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AFFINE COMPLETENESS OF SOME MODULES

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Abstract

In this paper we generalize some affine completeness properties of abelian groups to modules over commutative domains.

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1 Introduction

The question of affine completeness of modules over commutative rings has been studied by many authors. The case of abelian groups is well known thanks to W.Nöbauer [8], K.Kaarli [3] and A.Saks [9]. As pointed out in [1], most of the results on the affine completeness of abelian groups can be generalized to modules over commutative principal ideal domains because the abelian groups and these modules are similar due to the fact that the underlying ring structure of the ring of integers is that of a commutative principal ideal domain. According to K.Kaarli and A.Pixley, there is only one exception. Indeed, when proving that an abelian group of rank one with bounded torsion part is not affine complete, one relies on the countability of the ring of integers [1, Theorem 5.2.22]. This argument does not hold if it has to do with a ring which is uncountable. This leads to the following problem raised in [1, Problem 5.2.29].

Problem: *Does there exist an affine complete torsion free module of rank 1 over a commutative principal ideal domain?*

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The aim of this work is to answer this question. Moreover, we will give a generalization of some other theorems from abelian group's affine completeness theory to modules over a commutative domain. Throughout this paper R will designate a commutative ring with 1, and A a left module over R. For any a in A, the annihilator of a is designated by $\{r \in R : ra = 0\}$ and denoted by Ann(a) . It is clear that Ann(a) is an ideal of R as well as Ann(A)= $\cap_{a\in A}$ Ann(a). An element a of A for which Ann(a) is nontrivial, will be said to be a torsion element of A. Clearly the set T of torsion elements of A is a submodule of A. The module A is bounded if Ann(A) is nontrivial and in this case any nonzero element of Ann(A) is called an exponent of A. Otherwise we say that it is unbounded.

2 Preliminary results

2.1 Compatible functions

In this section, we collect some basic results about universal algebra and affine completeness of modules. We particularly follow [7] for the basic concepts of universal algebra and [1] for the notions about affine completeness of algebras.

An operation on a nonempty set A is a function $f : A^n \longrightarrow A$ where n is a natural integer. We then say that f is an n-ary operation or an operation of rank n on A. For $n = 0$, f has exactly one value, it is a constant. Thus we call 0-ary operations constants and identify them with their unique values. We call operations of rank 1 unary operations and identify them with functions from A into A. Binary operations are operations of rank 2.

Let $\mathcal{R}$ be a binary relation on A and n a natural number. A map $g : X \longrightarrow A$ where X is a subset of A^n is said to be compatible with $\mathcal{R}$ if the following condition is satisfied: for any n-tuples $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ in X such that $(x_i, y_i) \in \mathcal{R}$ for $i = 1, \dots, n$ then $(g(x_1, \dots, x_n), g(y_1, \dots, y_n)) \in \mathcal{R}$. The map g is compatible with a set E of binary relations on A if it is compatible with each relation $\mathcal{R} \in E$. We also say that g is E-compatible.

An algebra is an ordered pair $\mathcal{A} =< A, F >$ such that A is a nonempty set and $F =< f_i : i \in I >$ where f_i is an operation on A for each $i \in I$. A is called the universe of $\mathcal{A}$ and f_i is referred to as a fundamental operation of $\mathcal{A}$ for each $i \in I$. A congruence of the algebra $\mathcal{A}$ is an equivalence relation θ on the set A, that is compatible with the fundamental operations of $\mathcal{A}$. We denote by Con($\mathcal{A}$) the set of congruences of the algebra $\mathcal{A}$. When the context is clear, the algebra $\mathcal{A} =< A, F >$ is simply called the algebra A.

A commutative ring R with unit is an algebra $< R, +, \cdot, -, 0, 1 >$, such that $+$ and $\cdot$ are binary operations on R which satisfy the usual properties of associativity, commutativity and distributivity, 0 is a constant operation representing the neutral element of the operation $+$, 1 is a constant operation representing the neutral element of the operation $\cdot$, $-$ is a unary operation on R sending each element a of R to the element $-a$ such that $a + (-a) = 0$. An R-module M over the ring R is an algebra $< M, +, -, 0, (f_r)_{r\in R} >$, where $< M, +, -, 0 >$ is an abelian group and each f_r is a unary function on M such that for all $a, b \in M$ and for all r, s

and $t \in R$ the following equalities hold:

$$\begin{aligned} f_r(f_s(a)) &= f_t(a) \text{ where } r \cdot s = t \text{ in } R \\ f_r(a+b) &= f_r(a) + f_r(b) \\ f_r(a) + f_s(a) &= f_t(a) \text{ where } r + s = t \text{ in } R \\ f_1(a) &= a. \end{aligned}$$

For a subset X of the R-module M we denote by $< X >$ the submodule of M generated by X. As noted in [1], the compatibility criterion takes the following form in the case of modules. An n-ary operation f on the R-module M is compatible with Con(M) if and only if

$$f(\mathbf{a}) - f(\mathbf{b}) \in < a_1 - b_1, \dots, a_n - b_n >$$

for all $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in A^n$.
Clearly this means that for all $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in A^n$ there exist $s_1, \dots, s_n \in R$ depending on $\mathbf{a}$ and $\mathbf{b}$, such that

$$f(\mathbf{a}) - f(\mathbf{b}) = s_1(a_1 - b_1) + \cdots + s_n(a_n - b_n).$$

An n-ary function $f : M^n \longrightarrow M$ is zero preserving if $f(0, \dots, 0) = 0$. Obviously every Con(M)-compatible operation of the R-module M can be represented as a sum of a zero preserving function and a constant function. Also, if f is an n-ary zero preserving Con(M)-compatible function on an R-module M and $(a_1, \dots, a_n) \in M^n$, then there exist $r_1, \dots, r_n$ in R such that

$$f(a_1, \dots, a_n) = r_1 a_1 + \cdots + r_n a_n$$

where the coefficients r_i depend on the n-tuple $(a_1, \dots, a_n)$.

Let us recall the compatible function extension property which has played an important role in the theory of affine complete algebras.

Definition 2.1. [2] An algebra A satisfies the finite extension property, if for any natural number n, any finite subset X of A^n and y in $A^n \backslash X$, each Con(A)-compatible map $f : X \longrightarrow A$ there exists a Con(A)-compatible map $g : X \cup \{y\} \longrightarrow A$ agreeing with f on X.

The above notion was originally discovered in [3] for unary operations on abelian groups and established for all operations on any algebra in [2] later on.

2.2 Polynomial operations and affine complete algebras

By composition of operations on the set A is meant the construction of an n-ary operation h from k given n-ary operations $f_1, \dots, f_k$ and a k-ary operation g, by the rule $h(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n))$. The operation h is called the composition of f and g. For each $i \leq n$, the n-ary projections $p_i^n : A^n \longrightarrow A$ are defined by $p_i^n(x_1, \dots, x_n) = x_i$ for all $x_1, \dots, x_n$ in A.

Definition 2.2. Let A be a nonempty set. A clone on A is a set of operations on A which contains all projection operations on A and is closed under composition of operations. As noted in [7], for each set C of operations on a set A, there exists the smallest clone containing it. This clone is called the clone generated by C. Of particular interest is the clone of polynomial operations of an algebra $\mathcal{A} =< A; F >$ which is defined as the clone generated by the fundamental operations and the constant 0-ary operations of $\mathcal{A}$. This clone is denoted by Pol($\mathcal{A}$)

Definition 2.3. An algebra $\mathcal{A}$ is affine complete if the set of Con($\mathcal{A}$)-compatible operations of $\mathcal{A}$ is exactly Pol($\mathcal{A}$).

Remark 2.4. Polynomial operations on an algebra $\mathcal{A}$ are known to be Con($\mathcal{A}$)-compatible [7].

Definition 2.5. For every $n \in \mathbb{N}$, an n-ary function f on an algebra $\mathcal{A}$ is said to be a local polynomial of $\mathcal{A}$ if it can be interpolated by a polynomial operation on every finite subset X of A^n, that is, there exists a polynomial operation $g : A^n \longrightarrow A$ whose restriction to X is equal to f. An algebra $\mathcal{A}$ is said to be locally affine complete if every Con($\mathcal{A}$)-compatible function on $\mathcal{A}$ is a local polynomial of $\mathcal{A}$.

Remark 2.6. It is clear from the above definition that an affine complete algebra is locally affine complete.

Using [7, Theorem 4.6], it is easy to see, as noted in [1], that for an R-module A an n-ary polynomial operation on A is an operation $f : A^n \longrightarrow A$ satisfying the following property: there exists $r_0 \in A, r_1, \dots, r_n \in R$ such that the following equality holds:

$$f(x_1, \dots, x_n) = r_1x_1 + \cdots + r_nx_n + r_0 \text{ for all } (x_1, \dots, x_n) \in A^n.$$

Remark 2.7. It is important to notice that $r_0, \dots, r_n$ do not depend on the n-tuple $(x_1, \dots, x_n)$. Since constant operations are polynomials, it follows that when studying affine completeness of modules, we may restrict to the case of zero preserving functions.

Remark 2.8. Polynomial operations on modules are just the linear functions.

We recall here the well known notion of semisimple modules and a result on affine complete modules that will be used below.
An R-module A is called semisimple if it satisfies the following equivalent conditions:
1. A is direct sum of simple R-modules;
2. Every submodule of A is a direct summand.

Theorem 2.9. [1] *A semisimple R-module A is locally affine complete if and only if it has no simple homogeneous component.*

3 Affine completeness of free modules of rank 1

We will now give a proof of [1, Theorem 5.2.22] which avoids the use of the compatible function extension property. We first recall that if A is a bounded abelian group, then Ann(A) is generated by a unique positive element e called the exponent of A.

Theorem 3.1. *An abelian group of rank* 1 *with a bounded torsion part is not locally affine complete.*

Proof. Let A be an abelian group of rank 1. We suppose that it is locally affine complete. We will construct a unary function on A, compatible with Con(A) and which cannot be interpolated by a polynomial operation in a specified finite subset of A. We will hence obtain a contradiction. Let T be the torsion part of A so that A/T is a torsion free group of rank 1. Therefore A/T is an infinite cyclic group. Let $a+T$ be one of its generators. Define the function $g : A/T \longrightarrow A/T$ by setting $g(k(a+T)) = k^2a+T$, for each $k \in \mathbb{Z}$. It is easy to see that g is a zero preserving function on A/T which is Con(A/T)-compatible. Now, since T is bounded, let $\exp(T) = e$ be its exponent. It follows that $e(x+t) = ex$ for all $(x,t) \in A \times T$. Hence $e(x+T)$ is the set $\{ex\}$ for every $x \in A$. This fact allows us to define a function $f : A \longrightarrow A$ by sending x to the unique element of $e(g(x+T))$. For simplicity, we choose $f(x) = e(g(x+T))$. This function is a zero preserving function and it induces a function eg on the quotient group A/T. If $ka+T$ is an element of A/T, then $eg(ka+T) = ek^2a+T$. So if f were a local polynomial on A, then eg would be a local polynomial on A/T. Let $X = \{x_1+T,\dots,x_n+T\}$ be a finite subset of A/T. Since f is a local polynomial then there exists an integer s in R such that the restriction of the polynomial operation $y \longmapsto sy$ to $\tilde{X} = \{x_1,\dots,x_n\}$ is equal to f on $\tilde{X}$. For $i = 1,\dots,n$, we therefore have $eg(x_i+T) = ek_i^2(a+T) = sk_ia+T$, where $x_i+T = k_ia+T$. Taking $\tilde{X} = \{ka|k = 0,1,2\}$ we obtain $sk(a+T) = ek^2(a+T)$, $k \in \{0,1,2\}$. Since $a+T$ is an element of infinite order, the latter equality would imply that $s = e$ and $2s = 4e$, which is clearly absurd since e is a nonzero integer.

It now remains to prove that f is a Con(A)-compatible function on A. Let $b,c \in A$ and $g(b+T) = b_1+T, g(c+T) = c_1+T$ with $b_1,c_1 \in A$. Then $f(b) = eb_1$ and $f(c) = ec_1$. Since g is Con(A)-compatible and zero preserving, there exists an integer r such that

$$g(b+T)-g(c+T) = r(b-c+T).$$

Consequently there also exists $t \in T$ such that

$$b_1-c_1 = r(b-c)+t.$$

Then

$$f(b)-f(c) = e(r(b-c)+t) = er(b-c) \in < b-c >$$

proving that f is Con(A)-compatible. □

We will use the idea of the above proof in other situations below.

Theorem 3.2. *A torsion free module of rank* 1 *over a commutative domain is not locally affine complete.*

Proof. Again we proceeed by contradiction. Let A be a torsion free module of rank 1 over a commutative domain R. Since by [1, Theorem 5.2.9] any 1-dimensional vector space is not affine complete, we can suppose that R is not a field of two elements. Let (x) be a basis of A. Then each element $a \in A$ has the form $a = rx$ for some $r \in R$. Let us define the unary function $f : A \longrightarrow A, rx \longrightarrow r^2x$. Then f is Con(A)-compatible on A. Indeed if $a_1 = r_1x$ and $a_2 = r_2x$ are elements of A, then

$$f(a_1) - f(a_2) = r_1^2x - r_1^2x = (r_1 + r_2)(a_1 - a_2) \in < a_1 - a_2 > .$$

Let us assume that A is locally affine complete, which implies that f is a local polynomial on A. Since R is not a field with two elements we choose $\alpha \in R\backslash\{0,1\}$ and consider the finite set $X = \{0, x, \alpha x\}$. Since f is interpolated by a polynomial operation on X, we can find $s,t \in R$ such that $f(a) = sa + t$ for all $a \in X$. Therefore

$$\begin{cases} f(0) = t = 0 \\ f(x) = sx + t = x \\ f(\alpha x) = s\alpha x + t = \alpha^2 x. \end{cases} \tag{3.1}$$

Putting $t = 0$ and using the fact that (x) is a basis, we have $s = 1$ and therefore $\alpha = \alpha^2$. Now since R is a domain it follows that $\alpha = 0$ or $\alpha = 1$, which yields a contradiction. □

Corollary 3.3. *A torsion free module of rank 1 over a commutative domain is not affine complete.*

Proof. A non trivial cyclic module over a commutative principal ideal domain R is not locally affine complete. □

Remark 3.4. Actually, we do not need the compatible function extension property to prove that torsion free abelian groups of rank 1 are not locally affine complete. This is the main result of the present paper. This property is revelant because the compatible function extension property was originally introduced in order to prove that a free abelian group of rank 1 is not affine complete. Now [1, Problem 5.2.29] is solved by the above theorem.

We will now prove the following theorem that gives a more general result about local affine completeness of modules with one generator.

Theorem 3.5. *A nontrivial cyclic module over a commutative principal ideal domain R is not locally affine complete.*

Proof. Let A be a nontrivial cyclic module over a commutative principal ideal domain R. Then the R-module A is isomorphic to $R/(r)$ for some $r \in R$, and is generated by $x = 1 + (r)$. Let us define $f : A \longrightarrow A$ by setting $f(a) = s^2x$ where $a = s + (r) = s(1 + (r)) = sx$ and $s \in R$. We will show by contradiction that f is Con(A)-compatible but is not a local polynomial of A. The compatibility of f with Con(A) is straightforward. Let us first assume that there exists an element $\alpha \in R$ such that $\alpha(\alpha - 1) \notin R$. Set $X = \{0, x, \alpha x\}$ and suppose that f is a local polynomial on A. Then there exist $t_1, t_2 \in R$ such that $f(a) = t_1a + t_2$ for all $a \in X$. It is then clear that $t_2 = 0$, so that $f(x) = t_1x = x$ and $f(\alpha x) = \alpha^2x = t_1\alpha x$. Hence $\alpha x = \alpha^2x$ and $\alpha - \alpha^2 \in (r)$ which is a contradiction.

To complete this proof, we now suppose that for every $\alpha \in R$ we have $\alpha(\alpha - 1) \in (r)$. Let $r = p_1^{\beta_1} \cdots p_n^{\beta_n}$ be the factorisation of r into irreducible elements. Suppose that $\beta_i > 1$ for some $i \in \{1, \dots, n\}$. Then, since $p_j(p_j - 1) \in (r)$ for all $j \in \{1, \dots, n\}$, there exists $t_i \in R$ such that $p_i(p_i - 1) = rt_i$. But $r = p_iu$ where $u = p_1^{\beta_1} \cdots p_i^{\beta_i - 1} \cdots p_n^{\beta_n}$. This yields $p_i(p_i - 1) = p_iut_i$.

Consequently $p_i - 1 = ut_i$ and thus p_i cannot divide u, which contradicts the assumption that $\beta_i > 1$. Therefore, the factorisation of r is of the form $r = p_1 \cdots p_n$. This factorisation implies that

$$R/(r) \cong R/(p_1) \oplus \cdots \oplus R/(p_n)$$

which proves that A is a semisimple module with simple homogeneous components. Hence A cannot be locally affine complete [1, Theorem 5.2.13]). □

The next question is what can be said about modules of rank 1 whose torsion part is bounded. The answer is given by the following theorem which generalizes [1, Theorem 5.2.22] to modules over a commutative domain. We will use the same idea as in the proof of Theorem 3.1.

Theorem 3.6. *Let A be a module of rank* 1 *over a commutative domain with a nonzero torsion part T such that Ann(T) is nontrivial. Then A is not locally affine complete.*

Proof. We will construct a Con(A)-compatible unary function on A which is not a local polynomial. First, R is not a field with two elements since a vector space has no nontrivial torsion part. Denoting by T the torsion part of the R-module A, then A/T is a torsion free R-module of rank 1. Let $(a+T)$ be a basis of A/T, and define $f : A/T \longrightarrow A/T$ by setting $f(ka+T) = k^2a+T, k \in R$. Then f is clearly Con(A/T)-compatible. Let $r \in$ Ann(T) be nonzero. Then for all x in A and t in T, we have $r(x+t) = rt$, so that $r(x+T)$ is a well defined element of A. This fact allows us to define a function $g : A \longrightarrow A$ by the formula $g(x) = r(f(x+T))$. This function induces a function rf on the quotient A/T. Indeed if g were a local polynomial on A then rf would be a local polynomial on A/T. Since f is Con(A/T)-compatible rf is Con(A/T)-compatible and g is Con(A)-compatible. Suppose that A is locally affine complete. Then choose $\alpha \in R$ such that $\alpha \notin \{0,1\}$ and set $X = \{T, a+T, \alpha a+T\}$. By the local affine completeness of A, rf can be interpolated by a polynomial operation on X, so there exists $s \in R$ such that

$$rk^2(a+T) = sk(a+T), \text{ for } k = 0, 1, \alpha.$$

Since $a+T$ is not a torsion element, this implies that $r = s$ and $r\alpha^2 = s\alpha$, thus $r = 0$. This is impossible since we have assumed that $r \neq 0$. □

4 Affine completeness of modules of rank greater than one

In this section we generalize some affine completeness results for modules of rank 1 over a commutative domain to modules over a commutative domain. We first give preliminary lemmas that we will need for our main result given by Theorem 4.4.

Lemma 4.1. *Let A be a module over a ring R. Assume that for any d in A the annihilator of the quotient A/Rd is the same as the annihilator of A. Then A is affine complete if and only if any unary Con(A)-compatible function on A is a polynomial.*

Proof. We only have to prove that binary Con(A)-compatible operations on A are polynomials [1, Theorem 5.2.3]. Let f be a binary Con(A)-compatible operation. Without loss of generality, we may assume that $f(0,0) = 0$. Since unary Con(A)-compatible operations are polynomials, for each $x, y \in A$ there exist $k_x, l_y, b_x, c_y \in R$ such that

$$f(x,y) = k_y x + b_y \tag{4.1}$$

$$f(x,y) = l_x y + c_x. \tag{4.2}$$

Since $f(0,0) = 0$, we must have $b_0 = c_0 = 0$. This shows that $f(x,0) = c_x = k_0 x$ and $f(0,y) = b_y = l_0 y$ for all x, y in A. From (4.1) and (4.2) we obtain the equality:

$$(k_y - k_0)x = (l_x - l_0)y. \tag{4.3}$$

We want to prove that the both sides of equation (4.3) are 0. Suppose that this condition is not satisfied, then for example the left side is not identically zero. Therefore there exists d in A such that the left side of (4.3) is not 0. Thus $k_d x \neq k_0 x$ for some x. In A/Rd the right side of (4.3) vanishes and the left side must also vanish. This shows that $k_d - k_0$ is in the annihilator of A/Rd. By hypothesis this annihilator is the same as the annihilator of A which contradicts the fact that the left side of (4.3) is nontrivial for $x \neq d$. □

Lemma 4.2. *Let A be a module over a commutative domain R. If A contains a free submodule of rank ≥ 2, then A is affine complete if and only if each unary Con(A)-compatible function on A is a polynomial.*

Proof. We only need to prove that for each d in A the annihilator of A is the same as the annihilator of A/Rd, that is, they are all trivial. Let a be an element of the annihilator of A/Rd. Choose a free pair $\{x, y\}$ in A. We have $ax = td$ and $ay = sd$ for some t and s in R. Clearly $sax - tay = 0$. Hence, due to the freeness of $\{x, y\}$ we have $ta = sa = 0$. Observe that, since R is a domain, then $a \neq 0$ implies that $t = s = 0$. We conclude that $ax = 0$ which is absurd. Hence $a = 0$ and this leads to the result that the annihilator of A/Rd is trivial. □

The next corollary is direct.

Corollary 4.3. *Let A be a module over a commutative domain R. If A contains a free submodule of rank ≥ 2, then every quotient of A by a cyclic submodule is unbounded.*

We are now able to prove a generalized result.

Theorem 4.4. *Let A be a module over a commutative domain. If A contains a free direct summand of rank ≥ 2, then A is affine complete.*

Proof. Suppose that A has a free direct summand of rank at least 2. Then A contains a submodule of rank at least 2. Hence A satisfies the hypothesis of lemma 4.2; so it is sufficient to prove that every unary function on A, compatible with Con(A), is a polynomial operation. From the hypothesis we know that

$$A = A_1 \oplus F$$

where F is a free module whose rank ≥ 2. Because of the compatibility of f with Con(A), there are functions $g : A_1 \longrightarrow A_1$ and $h : F \longrightarrow F$ compatible with Con(A_1) in A_1 and with Con(F) in F respectively, such that :

$$f(x+y) = g(x) + h(y) \tag{4.4}$$

for every $x \in A_1$ and $y \in F$. We may also suppose that f is zero preserving, which is the case for g and h. But F is affine complete by [1, Theorem 5.2.8], hence there exists r in R such that $h(y) = ry$ for every y in F. Moreover, the compatibility of f and g respectively with Con(A) and Con(A_1) implies that, for every $x \in A_1$ and $y \in F$, there exists $s_{x+y}, t_x \in R$ such that

$$f(x+y) = s_{x+y}(x+y), \quad g(x) = t_x x$$

Equation (4.4) thus implies that

$$s_{x+y}x = t_x x, \quad s_{x+y}y = ry$$

for all $x_1 \in A_1$ and $y \in F$. Taking y as a nonzero element we see that $s_{x+y} = r$ for all $x \in A_1$ and $y \in F$. We hence get that

$$f(x+y) = r(x+y)$$

for all $x \in A_1$ and $y \in F$. □

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