AN EXTENSION OF FUGLEDE-PUTNAM THEOREM FOR W-HYPONORMAL OPERATORS

M.H.M.Rashid*
Department of Mathematics & Statistics, Faculty of Science P.O.Box(7)
Mu’tah University
Al-Karak, Jordan

Abstract

In this paper, we prove the following: assume that either (i) $T^*$ is $w$-hyponormal and $S$ is $w$-hyponormal such that $\ker(T^*) \subset \ker(T)$ and $\ker(S) \subset \ker(S^*)$ or (ii) $T^*$ is $p$-hyponormal or log-hyponormal and $S$ is $w$-hyponormal such that $\ker(S) \subset \ker(S^*)$ or (iii) $T^*$ is an injective $w$-hyponormal and $S$ is a dominant holds. Then the pair $(T, S)$ satisfy Fuglede-Putnam theorem. Also, other related results are given.

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1 Introduction

For complex infinite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $L(\mathcal{H})$, $L(\mathcal{K})$ and $L(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators on $\mathcal{H}$, the set of bounded linear operators on $\mathcal{K}$ and the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is called positive (in symbol $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in L(\mathcal{H})$ is called normal if $T^*T = TT^*$. Following [24, 28], an operator $T \in L(\mathcal{H})$ is called dominant if

$$\Re(T - \lambda) \subset \Re(T - \lambda)^* \quad \text{for all } \lambda \in \mathbb{C}.$$

This condition is equivalent to the existence of a positive constant $M_\lambda$ for each $\lambda \in \mathbb{C}$ such that

$$(T - \lambda)(T - \lambda)^* \leq M_\lambda(T - \lambda)^*(T - \lambda).$$

If there exists a constant $M$ such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, then $T$ is called $M$-hyponormal, and if $M = 1$, $T$ is hyponormal. Hence the following inclusion relations hold:

$$\{\text{Normal} \} \subset \{\text{Hyponormal} \} \subset \{M\text{-hyponormal} \} \subset \{\text{Dominant} \}.$$
According to [1, 3, 11], an operator \( T \in \mathcal{L}(\mathcal{H}) \) is called \( p \)-hyponormal for \( p \in (0,1] \) if \( |T|^{2p} \geq |T^*|^{2p} \), when \( p = 1 \), \( T \) is called hyponormal, when \( p = \frac{1}{2} \), \( T \) is called semi-hyponormal. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called log-hyponormal if \( T \) is invertible and \( \log(T^*T) \geq \log(TT^*) \). And \( T \in \mathcal{L}(\mathcal{H}) \) is called paranormal if \( \|T^2x\| \geq \|Tx\|^2 \) for every unit vector \( x \in \mathcal{H} \).

In order to discuss the relations between paranormal and \( p \)-hyponormal and log-hyponormal operators, Furuta el al. [12] introduced a class \( A \) defined by \( |T^2| \geq |T|^2 \) and they showed that class \( A \) is a subclass of paranormal and contains \( p \)-hyponormal and log-hyponormal operators. Class \( A \) operators have been studied by many researchers, for example [12, 10]. Fujii et al. [10] introduced a new class \( A(t,s) \) of operators: For \( t > 0 \) and \( s > 0 \) an operator \( T \) belongs to class \( A(s,t) \) if it satisfies an operator inequality

\[
\left( |T^*|^t |T|^2 |T^*|^t \right)^{\frac{1}{2t}} \geq |T^*|^{2t}.
\]

Recall from [2] that an operator \( T \in \mathcal{L}(\mathcal{H}) \) is called \( w \)-hyponormal if \( |T| \geq |T^*| \), where \( \overline{T} = |T| U |T|^2 \) is the Aluthge transformation. As a generalization of \( w \)-hyponormal and class \( A(s,t) \), Ito [15] introduced a class of operators called \( wA(s,t) \): For \( t > 0 \) and \( s > 0 \) an operator \( T \) belongs to class \( wA(s,t) \) if it satisfies an operator inequality

\[
\left( |T^*|^t |T|^2 |T^*|^t \right)^{\frac{1}{2t}} \geq |T^*|^{2t}.
\]

and

\[
|T|^{2s} \geq \left( |T|^t |T^*|^2 |T|^t \right)^{\frac{1}{2t}}.
\]

In [14], they showed that class \( w \)-hyponormal coincides with class \( w\mathcal{A}(\frac{1}{2}, \frac{1}{2}) \), class \( A \) coincides with class \( w\mathcal{A}(1,1) \) and class \( A(s,t) \) coincides with class \( wA(s,t) \) for each \( s > 0 \), and \( t > 0 \). Inclusion relations among these classes are known as follows:

\[
\text{[hyponormal operators]} \subset \text{[\( p \)-hyponormal operators for \( 0 < p \leq 1 \)]} \\
\subset \text{[class \( A(s,t) \) operators for \( s, t \in [0,1] \)]} \\
= \text{[class \( w\mathcal{A}(s,t) \) operators for \( s, t \in [0,1] \)]} \\
\subset \text{[class \( A \) operators]} \\
\subset \text{[paranormal operators]}.
\]

A pair \((T, S)\) is said to have the Fuglede-Putnam property if \( T^*X = XS^* \) whenever \( TX = XS \) for every \( X \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \). The Fuglede-Putnam theorem is well-known in the operator theory. It asserts that for any normal operators \( T \) and \( S \), the pair \((T, S)\) has the Fuglede-Putnam property. There exist many generalization of this theorem which most of them go into relaxing the normality of \( T \) and \( S \), see [4, 5, 8, 9, 19, 20, 21, 22, 24, 25, 27, 28, 29, 30, 31] and references therein. The two next lemmas are concerned with the Fuglede-Putnam theorem and we need them in the future.

**Lemma 1.1.** ([30]) Let \( T \in \mathcal{L}(\mathcal{H}) \) and \( S \in \mathcal{L}(\mathcal{H}) \). Then the following assertions are equivalent.

(i) The pair \((T, S)\) has the Fuglede-Putnam property.
Lemma 2.2. Let $A, B$ and $C$ be positive operators. Then

$$\left(B^\frac{1}{2}AB^\frac{1}{2}\right)^\alpha \geq B \text{ and } B \geq C \implies \left(C^\frac{1}{2}AC^\frac{1}{2}\right)^\alpha \geq C, \text{ for all } 0 < \alpha \leq 1.$$ 

Proof. There exists an operator $X$ such that

$$C^\frac{1}{2} = B^\frac{1}{2}X = X^*B^\frac{1}{2} \quad \text{and} \quad \|X\| \leq 1$$

by Douglas theorem [7]. Then with $C^\frac{1}{2} = B^\frac{1}{2}X$ we have

$$\left(C^\frac{1}{2}AC^\frac{1}{2}\right)^\alpha = \left(X^*B^\frac{1}{2}AB^\frac{1}{2}X\right)^\alpha \geq X^*\left(B^\frac{1}{2}AB^\frac{1}{2}\right)^\alpha X \geq X^*BX = C$$

by Lemma 2.1. \qed

Theorem 2.3. Let $0 < s, t \leq 1$. Let $T \in \mathcal{L}(\mathcal{H})$ be a class $A(s,t)$ operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_{\mathcal{M}}$ of $T$ to $\mathcal{M}$ is also class $A(s,t)$ operator.

Proof. Let $P$ be the projection onto $\mathcal{M}$, and $T_1 = TP$. Then

$$|T_1|^s = (P|T|^2P)^s \geq P|T|^2P$$

by Lemma 2.1, so that $|T_1|^s|T_1|^{2s}|T_1|^s \geq |T_1|^s|T|^2|T|^s$. And also,

$$|T_1|^2s = (TPT^*)^s \leq (TT^*)^s = |T|^2s$$

by Löwner-Heinz theorem [23]. Since $T$ belongs to class $A(s,t)$, we have

$$\left(|T|^s|T|^2|T|^s\right)^\frac{1}{2s} \geq |T|^s|T|^{2s},$$

it follows from Lemma 2.2 that

$$\left(|T|^s|T|^2|T|^s\right)^\frac{1}{2s} \geq |T|^s|T|^{2s},$$

and so

$$\left(|T|^s|T|^2|T|^s\right)^\frac{1}{2s} \geq |T|^s|T|^{2s}, \quad (2.1)$$

by Löwner-Heinz theorem. That is, the restriction $T|_{\mathcal{M}}$ of $T$ to $\mathcal{M}$ is class $A(s,t)$ operator. \qed
Since class $A(s,t)$ operators coincides with class $wA(s,t)$ for each $s > 0$ and $t > 0$, we have the following corollary.

**Corollary 2.4.** Let $0 < s, t \leq 1$. Let $T \in \mathcal{L}(\mathcal{H})$ be a class $wA(s,t)$ operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_{\mathcal{M}}$ of $T$ to $\mathcal{M}$ is also class $wA(s,t)$ operator.

Since class $wA(\frac{1}{2}, \frac{1}{2})$ operators coincides with class $w$-hyponormal operators, we have the following corollary.

**Corollary 2.5.** Let $T \in \mathcal{L}(\mathcal{H})$. If $T$ is $w$-hyponormal operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_{\mathcal{M}}$ of $T$ to $\mathcal{M}$ is also $w$-hyponormal operator.

**Lemma 2.6.** Let $0 < s, t \leq 1$. Let $T \in \mathcal{L}(\mathcal{H})$ belongs to class $wA(s,t)$ and $T = U|T|$ be the polar decomposition of $T$. If $\mathcal{M}$ is an invariant subspace of $T$ and $T|_{\mathcal{M}}$ is an injective normal operator, then the generalized Aluthge transformation has the form \( \widetilde{T}_{s,t} = N \oplus R \) on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $N$ is a normal operator on $\mathcal{M}$.

**Proof.** First, we show that if $T$ is a class $wA(s,t)$, then the generalized Aluthge transformation $\widetilde{T}_{s,t}$ has the form $\widetilde{T}_{s,t} = N \oplus R$. Since $T$ is a class $wA(s,t)$, it follows from [15] that $\widetilde{T}_{s,t}$ is a $p$-hyponormal operator, where $p = \frac{\min(s,t)}{s + t}$. By Lemma 5 and Lemma 11 of [31], $\widetilde{T}_{s,t}$ has the form $\begin{pmatrix} N & S \\ 0 & R \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $N$ is normal and $\mathcal{R}(S) \subseteq \ker(N)$. Then

$$\begin{pmatrix} N & S \\ 0 & R \end{pmatrix}^* \begin{pmatrix} N & S \\ 0 & R \end{pmatrix} = \begin{pmatrix} |N|^2 & 0 \\ 0 & \rho(S)^2 + |R|^2 \end{pmatrix} \geq \begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix}. $$

We have $\mathcal{R}(Y) \subseteq \mathcal{R}(X) \subseteq \mathcal{R}(N)$ by Lemma 9 of [31] and $\mathcal{R}(X) \subseteq \mathcal{R}(|N|^p)$ by Lemma 8 of [31]. Hence we have $\mathcal{R}(X) \cup \mathcal{R}(Y) \subseteq \mathcal{R}(X) \subseteq \mathcal{R}(|N|^p)$. Put $\widetilde{T}_{s,t} (\widetilde{T}_{s,t})^* = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}$. Hence

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} \begin{pmatrix} A & B' \\ B & C \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. $$

This implies that $|N|^2 + SS^* = XA + YB^*$. Therefore,

$$\mathcal{R}(SS^*) \subseteq \mathcal{R}(|N|^2) + \mathcal{R}(X) + \mathcal{R}(Y) \subseteq \mathcal{R}(|N|^p) \subseteq \mathcal{R}(N),$$

while, $\mathcal{R}(SS^*) \subseteq \mathcal{R}(S) \subseteq \ker(N)$. This shows that $\mathcal{R}(SS^*) = \{0\}$ and therefore $S = 0$. That is, $\widetilde{T}_{s,t} = N \oplus R$. 

\[\Box\]
Lemma 2.7. Let \( T \in \mathcal{L}(\mathcal{H}) \) be w-hyponormal operator and \( T = U|T| \) be the polar decomposition of \( T \). If \( \mathcal{M} \) is an invariant subspace of \( T \) and \( T|_\mathcal{M} \) is an injective normal operator, then \( \mathcal{M} \) reduces \( T \).

Proof. Since \( T \) is w-hyponormal operator
\[
|\overline{T}^*| \leq |T| \leq |\overline{T}|
\]
Hence we have
\[
|N| \oplus |R^*| \leq |T| \leq |N| \oplus |R|
\]
by assumption. This implies that \( |T| \) is of the form \( |N| \oplus L \) for some positive operator \( L \).

Let \( U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \) be the matrix representation of \( U \) with respect to the decomposition \( \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \). Then the definition \( \overline{T} = |T|^{1/2}U|T|^{1/2} \) means that
\[
\begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} |N|^{1/2} & 0 \\ 0 & L^{1/2} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^{1/2} & 0 \\ 0 & L^{1/2} \end{pmatrix}
\]
Hence we have
\[
N = |N|^{1/2}U_{11}|N|^{1/2}, \quad |N|^{1/2}U_{12}L^{1/2} = 0, \quad L^{1/2}U_{21}|N|^{1/2} = 0.
\]
Since \( \ker(U) = \ker(T) = \ker(|T|) \), we have
\[
\ker(N) \subset \ker(U_{11}), \quad \ker(N) \subset \ker(U_{21}), \quad \ker(L) \subset \ker(U_{12}), \quad \ker(N) \subset \ker(U_{22}).
\]
Let \( N = V|N| \) be the polar decomposition of \( N \). Then \( \Re(U_{11} - V) \subset \ker(N) \). Hence for arbitrary \( x \in \Re(N) \), we have
\[
||x||^2 \geq ||Vx||^2 + ||U_{11} - V||^2, \quad \text{by Pythagoras's theorem},
\]
\[
= ||x||^2 + ||U_{11} - V||^2, \quad \text{since } V \text{ is unitary on } \Re(N).
\]
Therefore, we obtain \( V = U_{11} \). Since
\[
||x||^2 = ||Ux||^2 + ||U_{21}x||^2 = ||x||^2 + ||U_{21}x||^2 \quad \text{for } x \in \Re(N),
\]
we have \( U_{21} = 0 \). Also, we see that \( \Re(U_{12}) \subset \ker(N) \) by (2.3) and (2.6). Hence,
\[
T = U|T| = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} |N| & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} N & U_{12}L \\ 0 & U_{22}L \end{pmatrix}
\]
Since \( \Re(U_{12}) \subset \ker(N) = \{0\} \), we have \( U_{12} = 0 \) and so \( T = N \oplus T_1 \) on \( \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \). That is, \( \mathcal{M} \) reduces \( T \). \( \square \)
The following example shows that there exists a $w$-hyponormal operator $T$ such that $T|_{\mathcal{M}}$ is quasinormal but $M$ does not reduce $T$.

**Example 2.8.** Let $T$ be a bilateral shift on $\ell^2(\mathbb{Z})$ defined by $Te_n = e_{n+1}$ and $\mathcal{M} = \oplus_{n \geq 0} Ce_n$. Then $T$ is unitary and $T|_{\mathcal{M}}$ is isometry. However, $\mathcal{M}$ does not reduce $T$.

**Lemma 2.9.** Let $0 < s, t \leq 1$. Let $T = \begin{pmatrix} A & S \\ 0 & B \end{pmatrix}$ be a class $A(s, t)$ operator on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}$ is a $T$-invariant subspace such that the restriction $A = T|_{\mathcal{M}}$ is an injective normal operator. Then $\mathcal{M}$ reduces $T$.

**Proof.** Since $T$ belongs to class $A(s, t)$ and $0 < s, t \leq 1$, $T$ belongs to class $A$. Let $P$ be the orthogonal projection onto $\mathcal{M}$. Then we have

$$\begin{pmatrix} A^*A & 0 \\ 0 & 0 \end{pmatrix} = PT^*TP \leq P|T^2|P \quad \text{(since $T \in$ class $A$)}$$

$$\leq \begin{pmatrix} (A^*A)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(by Lemma 2.1)}$$

$$= \begin{pmatrix} A^*A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(since $A$ is normal).}$$

Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ be the $2 \times 2$ matrix representation of $|T^2|$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then we have $X = A^*A$ by the equality above. Since $|A^2| = T^*T^2$, we have

$$\begin{pmatrix} X^2 + YY^* & XY + YZ \\ ZY^* + Y^*X & Y^*Y + Z^2 \end{pmatrix} = \begin{pmatrix} A^2A^2 & A^2AS \\ S^*A^*A^2 & S^*S + B^*B^2 \end{pmatrix},$$

and hence $X^2 + YY^* = A^2A^2 = (A^*A)^2 = X^2$. This implies that $Y = 0$. Thus we have

$$\begin{pmatrix} |A|^4 & 0 \\ 0 & Z^2 \end{pmatrix} = |T^2|^2 = T^*T^TT$$

$$= \begin{pmatrix} A^*A^*AA & A^*(AS + SB) \\ (S^*A^* + B^*S^*)AA & (AS + SB)^*(AS + SB) + B^*B^*BB \end{pmatrix}$$

Since $A$ is an injective normal operator, we have $AS + SB = 0$ and $Z = |B^2|$. Now, since $T$ is a class $A$, we have

$$0 \leq |T^2| - |T|^2$$

$$= \begin{pmatrix} 0 & -A^*S \\ -S^*A & -S^*S + (|B^2| - |B|^2) \end{pmatrix}$$

and hence $A^*S = 0$. Thus the range of $S$ is included in $\ker(A^*) = \ker(A) = \{0\}$. Therefore, $S = 0$ and so $\mathcal{M}$ reduces $T$. $\square$

An operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is called quasiaffinity if $X$ is both injective and has a dense range. For $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$, if there exist quasiaffinities $X \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and
Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint which satisfies $T L = \hat{H}$. Hence $T$ is normal by \[28, \text{Theorem 1}\], and so $T$ is normal by \[6, \text{Theorem 1}\].

**Lemma 2.10.** Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ be normal operators. If there exist injective operators $X \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $Y \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ such that $TX = XS$ and $YT = SY$, then $T$ and $S$ are unitarily equivalent.

**Corollary 2.11.** Let $T \in \mathcal{L}(\mathcal{H})$ be $w$-hyponormal operator. Then $T = T_1 \oplus T_2$ on the space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T_1$ is normal and $T_2$ is pure and $w$-hyponormal; i.e., $T_2$ has no invariant subspace $M$ such that $T_2\vert_M$ is normal.

The next lemma was proved for dominant operators in \[28, \text{Theorem 1}\], for $p$-hyponormal operators in \[17\] and for log-hyponormal operators in \[16, \text{Lemma 3}\].

**Lemma 2.12.** Let $T \in \mathcal{L}(\mathcal{H})$ be $w$-hyponormal operator and let $S \in \mathcal{L}(\mathcal{H})$ be a normal operator. If there exists an operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ with dense range such that $TX = XS$, then $T$ is normal.

**Proof.** First, we decompose $T$ into normal and pure parts by $T = T_1 \oplus T_2$ with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $T_2 = U_2 |T_2|$ be the polar decomposition of $T_2$ and $\tilde{T}_2 = |T_2|^{\frac{1}{2}}U_2 |T_2|^{\frac{1}{2}}$. Let $\tilde{T}_2 = V_2 |\tilde{T}_2|$ be the polar decomposition of $\tilde{T}_2$ and $\tilde{T}_2 = |\tilde{T}_2|^{\frac{1}{2}}V_2 |\tilde{T}_2|^{\frac{1}{2}}$. Since $T_1$ is normal, we have $\tilde{T} = T_1 \oplus \tilde{T}_2$ and $\tilde{T} = T_1 \oplus \tilde{T}_2$. Let $W = |\tilde{T}_2|^{\frac{1}{2}}|T_2|^{\frac{1}{2}}$. Since $\ker(|T_2|) = \ker(T_2) = \{0\}$, by Corollary 2.11, $|T_2|^{\frac{1}{2}}$ is a quasiaffinity. Hence $T_2$ is injective and $W$ is a quasiaffinity such that $TW = WT_2$. Let $Y = I_{\mathcal{H}} \oplus W$. Then $\tilde{T}$ is hyponormal and $Y$ is a quasiaffinity such that $\tilde{T}Y = YT$. Thus we have $\tilde{T}(XY) = (XY)S$ and $XY$ has dense range. Hence $\tilde{T}$ is normal, by \[28, \text{Theorem 1}\], and so $\tilde{T}$ is normal by \[6, \text{Theorem 1}\].

**3 The Fuglede-Putnam Theorem**

In this section, we present some results concerning the Fuglede-Putnam theorem.

**Theorem 3.1.** Let $T \in \mathcal{L}(\mathcal{H})$ be $w$-hyponormal such that $\ker(T) \subset \ker(T^*)$ and $L \in \mathcal{L}(\mathcal{H})$ be a self-adjoint which satisfies $TL = LT^*$. Then $T^*L = LT$.

**Proof.** We first show that if $TL = LT^* = 0$, then $T^*L = LT = 0$. Since $\ker(T) \subset \ker(T^*)$, $\ker(T)$ reduces $T$ by \[4\]. $TL = 0$ implies that $\Re(L) \subseteq \ker(T) \subset \ker(T^*)$ and by taking the orthogonal complement, we obtain $\Re(T) \subset \ker(L)$. Hence we have $T^*L = LT = 0$.

Next, we prove the case in which $TL \neq 0$. Since $T$ is $w$-hyponormal, the Aluthge transform $\tilde{T}$ of $T$ is semi-hyponormal. Moreover, it satisfies

$$|\tilde{T}| \geq |T| \geq |T^*|.$$  \hspace{1cm} (3.1)
Put \( W = |L|^\frac{1}{2} |T|^\frac{1}{2} \). Then \( W \) is self-adjoint and satisfies
\[
\overline{T}W = W\overline{T}^*.
\] (3.2)

By the argument in the proof of Theorem 2 of [31], we have that the restriction \( \overline{T}|_{\mathcal{R}(W)} \) of \( \overline{T} \) to its invariant subspace \( \mathcal{R}(W) \) is normal and
\[
\overline{T}^*W = W\overline{T}.
\] (3.3)

Hence \( \mathcal{R}(W) \) reduces \( \overline{T} \), by Lemma 2.7, and so \( \overline{T} \) is of the form \( \overline{T} = N \oplus S \) on \( \mathcal{R}(W) \oplus \ker(W) \), where \( N \) is normal. By Corollary 2.5 and Lemma 2.7, \( T = N \oplus B \), for some \( w \)-hyponormal operator \( B \). Let \( W = W_1 \oplus 0 \) and \( L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \) on \( \mathcal{R}(W) \oplus \ker(W) \). Then \( L_2 = L_3 = 0 \) and \( L_4 = 0 \) follows from the equality \( W = |L|^\frac{1}{2} |T|^\frac{1}{2} \). By assumption, \( NL_4 = L_1 N^* \), we have \( N^*L_1 = L_1 N \) by Fuglede-Putnam theorem and so \( T^*L = LT \).

\( \square \)

Example 3.2. Let \( \mathcal{H} = \bigoplus_{n=0}^\infty \mathbb{C}^2 \) and define an operator \( R \) on \( \mathcal{H} \) by
\[
R(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus A \cdot x_{-2} \oplus A x_{-1}^{(0)} \oplus B x_0 \oplus B x_1 \oplus \cdots,
\]
where \( A = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Then \( R \) is \( w \)-hyponormal. Moreover, \( \mathcal{R}(E) = \ker(R) \), \( E \) is not a self-adjoint and \( \ker(R) \neq \ker(R^*) \), where \( E \) is the Riesz idempotent with respect to 0, see [32, Example 13]. Let \( T = R \) and \( L = P \) be the orthogonal projection onto \( \ker(T) \). Then \( T \) is \( w \)-hyponormal operator and \( TL = 0 = LT^* \), but \( T^*L \neq LT \). Hence the kernel condition \( \ker(T) \subset \ker(T^*) \) is necessary for Theorem 3.1.

Corollary 3.3. Let \( T \in \mathcal{L}(\mathcal{H}) \) be \( w \)-hyponormal such that \( \ker(T) \subset \ker(T^*) \). If \( X \in \mathcal{L}(\mathcal{H}) \) and \( TX = XT^* \), then \( T^*X = XT \).

Proof. Let \( X = L + iK \) be the cartesian decomposition of \( X \). Then we have \( TL = LT^* \) and \( TK = KT^* \), by the assumption. By Theorem 3.1, we have \( T^*L = LT \) and \( T^*K = KT \). This implies that \( T^*X = XT \).

\( \square \)

If we use the \( 2 \times 2 \) matrix trick, we easily deduce the following result.

Corollary 3.4. Let \( T^* \in \mathcal{L}(\mathcal{H}) \) be \( w \)-hyponormal and \( S \in \mathcal{L}(\mathcal{H}) \) be \( w \)-hyponormal with \( \ker(T^*) \subset \ker(T) \) and \( \ker(S) \subset \ker(S^*) \). If \( X \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \) and \( XT = SX \), then \( XT^* = S^*X \).

Proof. Put \( A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \) on \( \mathcal{H} \oplus \mathcal{H} \). Then \( A \) is a \( w \)-hyponormal operator on \( \mathcal{H} \oplus \mathcal{H} \) that satisfies \( BA^* = AB \) and \( \ker(A) \subset \ker(A^*) \). Hence we have \( BA = A^*B \), by Corollary 3.3, and so \( XT^* = S^*X \).

\( \square \)

Example 3.5. Let \( S = T^* = R \) as in Example 3.2 and \( X = P \) be the orthogonal projection onto \( \ker(S) \). Then \( SX = 0 = XT \), but \( S^*X = XT^* \). Hence the kernel condition is necessary for Corollary 3.4.
Theorem 3.6. Let $T \in \mathcal{L}(\mathcal{H})$ be such that $T^*$ is $p$-hyponormal or log-hyponormal. Let $S \in \mathcal{L}(\mathcal{K})$ be w-hyponormal with ker$(S) \subset$ ker$(S^*)$. If $XT = SX$, for some $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then $XT^* = S^*X$.

Proof. Let $T^*$ be a $p$-hyponormal operator for $p \geq \frac{1}{2}$ and let $U|T|$ be the polar decomposition of $T$. Then the Aluthge transform $T^*$ of $T^*$ is hyponormal and satisfies

$$|\widetilde{T^*}| \geq |T|^2 \geq |\widetilde{T}|,$$

(3.4)

$$X^\prime \widetilde{T} = SX^\prime,$$

(3.5)

where $X^\prime = XU|T|^\frac{1}{2}$. Using the decompositions $\mathcal{H} = \ker(X^\prime) \oplus \ker(X^\prime)$ and $\mathcal{K} = \overline{\mathbb{R}(X^\prime)} \oplus \mathbb{R}(X^\prime)^\perp$, we see that $\widetilde{T}, S$ and $X^\prime$ are of the form

$$\widetilde{T} = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, \quad X^\prime = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $T_1^*$ is hyponormal, $S_1$ is w-hyponormal with ker$(S_1) \subset$ ker$(S_1^*)$ and $X_1$ is a one-one operator with dense range. Since $X^\prime \widetilde{T} = SX^\prime$, we have

$$X_1T_1 = S_1X_1.$$

(3.6)

Hence $T_1$ and $S_1$ are normal by Theorem 3.6 of [4], so that $T_2 = 0$, by Lemma 12 of [31] and $S_2 = 0$ by Lemma 2.7. Then $|T| = |T_1| \oplus P$, for some positive operator $P$, by (3.4) and $U = \begin{pmatrix} U_1 & U_2 \\ 0 & U_3 \end{pmatrix}$ by Lemma 13 of [31]. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ be a $2 \times 2$ matrix representation of $X$ with respect to the decomposition $\mathcal{H} = \ker(X^\prime) \oplus \ker(X^\prime)$ and $\mathcal{K} = \overline{\mathbb{R}(X^\prime)} \oplus \mathbb{R}(X^\prime)^\perp$. Then $X^\prime = XU|T|^\frac{1}{2}$ implies that $X_1 = X_{11}U_1|T_1|^\frac{1}{2}$ and hence ker$(T_1) \subset$ ker$(X_1) = \{0\}$. This shows that $T_1$ is one-one and hence it has dense range, so that $U_2 = 0$ and $T = T_1 \oplus T_4$ for some hyponormal operator $T_4^*$ by [31, Lemma 13]. Since

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X^\prime = XU|T|^\frac{1}{2} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} U_1|T_1|^\frac{1}{2} & 0 \\ 0 & U_3|T_4|^\frac{1}{2} \end{pmatrix},$$

we deduce the following assertions.

$$X_{12}U_2|T_4|^\frac{1}{2} = 0; \text{ hence } X_{12}T_3 = 0 \text{ because } T_4 = U_3|T_4|.$$  

$$X_{21}U_1|T_1|^\frac{1}{2} = 0; \text{ hence } X_{12} = 0 \text{ because } U_1|T_1|^\frac{1}{2} \text{ has dense range.}$$

$$X_{22}U_3|T_4|^\frac{1}{2} = 0; \text{ hence } X_{22}T_3 = 0.$$

The assumption $XT = SX$ tell us that,

$$X_{11}T_1 = S_1X_{11},$$

$$X_{12}T_4 = S_1X_{12} = 0,$$

$$X_{22}T_4 = S_3X_{22} = 0.$$

Since $T_1$ and $S_1$ are normal, we have $X_{11}T_1^* = S_1^*X_{11}$, by Fuglede-Putnam theorem. The $p$-hyponormality of $T_4^*$ shows that $\mathbb{R}(T_4^*) \subset \mathbb{R}(T_4)$. Also, we have ker$(S_3) \subset$ ker$(S_3^*)$. Hence,
we also have $X_{12}T^*_4 = S^*_1X_{12} = 0$ and $X_{22}T^*_4 = S^*_3X_{22} = 0$. This implies that $XT^* = X_{11}T^*_1 \oplus 0 = S^*_1X_{11} \oplus 0 = S^*X$.

Next, we prove the case where $T^*$ is $p$-hyponormal for $0 < p \leq \frac{1}{2}$. Let $X'$ be as above. Then $T^*$ is $(p + \frac{1}{4})$-hyponormal and satisfies $X'\bar{T} = SX'$. Use the same argument as above.

We obtain $\bar{T} = T_1 \oplus T_3$ on $\mathcal{H} = \ker(X')^\perp \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where $T_1$ is an injective normal operator and $S_1$ is also normal. Hence we have $T = T_1 \oplus T_4$ for some $p$-hyponormal $T_4$, by Lemma 13 of [31]. Again using the same argument as above, we obtain $X_{21} = 0, X_{11}T^*_1 = S^*_1X_{11}, X_{12}T^*_4 = S^*_1X_{12} = 0$ and $X_{22}T^*_4 = S^*_3X_{22} = 0$. Hence we have $XT^* = S^*X$.

Finally, we assume that $T^*$ is log-hyponormal. Let $\bar{T}$ and $X'$ be as above. Then $X'\bar{T} = SX'$ and $\bar{T}$ is semi-hyponormal and satisfies

$$|\bar{T}| \leq |T| \leq |\bar{T}^*|.$$ 

By the same argument as above, we have $\bar{T} = T_1 \oplus T_3$ on $\mathcal{H} = \ker(X')^\perp \oplus \ker(X')$ and $S = S_1 \oplus S_3$ on $\mathcal{H} = \mathfrak{R}(X')^\perp \oplus \mathfrak{R}(X')^\perp$, where $T_1$ is an injective normal operator, $S_1$ is normal, $T^*_3$ is invertible semi-hyponormal and $S_3$ is $w$-hyponormal with $\ker(S_3) \subset \ker(S^*_3)$.

By Lemma 13 of [31], we have that $T$ is of the form $T = T_1 \oplus T_4$, for some log-hyponormal $T_4$. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $X' = XU[T]^\frac{1}{2}$ implies that $X_{12} = 0, X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11},$ hence $X_{11}T^*_1 = S^*_1X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T^*_1 \oplus 0 = S^*_1X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved. 

\[\square\]

**Example 3.7.** Let $R$ be an operator such that $\ker(R)$ does not reduce $R$ and let $P$ be the orthogonal projection onto $\ker(R)$. Then $P$ does not commute with $T$; otherwise $\mathfrak{R}(R) = \ker(R)$ reduce $T$. Hence $PR \neq 0 = RP$. It is easy to see that $RP = PR^* = 0$ but $R^*P \neq PR(\neq 0)$ because $\mathfrak{R}(R^*P) \subset \mathfrak{R}(R^*) \subset \ker(R^\perp) = I - P$. If we put $T = R$, then the assertion of Theorem 3.1 does not hold for such $T$. Also, if we put $T = R^*,$ $S = I - P$ and $X = P$, then $XT = PR^* = 0 = (I - P)P = SX$. However, $XT^* = PR \neq 0 = (I - P)P = S^*X$. Hence the assertion of Theorem 3.6 does not hold for such $T$.

**Theorem 3.8.** Let $T \in \mathcal{L}(\mathcal{H})$ be such that $T^*$ is an injective $w$-hyponormal. Let $S \in \mathcal{L}(\mathcal{H})$ be dominant. If $XT = SX$, for some $X \in \mathcal{L}(\mathcal{H},\mathcal{H})$. Then $XT^* = S^*X$.

**Proof.** Assume that $T^*$ is an injective $w$-hyponormal and let $U[T]$ be the polar decomposition of $T$. Let $\bar{T}$ be the aluthge transform of $T$ and $X' = XU[T]^\frac{1}{2}$. Then $X'\bar{T} = SX'$ and $\bar{T}$ is semi-hyponormal and satisfies

$$|\bar{T}| \leq |T| \leq |\bar{T}^*|.$$ 

By the same argument in the proof of Theorem 3.6, we conclude that $\bar{T} = T_1 \oplus T_3$ on $\mathcal{H} = \ker(X')^\perp \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where $T_1$ is an injective normal operator and $S_1$ is also normal, $T^*_3$ is invertible $w$-hyponormal and $S_3$ is dominant. Hence by Lemma 2.7, we have that $T$ is of the form $T = T_1 \oplus T_4$ for some $w$-hyponormal $T_4$. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$
Then $X' = XU[T]^\frac{1}{2}$ implies that $X_{12} = 0, X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved. □

Example 3.9. Let $T^* = R$ as in Example 3.2. Let $X = P$ be the orthogonal projection onto $\ker(T^*)$ and $S = I - P$. Then $SX = 0 = XT^*$, but $0 \neq S^*X \neq XT^*$. Hence the injectivity condition is necessary for Theorem 3.8.

Example 3.10. Let $\{e_n\}_{n=-\infty}^\infty$ be a complete orthonormal system for $H$. We denote the orthogonal projection onto $\mathbb{C}e_n$ by $P_n$. Let $W$ be a weighted shift on $H$ defined by

$$W e_n = \begin{cases} \sqrt{2}e_{n+1}, & \text{if } n \geq 0; \\ e_{n+1}, & \text{if } n < 0. \end{cases}$$

Then $W^*W - WW^* = P_0$. Define an operator $T$ on a Hilbert space $H = H \oplus \mathbb{C}e_0$ by

$$T = \begin{pmatrix} W & P_0 \\ 0 & 0 \end{pmatrix}.$$ 

Then $T$ is class $A$, see [31, Example 1]. It is easy to see that

$$\ker(T) = \mathbb{C}(-e_{-1} \oplus e_0) \quad \text{and} \quad \ker(T^*) = \{0\} \oplus \mathbb{C}e_0.$$

Hence $T$ does not reduces $T$ and therefore the assertions of Theorems 3.8, 3.6 and Corollary 3.4 are not necessarily true for class $A$ operators.

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References


An Extension of Fuglede-Putnam Theorem for w-Hyponormal Operators


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