&\sup_{x\in D} \left|\tfrac{Ax(t)}{1+t} - \lim_{t\to+\infty} \tfrac{Ax(t)}{1+t}\right| \\
=\ & \sup_{x\in D} \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x(\tau), x'(\tau)) d\tau\right) ds}{1+t} \\
=\ & \sup_{x\in D} \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau) g(\tau, \frac{x(\tau)}{1+\tau}, x'(\tau)) d\tau\right) ds}{1+t} \\
\leq\ & \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau) g_r(\tau) d\tau\right) ds}{1+t}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{x\in D} |(Ax)'(t) - \lim_{t\to+\infty} (Ax)'(t)| &= \sup_{x\in D} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x(\tau), x'(\tau)) d\tau\right) ds \\
&\leq \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g_r(\tau) d\tau\right) ds
\end{aligned}$$

which implies that

$$\lim_{t\to+\infty}\sup_{x\in D}\left|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}\right|=0$$

and

$$\lim_{t\to+\infty}\sup_{x\in D}|(Ax)'(t)-\lim_{t\to+\infty}(Ax)'(t)|=0.$$

By Lemma 2.5, $A(D)$ is relatively compact in E. Therefore $A:\mathcal{P}\longrightarrow\mathcal{P}$ is completely continuous.

3.1 Existence of a single solution

Theorem 3.2. *Assume that $(\mathcal{H}_1)-(\mathcal{H}_2)$ hold. Then Problem (1.1) has at least one positive solution.*

Proof. Let $R>0$ be as in $(\mathcal{H}_2)$ and consider the open ball

$$\Omega_1=\{x\in E:\ \|x\|<R\}.$$

We claim that $x\neq\lambda Ax$ for any $x\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda\in(0,1]$. On the contrary, suppose that there exists $x_0\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda_0\in(0,1]$ such that $x_0=\lambda_0Ax_0$. By Lemma 2.9, we have

$$\begin{aligned}
R &= \|x_0\|=\|\lambda_0Ax_0\|\\
&\leq M\|Ax_0\|_2\\
&\leq M\sup_{t\geq0}\int_t^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}m(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds,\\
&\leq M\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}m(\tau)g(\tau,\tfrac{x(\tau)}{1+\tau},x'(\tau))d\tau\Big)ds,\\
&\leq M\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}m(\tau)g_R(\tau)d\tau\Big)ds,
\end{aligned}$$

which is a contradiction to (3.1). Owing to Lemma 2.2, we deduce that

$$i(A,\Omega_1\cap\mathcal{P},\mathcal{P})=1. \tag{3.2}$$

Then there exists an $x_0\in\Omega_1\cap\mathcal{P}$ such that $Ax_0=x_0$. Since $f(t_0,0,0)\not\equiv0$ and $x_0(t)\geq\frac{\beta}{\alpha+\beta}\rho(t)\|x_0\|$, we deduce that x_0 is a positive solution of (1.1).

3.2 Two positive solutions

Theorem 3.3. *Further to the hypotheses in Theorem 3.2, suppose that ϕ^{-1} is super-multiplicative and*

$(\mathcal{H}_3)$ there exist $R'>R$ and two real numbers $0<a<b$ such that

$$g(t,x,y)>N^*\phi(x),\quad \textit{for all } t\in[a,b],\ x\geq R',\ \textit{and } y\geq0,$$

where $N^* = 1+\phi(\frac{1}{c\Delta})$, $c = \frac{\beta}{\alpha+\beta}\min_{t\in[a,b]}\widetilde{\rho}(t)$, *and*

$$\Delta = \min_{t\in[a,b]}\int_a^{\frac{a+b}{2}}\frac{G(t,s)}{1+t}\phi^{-1}\left(\int_{\frac{a+b}{2}}^b m(\tau)d\tau\right)ds.$$

Then Problem (1.1) has at least two positive solutions.

Remark 3.4. A sufficient condition for $(\mathcal{H}_3)$ be satisfied is the super-linear case:

$$\lim_{x\to+\infty}\frac{g(t,x,y)}{\phi(x)} = +\infty, \text{ uniformly for } t\in[a,b] \text{ and } y\geq 0.$$

Proof. Choosing the same R as in the proof of Theorem 3.2 yields

$$i(A,\Omega_1\cap\mathcal{P},\mathcal{P}) = 1, \tag{3.3}$$

and thus Theorem 3.2 guarantees the existence of a solution x_0 of Problem (1.1) in Ω_1. Now define the open ball

$$\Omega_2 = \{x\in E: \|x\| < R'/c\}.$$

Since $0<c<1$ and $R<R'$ then $\Omega_1\subset\Omega_2$. We show that $Ax\nleq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$. Suppose on the contrary that there exists $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $Ax_0\leq x_0$. Since $x_0\in\mathcal{P}\cap\partial\Omega_2$, we have the estimates for $t\in[a,b]$

$$\frac{x_0(t)}{1+t}\geq\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)\|x_0\|\geq\min_{t\in[a,b]}\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)\frac{R'}{c} = c\frac{R'}{c}\geq R'$$

as well as

$$\begin{aligned}
\frac{x_0(t)}{1+t} &\geq \frac{Ax_0(t)}{1+t}\\
&= \frac{\int_0^{+\infty}G(t,s)\phi^{-1}\left(\int_s^{+\infty}m(\tau)f(\tau,x_0(\tau),x_0'(\tau))d\tau\right)ds}{1+t}\\
&= \frac{\int_0^{+\infty}G(t,s)\phi^{-1}\left(\int_s^{+\infty}m(\tau)g(\tau,\frac{x_0(\tau)}{1+\tau},x_0'(\tau))d\tau\right)ds}{1+t}\\
&\geq \frac{\int_a^{\frac{a+b}{2}}G(t,s)\phi^{-1}\left(\int_{\frac{a+b}{2}}^b m(\tau)N\phi(\frac{x_0(\tau)}{1+\tau})d\tau\right)ds}{1+t}\\
&\geq \frac{\int_a^{\frac{a+b}{2}}G(t,s)\phi^{-1}\left(N\phi(R')\int_{\frac{a+b}{2}}^b m(\tau)d\tau\right)ds}{1+t}\\
&\geq \frac{\int_a^{\frac{a+b}{2}}G(t,s)\phi^{-1}(N)R'\phi^{-1}\left(\int_{\frac{a+b}{2}}^b m(\tau)d\tau\right)ds}{1+t}\\
&\geq R'\phi^{-1}(N)\min_{t\in[a,b]}\int_a^{\frac{a+b}{2}}\frac{G(t,s)}{1+t}\phi^{-1}\left(\int_{\frac{a+b}{2}}^b m(\tau)d\tau\right)ds\\
&= R'\phi^{-1}(N^*)\Delta\\
&> R'/c.
\end{aligned}$$

Hence $\|x_0\| > \frac{R'}{c}$, contradicting $\|x_0\| = \frac{R'}{c}$. Finally, Lemma 2.3 yields

$$i(A,\Omega_2 \cap P,\mathcal{P}) = 0, \tag{3.4}$$

while (3.3) and (3.4) imply that

$$i(A,(\Omega_2 \setminus \overline{\Omega}_1)\cap\mathcal{P},\mathcal{P}) = -1. \tag{3.5}$$

Then A has another fixed point $y_0 \in (\Omega_2 \setminus \overline{\Omega}_1)\cap\mathcal{P}$. Moreover

$$y_0(t) \geq \frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)R,\ \|x_0\| < R < \|y_0\| < R'/c$$

imply that x_0 and y_0 are two distinct positive solutions of (1.1).

Example 3.5. Consider the boundary value problem

$$\begin{cases} (\phi(-x''(t)))' + \frac{e^{-\delta t}}{(1+t)^3}(x^3(t)+(1+t)^3x'^2(t)) = 0, \\ \alpha x(0)-\beta x'(0) = \lim\limits_{t\to+\infty} x'(t) = \lim\limits_{t\to+\infty} x''(t) = 0, \end{cases} \tag{3.6}$$

where $f(t,x,y) = \frac{e^{-\delta t}}{(1+t)^3}(x^3+(1+t)^3y^2)$, $m(t) = 1$ ($\delta > 0$), and

$$\phi(x) = \begin{cases} \frac{x^5}{1+x^2}, & x \leq 0, \\ x^2, & x \geq 0. \end{cases}$$

Then ϕ is continuous increasing and $\phi(0) = 0$. Moreover $g(t,x,y) = f(t,(1+t)x,y) = e^{-\delta t}(x^3 + y^2)$ and $g_r(t) = e^{-\delta t}(r^3+r^2)$. We check the main assumptions of Theorem 3.3:

($\mathcal{H}_1$) For every $r > 0$

$$\int_0^{+\infty} m(\tau)g_r(\tau)d\tau = \frac{r^3+r^2}{\delta} < +\infty$$

and

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)g_r(\tau)d\tau\right)ds = \frac{2\sqrt{r^3+r^2}}{\delta\sqrt{\delta}} < +\infty.$$

($\mathcal{H}_2$)

$$\sup_{c>0} \frac{c}{M\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} m(\tau)g_r(\tau)d\tau\right)ds} = \frac{\delta\sqrt{\delta}}{2M}\sup_{c>0}\frac{c}{\sqrt{c^3+c^2}}.$$

If we choose $\delta > 0$ large enough, then condition ($\mathcal{H}_2$) holds.

($\mathcal{H}_3$) We have

$$\lim_{x\to+\infty}\frac{g(t,x,y)}{\phi(x)} = +\infty, \text{ uniformly in } t,y \in [0,\infty).$$

Then Theorem 3.3 implies that Problem (3.6) has at least two positive solutions.

4 The singular case

In this section, we suppose that $f:\mathbb{R}^+\times I\times\mathbb{R}^+\to\mathbb{R}^+$ is such that $\lim\limits_{x\to 0^+} f(t,x,y)=+\infty$, i.e., $f(t,x,y)$ may present a space singularity at the origin $x=0$. Assume that

($\mathcal{H}_4$) there exist $p,q\in C(I,I)$ and $h\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ such that q is a decreasing function and $\frac{p}{q}$ is increasing function with

$$g(t,x,y)\le p(x)h(t,y),\quad \forall\, t,y\ge 0,\ \forall\, x>0 \tag{4.1}$$

and for every $r,r'>0$,

$$\int_0^{+\infty} m(\tau)q(r'\widetilde{\rho}(\tau))h_r(\tau)d\tau<+\infty$$

and

$$\int_0^{+\infty}\phi^{-1}\left(\frac{p(r)}{q(r)}\int_s^{+\infty} m(\tau)q(r'\widetilde{\rho}(\tau))h_r(\tau)d\tau\right)ds<+\infty,$$

where $h_r(t)=\sup\{h(t,y),\,0\le y\le r\}$.

($\mathcal{H}_5$) There exists

$$R>M\int_0^{+\infty}\phi^{-1}\left(\frac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds. \tag{4.2}$$

($\mathcal{H}_6$) There exist $\psi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and an interval $J\subset(0,+\infty)$ such that $\psi(t)>0$ on J and

$$g(t,x,y)\ge\psi(t),\quad \forall\, t,y\ge 0,\ \forall\, x\in(0,R]$$

with

$$\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} m(\tau)\psi(\tau)d\tau\right)ds<+\infty. \tag{4.3}$$

Now given $f\in C(\mathbb{R}^+\times I\times\mathbb{R}^+,\mathbb{R}^+)$, define a sequence of approximating functions $\{f_n\}_{n\ge 1}$ by

$$f_n(t,x,y)=f(t,\max\{(1+t)/n,x\},y),\quad n\in\{1,2,\ldots\}$$

and define a sequence of operators on $\mathcal{P}$ by

$$A_nx(t)=\int_0^{+\infty}G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau)f_n(\tau,x(\tau),x'(\tau))d\tau\right)ds.$$

We have

Lemma 4.1. *Suppose that ($\mathcal{H}_4$) holds. Then, for every $n\ge 1$, the operator A_n sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

Proof. For $n \geq 1$, we have

$$\begin{aligned}
& \int_0^{+\infty} m(\tau) f_n(\tau, x(\tau), x'(\tau)) d\tau \\
= & \int_0^{+\infty} m(\tau) g(\tau, \max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}, x'(\tau)) d\tau \\
\leq & \int_0^{+\infty} m(\tau) p(\max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}) h(\tau, x'(\tau)) d\tau \\
\leq & \int_0^{+\infty} m(\tau) q(\max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}) \frac{p(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})}{q(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})} h(\tau, x'(\tau)) d\tau \\
\leq & \frac{p(\max\{\frac{1}{n}, \|x\|\})}{q(\max\{\frac{1}{n}, \|x\|\})} \int_0^{+\infty} m(\tau) q(\tfrac{1}{n}) h_{\|x\|}(\tau) d\tau \\
\leq & \frac{p(\max\{\frac{1}{n}, \|x\|\})}{q(\max\{\frac{1}{n}, \|x\|\})} \int_0^{+\infty} m(\tau) q(\tfrac{1}{n}\widetilde{\rho}(\tau)) h_{\|x\|}(\tau) d\tau < \infty
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f_n(\tau, x(\tau), x'(\tau)) d\tau\right) ds \\
= & \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g(\tau, \max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}, x'(\tau)) d\tau\right) ds \\
\leq & \int_0^{+\infty} \phi^{-1}\left(\frac{p(\max\{\frac{1}{n}, \|x\|\})}{q(\max\{\frac{1}{n}, \|x\|\})} \int_s^{+\infty} m(\tau) q(\tfrac{1}{n}\widetilde{\rho}(\tau)) h_{\|x\|}(\tau) d\tau\right) ds < +\infty.
\end{aligned}$$

Then the conditions of Lemma 2.11 are fulfilled, hence $A_n\mathcal{P} \subseteq \mathcal{P}$. The proof that A_n is completely continuous is similar to that of the operator A in Theorem 3.2 and is omitted.

4.1 Existence of a single solution

Theorem 4.2. *Assume that Assumptions* $(\mathcal{H}_4)-(\mathcal{H}_6)$ *hold. Then Problem (1.1) has at least one positive solution.*

Proof.

Step 1: An approximating solution. Let $R > 0$ be as in $(\mathcal{H}_5)$ and put

$$\Omega_1 = \{x \in E : \|x\| < R\}.$$

We claim that $x \neq \lambda A_n x$ for all $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0, 1]$, and $n \geq n_0$ for some $n_0 > 1/R$. On the contrary, assume that there exist $n_1 \geq n_0$, $x_1 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_1 \in (0, 1]$ such that $x_1 = \lambda_1 A_{n_1} x_1$. By Lemma 2.9, $x_1(t) \geq \frac{\beta}{\alpha+\beta}\rho(t)\|x_1\| = \frac{\beta}{\alpha+\beta}\rho(t)R$, for all $t \in \mathbb{R}^+$. Then $\frac{x_1(t)}{1+t} \geq \frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)R$. As a consequence, we have the estimates

$$\begin{aligned}
R &= \|x_1\| \\
&= \|\lambda_1 A_{n_1} x_1\| \\
&\leq \|A_{n_1} x_1\| \\
&\leq M\|A_{n_1} x_1\|_2 \\
&\leq M \sup_{t\geq 0} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) g(\tau, \max\{\tfrac{1}{n}, \tfrac{x_1(\tau)}{1+\tau}\}, x_1'(\tau)) d\tau\right) ds \\
&\leq M \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) p(\max\{\tfrac{1}{n}, \tfrac{x_1(\tau)}{1+\tau}\}) h(\tau, x_1'(\tau)) d\tau\right) ds \\
&\leq M \int_0^{+\infty} \phi^{-1}\left(\tfrac{p(R)}{q(R)} \int_s^{+\infty} m(\tau) q(\tfrac{x_1(\tau)}{1+\tau}) h_R(\tau) d\tau\right) ds \\
&\leq M \int_0^{+\infty} \phi^{-1}\left(\tfrac{p(R)}{q(R)} \int_s^{+\infty} m(\tau) q(\tfrac{\beta}{\alpha+\beta} R\widetilde{\rho}(\tau)) h_R(\tau) d\tau\right) ds,
\end{aligned}$$

contradicting (4.2). By Lemma 2.2, we conclude that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \text{ for all } n \in \{n_0, n_0+1, \ldots\}. \tag{4.4}$$

Hence there exists an $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$, $\forall\, n \geq n_0$.

Step 2: A compactness argument. (a) Since $\frac{1}{n}, \|x_n\| < R$, by $(\mathcal{H}_6)$ we have

$$f_n(t, x_n(t), x_n'(t)) = g\left(t, \max\left\{\frac{1}{n}, \frac{x_n(t}{1+t}\right\}, x_n'(t)\right) \geq \psi(t), \quad \forall\, t \in I.$$

Let

$$c^* = \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)\psi(\tau) d\tau\right) ds > 0.$$

Then

$$\begin{aligned}
x_n(t) &= A_n x_n(t) \\
&= \int_0^{+\infty} G(t,s) \phi^{-1}\left(\int_s^{+\infty} m(\tau) f_n(\tau, x_n(\tau), x_n'(\tau)) d\tau\right) ds \\
&\geq \tfrac{\beta}{\alpha} \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f_n(\tau, x_n(\tau), x_n'(\tau)) d\tau\right) ds \\
&\geq \tfrac{\beta}{\alpha} \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)\psi(\tau) d\tau\right) ds \\
&\geq \tfrac{\beta}{\alpha+\beta} c^* \\
&\geq \tfrac{\beta}{\alpha+\beta} c^* \rho(t).
\end{aligned}$$

Hence $\frac{x_n(t)}{1+t} \geq \frac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(t),\ \forall\, t \in \mathbb{R}^+,\ \forall\, n \geq n_0.$

(b) For every $T > 0$ and all $t, t' \in [0, T]$ $(t > t')$, the following estimates hold:

$$\begin{aligned}
&\left|\tfrac{x_n(t)}{1+t} - \tfrac{x_n(t')}{1+t'}\right| \\
\leq\ & \int_0^{+\infty}\left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right|\phi^{-1}\left(\int_s^{+\infty} m(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right)ds \\
\leq\ & \int_0^{+\infty}\left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right|\phi^{-1}\left(\int_s^{+\infty} m(\tau)g(\tau, \max\{\tfrac{1}{n}, \tfrac{x_n(\tau)}{1+\tau}\}, x_n'(\tau)d\tau\right)ds \\
\leq\ & \int_0^{+\infty}\left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right|\phi^{-1}\left(\tfrac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\tfrac{x_n(\tau)}{1+\tau})h_R(\tau)d\tau\right)ds \\
\leq\ & \int_0^{+\infty}\left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right|\phi^{-1}\left(\tfrac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\tfrac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds \\
& + \int_0^{T}\left|\tfrac{G(t,s)}{1+t} - \tfrac{G(t',s)}{1+t'}\right|\phi^{-1}\left(\tfrac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\tfrac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds \\
\leq\ & \left|\tfrac{t+\frac{\beta}{\alpha}}{1+t} - \tfrac{t'+\frac{\beta}{\alpha}}{1+t'}\right|\int_T^{+\infty}\phi^{-1}\left(\tfrac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\tfrac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds
\end{aligned}$$

as well as

$$|x_n'(t) - x_n'(t')| \leq \int_{t'}^{t}\phi^{-1}\left(\frac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\frac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds.$$

Then, for every $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}| < \varepsilon$ and $|x_n'(t) - x_n'(t')| < \varepsilon$ for $t, t' \in [0, T]$ such that $|t - t'| < \delta$.

(c) For $n \in \{1, 2, \ldots\}$, we have by $(\mathcal{H}_4)$ and L'Hopital's rule $\lim\limits_{t\to+\infty}\frac{x_n(t)}{1+t} = \lim\limits_{t\to+\infty} x_n'(t) = 0$. Therefore

$$\begin{aligned}
&\sup_{n\geq n_0}\left|\tfrac{x_n(t)}{1+t} - \lim_{t\to+\infty}\tfrac{x_n(t)}{1+t}\right| \\
=\ & \sup_{n\geq n_0}\frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\int_s^{+\infty} m(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right)ds}{1+t} \\
\leq\ & \frac{\int_0^{+\infty} G(t,s)\phi^{-1}\left(\frac{p(R)}{q(R)}\int_s^{+\infty} m(\tau)q(\frac{\beta}{\alpha+\beta}c^*\widetilde{\rho}(\tau))h_R(\tau)d\tau\right)ds}{1+t}.
\end{aligned}$$

By condition $(\mathcal{H}_4)$ and Lemma 2.11, the right-hand side tends to 0, as $t \to +\infty$. Also

$$\begin{aligned}
& \lim_{t\to+\infty} \sup_{n\geq n_0} |x_n'(t) - \lim_{t\to+\infty} x_n'(t)| \\
= & \lim_{t\to+\infty} \sup_{n\geq n_0} \int_t^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau) f_n(\tau, x_n(\tau), x_n'(\tau)) d\tau\right) ds \\
\leq & \lim_{t\to+\infty} \int_t^{+\infty} \phi^{-1}\left(\frac{p(R)}{q(R)} \int_s^{+\infty} m(\tau) q(\tfrac{\beta}{\alpha+\beta} c^* \widetilde{\rho}(\tau)) h_R(\tau) d\tau\right) ds \\
= & 0.
\end{aligned}$$

Therefore $\{x_n\}_{n\geq n_0}$ is relatively compact in E by Lemma 2.5. Consequently there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ converging to some limit x_0. Since $x_{n_k}(t) \geq \frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)c^*, \forall\, k \geq 1$, we deduce that $x_0(t) \geq \frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)c^*, \forall\, t \in \mathbb{R}^+$. (4.2) implies that $\|x_0\| < R$. By continuity of f, for all $s \in \mathbb{R}^+$, we deduce that

$$\begin{aligned}
\lim_{k\to+\infty} f_{n_k}(s, x_{n_k}(s), x_{n_k}'(s)) &= \lim_{k\to+\infty} f(s, \max\{(1+s)/n_k, x_{n_k}(s)\}, x_{n_k}'(s)) \\
&= f(s, \max\{0, x_0(s)\}, x_0'(s)) \\
&= f(s, x_0(s), x_0'(s)).
\end{aligned}$$

The continuity of G, ϕ^{-1} and the Lebesgue dominated convergence theorem guarantee that

$$\begin{aligned}
x_0(t) &= \lim_{k\to+\infty} x_{n_k}(t) \\
&= \lim_{k\to+\infty} \int_0^{+\infty} G(t,s) \phi^{-1}\left(\int_s^{+\infty} m(\tau) f_{n_k}(\tau, x_{n_k}(\tau), x_{n_k}'(\tau)) d\tau\right) ds \\
&= \lim_{k\to+\infty} \int_0^{+\infty} G(t,s) \phi^{-1}\left(\int_s^{+\infty} m(\tau) f(\tau, x_0(\tau), x_0'(\tau)) d\tau\right) ds.
\end{aligned}$$

Then x_0 is a positive solution of Problem (1.1) with $\|x_0\| \leq R$. Now, using $(\mathcal{H}_5)$ and arguing as in Step 1, we conclude that $\|x_0\| < R$.

4.2 Two positive solutions

Theorem 4.3. *Further to Hypotheses $(\mathcal{H}_4)$-$(\mathcal{H}_5)$, suppose that ϕ^{-1} is super-multiplicative and*

$(\mathcal{H}_7)$ there exist $R' > R$ and two positive numbers a, b with $a < b$ such that

$$g(t, x, y) > N^* \phi(x), \quad \text{for all } t \in [a,b],\ x \geq R', \text{ and } y \geq 0,$$

where $N^* = 1 + \phi\left(\frac{1}{c\Delta}\right)$, $c = \frac{\beta}{\alpha+\beta} \min_{t\in[a,b]} \widetilde{\rho}(t)$, and

$$\Delta = \min_{t\in[a,b]} \int_a^{\frac{a+b}{2}} \frac{G(t,s)}{1+t} \phi^{-1}\left(\int_{\frac{a+b}{2}}^b m(\tau) d\tau\right) ds.$$

In addition assume that $(\mathcal{H}_6)$ holds for each $x \in (0, \frac{R'}{c}]$. Then Problem (1.1) has at least two positive solutions.

The proof is identical to that of Theorem 3.3 and is omitted.

Example 4.4. Consider the singular boundary value problem

$$\begin{cases} (\phi(-x''(t)))' + \frac{te^{-\delta t}(x^2(t)+(1+t)^2)(e^{x'(t)}+1)}{(1+t)^3 x(t)} = 0, \\ \alpha x(0) - \beta x'(0) = \lim\limits_{t\to+\infty} x'(t) = \lim\limits_{t\to+\infty} x''(t) = 0, \end{cases} \tag{4.5}$$

where

$$\phi(x) = \begin{cases} x^p + x^q, & x \le 0, \\ x^\theta, & x \ge 0, \end{cases}$$

where p and q are two odd numbers and $0 < \theta < 1$. Also $m(t) = \frac{t}{1+t}$ and

$$f(t,x,y) = \frac{e^{-\delta t}(x^2 + (1+t)^2)(e^y + 1)}{(1+t)^2 x} \; (\delta > 0).$$

Then $g(t,x,y) = \frac{e^{-\delta t}(x^2+1)(e^y+1)}{x}$,

$$p(x) = \frac{x^2+1}{x}, \; q(x) = \frac{1}{x}, \; \frac{p(x)}{q(x)} = x^2 + 1,$$

$$h(t,y) = e^{-\delta t}(e^y + 1), \text{ and } h_r(t) = e^{-\delta t}(e^r + 1).$$

($\mathcal{H}_4$) For all $r, r' > 0$,

$$\int_0^{+\infty} m(\tau) q(r'\widetilde{\rho}(\tau)) h_r(\tau) d\tau \le \int_0^{+\infty} \frac{(e^r+1)}{r'} e^{-\delta\tau} d\tau = \frac{(e^r+1)}{r'\delta} < +\infty$$

and

$$\begin{aligned} \int_0^{+\infty} \phi^{-1}\left(\frac{p(r)}{q(r)} \int_s^{+\infty} m(\tau) q(r'\widetilde{\rho}(\tau)) h_r(\tau) d\tau\right) ds &\le \frac{\theta}{\delta}\left[\frac{(r^2+1)(e^r+1)}{r'\delta}\right]^{\frac{1}{\theta}} \\ &< +\infty. \end{aligned}$$

($\mathcal{H}_5$)

$$\sup_{r>0} \frac{r}{M\int_0^{+\infty} \phi^{-1}\left(\frac{p(r)}{q(r)} \int_s^{+\infty} m(\tau) q(r\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau)) h_r(\tau) d\tau\right) ds}$$

$$\ge \frac{\delta\delta^{\frac{1}{\theta}}\beta^{\frac{1}{\theta}}}{M\theta(\alpha+\beta)^{\frac{1}{\theta}}} \sup_{r>0} \frac{rr^{\frac{1}{\theta}}}{[(r^2+1)(e^r+1)]^{\frac{1}{\theta}}}.$$

If we choose δ large enough, then condition ($\mathcal{H}_5$) holds.

($\mathcal{H}_6$) For every $c > 0$, we have

$$g(t,x,y) = \frac{e^{-\delta t}(x^2+1)(e^y+1)}{x} \ge \frac{e^{-\delta t}}{x} \ge \frac{e^{-\delta t}}{c} = \psi_c(t), \; \forall\, x \in (0,c]$$

and

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} m(\tau)\psi_c(\tau) d\tau\right) ds \le \frac{\delta}{\theta c^{\frac{1}{\theta}}}.$$

($\mathcal{H}_7$) For every $0 < a < b$ and all $t \in [a,b]$, and $y \ge 0$, we have

$$\lim_{x\to+\infty} \frac{g(t,x,y)}{\phi(x)} \ge \lim_{x\to+\infty} \frac{e^{-\delta b}(x^2+1)}{x^{1+\theta}} = +\infty.$$

Then all conditions of Theorem 4.3 are satisfied which implies that Problem (4.5) has at least two positive solutions.

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