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Transfer of Multiplication-Like Conditions in Amalgamated Algebra Along an Ideal

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Abstract

Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we study the multiplication-like conditions in $A \bowtie^f J$.

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1 Introduction

Throughout this paper, all rings are commutative with unity.

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finacchiaro, and Fontana in [8, 9]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [10, 11, 12]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([8, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (i.e, trivial ring extension) [16, page 2], and the CPI extensions (in the sense of Boisen and Sheldon [3]) are strictly related to it [8, Example 2.7 and Remark 2.8].

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [8, Section 4]. This point of view allows the authors in [8, 9] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [8], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [9], they pursue the investigation on the

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structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Recall that an ideal I of a ring R is called a multiplication ideal if for every ideal $J \subseteq I$ there exists an ideal K such that $J = IK$, we say that J is a multiple of I. A ring R is called a multiplication ring if all its ideals are multiplication ideals. In [2, Section 4], Anderson proved that a ring R is a multiplication ring if and only if $R(X)$ is a multiplication ring. Examples of multiplication rings include principal rings and a finite direct product of fields R. Moreover, in [2, Theorem 1], the authors proved that multiplication rings and principal rings coincide.

Let A and B be rings, J an ideal of B, I an ideal of A and let $f : A \to B$ be a ring monomorphism. We set

$$I \bowtie^f J := \{(f(i), f(i) + j \mid i \in I,\ j \in J\}$$

which is clearly an ideal of $A \bowtie^f J$.

In this paper, we give several examples of non-multiplication rings. We also study, under some conditions, when $I \bowtie^f J$ is a multiplication ideal, a faithful ideal and cancellation ideal.

2 Main results

In the next section, we study the asymptotic properties of our wavelet estimator and establish strong uniform consistency which is an extension of results obtained by the previous investigations in the topic.

It is easily seen that if I is a multiplication ideal and S is a multiplicatively closed subset of R, then $S^{-1}I$ is a multiplication ideal of $S^{-1}R$. Hence, a multiplication ideal is locally principal. In what follows, we prove that the set of multiplication rings is closed under homomorphic image.

Proposition 2.1. *Let A be a ring, I an ideal of A. If A is a multiplication ring then A/I is a multiplication ring.*

Proof. Consider J and K ideals of A with $I \subset J \subset K$. Clearly, $J/I \subset K/I$ are ideals of A/I. Since A is a multiplication ring, there exists an ideal K' of A such that $J = KK'$. Then $J/I = (KK')/I = (K/I)(K'/I)$. Therefore, A/I is a multiplication ring. □

Corollary 2.2. *Let A and B be a ring, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If $A \bowtie^f J$ is a multiplication ring then so is A.*

Proof. Let $p_A : A \bowtie^f J \to A$ be the natural projection of $A \bowtie^f J \subseteq A \times B$ into A. Then, p_A is surjective and $Ker(p_A) = \{0\} \times J$. Hence, the canonical isomorphism hold : $\dfrac{A \bowtie^f J}{(0 \times J)} \cong A$ by [9, Proposition 2.1]. If $A \bowtie^f J$ is a multiplication ring then $\dfrac{A \bowtie^f J}{(0 \times J)}$ is a multiplication ring by Proposition 2.1. Therefore A is a multiplication ring. □

The converse of Corollary 2.2 is not true in general as showing by the following examples.

Example 2.3. Consider $A = K[X]$ where K is a field and set $J = (X)$. A is a multiplication ring but $A \bowtie J$ is not a multiplication ring.

The proof of this Example involves the following Lemma.

Lemma 2.4. *Let A be a ring and let J be an ideal of A. If $I \subset J' \subset J$ are two ideals of A, then $I \bowtie J'$ is an ideal of $A \bowtie J$.*

Proof. Let $(a, a+j) \in A \bowtie J$, $(i, i+j') \in I \bowtie J'$ and $(i_1, i_1 + j'_1) \in I \bowtie J'$.
Then, $(i, i+j') - (i_1, i_1 + j'_1) = (i - i_1, i + j' - i_1 - j'_1) = (i - i_1, i - i_1 + j' - j'_1) \in I \bowtie J'$. And $(a, a+j)(i, i+j') = (ai, (a+j)(i+j')) = (ai, ai + aj' + ji + jj') \in I \bowtie J'$. □

Proof of Example 2.3. A is a principal ring then it is a multiplication ring. Let $J' = (X^3) \bowtie (X^2)$. J' is an ideal of $A \bowtie J$ by lemma 2.4. Let $I = (X^3) \bowtie (X^3) \subset J'$. Assume that J' is a multiplication ideal of $A \bowtie J$, then there exist an ideal K of $A \bowtie J$ such that $I = KJ'$. Therefore $(X^3, X^3) = (a, a + \alpha X)(\beta X^3, \beta X^3 + \gamma X^2)$ for some $a, \beta, \gamma \in A$ and $(a, a + \alpha X) \in K$. Hence $X^3 = a\beta X^3$. Then $a\beta = 1$. Consequently, a, β are invertible in R. Or $X^3 = (a + \alpha X)(\beta X^3 + \gamma X^2) = a\beta X^3 + a\gamma X^2 + \alpha\beta X^4 + \alpha\gamma X^3$, then $\alpha\beta = 0$. Since β is invertible in R, then $\alpha = 0$. Therefore, $(a, a + \alpha X) = (a, a)$ is invertible. Consequently, $K = A \bowtie J$ and $(X^3) \bowtie (X^3) = (X^3) \bowtie (X^2)$, a contradiction. Hence, $K = A \bowtie J$ is not a multiplication ring. □

To give more examples, we need the following Lemma.

Lemma 2.5. *Let (A, M) be a local ring, E an $A-$module and let $R = A \propto E$. If R is a multiplication ring. Then :*

1. *M is principal,*
2. *E is principal, and*
3. *$E = ME$.*

Proof. (1) R is a multiplication ring then A i a multiplication ring by Proposition 2.2. Therefore, M i a principal ideal by [2, Theoreme 1]. Then $M = Am$ such that $m \in M$.

(2) R is a multiplication local ring and $(0 \propto E)$ is an ideal of R. Then $(0 \propto E)$ is a principal ideal. Therefore, $(0 \propto E) = R(0, e)$ such that $e \in E$. Consequently, $E = Ae$ is a principal ideal.

(3) The ideal $J := M \propto E = R(m, 0) + R(0, e)$ is maximal ideal of R . Then, $J = R(n, f)$ such that $(n, f) \in R$. Since (n, f) is not invertible, n is not invertible. Hence, $n \in M = Am$. Since $(0, e) \in J = (n, f)R$. Hence, there exists $(\alpha, \beta) \in R$ such that $(0, e) = (\alpha, \beta)(n, f) = (\alpha n, \alpha f + \beta n)$. Then, $\alpha n = 0$ and so $\alpha \in Ann(n) \subseteq M$. Therefore, $e = \alpha f + \beta n \in Em + Em = Em$. However, $E = Ae \subseteq Em = ME \subseteq E$. Consequently, $E = ME$. □

Example 2.6. Let (A,M) be a local ring, $E \neq 0$ an $A-$module such that $ME = 0$ and let $R = A \propto E$. Then, R is not a multiplication ring.

Proof. Assume that R is a multiplication ring. Then, $E = ME = 0$, a contradiction. Hence, R is not a multiplication ring. □

Example 2.7. Let A be a ring and $R := A \propto A$. Then R is not a multiplication ring.

Proof. Let $P \in Spec(A)$ and $S = A \backslash P$. Then, S is a multiplicatively closed set in A and in R. Therefore, $S^{-1}R := A_P \propto A_P$ is a local ring and PA_P is a maximal ideal of A_P. Since $PA_PA_P = PA_P \neq A_P$, $S^{-1}R$ is not a multiplication ring (by Lemma 2.5). Consequently, R is not a multiplication ring by [2, p.761]. □

We say that an ideal I of a ring R is idempotent if $I^2 = I$. Also, an ideal I of $A \bowtie^f J$ is called homogeneous if $I = K \bowtie^f J$ for some ideal K of A.

Proposition 2.8. *Let A and B be a rings, J an idempotent ideal of B, $f : A \to B$ be a ring homomorphism and let I a multiplication ideal of A. Then, every homogeneous ideal contained in $I \bowtie^f J$ is a multiple of $I \bowtie^f J$.*

To prove this Proposition, we need the followings Lemmas.

Lemma 2.9. *Let A and B be a rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If K is an ideal of $A \bowtie^f J$ such that $0 \times J \subset K$ then K is homogeneous.*

Proof. Let $p_A : A \bowtie^f J \to A$ be the natural projection of $A \bowtie^f J \subseteq A \times B$ into A. Let K be an ideal of $A \bowtie^f J$ then $K' = \{a \in A/(a, f(a)+e) \in K\} = p_A(K)$ is an ideal of A. If $a \in K'$, there exists $e_a \in J$ such that $(a, f(a)+e_a) \in K$. Therefore, $(a, f(a)+e) = (a, f(a)+e_a)+(0, e-e_a) \in K$ for each $e \in J$. Hence $K' \bowtie^f J \subset K$.
Conversely, let $(a, f(a)+e) \in K$. Then, $a \in K'$, since $(a, f(a)+e) \in K' \bowtie^f J$. Consequently, $K \subset K' \bowtie^f J$. □

Lemma 2.10. *Let A and B be a rings, J an idempotent ideal of B and let $f : A \to B$ be a ring homomorphism. If K and L are ideals of A. Then, $(KL) \bowtie^f J = (K \bowtie^f J)(L \bowtie^f J)$ (is the product of the two ideals $K \bowtie^f J$ and $L \bowtie^f J$).*

Proof. Consider $a \in K$, $b \in L$, and $e \in J$. Since $J^2 = J$ there exists $e', e'' \in J$ such that $e = e'e''$. We have:

$$\begin{aligned}(ab, f(ab)+e) &= ((a, f(a).(b, f(b))+(0,e) = ((a, f(a).(b, f(b))+(0, e'e'') \\ &= ((a, f(a).(b, f(b))+(0,e').(0,e'') \in (K \bowtie^f J)(L \bowtie^f J).\end{aligned}$$

Conversely, if $(a, f(a)+e)$ and $(b, f(b)+g)$ are, respectively, elements of $K \bowtie^f J$ and $L \bowtie^f J$, then $(a, f(a)+e)(b, f(b)+g) = (ab, f(ab)+f(a)g+f(b)e+eg) \in (K \bowtie^f J)(L \bowtie^f J) \subset (KL) \bowtie^f J$. Thus, we have the desired equality. □

Lemma 2.11. *Let A and B be a rings, J an idempotent ideal of B and let $f : A \to B$ be a ring homomorphism. If K and L are homogeneous ideals then so KL.*

Proof. Let K and L be a homogeneous ideals. Then, $K = K' \bowtie^f J$ and $L = L' \bowtie^f J$ for some ideals K' and L' ideals of A. However, J is idempotent ideal of B. Then, $KL = (K' \bowtie^f J)(L' \bowtie^f J) = K'L' \bowtie^f J$, by Lemma 2.10. Therefore, KL is homogeneous ideal of $A \bowtie^f J$. □

Proof of Proposition 2.8. Let $I' \subseteq I \bowtie^f J$ be a homogeneous ideal. Then, $I' = K \bowtie^f J$ for some ideal K of A such that $K \subseteq I$. Therefore, $K = LI$ for some ideal L of A. Hence, $I' = K \bowtie^f J = LI \bowtie^f J = (L \bowtie^f J)(I \bowtie^f J)$ by lemma 2.10. □

Let R be a ring. An ideal I of R is called faithful if

$$ann(I) := [0 : I] = \{x \in R \mid xI = \{0\}\} = \{0\}$$

In what follows, we study when an ideal $I \bowtie^f J$ is a faithful ideal of $A \bowtie^f J$ provided $f(ann(I)) \subseteq ann(J)$.

Proposition 2.12. *Let A and B be a rings, J a faithful ideal of B, I an ideal of A and let $f : A \to B$ be a ring homomorphism. If $f(ann(I)) \subseteq ann(J)$ then, $I \bowtie^f J$ is faithful ideal of $A \bowtie^f J$ if and only if I is faithful ideal of A.*

Proof. If $a \in ann(I)$ then $(a; f(a))(b; f(b)+e) = (ab; f(a)(f(b)+e)) = (0; f(a)e) = (0; f(a)f(e')) = (0; 0)$. Such that $e = f(e') \in J$ and $b \in I$. Therefore I is faithful ideal of A.
Conversely, let $(a; f(a)+e) \in ann(I \bowtie^f J)$. Then $(a; f(a)+e)(b; f(b)+e') = 0$ for any $(b; f(b)+e') \in I \bowtie^f J$. Therefore, $ab = 0$ for any $b \in I$. Since $a = 0$ and $(a; f(a)+e) = (0; e)$. Then, $(0; e)(0; j) = 0$ for any $j \in J$. Therefore, $e \in ann(J) = 0$. Thus, we have $(a; f(a)+e) = (0; 0)$. Therefore, $I \bowtie^f J$ is faithful ideal of $A \bowtie^f J$. □

An ideal I of R is called a cancellation ideal if whenever $IK = IL$ for ideals K and L of R, we have $K = L$. A principal ideal of R is cancellation ideal if and only if it is regular. An invertible ideal is a cancellation ideal and a cancellation ideal is faithful. If I is a finitely generated ideal of A, then, I is a faithful multiplication ideal if and only if I is a cancelation ideal by [5, Lemma 1.3].

Proposition 2.13. *Let A and B be a rings, J an ideal of B, I an ideal of A and let $f : A \to B$ be a ring monomorphism such that $f(A) \cap J = \{0\}$. If $f(I) + J$ is a cancelation ideal of B, then, $I \bowtie^f J$ is a cancelation ideal of $A \bowtie^f J$.*

Proof. Let K and L be ideals in $A \bowtie^f J$ such that $(I \bowtie^f J)K = (I \bowtie^f J)L$. And $p_B : A \bowtie^f J \to B$ be the naturel projection of $A \bowtie^f J \subseteq A \times B$ into B. Then $(f(I)+J)p_B(K) = (f(I)+J)p_B(L)$. Or $f(I)+J$ is a cancellation ideal of B. Then, $p_B(K) = p_B(L)$. let $f(a)+e \in$

$p_B(K) = p_B(L)$. There exist $b \in A$ and $e' \in J$ such that $(a; f(a)+e) \in K$ and $(b; f(b)+e') \in L$ and $f(a)+e = f(b)+e'$. Then, $f(a)-f(b) = e'-e \in f(A)\cap J = \{0\}$. Therefore, $f(a)-f(b) = e'-e = 0$. Hence $a = b$ and $e = e'$. Since $K = L$. Then, $I \bowtie^f J$ is a cancellation ideal of $A \bowtie^f J$. □

We remark that for a finitely generated ideal I of A. I is a faithful multiplication ideal if and only if I is a cancelation ideal by [5, Lemma 1.3]. Therefore, if I is finitely generated ideal of A and J is finitely generated ideal of B, then, $I \bowtie^f J$ is a faithful multiplication ideal if and only if $I \bowtie^f J$ is a cancellation ideal of $A \bowtie^f J$. □

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