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On Derivations of Prime Near-Rings

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Abstract

In this paper we investigate derivations satisfying certain differential identities on 3-prime near-rings, and we provide examples to show that the assumed restrictions cannot be relaxed.

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1 Introduction

Throughout this paper N will denote a zero symmetric left near-ring (i.e.; a left near ring N satisfying the property $0 \cdot x = 0$ for all $x \in N$). Note that the left distributivity in N gives $x \cdot 0 = 0$ for all $x \in N$. For any $x, y \in N$ the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. Unless otherwise specified, we will use the word near-ring to mean zero symmetric left near-ring. An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [13], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. A near-ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$; and N is said to be 2-torsion free if $x \in N$ and $x + x = 0$ implies $x = 0$. Recently, there has been a great deal of

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work concerning commutativity of prime and semi-prime rings with derivations satisfying certain differential identities (see [1], [7], [9],[11], where further references can be found). In view of these results many authors have investigated commutativity of 3-prime near-rings satisfying certain conditions involving derivations (see [2],[3],[4], [8],[9], [13] etc). In this paper we continue this line of investigation.

2 Main results

Theorem 2.1. *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d such that $[d(x),y] = [x,d(y)]$ for all $x,y \in N$, then N is a commutative ring.*

Proof. We are given that

$$[d(x),y] = [x,d(y)] \quad \text{for all } x,y \in N. \tag{2.1}$$

Replacing x by $d(y)x$ in (2.1), we obtain

$$[d(d(y)x),y] = [d(y)x,d(y)] = d(y)[x,d(y)] = d(y)[d(x),y] \quad \text{for all } x,y \in N,$$

so that

$$d(d(y)x)y - yd(d(y)x) = d(y)d(x)y - d(y)yd(x) \quad \text{for all } x,y \in N. \tag{2.2}$$

In view of ([8], Lemma 1), equation (2.2) implies

$$d(y)d(x)y + d^2(y)xy - yd^2(y)x - yd(y)d(x) = d(y)d(x)y - d(y)yd(x).$$

Since $yd(y) = d(y)y$ by (2.1), the last equation reduces to

$$d^2(y)xy = yd^2(y)x \quad \text{for all } x,y \in N. \tag{2.3}$$

Replacing x by tx in (2.3), we get

$$d^2(y)txy = yd^2(y)tx = d^2(y)tyx \quad \text{for all } x,y,t \in N,$$

which leads to

$$d^2(y)N[x,y] = \{0\} \quad \text{for all } x,y \in N.$$

Since N is 3-prime, we conclude that for each $y \in N$

$$d^2(y) = 0 \text{ or } y \in Z(N);$$

and substituting $d(y)$ for y in (2.1) gives

$$[d(x),d(y)] = 0 \quad \text{for all } x,y \in N.$$

Therefore, N is a commutative ring by Theorem 3 of [8].

Theorem 2.2. *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d satisfying*
(i) $[d(x),y)] = [d(x),d(y)]$ for all $x,y \in N$,
or
(ii) $[x,d(y)] = [d(x),d(y)]$ for all $x,y \in N$,
then N is a commutative ring.

Proof. (*i*) We are given that

$$[d(x),y] = [d(x),d(y)] \quad \text{for all} \quad x,y \in N. \tag{2.4}$$

Replacing y by $d(x)y$ in (2.4), we obtain

$$[d(x),d(d(x)y)] = [d(x),d(x)y] = d(x)[d(x),y] = d(x)[d(x),d(y)] \quad \text{for all} \quad x,y \in N,$$

so that

$$d(x)d(d(x)y) - d(d(x)y)d(x) = d(x)d(x)d(y) - d(x)d(y)d(x) \quad \text{for all} \quad x,y \in N. \tag{2.5}$$

In view of ([8], Lemma 1), equation (2.5) implies

$$d(x)d(x)d(y) + d(x)d^2(x)y - d^2(x)yd(x) - d(x)d(y)d(x) = d(x)d(x)d(y) - d(x)d(y)d(x).$$

The last equation reduces to

$$d(x)d^2(x)y = d^2(x)yd(x) \quad \text{for all} \quad x,y \in N. \tag{2.6}$$

Replacing y by ty in (2.6), we get

$$d^2(x)tyd(x) = d(x)d^2(x)ty = d^2(x)td(x)y \quad \text{for all} \quad x,y,t \in N,$$

which leads to

$$d^2(x)N[d(x),y] = \{0\} \quad \text{for all} \quad x,y \in N.$$

Since N is 3-prime, we conclude that for each $x \in N$

$$d^2(x) = 0 \;\text{ or }\; d(x) \in Z(N) \quad \text{for all} \quad y \in N,$$

so that

$$d^2(x) \in Z(N) \quad \text{for all} \quad x \in N.$$

Substituting $d(y)$ for y in (2.4) gives

$$[d(x),d(y)] = 0 \quad \text{for all} \quad x,y \in N.$$

Therefore, N is a commutative ring by Theorem 3 of [8].
(*ii*) Using similar techniques as above, we can prove that N is a commutative ring.

Theorem 2.3. *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d satisfying*
(*i*) $[d(x),y)] = -[d(x),d(y)]$ *for all* $x,y \in N$,
or
(*ii*) $[x,d(y)] = -[d(x),y]$ *for all* $x,y \in N$,
then N is a commutative ring.

Proof. (i) Assume that

$$[d(x),d(y)] = -[d(x),y] \quad \text{for all } x,y \in N. \tag{2.7}$$

Replacing y by $d(x)y$ in (2.7), we obtain

$$[d(x),d(d(x)y)] = -(d(x)[d(x),y]) = d(x)(-[d(x),y]) = d(x)[d(x),d(y)] \quad \text{for all } x,y \in N,$$

so that

$$d(x)d(d(x)y) - d(d(x)y)d(x) = d(x)d(x)d(y) - d(x)d(y)d(x) \quad \text{for all } x,y \in N. \tag{2.8}$$

Since equation (2.8) is the same as equation (2.5), using the same techniques as we have used in the proof of Theorem 2.2, we get the required result.
(ii) Using similar arguments as above, we can prove that N is a commutative ring.

We now consider differential identities involving anti-commutators instead of commutators. Our results are of a different kind.

Theorem 2.4. *If N is a 2-torsion free 3-prime near-ring, N admits no nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x,y \in N$.*

Proof. Suppose d is a nonzero derivation such that

$$d(x)d(y) + d(y)d(x) = 0 \quad \text{for all } x,y \in N. \tag{2.9}$$

Then

$$d(x) \circ d(x) = 2d(x)^2 = 0, \text{ so } d(x)^2 = 0 \quad \text{for all } x \in N.$$

Thus,

$$d(d(x)d(x))y = 0 = (d^2(x)d(x) + d(x)d^2(x))y = d^2(x)d(x)y + d(x)d^2(x)y \quad \text{for all } x,y \in N;$$

and in particular,

$$d^2(x)d(x)d(y) = -d(x)d^2(x)d(y) \quad \text{for all } x,y \in N. \tag{2.10}$$

Now replace y by $d(x)y$ in (2.9), thereby obtaining

$$d(x)(d^2(x)y + d(x)d(y)) + (d(x)d(y) + d^2(x)y)d(x) = 0 \quad \text{for all } x,y \in N,$$

which can be rewritten as

$$d(x)d^2(x)y + d(x)(d(x)d(y) + d(y)d(x)) + d^2(x)yd(x) = 0 \quad \text{for all } x,y \in N. \tag{2.11}$$

In light of (2.9), equation (2.11) yields $d^2(x)yd(x) = -d(x)d^2(x)y$ for all $x,y \in N$, so that $d^2(x)d(y)d(x) = -d(x)d^2(x)d(y)$ and hence

$$-d^2(x)d(x)d(y) = -d(x)d^2(x)d(y) \quad \text{for all } x,y \in N. \tag{2.12}$$

In view of (2.10) and (2.12), we now have $d(x)d^2(x)d(y) = 0$ for all $x,y \in N$; and by Lemma 1.4(iii) of [6],

$$d(x)d^2(x) = 0 \quad \text{for all } x \in N.$$

It now follows from (2.11) that $d^2(x)Nd(x) = \{0\}$ for all $x \in N$; and by 3-primeness of N, $d^2(x) = 0$ for all $x \in N$. By Lemma 3 of [8], we get $d = 0$, which contradicts our original assumption that $d \neq 0$.

Theorem 2.5. *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d such that $d(x)\circ d(y) = x\circ y$ for all $x,y \in N$, then $Z(N) = \{0\}$ and N is not a ring.*

Proof. Assume that

$$d(x)\circ d(y) = x\circ y \quad \text{for all } x,y\in N. \tag{2.13}$$

Replacing y by yz in (2.13), where $z \in Z(N)$, we get

$$d(x)\circ d(yz) = (x\circ y)z = (d(x)\circ d(y))z$$

and therefore

$$d(x)(d(y)z + yd(z)) + (yd(z) + d(y)z)d(x) = d(x)d(y)z + d(y)d(x)z. \tag{2.14}$$

Using ([8], Lemma 1), equation (2.14) becomes

$$d(x)yd(z) + yd(z)d(x) = 0 \quad \text{for all } x,y\in N, z\in Z(N). \tag{2.15}$$

Since $d(Z(N)) \subset Z(N)$, from equation (2.15) it follows that

$$(d(x)y + yd(x))Nd(z) = 0 \quad \text{for all } x,y\in N, z\in Z(N). \tag{2.16}$$

From (2.16) we have that $d(Z(N)) = \{0\}$ or $d(x)y + yd(x) = 0$ for all $x,y \in N$. In the latter case, $d(x)d(y) + d(y)d(x) = 0$ for all $x,y\in N$, so this case cannot occur by Theorem 2.4.
If $d(Z(N)) = \{0\}$, then for $y \in Z(N)$ equation (2.13) implies that $2xy = x\circ y = d(x)\circ d(y) = 0$ for all $x\in N$. Since N is 2-torsion free, then $xy = 0$ for all $x \in N$. Hence $yNy = 0$ for all $y \in Z(N)$ and the 3-primeness of N yields $Z(N) = \{0\}$.
Suppose N is a ring. Then from $2d(x)^2 = 2x^2$ we have $d(x)^2 = x^2$ for all $x\in N$. Now $d(x)\circ d(x^2) = 2x^3$, hence

$$d(x)(d(x)x + xd(x)) + (d(x)x + xd(x))d(x) = 2x^3 \quad \text{for all } x\in N,$$

so that

$$d(x)^2x + d(x)xd(x) + d(x)xd(x) + xd(x)^2 = 2x^3 \quad \text{for all } x\in N.$$

Accordingly,

$$x^3 + 2d(x)xd(x) + x^3 = 2x^3 \quad \text{for all } x\in N,$$

so $2d(x)xd(x) = 0$ and therefore

$$d(x)xd(x) = 0 \quad \text{for all } x\in N.$$

Pre-multiplying and post-multiplying by $d(x)$ gives $d(x)^2xd(x)^2 = 0$, so $x^5 = 0$ for all $x\in N$. But this cannot happen, for by a well-known result of Levitzki (cf. [12]) a prime ring cannot be nil of bounded index. Thus, our assumption that N is a ring yields a contradiction.

Corollary 2.6. *Let N be a 2-torsion free 3-prime near-ring with $Z(N) \neq \{0\}$. Then N admits no nonzero derivation d such that $d(x)\circ d(y) = x\circ y$ for all $x,y\in N$.*

Theorem 2.7. *Let N be a 2-torsion free 3-prime near-ring with $Z(N) \neq \{0\}$. Then N admits no nonzero derivation such that $d(x)\circ y = x\circ d(y)$ for all $x,y\in N$.*

Proof. Suppose there does exist such a nonzero derivation. Let $0 \neq z \in Z(N)$. From $x \circ d(yz) = d(x) \circ yz = 0$ it follows that

$$x \circ d(yz) = d(x) \circ yz = (d(x) \circ y)z = (x \circ d(y))z \quad \text{for all } x, y \in N. \tag{2.17}$$

Therefore

$$x(d(y)z + yd(z)) + (yd(z) + d(y)z)x = xd(y)z + d(y)xz \quad \text{for all } x, y \in N. \tag{2.18}$$

In view of ([8], Lemma 1), equation (2.18) reduces to

$$xyd(z) + yd(z)x = 0 \quad \text{for all } x, y \in N,$$

and thus

$$(x \circ y)Nd(z) = 0 \quad \text{for all } x, y \in N. \tag{2.19}$$

Since N is 3-prime, from equation (2.19) it follows that

$$d(Z(N)) = \{0\} \text{ or } x \circ y = 0 \quad \text{for all } x, y \in N. \tag{2.20}$$

Assume that $d(Z(N)) = \{0\}$ and let $y \in Z(N)$. As $d(x) \circ y = d(x) \circ d(y) = 0$, then $d(x)y = 0$ and therefore

$$d(x)Ny = 0 \quad \text{for all } x \in N,\ y \in Z(N).$$

Using the 3-primeness of N, because of $d \neq 0$, we get $Z(N) = \{0\}$, a contradiction.
If $x \circ y = 0$ for all $x, y \in N$, then $xy = y(-x)$ and therefore

$$xyt = y(-x)t = yt(-x) \quad \text{for all } x, y, t \in N.$$

Replacing x by $-x$ in the last equation we arrive at

$$yN[x, t] = \{0\} \quad \text{for all } x, y, t \in N. \tag{2.21}$$

Since N is 3-prime, equation (2.21) yields $x \in Z(N)$ for all $x \in N$ and thus $d(x) \in Z(N)$. Hence $d(N) \subset Z(N)$ and ([6], Theorem 2.1) implies that N is a commutative ring; and we have

$$d(x)y = xd(y) \quad \text{for all } x, y \in N. \tag{2.22}$$

Replacing y by y^2 gives

$$d(x)y^2 = xd(y^2) = xd(y)y + xyd(y) \quad \text{for all } x, y \in N,$$

and applying (2.22) gives $xyd(y) = 0$ for all $x, y \in N$.
Since a commutative prime ring has no nonzero divisors of zero, this yields $d(y) = 0$ for all $y \neq 0$. Therefore $d = 0$, a contradiction.

Using similar techniques to those used in Theorem 2.7, we can prove the following.

Theorem 2.8. *Let N be a 2-torsion free 3-prime near-ring with $Z(N) \neq \{0\}$. Then N admits no nonzero derivation such that*
(i) $d(x) \circ y = d(x) \circ d(y)$ for all $x, y \in N$, or
(ii) $x \circ d(y) = d(x) \circ d(y)$ for all $x, y \in N$.

The following example proves that the primeness hypothesis in Theorems 2.1, 2.2, 2.3, 2.4, 2.7 and 2.8 is not superfluous.

Example 2.9. Let S be a noncommutative 2-torsion free left near-ring. Let us consider $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}$ and $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is straightforward to check that N is a 2-torsion free non-3-prime left near-ring with $Z(N) \neq \{0\}$. Moreover, d is a nonzero derivation of N satisfying the conditions:

$$\text{(i) } [d(A), B] = [A, d(B)], \quad \text{(ii) } [d(A), B] = [d(A), d(B)], \quad \text{(iii) } [A, d(B)] = [d(A), d(B)]$$

$$\text{(iv) } [d(A), B] = -[d(A), d(B)], \quad \text{(v) } [A, d(B)] = -[d(A), B], \quad \text{(vi) } d(A) \circ d(B) = 0$$

$$\text{(vii) } d(A) \circ B = A \circ d(B), \quad \text{(viii) } d(A) \circ B = d(A) \circ d(B), \quad \text{(ix) } A \circ d(B) = d(A) \circ d(B)$$

for all $A, B \in N$, and N is not a commutative ring.

The following example proves that the primeness hypothesis in Theorem 2.5 cannot be omitted.

Example 2.10. Let S be a noncommutative 2-torsion free left near-ring. Let us consider $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in S \right\}$ and $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$. It is obvious that N is a non-3-prime left near-ring. Furthermore, d is a nonzero derivation of N such that

$$d(A) \circ d(B) = A \circ B \quad \text{for all } A, B \in N,$$

but $Z(N) \neq \{0\}$.

Theorem 2.11. *Let N be a 2-torsion free 3-prime near-ring. Then N admits no nonzero derivation d such that $[d(x), d(y)] = x \circ y$ for all $x, y \in N$.*

Proof. Suppose that d is a derivation such that

$$[d(x), d(y)] = x \circ y \quad \text{for all } x, y \in N.$$

Taking $y = x$ gives $x^2 = 0$ for all $x \in N$. Therefore $0 = x(x+y)^2 = xy(x+y) = xyx$ for all $x, y \in N$, so that $xNx = \{0\}$ for all $x \in N$. By primeness, we get $N = \{0\}$ and hence $d = 0$.

References

[1] M. Ashraf and N. Rehman, On commutativity of rings with derivations. *Result. Math.* **12** (2002), 3-8.

[2] M. Ashraf and A. Shakir, On (σ, τ)-derivations of prime near-rings. *Arch. Math. (Brno)* **40** (2004), no. 3, 281-286.

[3] M. Ashraf and A. Shakir, On (σ, τ)-derivations of prime near-rings-II. *Sarajevo J. Math.* **4** (2008), no. 16, 23-30.

[4] K. I. Beidar, Y. Fong and X. K. Wang, Posner and Herstein theorems for derivations of 3-prime near-rings. *Comm. Algebra* **24** (1996), no. 5, 1581-1589.

[5] H. E. Bell, Certain near-rings are rings. *J. London Math. Soc.* **4** (1971),264-270.

[6] H. E. Bell, On derivations in near-rings II. *Kluwer Academic Publishers Netherlands* (1997), 191-197.

[7] H. E. Bell and M. N. Daif, Commutativity and strong commutativity preserving maps. *Canad. Math. Bull.* **37** (1994), 443-447.

[8] H. E. Bell and G. Mason, On derivations in near-rings. *North-Holland Mathematics Studies* **137** (1987), 31-35.

[9] H. E. Bell and G. Mason, On derivations in near-rings and rings. *Math. J. Okayama Univ.* **34** (1992), 135-144.

[10] A. Boua and L. Oukhtite, Derivations on prime near-rings. *Int. J. Open Probl. Comput. Sci. Math.* **4** (2011), no. 2, 162-167.

[11] M. N. Daif and H. E. Bell, Remarks on derivations on semiprime rings. *Int. J. Math. & Math. Sci.* **15** (1992), 205-206.

[12] A. A. Klein, A new proof of a result of Levitzki. *Proc. Amer. Math. Soc.* **81** (1981), no. 1, 8.

[13] X. K. Wang, Derivations in prime near-rings. *Proc. Amer. Math. Soc.* **121** (1994), 361-366.