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Quasicrystals, Almost Periodic Patterns, Mean-periodic Functions and Irregular Sampling

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Abstract

Three properties of *quasicrystals* will be proved in this essay. Quasicrystals are *almost periodic patterns* (such patterns are carefully defined below). Every *mean-periodic function* whose spectrum is contained in a quasicrystal is almost periodic. Finally simple quasicrystals are *universal sampling sets*.

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Introduction

Quasicrystals have been discovered by Dan Shechtman in 1982 [32] but a mathematical model was needed to explain this controversial discovery. In 1985 M. Duneau and A. Katz [7] elaborated the *cut and projection* model of quasicrystals. They constructed remarkable point sets yielding the extraordinary diffraction pictures found by Shechtman. These point sets were also named quasicrystals. Penrose tilings, as constructed by R. Penrose in 1974, provide beautiful examples of quasicrystals. N.G. de Bruijn found in 1981 that some Penrose tilings could be constructed by the *cut and projection* scheme [2].

The same *cut and projection* construction of *model sets* was already discussed in my book [20]. This book has been published ten years before Shechtman's discovery. In a sense my *model sets* preluded quasicrystals. But I should not be praised for this finding since, as Pierre Deligne once said, the *cut and projection* scheme was already implicit in algebraic number theory (see the definition of Pisot numbers, for example).

My *model sets* were welcomed as models of quasicrystals but the main message in my book went unnoticed. I suggested that *model sets* provide us with new and improved *grids*. In numerical analysis *grids* are used to *sample functions or distributions*. Grids can be uniform (then a grid is a lattice) or non-uniform. *Model sets* are non-uniform grids.

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These grids improve on uniform grids as it will be shown in the seventh section of this essay. Mean-periodic functions, the problem of the uniqueness of trigonometric expansion, the problem of spectral synthesis, and the problem of the irregular sampling of signals and images are four instances where these new grids are playing an important role, as it was already observed in [16] and [20].

In the first part of this essay a new definition of almost periodicity will be proposed and it will be proved that model sets are *almost periodic patterns.* Connecting *model sets* to pure and applied mathematics is an exciting challenge which will be met in the second part. The first part is entirely new while the results of the second part have already been published elsewhere [16], [20]. They are given a new perspective in this essay.

Some standard facts on almost periodic functions are listed for the reader's convenience in the first section. In the second section *generalized almost periodic measures* will be defined and studied. We aim at relating the arithmetical properties of a Delone set Λ to the analytical properties of the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$. We say that Λ is an almost periodic pattern if the measure σ_Λ is a *generalized almost periodic measure* (Definition 2.32). We prove that *model sets are almost periodic patterns*.

In Sections 6, 7 and 8 *model sets* are used as a tool to solve some problems in pure mathematics and in signal processing. The first problem deals with mean-periodic functions. Let Λ be a set of points in $\mathbb{R}^n$ and let C_Λ be the vector space of all *mean-periodic functions* f whose spectrum is simple and contained in Λ. It will be shown that the behavior at infinity of the functions $f \in C_\Lambda$ strongly depends on the *arithmetical properties* of Λ. It will be proved that if Λ is a model set then every $f \in C_\Lambda$ is an almost periodic function in the sense given by H. Bohr. A stronger statement is obtained in Section 6. The recently discovered sampling properties of simple quasicrystals are unveiled in Section 7. This essay ends with a beautiful theorem by Raphaël Salem and Antoni Zygmund. This theorem says that a Cantor set E_θ constructed with a dissection ratio $1/\theta$ is a set of uniqueness for the trigonometric expansion if and only if θ is a Pisot number. Model sets are seminal in the simple proof of the uniqueness which will be given in Section 8. The proof relies on the existence of the embedded grids $\Gamma_j = \theta^{-j}\Lambda$, $j \geq 0$, where $\Lambda = \Lambda_\theta$ is the associated model set. We also consider the problem of spectral synthesis. Here also the ladder of embedded grids Γ_j is playing a seminal role but the proof of the main theorem is only sketched.

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1 Almost periodic functions and measures

Model sets are almost periodic patterns. As it was stressed by J. Lagarias in [11] this statement cannot be true if a naïve definition of almost periodicity is being used. The definition of almost periodic patterns will be unveiled in Section 2.

1.1 Almost periodic functions

The reader who is familiar with the theory of almost periodic functions in the sense of Bohr is urged to skip this subsection. The next one should also be skipped if one has previously read *La théorie des distributions* by Laurent Schwartz. Section 2 will contain new and original material. In Lagarias [11] almost periodic functions in the sense of Bohr are called "uniformly almost periodic functions".

The Fourier transform $\mathcal{F}(f) = \hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-ix\cdot\xi) f(x)\,dx \tag{1.1}$$

and the Fourier inversion formula reads

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(ix\cdot\xi)\hat{f}(\xi)\,d\xi \tag{1.2}$$

whenever this makes sense.

A continuous function $f:\mathbb{R}^n \to \mathbb{C}$ is almost periodic (in the sense given by Harald Bohr) if for each positive ε there exists a *relatively dense* set Λ_ε of ε-*almost periods* τ of f. These two concepts (relatively dense and ε-almost period) are now defined.

A subset $\Lambda \subset \mathbb{R}^n$ is *relatively dense* if there exists a positive R such that each ball with radius R (whatever be its center) contains at least a point λ in Λ. This definition was introduced by Besicovitch.

We say that τ is an ε-*almost period* of a function $f : \mathbb{R}^n \mapsto \mathbb{C}$ if

$$\sup_{x\in\mathbb{R}^n} |f(x+\tau) - f(x)| \le \varepsilon \tag{1.3}$$

The space of almost periodic functions on $\mathbb{R}^n$ equipped with the norm $\|f\|_\infty$ is a Banach space which will be denoted by $\mathcal{E}$. Here and in what follows, $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$.

Let f be an almost periodic function. The orbit of f under translations is the collection $\mathcal{E}$ of all functions $f(\cdot - y)$, $y \in \mathbb{R}^n$.

Lemma 1.1. *The orbit $\mathcal{E}$ of an almost periodic function f is a precompact set for the topology of uniform convergence on $\mathbb{R}^n$.*

In other words for every sequence $x_j \in \mathbb{R}^n$ there exists a subsequence x_{j_k} such that $f(x - x_{j_k})$ converges to an almost periodic function g uniformly on $\mathbb{R}^n$.

Every finite trigonometric sum $P(x) = \sum_{\lambda\in S} c(\lambda)\exp(i\lambda\cdot x)$ is an almost periodic function (S being an arbitrary finite subset of $\mathbb{R}^n$). H. Bohr proved that a function $f:\mathbb{R}^n \to \mathbb{C}$ is almost periodic if and only if, for each $\varepsilon > 0$, there exists a finite trigonometric sum $P_\varepsilon(x) = \sum_{\lambda\in S(\varepsilon)} c(\lambda,\varepsilon)\exp(i\lambda\cdot x)$ such that $\|f - P_\varepsilon\|_\infty \le \varepsilon$.

A detour to the Bohr compactification of $\mathbb{R}^n$ is needed to better understand what an almost periodic function looks like.

The Bohr compactification of $\mathbb{R}^n$ is denoted by $\mathcal{G}_n$. It is the dual group (in the sense of Pontryagin duality) of the group $\mathbb{R}^n$ equipped with the discrete topology. The elements of the compact group $\mathcal{G}_n$ are the characters χ on $\mathbb{R}^n$ which are defined now.

Definition 1.2. *A function $\chi : \mathbb{R}^n \mapsto \mathbb{T}$ is a character on $\mathbb{R}^n$ if it maps the additive group $\mathbb{R}^n$ to the multiplicative group $\mathbb{T}$ of complex numbers of modulus 1 and if it is a group homomorphism:* $\chi(x+y) = \chi(x)\chi(y)$ $(\forall x, y \in \mathbb{R}^n)$.

A character does not need to be continuous. The product $\chi\chi'$ between two characters is still a character. As it was said above the group $\mathcal{G}_n$ is the multiplicative group of all such characters. Then $\mathbb{R}^n$ is a subgroup of $\mathcal{G}_n$ since every continuous character is a character. Here each $\omega \in \mathbb{R}^n$ is identified to the character χ_ω defined by $\chi_\omega(x) = \exp(i\omega \cdot x)$. Moreover $\mathbb{R}^n$ is dense in $\mathcal{G}_n$. The canonical embedding of $\mathbb{R}^n$ into $\mathcal{G}_n$ will be denoted by $\mathcal{J}_n$. We obviously have $\mathcal{G}_n = \mathcal{G}_1 \times \ldots \times \mathcal{G}_1$. With these notations we have

Lemma 1.3. *Let F be a continuous function on $\mathcal{G}_n$. Then its restriction $f = F \circ \mathcal{J}_n$ to $\mathbb{R}^n$ is an almost periodic function. Conversely any almost periodic function f on $\mathbb{R}^n$ is the restriction to $\mathbb{R}^n$ of a continuous function F on $\mathcal{G}_n$. This F is unique and is the extension of f to $\mathcal{G}_n$.*

In [8] the mapping $f \mapsto F$ is named the Bohr mapping. The Bohr mapping is an isometric isomorphism between the Banach space of almost periodic functions on $\mathbb{R}^n$ and the Banach space of all continuous functions on $\mathcal{G}_n$.

Let f be an almost periodic function. The ball centered at x, with radius R is denoted by $B(x,R)$ and the constant c_n is the inverse of the volume of the unit ball. Then the limit

$$\mathcal{M}(f) \;=\; \lim_{R\to+\infty} c_n R^{-n} \int_{B(x,R)} f(y)\,dy \tag{1.4}$$

is attained uniformly in x.

Moreover $\mathcal{M}(f) = \int_{\mathcal{G}_n} F(x)\,dx$ when f and F are related by Lemma 1.3.

Is the Bohr compactification of $\mathbb{R}^n$ actually needed in Lemma 1.3 ? Two definitions will be used to answer this question.

Definition 1.4. *A compact abelian group G is a compactification of $\mathbb{R}^n$ if G is the dual group (in the sense of Pontryagin duality) of a dense subgroup $\Gamma \subset \mathbb{R}^n$. We then denote by $J : \mathbb{R}^n \mapsto G$ the canonical embedding.*

In other words G is the compact group of all weak characters $\chi : \Gamma \mapsto \mathbb{T}$. For answering our question we need to define the spectrum S of an almost periodic function f (see Definition 1.5 below). The "smallest" group G on which f extends continuously is the dual of the additive group Γ generated by the spectrum S of f.

For each $\omega \in \mathbb{R}^n$, $\exp(-i\omega \cdot x) f(x)$ is also an almost periodic function. This remark paves the road to the definition of the Fourier coefficient of f at the frequency $\omega \in \mathbb{R}^n$. This Fourier coefficient is denoted by $\hat{f}(\omega)$ and is defined as

$$\hat{f}(\omega) \;=\; \mathcal{M}[\exp(-i\omega \cdot x) f(x)] \tag{1.5}$$

The notation $\hat{f}(\omega)$ is confusing since $\hat{f}(\omega)$ is not the value at ω of the distributional Fourier transform of f. However we have $\hat{f}(\omega) = \hat{F}(\omega)$ when $f = F \circ \mathcal{J}_n$ as in Lemma 1.3. Here $\hat{F}(\omega)$ is the ordinary Fourier coefficient at the frequency ω of the continuous function F.

If f is almost periodic, so is $|f|^2$, and one has

$$\mathcal{M}(|f|^2) = \sum |\hat{f}(\omega)|^2 . \tag{1.6}$$

Definition 1.5. *The set S of frequencies ω for which $\hat{f}(\omega) \neq 0$ is at most a countable set. This set S is named the spectrum of f.*

In a sense f is the sum of its Fourier series expansion:

$$f(x) \sim \sum_{\omega \in S} \hat{f}(\omega)\, e^{i\omega\cdot x} . \tag{1.7}$$

The Fourier series expansion (2.7) of f becomes an ordinary Fourier expansion when f is viewed as a continuous function F on $\mathcal{G}_n$.

Lemma 1.6. *Let ϵ_k, $k \in \mathbb{Z}$ be a sequence of real numbers tending to 0 as $|k|$ tends to infinity and $\lambda_k = k + \epsilon_k$. Let $\phi(x)$ be a continuous and compactly supported function of the real variable x. Then $f(x) = \sum_{-\infty}^{\infty} \phi(x - \lambda_k)$ is never an almost periodic function unless $\epsilon_k = 0, k \in \mathbb{Z}$.*

The proof is obvious. We set $g(x) = \sum_{-\infty}^{\infty} \phi(x-k)$ and observe that $\mathcal{M}|f-g| = 0$. But g is a periodic function. Therefore $f-g = 0$ since $f-g$ is assumed to be almost periodic. We will see that f is a generalized almost periodic function (Section 2).

One might be tempted to say that the distributional Fourier transform $\hat{f}$ of f is $(2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$ where δ_ω is the Dirac mass at ω. This is not true at this naïve level. We cannot write $\hat{f} = (2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$ since this sum of Dirac masses is not defined unless $\sum_{\omega\in S} |\hat{f}(\omega)|$ is finite. If it is the case the two definitions of the Fourier transform of an almost periodic functions agree as indicated in the following lemma:

Lemma 1.7. *Let us assume that f is an almost periodic function and that $\sum_{\omega\in S} |\hat{f}(\omega)|$ is finite. Then the distributional Fourier transform of f is $(2\pi)^n \sum_{\omega\in S} \hat{f}(\omega)\delta_\omega$. Conversely if the distributional Fourier transform of a function $f \in L^\infty$ is a finite atomic measure $\sigma = \sum_{\omega\in S} c(\omega)\delta_\omega$ then f is an almost periodic function and its Fourier coefficients are $\hat{f}(\omega) = (2\pi)^{-n} c(\omega)$.*

The convolution product (named the Eberlein convolution in [8]) between two almost periodic functions is defined by

$$(f \star g)(x) = \mathcal{M}[f(x-\cdot)g(\cdot)] \tag{1.8}$$

and we have

$$\mathcal{M}(f \star g) = \mathcal{M}(f)\mathcal{M}(g) \tag{1.9}$$

If f and g are two almost periodic functions on $\mathbb{R}^n$ and if F and G denote their extensions to $\mathcal{G}_n$ then the restriction of $F * G$ to $\mathbb{R}^n$ is $f \star g$. The identity (1.9) becomes obvious since $\int_{\mathcal{G}_n} (F * G)\, dx = (\int_{\mathcal{G}_n} F\, dx)(\int_{\mathcal{G}_n} G\, dx)$.

We now compute the Fourier coefficients of the convolution product between two almost periodic functions.

Lemma 1.8. *The convolution product $h = f \star g$ between two almost periodic functions is an almost periodic function and the Fourier coefficients of h are given by*

$$\hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \tag{1.10}$$

We write $f_\omega(x) = \exp(-i\omega \cdot x) f(x)$ and use the same notations for g and $h = f \star g$. Then it suffices to observe that $h_\omega = f_\omega \star g_\omega$ and to use (1.9). The Fourier series of h is absolutely convergent: $\sum |\hat{h}(\omega)|$ is finite.

The following simple remark will be used later on. One defines an almost periodic function on $\mathbb{Z}$ the same way we used on $\mathbb{R}$ and we have

Lemma 1.9. *The restriction to $\mathbb{Z}$ of an almost periodic function f on $\mathbb{R}$ is an almost periodic function g on $\mathbb{Z}$. Moreover if for every $k \in \mathbb{Z}$ we set $h(k) = \int_k^{k+1} f(x)\,dx$ then the mean $\mathcal{M}_{\mathbb{R}}(f)$ of f on $\mathbb{R}$ is equal to the mean $\mathcal{M}_{\mathbb{Z}}(h)$ of h on $\mathbb{Z}$.*

Conversely let g be an almost periodic function on $\mathbb{Z}$ and let f be the function which is continuous, affine on each interval $[k, k+1]$, $k \in \mathbb{Z}$, and which coincides with g on $\mathbb{Z}$. Then f is almost periodic on $\mathbb{R}$.

If $\Lambda \subset \mathbb{R}^n$ is a model set and ϕ a compactly supported continuous function, then $f(x) = \sum_{\lambda \in \Lambda} \phi(x - \lambda)$ is not an almost periodic function. This was observed by J. Lagarias [11] and paves the road to the definitions which are given in Section 2.

1.2 Almost periodic measures

The reader who is familiar with the theory of distributions by Laurent Schwartz is invited to skip this subsection and to jump to the next one. Almost periodic measures are called "uniformly almost periodic measures" by J. Lagarias.

Schwartz proposed the following definition of an almost periodic distribution.

Definition 1.10. *A distribution S is almost periodic if for every test function $\phi \in \mathcal{D}$ the convolution product $S * \phi$ is an almost periodic function in the sense of Bohr.*

Here $\mathcal{D}$ stands for the vector space of compactly supported C^∞ functions. This definition can be adapted to almost periodic measures. The only difference is that the class $\mathcal{D}$ of test functions is replaced by the class $\mathcal{E}$ of compactly supported continuous functions. It will be proved in this essay that this definition of almost periodic measures is too demanding since the sum σ_Λ of Dirac masses on a model set Λ is not an almost periodic measure in general. This was already observed by J. Lagarias. In contrast σ_Λ is a *generalized almost periodic measure* which motivates the definition given in Section 2. Lagarias proved that for every compactly supported continuous function ϕ the convolution product $\sigma_\Lambda * \phi$ is a Besicovitch almost periodic function. This can be found in [11] and this result will be improved in this essay. In this subsection the well known properties of almost periodic measures are listed for the reader's convenience.

Definition 1.11. *A Borel measure μ on $\mathbb{R}^n$ is almost periodic if for every compactly supported continuous function g the convolution product $\mu * g = f$ is an almost periodic function.*

If μ is an almost periodic measure the closed graph theorem implies

$$\sup_{x\in\mathbb{R}^n}\int_{B(x)}|d\mu| = C < \infty \tag{1.11}$$

Here $B(x)$ is the ball centered at x with radius 1. We then say that μ is a *translation bounded measure*. We then have

Lemma 1.12. *A translation bounded measure μ is almost periodic if and only if μ is an almost periodic distribution.*

The Banach space of translation bounded measures is equipped with the norm $\sup_{x\in\mathbb{R}^n}|\mu|[B(x)]$. The weak convergence of a bounded sequence μ_j of translation bounded measures is defined by the duality with compactly supported continous functions.

Here is an example of an almost periodic measure. Let w be a real valued continous function of $x\in\mathbb{R}$ and let us assume that $w(x+1) = w(x)$. Let Λ_w consist of all real numbers $k + w(\sqrt{2}k),\ \ k\in\mathbb{Z}$. We define a measure σ_w by $\sigma_w = \sum_{\lambda\in\Lambda_w}\delta_\lambda$ where δ_a is the Dirac mass at a.

Lemma 1.13. *The measure σ_w is almost periodic.*

The set Λ_w is an almost periodic pattern (see Definition 2.32 below). However this set is not a model set in any sense of this word.

The proof of Lemma 1.13 relies on the following Diophantine approximation property.

Lemma 1.14. *Let ϵ be a positive number. Then the set*

$$M_\epsilon = \{\tau\in\mathbb{Z};\ \inf_{k\in\mathbb{Z}}|\sqrt{2}\tau - k| \le \epsilon\}$$

is relatively dense.

This being recalled, let ϕ be a compactly supported continuous function and let $f(x) = \sum\phi(x - k - w(\sqrt{2}k))$. We need to show that $f(x)$ is an almost periodic function. If $\tau\in\mathbb{Z}$ we obviously have

$$f(x+\tau) = \sum_{k\in\mathbb{Z}}\phi(x - k - w(\sqrt{2}(k+\tau))).$$

If $\tau\in M_\epsilon$ then $|w(\sqrt{2}(k+\tau)) - w(\sqrt{2}k)| \le \eta(\epsilon)$ uniformly in k by the the continuity of w and the definition of M_ϵ. It now suffices to observe that $\eta(\epsilon)$ tends to 0 with ϵ and that the series defining $f(x)$ is locally finite. This concludes the proof.

Here is an example of a measure which is not almost periodic.

Lemma 1.15. *Let ϵ_k, $k\in\mathbb{Z}$ be a sequence of real numbers tending to 0 as $|k|$ tends to infinity and let $\lambda_k = k + \epsilon_k$. Then the sum of Dirac masses $\sigma(x) = \sum_{-\infty}^{\infty}\delta_{\lambda_k}$ is never an almost periodic measure unless $\epsilon_k = 0$, $k\in\mathbb{Z}$.*

Lemma 1.15 follows immediately from Lemma 1.6.

We now define the mean value of an almost periodic measure as in [8].

Lemma 1.16. *If the closed ball centered at x with radius R is denoted by $B(x,R)$, and if c_n denotes the inverse of the volume of the unit ball then the mean value $\mathcal{M}(\mu)$ of an almost periodic measure μ is defined by*

$$\mathcal{M}(\mu) = \lim_{R\to+\infty} c_n R^{-n} \mu[B(x,R)] \tag{1.12}$$

and this limit is attained uniformly in x.

The convolution product $\mu * g$ between g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and an almost periodic measure μ is an almost periodic function. Therefore the distributional Fourier transform of an almost periodic measure makes sense. The Fourier coefficients of an almost periodic measure μ are not defined through its distributional Fourier transform $\hat{\mu}$ but as follows.

Definition 1.17. *The Fourier coefficients of an almost periodic measure μ are*

$$\hat{\mu}(\omega) = \mathcal{M}[\mu(x)\exp(-ix\cdot\omega)],\ \omega \in \mathbb{R}^n. \tag{1.13}$$

An almost periodic measure is uniquely defined by its Fourier coefficients $\hat{\mu}(\omega)$ and one can write a formal expansion

$$d\mu(x) \sim \sum_{\omega\in S} \hat{\mu}(\omega)\, e^{i\omega\cdot x}. \tag{1.14}$$

as it was the case for almost periodic functions.

Lemma 1.18. *If μ and ν are two almost periodic measures, their convolution product $\tau = \mu\star\nu$ is still an almost periodic measure.*

We shall define this convolution product. The indicator function of the closed ball $B(0,R)$ centered at 0 with radius R is denoted by χ_R and we write $\mu_R = \mu\chi_R$. Next the measure τ_R is defined by

$$\tau_R = c_n R^{-n} \mu_R * \nu \tag{1.15}$$

where c_n is the inverse of the volume of the unit ball. The convolution product $\tau = \mu\star\nu$ is the limit in the distributional sense of τ_R as R tends to ∞.

Lemma 1.19. *If μ and ν are two almost periodic measures we have*

$$\mathcal{M}(\mu\star\nu) = \mathcal{M}(\mu)\mathcal{M}(\nu) \tag{1.16}$$

The proof is left to the reader.

Here is an example of the convolution product between two almost periodic measures. We consider the Dirac comb $\mu = \sum_{k\in\mathbb{Z}} \delta_k$ and for some positive $\alpha \notin \mathbb{Q}$ we consider $\nu = \alpha \sum_{k\in\mathbb{Z}} \delta_{\alpha k}$. Then the convolution product $\mu\star\nu$ is the Lebesgue measure on the real line.

A specific example of an almost periodic measure will be detailed now. This example is aimed at proving the following fact

Proposition 1.20. *There exists an almost periodic measure μ such that $|\mu|$ is not an almost periodic measure.*

Two constructions of μ will be given. Here is the first one. We consider the set $\Gamma_0 = 2\mathbb{Z}+1$ of odd integers and write $\Gamma_j = 2^j\Gamma_0,\ j \in \mathbb{N}$. Then $\mathbb{Z}\setminus\{0\}$ is the disjoint union of Γ_j, $j \geq 0$. We let σ_j be the sum $\sum_{k\in\Gamma_j}\delta_k$ of Dirac masses at $k \in \Gamma_j$. Then σ_j is a $2^{j+1}-$periodic measure. Let τ be the Dirac comb $\sum_{k\in\mathbb{Z}}\delta_k$. Then $\sigma = \sigma_0 + \sigma_1 + \ldots = \tau - \delta_0$. Therefore σ is not an almost periodic measure.

Lemma 1.21. *Let $\mu_j = \sigma_j * (\delta_0 - \delta_{2^{-j-1}})$. Then the sum $\mu = \mu_0 + \mu_1 + \ldots$ is an almost periodic measure.*

We first observe that μ is translation bounded since $|\mu|([k, k+1]) = 2$ for $k \in \mathbb{Z}$. It then suffices to prove that $\mu * g$ is an almost periodic function for every smooth test function g. We set $g_j = g * \mu_j$ and observe that g_j is $2^{j+1}-$periodic. Moreover $\|g_j\|_\infty \leq C2^{-j}$ which implies that $\sum_{j\geq 0} g_j$ is an almost periodic function.

Lemma 1.22. *The measure $|\mu|$ is not almost periodic.*

Indeed $|\mu| = \sum_{j\geq 0}|\mu_j| = \sum_{j\geq 0}\sigma_j * (\delta_0 + \delta_{2^{-j-1}})$. If $|\mu|$ was an almost periodic function, then the sum $\mu + |\mu| = 2\sigma$ would also be an almost periodic function which is not the case.

In the second construction we consider the set $\Gamma_0 = 3\mathbb{Z}+1$ and write $\Gamma_j = 3^j\Gamma_0,\ j \in \mathbb{N}$. We let σ_j be the sum $\sum_{k\in\Gamma_j}\delta_k$ of Dirac masses at $k \in \Gamma_j$. Then σ_j is a $3^{j+1}-$periodic measure.

Lemma 1.23. *The sum $\sigma = \sigma_0 + \sigma_1 + \ldots$ is not an almost periodic measure.*

The proof relies on the following observation

Lemma 1.24. *Let $\Lambda \subset \mathbb{Z}$. Then $\sigma = \sum_{k\in\Lambda}\delta_k$ is an almost periodic measure if and only if Λ is a periodic set.*

The proof is left to the reader. In our case $\cup_{j\geq 0}\Gamma_j$ is not a periodic set. The measure $\sigma = \sigma_0 + \sigma_1 + \ldots$ will be an example of a generalized almost periodic measure. We proceed to the construction of μ.

Lemma 1.25. *Let $\mu_j = \sigma_j * (\delta_0 - \delta_{3^{-j-1}})$. Then the sum $\mu = \mu_0 + \mu_1 + \ldots$ is an almost periodic measure but $|\mu|$ is not an almost periodic measure.*

The proofs are the same as above.

The duality between almost periodic measures and almost periodic functions is defined by

$$< f, \mu >= \mathcal{M}(f\mu) \tag{1.17}$$

This makes sense since the product between an almost periodic measure and an almost periodic function is an almost periodic measure. In other words an almost periodic measure μ defines a Borel measure $\mathcal{J}(\mu)$ on the Bohr compactification $\mathcal{G}_n$ of $\mathbb{R}^n$ and the Fourier coefficients of μ are identical to the Fourier coefficients of $\mathcal{J}(\mu)$. The mapping $\mu \mapsto \mathcal{J}(\mu)$ from the space of almost periodic measures to the space of Borel measures on $\mathcal{G}_n$ is named the Bohr mapping in [8].

Lemma 1.26. *If the Borel measure $\mathcal{J}(\mu)$ on the Bohr compactification $\mathcal{G}_n$ of $\mathbb{R}^n$ is non negative, then the almost periodic measure μ is non negative.*

The proof is left to the reader.

The Bohr mapping is injective but is not onto. This motivates a second definition of almost periodic measures which will be given in Section 2.

If a measure μ is a Poisson measure as defined below, then μ is an almost periodic measure and the distributional Fourier transform $\hat{\mu}$ agrees with the Fourier transform of μ.

Definition 1.27. *A Poisson measure is an almost periodic measure μ whose distributional Fourier transform $\hat{\mu}$ is also an almost periodic measure.*

Let μ be a Poisson measure and ϕ a function in the Schwartz class. Then $\tau = \hat{\mu}\hat{\phi}$ is a bounded measure and is the Fourier transform of $f = \mu * \phi$. Let us write τ as sum between a continuous component τ_1 and an atomic component τ_2. The inverse Fourier transform of τ is f. The inverse Fourier transform of τ_2 is an almost periodic function h with an absolutely convergent Fourier series. Let us show that $\tau_1 = 0$. We know that the inverse Fourier transform of τ_1 is the almost periodic function $g = f - h$. Since τ_1 is a continuous measure its inverse Fourier transform g satisfies $\mathcal{M}(|g|) = 0$. Since g is almost periodic we have $g = 0$ as announced. We just proved the following

Lemma 1.28. *A Poisson measure and its distributional Fourier transform are purely atomic measures. Conversely let us assume that both μ and its distributional Fourier transform are translation bounded atomic measures. Then μ is an almost periodic measure.*

Let μ be a Poisson measure. We have $\hat{\mu} = \sum_{\omega\in S} c(\omega)\delta_\omega$. With an abuse of language we say that S is the support of $\hat{\mu}$. Then we have

Lemma 1.29. *If $\omega \notin S$ we have $\mathcal{M}[\exp(-i\omega\cdot x)\,d\mu(x)] = 0$. If $\omega \in S$ then $\mathcal{M}[\exp(-i\omega\cdot x)\,d\mu(x)] = (2\pi)^{-n}\hat{\mu}(\{\omega\})$.*

In short one has $\hat{\mu}(\omega) = (2\pi)^{-n}\hat{\mu}(\{\omega\})$ where the left hand side is the Fourier coefficient of the Poisson measure μ at ω and the right hand side is the mass of the atomic measure $\hat{\mu}$ at ω.

To prove Lemma 1.29 it suffices to consider a test function ϕ whose Fourier transform satisfies $\hat{\phi}(\omega) = 1$ and to apply to $\phi * \mu$ the known properties of almost periodic functions with an absolutely convergent Fourier series.

A construction of Poisson measures will be given in Section 4. The measure described in Lemma 1.13 is not a Poisson measure.

2 Generalized almost periodic functions and measures

Generalized almost periodic functions are defined now and pave the way to generalized almost periodic measures which are the right tool to describe model sets.

The main theorem of this essay says that if Λ is a model set then the sum $\mu_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$ of Dirac masses on Λ is a generalized almost periodic measure. Here we differ from Lagarias [11] and from Laurent Schwartz. The definition given by Schwartz is too restrictive. The measure μ_Λ is not an almost periodic distribution in the sense given by Schwartz. It is a Besicovitch almost periodic measure as proved by Lagarias. It is much more demanding to be a generalized almost periodic measure than to be a Besicovitch almost periodic measure (see Proposition 2.4 below). Therefore our result improves on Lagarias theorem.

The notation $\mathcal{M}$ is defined by (1.4).

Definition 2.1. *A real valued function f defined on $\mathbb{R}^n$ is a generalized almost periodic (g-a-p) function if it is a Borel function and if for every positive ϵ there exist two uniformly almost periodic functions g_ϵ and h_ϵ such that*

$$g_\epsilon \le f \le h_\epsilon \tag{2.1}$$

and

$$\mathcal{M}(h_\epsilon - g_\epsilon) \le \epsilon \tag{2.2}$$

It is easily checked that the collection of all g-a-p functions is a vector space $\mathcal{X}$. One might infer from Definition 2.1 that the averaged L^1 norm norm defined by $\|f\| = \mathcal{M}(|f|)$ is the natural norm on $\mathcal{X}$. It is not the case since $\mathcal{X}$ is not a Banach space for this norm. The other extreme is the L^∞ norm and then $\mathcal{X}$ becomes a Banach space. But this choice of norm is also problematic for two reasons. Trigonometric polynomials are not dense in $\mathcal{X}$ if the L^∞ norm is adopted. Moreover the definition of almost periodicity given in [8] is not valid. Indeed an almost periodic function f belonging to a topological vector space X is defined by the fact that the closure in X of the orbit of f under the translation group is compact. If $\mathcal{X}$ is equipped with the L^∞ norm this definition of almost periodicity yields the almost periodic functions in the sense of Bohr. A third choice would be to consider the weak topology $\sigma(L^\infty, L^1)$. This does not work either since any function in L^∞ would be almost periodic. Thus our new definition of almost periodicity cannot be incorporated in the framework of [8]. We thank the anonymous referee for raising this important issue.

Returning to Definition 2.1 we set $\epsilon = 1/j$ and with an obvious abuse of notations we have $g_j \le f \le h_j$ and $\epsilon_j = \mathcal{M}(h_j - g_j) \to 0$. Replacing g_j by $\sup(g_1,\dots,g_j)$ and h_j by $\inf(h_1,\dots,h_j)$ we can further assume that g_j is an increasing sequence of almost periodic functions and that h_j is a decreasing sequence of almost periodic functions.

We now abbreviate "almost periodic" into a-p and "generalized almost periodic" into g-a-p. We have proved the following lemma

Lemma 2.2. *A real valued Borel function f is a g-a-p function if and only if there exist an increasing sequence g_j of a-p functions and a decreasing sequence h_k of a-p functions such that*

(a) $g_j \le f \le h_k \quad (j,k\in\mathbb{N})$

(b) $\mathcal{M}(h_j - g_j)$ *tends to 0 as j tends to infinity.*

We do not have in general $\lim_{j\to\infty} f_j(x) = f(x)$. If indeed $f(x)$ is a non negative continuous function on the real line with a compact support we can decide that $f_j = 0$ while g_j will be a suitable 2^j−periodic function. Then $\lim_{j\to\infty} f_j(x) = 0 \neq f(x)$. This example will be detailed below. Similarly we do not have in general $\lim_{j\to\infty} g_j(x) = f(x)$. A generalized almost periodic function f is not an almost periodic distribution in general. It is not true that $f * \phi$ is an almost periodic function when ϕ is a test function. An example is $f(x) = \sum_{\lambda\in\Lambda} w(x-\lambda)$ where Λ is a quasicristal which is not a lattice and w is a test function. Then $f*\phi$ cannot be almost periodic in the sense of Bohr.

Generalized almost periodic (g-a-p) functions have not been studied before. They sit in between almost periodic functions and Weyl or Besicovitch almost periodic functions. The class of generalized almost periodic functions is strictly included into the class of Besicovitch almost periodic functions as indicated in the following result.

Proposition 2.3. *If $f(x)$ is a 2π−periodic function of the real variable x then we have*

(a) $f(x)$ is a Besicovitch almost periodic function if and only if it belongs to $L^2[0,2\pi]$

(b) $f(x)$ is a g-a-p function if and only if it is Riemann integrable on $[0,2\pi]$

(c) $f(x)$ is almost periodic in the sense of Bohr if and only if $f(x)$ is continuous on $[0,2\pi]$ and $f(0) = f(2\pi)$.

Only (b) needs a proof. We assume that f is real valued and that

$$u_\epsilon(x) \leq f(x) \leq v_\epsilon(x) \tag{2.3}$$

where

$$\mathcal{M}(v_\epsilon - u_\epsilon) \leq \epsilon \tag{2.4}$$

The proof consists in replacing u_ϵ and v_ϵ by two 2π−periodic functions satisfying (2.3) and (2.4). We replace x by $x+2k\pi$ in (2.3) and average in $k\in\mathbb{Z}$. Since u_ϵ and v_ϵ are almost periodic functions these averages converge uniformy to $\tilde{u}_\epsilon$ and $\tilde{v}_\epsilon$ which are 2π periodic.

Here is another result.

Proposition 2.4. *Let $1 \leq m_1 < \ldots < m_k < \ldots$ be an increasing sequence of integers such that m_k/k tends to infinity with k and let $M = \{m_1,\ldots,m_k,\ldots\}$. Let ϕ be a non negative continuous function with compact support such that $\phi(0) = 1$ and let us consider the function of the real variable x defined by $f_M(x) = \sum_1^\infty \phi(x-m_k)$.*

Then f_M is a g-a-p function if and only if the closure $\overline{M}$ of M in the Bohr compactification $\mathcal{Z}$ of $\mathbb{Z}$ is a set of measure 0 for the Haar measure of $\mathcal{Z}$.

Let us stress that $\mathcal{M}(f_M) = 0$. Viewed as a Besicovitch almost periodic function f_M shall be identified to 0. However f_M may not be a g-a-p function. The arithmetical properties of M are playing a key role in the discussion. For instance if $m_k = k^2$ this condition is satisfied but it is not the case if M is the sequence consisting of all sums $m^2 + [\sqrt{2}n^3]$, $m,n\in\mathbb{N}$. Here $[x]$ is the integer part of x (the largest integer which does not exceed x).

We now prove Proposition 2.4. The mean value of f_M is zero since m_k/k tends to infinity with k. Let us assume that f_M is a g-a-p function. Then we have $\lim_{\epsilon\to 0}\mathcal{M}(v_\epsilon) = 0$. Next we use Lemma 1.9 and work on $\mathbb{Z}$. When $C > 0$ is large enough the sequence $c_\epsilon(k) = C\int_k^{k+1} v_\epsilon(x)dx$ satisfies $c_\epsilon \geq \mathbf{1}_M$ where $\mathbf{1}_M$ is the indicator function of M. We imbed $\mathbb{Z}$ into its Bohr compactification $\mathcal{Z}$. The function c_ϵ is continuous on $\mathcal{Z}$ and we have $c_\epsilon \geq \mathbf{1}_M$. Let us denote by $\overline{M}$ the closure of M in $\mathcal{Z}$. It implies $c_\epsilon \geq \mathbf{1}_{\overline{M}}$. Therefore the measure of $\overline{M}$ is less than ϵ as claimed.

Conversely let us assume that the measure of $\overline{M}$ is 0. Then for each positive ϵ there exists a continuous function V_ϵ on $\mathcal{Z}$ such that $V_\epsilon \geq \mathbf{1}_{\overline{M}}$. It then suffices to read backwards the proof of the direct implication.

A g-a-p function f belongs to $L^\infty(\mathbb{R}^n)$. Keeping the preceding notations we have $\|f\|_\infty \leq \sup(\|g_1\|_\infty, \|h_1\|_\infty)$.

The following theorem says that the orbit of a g-a-p function is weakly sequencially compact. This is close to the defintion of almost periodicity in [8].

Theorem 2.5. *Let f be a g-a-p function. For every sequence $x_j \in \mathbb{R}^n$ there exist a subsequence x_{j_k} and a g-a-p function g such that*

(a) $f(x - x_{j_k}) \rightharpoonup g(x)$ in the weak topology defined by the duality between L^∞ and L^1

(b) $\mathcal{M}[|g(x) - f(x - x_{j_k})|] \to 0,\ k \to \infty$.

We know that $u_\epsilon \leq f \leq v_\epsilon$ where u_ϵ and v_ϵ are almost periodic functions and $\mathcal{M}(v_\epsilon - u_\epsilon) \leq \epsilon$. We extract subsequences in such a way that (a) $f(x - x_{j_k}) \rightharpoonup g(x)$ where the weak convergence is defined by the duality between L^∞ and L^1 together with (b) $u_\epsilon(x - x_{j_k}) \to U_\epsilon$ where the convergence is uniform on $\mathbb{R}^n$ and (c) $v_\epsilon(x - x_{j_k}) \to V_\epsilon$ where the convergence is also uniform. We still have $\mathcal{M}(V_\epsilon - U_\epsilon) \leq \epsilon$ and passing to the weak limits we obtain $U_\epsilon \leq g \leq V_\epsilon$. Therefore g is a g-a-p function. The proof of the second claim in Theorem 2.5 is immediate. We have $|g(x) - f(x - x_{j_k})| \leq |g(x) - U_\epsilon(x)| + |U_\epsilon(x) - u_\epsilon(x - x_{j_k})| + |u_\epsilon(x - x_{j_k}) - f(x - x_{j_k})|$. But $|U_\epsilon(x) - u_\epsilon(x - x_{j_k})| \leq \epsilon$ if $k \geq k_0$ while the mean values of the two other terms in the RHS do not exceed ϵ. This ends the proof.

A complex valued function f is a generalized almost periodic function if $\Re f$ and $\Im g$ are generalized almost periodic functions. In the following lemma, $B(x, R)$ denotes the ball centered at x with radius R and c_n is the inverse of the volume of $B(0, 1)$.

Lemma 2.6. *If f is a g-a-p function the limit*

$$\mathcal{M}(f) = \lim_{R\to+\infty} c_n R^{-n} \int_{B(x,R)} f(y)\,dy. \tag{2.5}$$

is attained uniformly in x.

The function f will be assumed to be real valued which suffices for proving Lemma 2.6 in full generality. One writes $M(f, R, x) = c_n R^{-n} \int_{B(x,R)} f(y)\,dy$. Then (2.1) and Lemma 2.2 imply

(a) $M(g_j,R,x) \le M(f,R,x) \le M(h_k,R,x)$

(b) $\lim_{R\to\infty} M(g_j,R,x) = \mathcal{M}(g_j)$

(c) $\lim_{R\to\infty} M(h_k,R,x) = \mathcal{M}(h_k)$

(d) $\mathcal{M}(h_k) - \mathcal{M}(g_j) \le \epsilon$ if $j,k \ge j_0$.

(e) $\mathcal{M}(g_j)$ is an increasing sequence and $\mathcal{M}(h_k)$ is a decreasing sequence.

Properties (d) and (e) imply that the increasing sequence $\mathcal{M}(g_j)$ and the decreasing sequence $\mathcal{M}(h_j)$ are converging to the same limit λ and (a), (b) and (c) imply that $M(f,R,x)$ tends to λ uniformly in x as R tends to infinity.

Proposition 2.7. *If f is a g-a-p function, so is $|f|$. If f and g are two g-a-p functions, so are* $\sup(f,g)$ *and* fg.

The proof of the first statement is almost obvious. Let us assume that f is a g-a-p function and prove that $f^+ = \sup(f,0)$ is a g-a-p function. Indeed $g_j \le f \le h_j$ implies $g_j^+ \le f^+ \le h_j^+$ and $h_j^+ - g_j^+ \le h_j - g_j$. This yields the first claim and the second follows immediately.

Let us prove the last claim. Using the first or the second claim one can assume $f \ge 0$ and $g \ge 0$. Then we have with obvious notations $0 \le u_j \le f \le v_j$ and $0 \le g_j \le g \le h_j$ where u_j, v_j, g_j, h_j are a-p together with $\mathcal{M}(v_j - u_j) \le \epsilon_j$, $\mathcal{M}(h_j - g_j) \le \epsilon_j$. Then $u_j g_j \le fg \le v_j h_j$ and $v_j h_j - u_j g_j \le (v_j - u_j)\|h_1\|_\infty + (h_j - g_j)\|v_1\|_\infty$. Therefore $\mathcal{M}(v_j h_j - u_j g_j)$ tends to 0.

Corollary 2.8. *Let f and g be two g-a-p functions. Then the following limit exists*

$$\mathcal{M}(fg) = \lim_{R\to+\infty} c_n R^{-n} \int_{B(x,R)} f(y)g(y)\,dy \tag{2.6}$$

This follows from Lemma 2.6 and Proposition 2.7.

We need to define the Fourier coefficients of a g-a-p function f.

Definition 2.9. *The Fourier coefficient of a g-a-p function f at the frequency $\omega \in \mathbb{R}^n$ is*

$$\hat{f}(\omega) = \mathcal{M}[f(x)\exp(-i\omega\cdot x)] \tag{2.7}$$

A g-a-p function is not characterized by its Fourier coefficients. Indeed we have:

Lemma 2.10. *Any bounded Borel function θ with compact support is a g-a-p function. We have $\hat{\theta}(\omega) = 0$ identically.*

To prove the first claim it suffices to treat the case when θ is non negative and vanishes outside $[-1,1]^n$. We can assume $0 \le \theta \le 1$. We simply decide that $g_\epsilon = 0$ and construct a continuous function h_ϵ with the following properties

(a) h_ϵ is $2/\epsilon$−periodic in each variable

(b) $h_\epsilon = 1$ on $[-1,1]^n$

(c) $0 \le h_\epsilon \le 1$ and $h_\epsilon = 0$ on $[-1/\epsilon, 1/\epsilon]^n \setminus [-2,2]^n$.

Then $\mathcal{M}(h_\epsilon) \le \epsilon^n$ which ends the proof. The second claim is obvious. This example is showing that a g-a-p function is not Riemann integrable in general (see Definition 2.16). It is also showing that we cannot expect f to be the limit of g_ϵ or of h_ϵ as ϵ tends to 0.

The same argument shows that a continuous function vanishing at infinity is a g-a-p function. However there exists a continuous function f such that (a) f is uniformly bounded on the real line, (b) $\mathcal{M}[|f(x)|] = 0$ but (c) f is not a g-a-p function. An example has already been given in Proposition 2.4.

Theorem 2.11. *Let f be a g-a-p function. Then for every positive ϵ there exists a relatively dense set Λ such that*

$$\tau \in \Lambda \Rightarrow \mathcal{M}(|f(\cdot + \tau) - f(\cdot)|) \le \epsilon \tag{2.8}$$

The proof is straightforward. The notations of Definition 2.1 being kept, we have $f_\epsilon(x) \le f(x) \le g_\epsilon(x)$. The relatively dense set M_ϵ is defined by the two conditions

(a) $\|f_\epsilon(x+\tau) - f_\epsilon(x)\|_\infty \le \epsilon$

(b) $\|g_\epsilon(x+\tau) - g_\epsilon(x)\|_\infty \le \epsilon$

Here we use the fact that the vector valued function (f_ϵ, g_ϵ) is almost periodic. We then have

$$|f(x+\tau) - f(x)| \le \sup(|g_\epsilon(x+\tau) - f_\epsilon(x)|, |f_\epsilon(x+\tau) - g_\epsilon(x)|) \tag{2.9}$$

But

$$|g_\epsilon(x+\tau) - f_\epsilon(x)| \le |g_\epsilon(x+\tau) - g_\epsilon(x)| + |g_\epsilon(x) - f_\epsilon(x)| \tag{2.10}$$

The first term in the RHS of (2.10) does not exceed ϵ as well as the mean value of the second term. The second term in the RHS of (2.9) is treated similarly.

Corollary 2.12. *Keeping the notations of Theorem 2.11 we have*

$$\tau \in \Lambda \Rightarrow \mathcal{M}(|f(\cdot + \tau) - f(\cdot)|^2) \le \epsilon \tag{2.11}$$

Therefore a g-a-p function is almost periodic in the sense of Besicovitch.

The convolution product $f \star g$ between two g-a-p functions is defined by

$$(f \star g)(x) = \mathcal{M}[f(x - \cdot)g(\cdot)] \tag{2.12}$$

Corollary 2.13, Corollary 2.15 and Lemma 2.14 below are valid in the more general context of Besicovitch almost periodicity and the reader is invited to skip their proofs.

Corollary 2.13. *Let f and g be two g-a-p functions. Then their convolution product $f \star g$ is an almost periodic function in the sense of Bohr.*

Indeed $|(f \star g)(x+\tau) - (f \star g)(x)| \le \|g\|_\infty \mathcal{M}[|f(\cdot + \tau) - f(\cdot)|]$ and it suffices to apply Theorem 2.11.

Lemma 2.14. *Keeping the same notations we have*

$$\mathcal{M}(f\star g) = (\mathcal{M}f)(\mathcal{M}g) \tag{2.13}$$

By linearity we can assume $f,g \geq 0$. We have by Lemma 2.2, $u_j \leq f \leq v_j$ and $g_j \leq g \leq h_j$. Then

$$\mathcal{M}[u_j(x-\cdot)g_j(\cdot)] \leq \mathcal{M}[f(x-\cdot)g(\cdot)] \leq \mathcal{M}[v_k(x-\cdot)h_k(\cdot)].$$

In other words

$$u_j\star g_j \leq f\star g \leq v_k\star h_k.$$

Next

$$\int (u_j\star g_j)\chi_R\,dx \leq \int (f\star g)\chi_R\,dx \leq \int (v_k\star h_k)\chi_R\,dx.$$

Multiplying by c_nR^{-n}, passing to the limit as R tends to infinity and using (2.6) we obtain

$$\mathcal{M}(u_j)\mathcal{M}(g_j) \leq \mathcal{M}(f\star g) \leq \mathcal{M}(v_k)\mathcal{M}(h_k).$$

Finally

$$\mathcal{M}(v_k)\mathcal{M}(h_k) - \mathcal{M}(u_j)\mathcal{M}(g_j) \leq \epsilon$$

if $j,k \geq j_0$. Indeed these four sequences are bounded and satisfy property (b) in Lemma 2.2. Then the proof ends as the one in Lemma 2.6.

Corollary 2.15. *Let $\hat{h}(\omega)$ be the Fourier coefficients of $h = f\star g$. We then have $\hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$.*

It suffices to notice that $\hat{f}(\omega) = \mathcal{M}(f(x)\exp(-i\omega\cdot x))$ and to apply Lemma 2.14.

Almost periodic functions extend by continuity to the Bohr compactification $\mathcal{G}_n$ of $\mathbb{R}^n$. In a sense to be made precise the extension to $\mathcal{G}_n$ of a generalized almost periodic function is a Riemann integrable function (see Theorem 2.18 below for a more precise statement). The definition of Riemann integrable functions is given for the reader's convenience:

Definition 2.16. *Let G be a topological space and let μ be a non negative Borel measure on G. A real valued function f on (G,μ) is Riemann integrable if and only if for every positive ϵ there exist two functions f_ϵ and g_ϵ which are continuous on G and satisfy the following*

(a) $f_\epsilon \leq f \leq g_\epsilon$ *everywhere on G*

(b) $\int_G (g_\epsilon - f_\epsilon)\,d\mu \leq \epsilon$

A function f is Riemann integrable if and only if the set of points $x \in G$ where f is not continuous is a set of zero measure. Returning to Definition 2.16 it suffices to check (a) and (b) when $\epsilon = 1/j$, $j = 1,2,\ldots$. Replacing f_j by $\sup(f_1, f_2, \ldots, f_j)$ we can assume that f_j is an increasing sequence of continuous functions. Similarly we can assume that g_k is a decreasing sequence of continuous functions and we have

(i) $f_j \leq f \leq g_j$ everywhere on G

(ii) $\int_G (g_j - f_j)\,dx \to 0$ as j tends to infinity.

In general we cannot expect f to be at the same time the pointwise limit of the sequence f_j and of the sequence g_j. More precisely if for every $x \in G$ we have $g_j(x) \to f(x)$ and $f_j(x) \to f(x)$, then f is a continuous function. The pointwise convergence of the sequence $f_j(x)$ to $f(x)$ holds only almost everywhere. This seems inconsistent with the fact that a Riemann integrable function is defined everywhere. If $f(x) = g(x)$ almost everywhere and f is Riemann integrable this does not imply that g is Riemann integrable. Let us stress that a Riemann integrable function can be restricted to a set of measure 0 since it is defined everywhere.

Let Γ be a dense subgroup of $\mathbb{R}^n$ and let G be the dual group of Γ. This dual group is the multiplicative group consisting of all characters on Γ. As it was said above, a character χ is a mapping $\Gamma \mapsto \mathbb{T}$ which satisfies the identity $\chi(x+y) = \chi(x)\chi(y)$ $(\forall x, y \in \Gamma)$. Then G is a compact abelian group. Each $y \in \mathbb{R}^n$ is a character on $\mathbb{R}^n$ defined by $\chi_y(x) = \exp(ix \cdot y)$. This remark implies that $\mathbb{R}^n$ can be viewed as a dense subgroup of the compact group G. Let us denote by J the canonical embedding of $\mathbb{R}^n$ into G defined by $J(y) = \chi_y$.

Lemma 2.17. *Let F be a Borel function on G. If F is Riemann integrable, then the function $F \circ J$ is g-a-p.*

Let us insist on the fact that a Riemann integrable function is defined everywhere. It is not a class of functions. Therefore $F \circ J$ makes sense.

We now prove Lemma 2.17. We have $A_\epsilon \le F \le B_\epsilon$ where A_ϵ and B_ϵ are continuous on G and have close integrals. It suffices to set $f_\epsilon = A_\epsilon \circ J$, $g_\epsilon = B_\epsilon \circ J$. Then f_ϵ and g_ϵ are two almost periodic functions and $\mathcal{M}(g_\epsilon - f_\epsilon) \le \epsilon$.

The converse statement is true. Let $\mathcal{G}_n$ be the Bohr compactification of $\mathbb{R}^n$ and $\mathcal{J}$ be the canonical embedding of $\mathbb{R}^n$ into $\mathcal{G}_n$.

Theorem 2.18. *Let f be a g-a-p function. Then f can be written $f = F \circ \mathcal{J} + r$ where*

(a) F is Riemann integrable on $\mathcal{G}_n$

(b) For every almost periodic function u on $\mathbb{R}^n$ we have

$$\mathcal{M}(uf) = \int_{\mathcal{G}_n} U(x)F(x)\,dx$$

where $U(x)$ is the extension of u to $\mathcal{G}_n$

(c) More generally we have $\hat{f}(\omega) = \hat{F}(\omega)$ for every $\omega \in \mathbb{R}^n$

(d) r satisfies $\mathcal{M}(|r|) = 0$.

In the LHS of (c) $\hat{f}(\omega)$ denotes the Fourier coefficient of f viewed as a g-a-p function while $\hat{F}(\omega)$ is the ordinary Fourier coefficient of F defined on the compact abelian group $\mathcal{G}_n$. The function F is not unique.

Definition 2.19. *With the notations of Theorem 2.18 we say that F is an extension of f to $\mathcal{G}_n$.*

If F_1 and F_2 are two such extensions of f then $F_1 = F_2$ almost everywhere on $\mathcal{G}_n$.

If for instance f is a continuous function with compact support, $F = 0$ and $f = r$.

For proving Theorem 2.18 we lift f to $\mathcal{J}(\mathbb{R}^n) \subset \mathcal{G}_n$, i.e. we consider the auxiliary function $\tilde{f}$ defined on $\mathcal{J}(\mathbb{R}^n) \subset \mathcal{G}_n$ by $\tilde{f} \circ \mathcal{J} = f$. The a-p function f_j which is defined by (i) and (ii) of Lemma 2.2 extends to a continuous function F_j on $\mathcal{G}_n$ and similarly the function g_j which is defined by (i) and (ii) of Lemma 2.2 extends to a continuous function G_j on $\mathcal{G}_n$. We have $F_j \circ \mathcal{J} = f_j$ and $G_j \circ \mathcal{J} = g_j$. Therefore $f_j \le f \le g_k$ implies $F_j \le \tilde{f} \le G_k$ on $\mathcal{J}(\mathbb{R}^n)$. But $\mathcal{J}(\mathbb{R}^n)$ is dense in $\mathcal{G}_n$ which implies $F_j(x) \le G_k(x)$, $\forall x \in \mathcal{G}_n$. Let $F(x)$ be the pointwise limit of the increasing sequence $F_j(x)$ and let $G(x)$ be the pointwise limit of the decreasing sequence $G_k(x)$. We know that $\|G_j - F_j\|_1 = \mathcal{M}(g_j - f_j)$ tends to 0 as j tends to infinity. It implies that the function F is Riemann integrable on $\mathcal{G}_n$ and $F = G$ almost everywhere. We would have $F(x) = G(x)$ everywhere if and only if F was continuous on $\mathcal{G}_n$ and this happens if and only if the function f we started with is almost periodic in the sense of Bohr. Every point x_0 where $F(x_0) = G(x_0)$ is a point of continuity of F. Then $F(x_0)$ is the limit of $\tilde{f}(x)$ as x tends to x_0 in $\mathcal{G}_n$.

It remains to relate F to f which is not entirely obvious. The function $F \circ \mathcal{J}$ makes sense since F is defined everywhere on $\mathcal{G}_n$. We set $r = f - F \circ \mathcal{J}$ and claim that $\mathcal{M}(|r|) = 0$. Indeed $f_j \le f \le g_j$ and $f_j = F_j \circ \mathcal{J} \le F \circ \mathcal{J} \le G_j \circ \mathcal{J} = g_j$ imply $f_j - g_j \le r \le g_j - f_j$ and $|r| \le g_j - f_j$. Our claim follows from $\mathcal{M}(g_j - f_j) \to 0$.

The reader may observe that in our proof the function G could have been used instead of F. Any Borel function U such that $F \le U \le G$ would have played the same role. We then say that U is an extension of f to $\mathcal{G}_n$. The proofs of (b) and (c) are straightforward. Indeed these properties hold for each pair (f_j, F_j) and it suffices to pass to the limit using Lebesgue's dominated convergence theorem on $\mathcal{G}_n$.

Lemma 2.20. *If f and g are two g-a-p functions and if F and G are their extensions to $\mathcal{G}_n$ then FG is an extension of fg to $\mathcal{G}_n$.*

It suffices to prove Lemma 2.20 when f and g are non negative. Then our claim follows from the proof of Theorem 2.18.

Lemma 2.21. *Let f be a g-a-p function. Then we have*

$$\sum |\hat{f}(\omega)|^2 = \mathcal{M}[|f|^2] \tag{2.14}$$

The proof is straightforward. An extension of the g-a-p function $|f|^2$ is $|F|^2$ if F is an extension of f to $\mathcal{G}_n$. Property (b) in Theorem 2.18 implies that the RHS of (2.14) equals $\int_{\mathcal{G}_n} |F(x)|^2\,dx$ and it then suffices to use (c) and Plancherel theorem on $\mathcal{G}_n$.

Definition 2.22. *The spectrum of a g-a-p function f is the set S of all ω such that $\hat{f}(\omega) \neq 0$.*

We know from (2.14) that S is a numerable set.

Theorem 2.23. *Let f be a g-a-p function, let S be the spectrum of f and H be the additive subgroup of $\mathbb{R}^n$ generated by S. Then the conclusion of Theorem 2.18 is valid when $\mathcal{G}_n$ is replaced by the compact abelian group G which is the dual group of H.*

Let G_0 be the annihilator of H in $\mathcal{G}_n$. A character $\chi \in \mathcal{G}_n$ belongs to G_0 if and only if $\chi(x) = 1$, $\forall x \in H$. Then G is the quotient group $\mathcal{G}_n/G_0$. With an abuse of notation we write $\omega(\chi) = \chi(\omega)$ for every $\omega \in H$ and $\chi \in \mathcal{G}_n$. Then ω is a character on $\mathcal{G}_n$. Keeping this notation we have $\omega(x) = 1$, $(\forall x \in G_0)$, for every $\omega \in H$. For any function $u \in L^1(\mathcal{G}_n)$ we denote by $v \in L^1(\mathcal{G}_n)$ the function defined by $v(x) = \int_{G_0} u(x+y)\,dy$. For almost every $x \in \mathcal{G}_n$ we have $v(x+y) = v(x)$, $\forall y \in G_0$. Then $v(x)$ defines a function on $G = \mathcal{G}_n/G_0$. This function will also be denoted by v by an abuse of notations and we have

$$\int_{\mathcal{G}_n} u(x)\,dx = \int_G v(x)\,dx \tag{2.15}$$

If $\omega \in H$ and if $\omega(x)$, $x \in \mathcal{G}_n$, denotes the corresponding character on $\mathcal{G}_n$ we apply (2.15) to the auxiliary function $u(x)\omega(x)$. We observe that $\omega(x+y) = \omega(x)$, $(\forall y \in G_0)$ and we obtain

$$\hat{u}(\omega) = \hat{v}(\omega),\ (\forall \omega \in H) \tag{2.16}$$

We return to the proof of Theorem 2.23. The notations of Theorem 2.18 are kept and $F \in L^1(\mathcal{G}_n)$ is an extension of f. We know from (c) that the Fourier coefficients of F vanish outside H. We define three functions F'_j, F', G'_j by

(a) $F'(x) = \int_{G_0} F(x+y)\,dy$

(b) $F'_j(x) = \int_{G_0} F_j(x+y)\,dy$

(c) $G'_j(x) = \int_{G_0} G_j(x+y)\,dy$

This makes sense for every $x \in \mathcal{G}_n$. These three functions F'_j, F' and G'_j are G_0-invariant and are therefore defined on $G = \mathcal{G}_n/G_0$. We then have $\hat{F}(\omega) = \hat{F}'(\omega)$ by (2.16) for every $\omega \in H$. Moreover the functions F'_j and G'_j are continuous on G. Then $F_j(x) \le F(x) \le G_j(x)$ on $\mathcal{G}_n$ implies $F'_j(x) \le F'(x) \le G'_j(x)$ everywhere on G. Moreover $\int_G (G'_j - F'_j)\,dx \to 0$. It follows that F' is Riemann integrable on G. We set $f' = F' \circ J$ where $J : \mathbb{R}^n \mapsto G$ is the canonical embedding. It remains to prove that $\mathcal{M}(|f - f'|) = 0$. For proving this claim we observe that the two g-a-p functions f and f' have the same Fourier coefficients. This is obvious if $\omega \notin H$ since $\hat{f}(\omega) = 0 = \hat{f}'(\omega)$. If $\omega \in H$ we have $\hat{f}(\omega) = \hat{F}(\omega)$ by (c) and similarly $\hat{f}'(\omega) = \hat{F}'(\omega)$. But $\hat{F}(\omega) = \hat{F}'(\omega)$. Finally we apply the following lemma to $f - f'$.

Lemma 2.24. *If f is a g-a-p function and if $\hat{f}(\omega) = 0$ everywhere, then $\mathcal{M}(|f|) = 0$.*

Lemma 2.24 follows from Lemma 2.21 and concludes the proof of Theorem 2.23.

Lemma 2.25. *If f and g are two g-a-p functions and if F and G are their extensions to some compactification of $\mathbb{R}^n$, then $F * G$ is an extension of $f \star g$ and $\mathcal{M}(f \star g) = \mathcal{M}(f)\mathcal{M}(g)$.*

The argument used in Lemma 2.20 applies here.

The conclusion of Lemma 5 is still valid for g-a-p functions as it will be proved now.

Lemma 2.26. *If f and g are two g-a-p functions, the Fourier coefficient of $f \star g$ at the frequency ω is the product $a(\omega)b(\omega)$ between $a(\omega) = \hat{f}(\omega)$ and $b(\omega) = \hat{g}(\omega)$.*

We write $f_\omega(x) = \exp(-i\omega \cdot x)$ and use the same notations for g and $h = f\star g$. Then it suffices to observe that $h_\omega = f_\omega\star g_\omega$ and to apply Lemma 2.25.

The error term r in Theorem 2.23 satisfies $\mathcal{M}[|r|] = 0$. This raises the following issue. Let f be a continuous function on the real line satisfying $\|f\|\|_\infty \le C$ and $\mathcal{M}[|f|] = 0$. Is it a g-a-p function ? The answer is no, as the following example shows.

Proposition 2.27. *Let $\Lambda \subset \mathbb{R}$ be the increasing sequence $\lambda_k = k^2 + k\sqrt{2},\ k \in \mathbb{N}$. Let ϕ be a non negative compactly supported continuous function with $\phi(0) = 1$. Then $f(x) = \sum_{\lambda\in\Lambda} \phi(x-\lambda)$ is not a g-a-p function.*

The proof relies on the fact that Λ is dense in the Bohr compactification $\mathcal{G}$ of $\mathbb{R}$. We will prove a stronger statement

Lemma 2.28. *For every almost periodic function u we have*

$$\sigma_N(u) = \frac{1}{N} \sum_{1\le k\le N} u(\lambda_k) \to \mathcal{M}(u) \tag{2.15}$$

For proving Lemma 2.28, it suffices by density to treat the case where $u(x) = \exp(i\omega x)$. We use a theorem by van der Corput saying that if P is a polynomial with at least one irrational coefficient (other than the constant term) then the sequence $P(n)$ is uniformly distributed modulo 1. If $u(x) = \exp(i\omega x)$, we have $\sigma_N(u) = A_N(\omega)$ where $A_N(\omega) = \frac{1}{N}\sum_{1\le k\le N} \exp(i\omega(k^2 + k\sqrt{2}))$. At least one among the two numbers $\omega, \omega\sqrt{2}$ is irrational and van der Corput's theorem applies. Then $A_N(\omega)$ tends to 0 for $\omega \neq 0$.

We now return to Proposition 2.27 and argue by contradiction. Let us assume $g_\epsilon \le f \le h_\epsilon$ with $\mathcal{M}[h_\epsilon - g_\epsilon] \le \epsilon$. Then $h_\epsilon \ge 1$ on Λ which implies $h_\epsilon \ge 1$ everywhere by density. But $\mathcal{M}[h_\epsilon - g_\epsilon] \ge \mathcal{M}[h_\epsilon - f] \ge \mathcal{M}[1 - f]$. We reach a contradiction since $\mathcal{M}[f] = 0$.

Here is a second counter example. We denote by $\theta > 2$ a real number which is not a Pisot number. For instance $\theta = 5/2$. We define Λ_θ as the set consisting of all finite sums $\sum_{k\ge 0} \epsilon_k \theta^k$ where $\epsilon_k \in \{0, 1\}$.

Lemma 2.29. *If θ is not a Pisot number, then Λ_θ is dense in the Bohr compactification of $\mathbb{R}$ and the density of Λ_θ is 0.*

Lemma 2.29 will be proved below (see Lemma 2.36). Then the argument used for the sequence $k^2 + k\sqrt{2}$ works if we replace it by Λ_θ and Proposition 2.27 is still true when $\Lambda = \Lambda_\theta$.

This example can be modified to produce a set of integers.

Lemma 2.30. *There exists a set $M \subset \mathbb{Z}$ whose density is 0 and which is dense in the Bohr compactification of $\mathbb{Z}$.*

The construction of M follows from the preceding one. We denote by M_0 the set of all integer parts $[\lambda]$ of $\lambda \in \Lambda_\theta$. We set $M = M_0 \cup \{M_0 + 1\}$. We assume that θ is not a Pisot number. Then M has the required properties. The density of M is 0. If an almost periodic function f defined on $\mathbb{Z}$ vanishes on M then we extend f to an almost periodic function F on $\mathbb{R}$ as in Lemma 1.9. Then F vanishes on Λ_θ. Therefore $F = 0$ identically and so does f. To construct a non negative continuous function f with a zero mean and which is not a g-a-p function it suffices to consider $f(x) = \sum_{m\in M} \phi(x - m)$ where ϕ is a no negative continuous function with compact support and such that $\phi(0) = 1$.

2.1 Generalized almost periodic measures

Definition 2.31. *A real valued Borel measure μ is a generalized almost periodic measure on $\mathbb{R}^n$ (a g-a-p measure) if the following property holds:*

For every $\epsilon > 0$ there exist two almost periodic measures μ_ϵ and ν_ϵ such that

$$\mu_\epsilon \leq \mu \leq \nu_\epsilon \tag{2.16}$$

and

$$\mathcal{M}(\nu_\epsilon - \mu_\epsilon) \leq \epsilon \tag{2.17}$$

We aim at relating the arithmetical properties of a Delone set $\Lambda \subset \mathbb{R}^n$ to the analytical properties of the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$.

Definition 2.32. *We say that Λ is an almost periodic pattern if the corresponding measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a generalized almost periodic measure.*

Before moving further let us give a motivating example. This example will hopefully convince the reader that *almost periodic patterns* have not been studied before. Let $\theta > 2$ be a real number and let Λ_θ be the set consisting of all finite sums $\sum_{k\geq 0} \epsilon_k \theta^k$ with $\epsilon_k \in \{0, 1\}$. For $j \geq 1$, $\Lambda_\theta^{(j)} \subset \Lambda_\theta$ is the set consisting of the 2^j finite sums $\sum_0^{j-1} \epsilon_k \theta^k$ with $\epsilon_k \in \{0, 1\}$. We denote by μ_j the measure which is the sum of the Dirac masses δ_λ, $\lambda \in \Lambda_\theta^{(j)}$. Finally $\sigma_\theta = \sigma_{\Lambda_\theta} = \sum_{\lambda\in\Lambda_\theta} \delta_\lambda$. The following definition is needed now.

Definition 2.33. *A Pisot-Vijayaraghavan number is a real number $\theta > 1$ with the following two properties :*

(a) θ is an algebraic integer of degree $n \geq 1$

(b) the $n-1$ conjugates $\theta_2, \dots, \theta_n$ of θ satisfy

$$|\theta_2| < 1, \dots, |\theta_n| < 1. \tag{2.18}$$

For example, the natural integers $2, 3, \dots$ are Pisot-Vijayaraghavan numbers and condition (b) is vacuous in that case. When the degree n of θ exceeds 1, the minimal polynomial of θ is $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ where $a_1 \in \mathbb{Z}, \dots, a_n \in \mathbb{Z}$. Then the conjugates $\theta_2, \dots, \theta_n$ of θ are the other solutions to this equation and can be either real or complex numbers. Raphaël Salem proved that the set S of all Pisot numbers is closed. The smallest Pisot number $\rho = 1.324717\dots$ is named the plastic number and is the real solution to the equation $x^3 - x - 1 = 0$. The two other solutions z_1 and z_2 to this equation are complex numbers. They satisfy $z_1 = \bar{z}_2$ and $z_1 z_2 = |z_1|^2 = |z_2|^2 = 1/\rho$ which is fully consistent with the fact that ρ is a Pisot number.

Salem numbers are defined the same way. On keeps condition (a) but replaces (b) by $|\theta_2| \leq 1, \dots, |\theta_n| \leq 1$ with, at least, equality somewhere. Then the degree n of θ is even. Up to some permutation between the conjugates we always have $\theta_2 = \frac{1}{\theta}$ and $|\theta_3| = \cdots = |\theta_n| = 1$.

With these notations we have

Theorem 2.34. *Let $\theta > 2$ be a real number. Then the set Λ_θ is an almost periodic pattern if and only if θ is a Pisot-Vijayaraghavan number.*

The proof of Theorem 2.34 relies on the assumption $\theta > 2$. We do not know if Theorem 2.34 remains true when $1 < \theta \leq 2$.

Let us first prove the easy part of Theorem 2.34. We are assuming that $\theta > 1$ is not a Pisot number and we shall prove that Λ_θ is not an almost periodic pattern. We argue by contradiction. We first observe that the sequence $P_j(x) = \cos(x)\cos(x\theta)\ldots\cos(x\theta^{j-1})$ tends to 0 for every $x \neq 0$. This follows from the following theorem by Charles Pisot:

Theorem 2.35. *The following two properties of a real number $\theta > 1$ are equivalent ones*

(a) There exists a real number $\alpha \neq 0$ such that $\alpha\theta^j = m_j + \epsilon_j$, $j \in \mathbb{N}$, where $m_j \in \mathbb{Z}$ and $\epsilon_j \in l^2$

(b) θ is a Pisot number.

Let $\mathcal{G}$ be the Bohr compactification of the real line and let $\tilde{\mu}_j$ be the image of the measure $2^{-j}\mu_j$ by the canonical embedding $\mathcal{J} : \mathbb{R} \mapsto \mathcal{G}$.

Lemma 2.36. *As j tends to infinity the measures $\tilde{\mu}_j$ weakly converge to the Haar measure on $\mathcal{G}$.*

Let $c_j(\omega)$ be the Fourier coefficients of $\tilde{\mu}_j$. We then have $|c_j(\omega)| = |P_j(\omega/2)|$. This remark and Theorem 2.35 imply the weak convergence of the probability measures $\tilde{\mu}_j$ to the Haar measure on $\mathcal{G}$. Lemma 2.36 is proved. Therefore Λ_θ is dense in $\mathcal{G}$ when θ is not a Pisot number.

We return to the proof of Theorem 2.34 and, as it was already said, we argue by contradiction. Let us assume that θ is not a Pisot number and that σ_θ is a g-a-p measure. We then follow the lines of the proof of Proposition 2.4. We denote by ϕ a non negative continuous function supported by $[-1, 1]$ such that $\phi(0) = 1$. Then $\sigma_\theta * \phi$ is a g-a-p function. There exist two a-p functions u_ϵ and v_ϵ such that $u_\epsilon \leq \sigma_\theta * \phi \leq v_\epsilon$ with $\mathcal{M}(v_\epsilon - u_\epsilon) \leq \epsilon$. The definition of ϕ implies $\sigma_\theta * \phi \geq 1$ on Λ_θ. Since Λ_θ is dense in $\mathcal{G}$ we have $v_\epsilon \geq 1$ everywhere. On the other hand $u_\epsilon \leq \sigma_\theta * \phi$ which implies $u_\epsilon^+ \leq \sigma_\theta * \phi$. The density of Λ_θ is 0 which implies that the mean value of $\sigma_\theta * \phi$ is also 0. Therefore $\mathcal{M}(u_\epsilon^+) = 0$. This yields the required contradiction since we have proved that $\mathcal{M}(v_\epsilon - u_\epsilon) \geq 1$.

The following lemma summarizes our discussion.

Lemma 2.37. *Let us assume that an almost periodic measure μ exists such that $\mu \geq \sigma_\theta$. Then θ is a Pisot number. Conversely if θ is a Pisot number such an almost periodic measure exists.*

If the hypothesis is relaxed and if one assumes that μ is a g-a-p measure such that $\mu \geq \sigma_\theta$ then the conclusion will be the same. Indeed there exists an almost periodic measure ρ such that $\rho \geq \mu \geq \sigma_\theta$.

We now prove the first assertion of Lemma 2.37. Once more let $\phi = \phi_\alpha$ be a continuous function of the real variable x supported by $[-\alpha, \alpha]$, such that $\phi(0) = 1$ and $0 \leq \phi(x) \leq 1$.

We have $\mu \geq \sigma_\theta$ which implies $\mu * \phi(x) \geq \sum_{\lambda\in\Lambda_\theta} \phi(x-\lambda)$. Therefore $\mu * \phi(x) \geq 1$ on Λ_θ and $\mu * \phi(x) \geq 1$ everywhere if θ is not a Pisot number. Indeed $\mu * \phi$ is an almost periodic function and Λ_θ is dense in the Bohr compactication of the real line. We now let α tend to 0 and we have $\mu * \phi_\alpha(x) \to \mu\{x\}$ everywhere. We arrive at a contradiction since $\mu\{x\} \geq 1$ everywhere is impossible.

We now assume that θ is a Pisot number and we shall prove that Λ_θ is an almost periodic pattern. The following lemma will be used to prove the second assertion of Lemma 2.37 and Theorem 2.34.

Lemma 2.38. *If θ is a Pisot number, then Λ_θ is contained in a model set M.*

The definition of a model set will be given in Section 3. Let $\mathcal{K}$ be the algebraic number field of degree n over $\mathbb{Q}$ generated by θ.

The ring $\Omega_{\mathcal{K}}$ of algebraic integers of $\mathcal{K}$ is isomorphic to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. In other words there exist n algebraic integers $\omega_1, \dots, \omega_n$ which are linearly independent over $\mathbb{Q}$ and such that

$$\Omega_{\mathcal{K}} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \,. \tag{2.19}$$

Let $\sigma_1 : \mathcal{K} \to \mathbb{C}, \dots, \sigma_n : \mathcal{K} \to \mathbb{C}$ be the n embeddings of $\mathcal{K}$ into $\mathbb{R}$ or $\mathbb{C}$.

The lattice $\Gamma \subset \mathbb{R}^n$ is defined by

$$\Gamma = \{\sigma_1(\omega), \dots, \sigma_n(\omega);\ \omega \in \Omega_{\mathcal{K}}\}$$

when the algebraic number field $\mathcal{K}$ is totally real. If $\mathcal{K}$ is not totally real, we then have $n = r_1 + 2r_2$, σ_j is real valued for $1 \leq j \leq r_1$ while σ_j and σ_{j+r_2} are complex conjugates. In this case the lattice $\Gamma \subset \mathbb{R}^n$ is defined as before but $\sigma_j(\omega)$, $r_1 + 1 \leq j \leq n$ are replaced by pairs $\Re[\sigma_j(\omega)], \Im[\sigma_j(\omega)]$, $r_1 + 1 \leq j \leq r_1 + r_2$.

If $\lambda \in \Lambda_\theta$ we have $\lambda = \sum_0 \epsilon_k \theta^k$ with $\epsilon_k \in \{0, 1\}$. We then have $\sigma_j(\lambda) = \sum_0 \epsilon_k \theta_j^k$ and $|\theta_j| < 1$ implies $|\sigma_j(\lambda)| \leq 1/(1 - |\theta_j|)$ for $2 \leq j \leq n$. Finally M is the set of all $m \in \Omega_{\mathcal{K}}$ such that $|\sigma_j(m)| \leq 1/(1 - |\theta_j|)$. This is exactly the definition of a model set. This ends the proof of Lemma 2.38.

The measure $\nu = \sum_{\lambda\in M} \delta_\lambda$ is a g-a-p measure (Theorem 3.3 below). Therefore $\nu \leq \mu$ where μ is an almost periodic measure. We obviously have $\mu \geq \sigma_\theta$. Therefore Lemma 2.37 is proved.

We complete the proof of Theorem 2.34. One has for every $j \geq 1$, $\Lambda_\theta = \theta^j \Lambda_\theta + F_j$ where the cardinality of F_j is 2^j. Let $\mu(x)$ be the almost periodic measure of Lemma 2.37. With an obvious abuse of notations we define the measure τ_j by $\tau_j(x) = \theta^{-j} \sum_{\lambda\in F_j} \mu(\theta^{-j}(x - \lambda))$. We then have $\tau_j \geq \sigma_\theta$ together with $\mathcal{M}(\tau_j) = C2^j\theta^{-j}$ which tends to 0 as j tends to infinity. Therefore the definition of a g-a-p measure is satisfied with $\tau_j \geq \sigma_\theta \geq 0$.

Let us return to the general properties of g-a-p measures.

Lemma 2.39. *If μ is a g-a-p measure and f is an almost periodic function, the product $f\mu$ is a g-a-p measure.*

The proof is obvious if f is non negative. The general case follows from the fact that any almost periodic function f can be written $f = f_1 - f_2$ where f_1 and f_2 are non negative and almost periodic. This argument does not work if f is a g-a-p function.

Lemma 2.40. *If μ is a g-a-p measure, then*

$$\mathcal{M}(\mu) \;=\; \lim_{R\to+\infty} c_n R^{-n} \mu(B(x_0, R) \tag{2.20}$$

exists uniformly in $x_0 \in \mathbb{R}^n$ and we have

$$\mathcal{M}(\mu) = \lim_{\epsilon\to 0} \mathcal{M}(\mu_\epsilon) \tag{2.21}$$

when μ_ϵ is defined by (2.16).

The proof is the same as the one of Lemma 2.6.

Corollary 2.41. *Let μ be a g-a-p measure. Then for every almost periodic function f the limit*

$$\mathcal{M}(f\mu) \;=\; \lim_{R\to+\infty} c_n R^{-n} \int_{B(x_0,R)} f\, d\mu \tag{2.22}$$

exists uniformly in $x_0 \in \mathbb{R}^n$.

It suffices to combine Lemma 2.39 and Lemma 2.40.

Definition 2.42. *Let μ be a g-a-p measure. The Fourier coefficients $\hat{\mu}(\omega), \omega \in \mathbb{R}^n$, are defined by $\mathcal{M}(e_\omega \mu)$ where $e_\omega(x) = \exp(-i\omega\cdot x)$.*

The following remark will be used in Section 4.

Lemma 2.43. *With the notations of Definition 2.31 we have*

$$\hat{\mu}(\omega) = \lim_{\epsilon\to 0} \hat{\mu}_\epsilon(\omega) \tag{2.23}$$

If moreover the measures $\hat{\mu}_\epsilon$ are Poisson measures we have

$$\hat{\mu}(\omega) = (2\pi)^{-n} \lim_{\epsilon\to 0} \hat{\mu}_\epsilon(\{\omega\}) \tag{2.24}$$

This follows from Lemma 2.40.

In the example treated in Section 4 the measures μ_ϵ are Poisson measures and the computation of $\hat{\mu}_\epsilon(\omega)$ will be trivial.

Definition 2.44. *A translation bounded measure μ is a weakly g-a-p measure if $\mu * f$ is a g-a-p function for every compactly supported continuous function f.*

Here is an example of a weakly g-a-p measure which is not a g-a-p measure. We consider in one dimension the series $\sigma = \sum_{k\in\mathbb{Z}}(\delta_{k+\epsilon_k} - \delta_k)$ where ϵ_k tends to 0 as k tends to infinity and $\epsilon_k \neq 0$ for every k. We then have

Lemma 2.45. *The measure $\sigma = \sum_{k\in\mathbb{Z}}(\delta_{k+\epsilon_k} - \delta_k)$ is a weakly g-a-p measure but is not a g-a-p measure.*

The measure σ is translation bounded. If ϕ is a test function the convolution product $g = \phi * \sigma$ is a continuous function tending to 0 at infinity. Therefore g is a g-a-p function.

We need to prove that σ is not a g-a-p measure. Our first and immediate observation is $\mathcal{M}(f\sigma) = 0$ for every almost periodic function f. This is obvious since $f(k+\epsilon_k) - f(k)$ tends to 0 as k tends to infinity. We then argue by contradiction and assume that $\mu_\epsilon \le \sigma \le \nu_\epsilon$ where μ_ϵ and ν_ϵ are two almost periodic measures which have close mean values. If f is a non negative almost periodic function we obtain $\mathcal{M}[f\mu_\epsilon] \le 0 \le \mathcal{M}[f\nu_\epsilon]$. Lemma 1.26 implies $\nu_\epsilon \ge 0$ and $\mu_\epsilon \le 0$.

Since $\sigma \le \nu_\epsilon$ and $0 \le \nu_\epsilon$ the same is true for the positive part $\tau = \sup(0,\sigma)$ of σ. Therefore $\tau = \sum_{k\in\mathbb{Z}} \delta_{k+\epsilon_k} \le \nu_\epsilon$. Similarly $\sigma \ge \mu_\epsilon$ and $0 \ge \mu_\epsilon$ yield $\rho = \inf(\sigma, 0) = -\sum_{k\in\mathbb{Z}} \delta_k \ge \mu_\epsilon$. Finally $\mathcal{M}(\nu_\epsilon - \mu_\epsilon) \le \epsilon$ implies $\mathcal{M}(\tau - \rho) \le \epsilon$ which is not the case since $\epsilon_k \ne 0$ for every k. This ends the proof of Lemma 2.45.

If μ is a weakly g-a-p measure the linear functional $f \mapsto \mathcal{M}(f\mu)$ is continuous on the space of all almost periodic functions. Therefore this functional defines a Borel measure $\tilde{\mu}$ on the Bohr compactification $\mathcal{G}_n$ of $\mathbb{R}^n$. The mapping $\mu \mapsto \tilde{\mu}$ is not injective.

3 Model sets.

A lattice $\Gamma \subset \mathbb{R}^N$ is a discrete subgroup with compact quotient. In other words $\Gamma = A(\mathbb{Z}^N)$ where A is an $N \times N$ invertible matrix. We define $\mathrm{vol}(\Gamma)$ as the volume of any fundamental domain V of Γ. A fundamental domain is any Borel set V such that $V + \gamma, \gamma \in \Gamma$, is a measurable partition of $\mathbb{R}^N$. Then $\mathrm{vol}(\Gamma) = |\det A|$. The dual lattice $\Gamma^* \subset \mathbb{R}^N$ is defined by $\exp(iy\cdot x) = 1$ for every $x \in \Gamma$ and every $y \in \Gamma^*$. We obviously have $\Gamma = (\Gamma^*)^*$.

The definition of a model set $\Lambda \subset \mathbb{R}^n$ is given now. One starts with an integer $m \ge 1$, we set $N = n+m$, $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and consider a lattice $\Gamma \subset \mathbb{R}^N$. If $X = (x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we write $x = p_1(X)$ and $y = p_2(X)$.

Let us assume that $p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_2(\Gamma)$ is a dense subgroup of $\mathbb{R}^m$. Let $\Gamma^* \subset \mathbb{R}^N$ be the dual lattice and p_1^*, p_2^* be defined as p_1, p_2. Then $p_1^* : \Gamma^* \to p_1^*(\Gamma^*)$ is a one-to-one mapping and $p_2^*(\Gamma^*)$ is a dense subgroup of $\mathbb{R}^m$. A set $K \subset \mathbb{R}^m$ is Riemann integrable if its boundary has a zero Lebesgue measure. The boundary of K is $\overline{K} \setminus L$ where $\overline{K}$ is the closure of K and L is the interior of K. The interior of K is the largest open set contained in K. If K is Riemann integrable, then K has a positive measure if and only if K has a non-empty interior.

Definition 3.1. *Let K be a Riemann integrable compact subset of $\mathbb{R}^m$ with a positive measure. Then the model set Λ defined by Γ and K is*

$$\Lambda = \{\lambda = p_1(\gamma);\, \gamma \in \Gamma,\, p_2(\gamma) \in K\} \tag{3.1}$$

A subset Λ of $\mathbb{R}^n$ is a model set if either Λ is a lattice or if one can find m, Γ, and K such that Λ is the model set defined by (3.1).

The compact set K is named the *window* of the model set Λ. The reader is referred to Definition 1.10, page 48 of [20]. But in this reference the compact window K is replaced by an open set Ω with a compact closure. For proving that a model set has a uniform density, we were assuming that the closure of Ω is Riemann integrable.

Let $\mathcal{P} : p_1(\Gamma) \mapsto p_2(\Gamma)$ be the mapping defined by $\mathcal{P}(p_1(\gamma)) = p_2(\gamma)$, $\gamma \in \Gamma$. This mapping satisfies $\mathcal{P}(x+y) = \mathcal{P}(x) + \mathcal{P}(y)$, $\forall x, \forall y \in p_1(\Gamma)$. It will be proved (see Corollary 4.5, Section 4) that $\mathcal{P}(\Lambda)$ is equidistributed in K.

In [20] the definition of model sets was slightly more general than the one which is given here and $\mathbb{R}^m$ was replaced by a locally compact abelian group.

Here is another definition of model sets. As above $\Gamma \subset \mathbb{R}^N$, $N = n+m$, is a lattice and Γ^* is the dual lattice. Let $\Delta = \mathbb{R}^N/\Gamma$ be the compact quotient group which is isomorphic to $\mathbb{T}^N$ and let $\zeta : \mathbb{R}^N \mapsto \Delta = \mathbb{R}^N/\Gamma$ be the quotient map. The dual lattice Γ^* is the dual group of Δ. In other words the Fourier series expansion of a $\Gamma-$periodic function f is $f(x) = \sum_{y\in\Gamma^*} c(y)\exp(ix\cdot y)$. Let us identify $\mathbb{R}^n$ with $L = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N$ and define $\Theta : \mathbb{R}^n \mapsto \Delta$ by $\Theta(x) = \zeta((x,0))$. Then Θ is injective if and only if $p_2 : \Gamma \mapsto \mathbb{R}^m$ is injective. Since $p_1 : \Gamma \mapsto \mathbb{R}^n$ is injective, the mapping ζ restricted to $\{0\}\times\mathbb{R}^m$ is one-to-one. Let W be the subgroup $\zeta[\{0\}\times\mathbb{R}^m]$ of Δ. As above $K \subset \mathbb{R}^m$ is a Riemann integrable compact set. We move K to $V = \zeta[\{0\}\times(-K)] \subset \Delta$. Let us stress that $V \subset W$. Let Λ be the model set defined by Γ and K. Then we have

Lemma 3.2. *With the preceding notations the model set* Λ *can be defined by*

$$\Lambda = \{x \in \mathbb{R}^n; \Theta(x) \in V\} \tag{3.2}$$

Indeed $x \in \Lambda$ reads $x = \gamma_1$ with $(\gamma_1, \gamma_2) \in \Gamma$, $\gamma_2 \in K$. Then $(\gamma_1, 0) - (\gamma_1, \gamma_2) = (0, -\gamma_2) \in \{0\}\times(-K)$. Therefore $\Theta(x) = \zeta(\gamma_1, 0) \in V$. Conversely $\Theta(x) \in V$ implies $(x,0) - (\gamma_1, \gamma_2) = (0, -y)$ for some $y \in K$. Therefore $x \in \Lambda$.

Theorem 3.3. *The sum* $\sigma_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$ *of Dirac masses on a model set* Λ *is a generalized almost periodic measure.*

The proof of this theorem relies on an Poisson summation formula which is detailed in the following section. Theorem 3.3 will be rephrased as Corollary 4.3 there.

4 Poisson summation formula and model sets.

Let Λ be a model set defined as above by a lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ and a compact set $K \subset \mathbb{R}^m$. We let H denote the group $p_1(\Gamma^*)$ where Γ^* is the dual lattice of Γ. Let us assume K to be Riemann integrable with a positive measure and let φ denote any $C_0^\infty(\mathbb{R}^m)$ function vanishing outside K.

The corresponding weight factors $w(\lambda)$, $\lambda \in \Lambda$, are defined on the model set Λ by $w(p_1(\gamma)) = \varphi(p_2(\gamma))$, $\gamma \in \Gamma$. If φ was the indicator function of K (this indicator function is not smooth), we would have $w(\lambda) = 1$ on Λ.

With these notations, one obtains [19], [20]

Theorem 4.1. *Let μ be the sum $\sum_{\lambda\in\Lambda} w(\lambda)\delta_\lambda$ of Dirac masses over Λ where the weight factors $w(\lambda)$ are defined as above. Then the distributional Fourier transform $\nu = \hat{\mu}$ of μ is the atomic measure ν defined by*

$$\nu = \sum_{\Gamma^*} \omega(p_2(\gamma^*))\,\delta_{p_1(\gamma^*)} \tag{4.1}$$

where the dual weights $\omega(p_2(\gamma^*))$ are

$$\omega(y) \;=\; \frac{(2\pi)^n}{\mathrm{vol}\,\Gamma}\,\hat{\varphi}(-y) \quad , \quad y = p_2(\gamma^*)\,,\ \gamma^* \in \Gamma^*\,. \tag{4.2}$$

For proving Theorem 4.1 it suffices to show that $\int \hat{u}\, d\mu = \int u\, d\nu$ for every test function u. This reads

$$\sum_{\lambda\in\Lambda} w(\lambda)\hat{u}(\lambda) = \sum_{\Gamma^*} \omega(p_2(\gamma^*))\, u(p_1(\gamma^*)) \tag{4.3}$$

But $w(\lambda) = w(p_1(\gamma)) = \varphi(p_2(\gamma))$ and one can forget the restriction $\lambda \in \Lambda$ which is given for free by the support of φ. Then (4.3) follows from the ordinary Poisson formula applied to the lattice Γ and the dual lattice Γ^*. A. Córdoba proved in [4], [5] that a Poisson summation formula cannot be true without such weight factors unless Λ is a lattice.

Corollary 4.2. *The measure μ defined by Theorem 4.1 is a Poisson measure and we have*

$$\mathcal{M}(\mu) = \frac{1}{\mathrm{vol}\,\Gamma} \int_K \varphi(x)\, dx \tag{4.4}$$

The first statement is obvious since ω belongs to the Schwartz class. The second statement is implied by Lemma 1.29.

We have more

Corollary 4.3. *Let Λ be a model set. Then the measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a generalized almost periodic measure.*

Since K is Riemann integrable for each positive ϵ we can construct two test functions u_ϵ and v_ϵ such that $0 \le u_\epsilon \le \chi_K \le v_\epsilon$ and $\int (v_\epsilon - u_\epsilon)\, dx \le \epsilon$. We then construct the measures μ_ϵ and ν_ϵ in terms of u_ϵ and v_ϵ as in Theorem 4.1 and we obviously have $\mu_\epsilon \le \sigma_\Lambda \le \mu_\epsilon$. It suffices to use (4.4) which yields $\mathcal{M}[\nu_\epsilon - \mu_\epsilon] = \frac{1}{\mathrm{vol}\,\Gamma} \int (v_\epsilon - u_\epsilon)\, dx \le \frac{\epsilon}{\mathrm{vol}\,\Gamma}$.

Proposition 4.4. *Let Λ be a model set. Then the Fourier coefficients of the generalized almost periodic measure $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$ are given by*

(a) $\hat{\sigma}_\Lambda(\omega) \;=\; \frac{1}{\mathrm{vol}\,\Gamma} \int_K \exp(i\omega\cdot x)\, dx$ *if* $\omega = p_2(\gamma^*)\,,\ \gamma^* \in \Gamma^*$

(b) $\hat{\sigma}_\Lambda(\omega) = 0$ *if* $\omega \notin p_2(\Gamma^*)$.

Proposition 4.4 follows immediately from Lemma 2.43. Assuming $n = 1$ in the definition of a model set and ordering Λ as an increasing sequence λ_j, $j \in \mathbb{Z}$ we have

Corollary 4.5. *The sequence $\tilde{\lambda}_j = \mathcal{P}(\lambda_j)$, $j \in \mathbb{Z}$, is equidistributed in K.*

For proving this fact it suffices to check that for every $\omega \in \mathbb{R}^n$ we have

$$\lim_{N\to\infty} \frac{1}{N} \exp(i\omega \cdot \tilde{\lambda}_j) = \int_K \exp(i\omega \cdot x)\, dx \tag{4.5}$$

By density it suffices to prove (4.5) when $\omega = p_2(\gamma)$, $\gamma \in \Gamma$. But if $(\omega, v) \in \Gamma^*$ we have $\exp(i(\omega \cdot \tilde{\lambda}_j + v\lambda_j)) = 1$. Therefore the mean value on j of $\exp(i(\omega \cdot \tilde{\lambda}_j)$ equals the mean value of $\exp(-iv\lambda_j)$ which is given by Proposition 4.4. This ends the proof. See [23], [24] for another approach to Corollary 4.5.

5 Harmonious sets.

Harmonious sets which are defined in this section are playing an important role in the study of the behavior of mean-periodic functions. Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary set of real numbers and let $\Gamma(\Lambda)$ be the subgroup of $\mathbb{R}^n$ generated by Λ, equipped with the discrete topology. A weak character on Λ is the restriction to Λ of an algebraic homomorphism from $\Gamma(\Lambda)$ into the group $\mathbb{T}$ of complex numbers of modulus 1. No continuity property is required on this algebraic homomorphism χ. It satisifies $\chi(x+y) = \chi(x)\chi(y)$, $x, y \in \mathbb{R}^n$. A strong character h is a restriction to $\Gamma(\Lambda)$ of a continuous homomorphism from $\mathbb{R}^n$ to $\mathbb{T}$ and is therefore given by $h(x) = \exp(i\xi \cdot x)$, $\quad \xi \in \mathbb{R}^n$.

Definition 5.1. *Let $\epsilon \in (0,2)$. A set $\Lambda \subset \mathbb{R}^n$ is $\epsilon-$harmonious if for every weak character χ, there exists a strong character h such that*

$$\sup_{\lambda\in\Lambda} |\chi(\lambda) - h(\lambda)| \le \epsilon. \tag{5.1}$$

Weak characters are no longer used in the equivalent definition which is given now.

Definition 5.2. *Given a subset Λ of $\mathbb{R}^n$ and $\epsilon \in (0,2)$, let $\Lambda^*_\epsilon \subset \mathbb{R}^n$ be the set of all $\xi \in \mathbb{R}^n$ such that $\sup_\Lambda |\exp(i\xi \cdot \lambda) - 1| \le \epsilon$. This set $\Lambda^*_\epsilon \subset \mathbb{R}^n$ is named the ϵ-dual set of Λ.*

We now have [20]

Lemma 5.3. *Let $\epsilon \in (0,2)$. A set $\Lambda \subset \mathbb{R}^n$ is $\epsilon-$harmonious if and only if Λ^*_ϵ is relatively dense in the sense of Besicovitch.*

Definition 5.4. *A subset Λ of $\mathbb{R}^n$ is harmonious if it is $\epsilon-$harmonious for all $\epsilon \in (0,2)$.*

A subset of harmonious set is a harmonious set. Any finite set F is harmonious. If Λ is harmonious and if F is finite, then $\Lambda + F$ is harmonious. If Λ is harmonious, so is $\Lambda - \Lambda$. The union of two harmonious sets is not harmonious in general. Indeed the lattices $\Lambda_1 = \mathbb{Z}$ and $\Lambda_2 = \sqrt{2}\mathbb{Z}$ are harmonious but their union $\Lambda_1 \cup \Lambda_2$ is not harmonious. If this union was harmonious, then $\Lambda_1 - \Lambda_2$ would also be a harmonious set. This is impossible since any harmonious set is uniformly discrete : there exists a positive β such that $\lambda \in \Lambda$, $\lambda' \in \Lambda$ and $\lambda' \ne \lambda$ imply $|\lambda' - \lambda| \ge \beta$. See [20].

The proof of the following can be found in [20].

Lemma 5.5. *Model sets are harmonious sets.*

6 Coherent sets of frequencies

In the late sixties I became interested in the behavior at infinity of mean-periodic functions. Let me briefly explain what mean-periodic functions are. A complex valued continuous function f defined on $\mathbb{R}^n$ is mean-periodic if the closed linear span of the translates $f(\cdot - y), y \in \mathbb{R}^n$, is not the space of all continuous functions on $\mathbb{R}^n$. Here the topology is defined by uniform convergence on compact sets. An equivalent definition is given by the following

Definition 6.1. *A mean-periodic function is a continuous solution f of a convolution equation*

$$f * \tau = 0 \tag{6.1}$$

where τ is a compactly supported Borel measure.

The case $\tau = 0$ is obviously excluded. We now restrict our attention to the one-dimensional case. The Fourier-Laplace transform of τ is the entire function $F(z) = \int_{\mathbb{R}} \tau(x) \exp(-izx)\, dx$. Let Λ denote the zero set of $F(z)$ where each zero λ is counted in Λ as many times as its multiplicity m_λ indicates. Then $f(x) = P(x)\exp(i\lambda x)$ is a solution to (6.1) if and only if $\lambda \in \Lambda$ and $P(x)$ is a polynomial of degree less than or equal to $m_\lambda - 1$. A finite sum of these building blocks is still a solution to (6.1). Finally any solution f to (6.1) has a Fourier series expansion

$$f(x) \sim \sum_{\lambda \in \Lambda} P_\lambda(x) \exp(i\lambda x) \tag{6.2}$$

where $P_\lambda(x)\exp(i\lambda x)$ are the above mentioned elementary solutions. It means that any solution to (6.1) is the limit of a sequence of finite linear combinations of elementary solutions to (6.1). The convergence is uniform on compact intervals. This does not mean that f is the limit of the sequence of partial Fourier sums in (6.2). Some summation procedures are needed.

The set $\Lambda \subset \mathbb{C}$ is not arbitrary since it is the zero set of the Fourier-Laplace transform of a function with compact support. Such sets Λ have been characterized by A. Beurling and P. Malliavin [1].

From now on multiplicities will be excluded in Λ. If $\Lambda \subset \mathbb{R}$ is the zero set of the Fourier-Laplace transform of a function with compact support, then the Frechet space consisting of all mean-periodic functions which are limits of finite trigonometric sums $\sum_{\lambda \in \Lambda} a_\lambda \exp(i\lambda x)$ will be denoted by C_Λ. The Frechet space C_Λ is equipped with the topology of uniform convergence on compact sets. By construction finite trigonometric sums $g(x) = \sum_{\lambda \in \Lambda} a_\lambda \exp(i\lambda x)$ are dense in C_Λ.

Is it possible to relate the arithmetical structure of the given set Λ to the growth at infinity of all $f \in C_\Lambda$? The most natural problem is given by the following definition.

Definition 6.2. *With the preceding notations and definitions we say that $\Lambda \subset \mathbb{R}$ is a coherent set of frequencies if every $f \in C_\Lambda$ is an almost periodic function in the sense of Bohr.*

A necessary and sufficient condition is given by the following lemma

Lemma 6.3. *The set of real numbers $\Lambda \subset \mathbb{R}$ is a coherent set of frequencies if and only if there exist a compact interval I and a constant C such that*

$$\sup_{x\in\mathbb{R}} |f(x)| \le C \sup_{x\in I} |f(x)| \tag{6.3}$$

holds for every trigonometric sum $f(x) = \sum_{\lambda\in\Lambda} c_\lambda \exp(i\lambda x)$.

It is interesting to find the shortest interval I for which (6.3) holds. It is even more interesting to try to replace I by an arbitrary compact set K of real numbers. This issue will be raised below. Property (6.3) cannot happen if polynomials are allowed in the definition of C_Λ and that explains why multiplicities have been excluded in Λ. The simplest case where this is true is given by $\Lambda = \mathbb{Z}$. Then every $f \in C_\Lambda$ is 2π-periodic. Another example is given when Λ is a finite set.

The generalization to n real variables is now obvious and a coherent set of frequencies $\Lambda \subset \mathbb{R}^n$ is defined by (6.3) where I is replaced by a suitable compact subset $K \subset \mathbb{R}^n$.

The characterization of coherent sets of frequencies given by Theorem 6.6 will be seminal in the proof of the property of spectral synthesis. Let $\Lambda \subset \mathbb{R}$ be a coherent set of frequencies and $V = [-\epsilon, \epsilon]$, $\epsilon > 0$, be an interval. We assume that the intervals $\lambda + V$, $\lambda \in \Lambda$, are pairwise disjoints. We want to analyze the functions f whose Fourier transform $\hat{f}$ is contained in $\Lambda + V$. Then

$$f = \sum_{\lambda\in\Lambda} a_\lambda(x) \exp(i\lambda x) \tag{6.4}$$

where the Fourier transform of each $a_\lambda(x)$ is contained in V. If ϵ is small enough, the functions a_λ go at a slow pace. Bernstein theorem yields a more precise information

Lemma 6.4. *If $f \in L^\infty$ and if the Fourier transform of f is contained in $[-\epsilon, \epsilon]$ then $\|\frac{d}{dx} f\|_\infty \le \epsilon \|f\|_\infty$.*

Therefore these functions $a_\lambda(x)$ go at a slow pace while the oscillating functions $\exp(i\lambda x)$ go at a quick pace. This motivates the following definition

Definition 6.5. *Let f be given by* (6.4)*. Then the unfolding of f is the function $g(x, y)$ of two variables x and y defined as*

$$g(x, y) = \sum_{\lambda\in\Lambda} a_\lambda(y) \exp(i\lambda x) \tag{6.5}$$

We have

Theorem 6.6. *Let $\Lambda \subset \mathbb{R}$ be a coherent set of frequencies. Then there exists an interval V and a constant C with the following property. For every sequence a_λ, $\lambda \in \Lambda$, of functions whose spectra are contained in V we have*

$$\|g(x, y)\|_{L^\infty(\mathbb{R}^2)} \le C \|f(x)\|_{L^\infty(\mathbb{R})} \tag{6.6}$$

when, as above, $g(x, y) = \sum_{\lambda\in\Lambda} a_\lambda(y) \exp(i\lambda x)$

Conversely if (6.6) *holds then Λ is a coherent set of frequencies.*

Theorem 6.6 is playing a seminal role in the problem of spectral synthesis (Section 8).

For proving the first implication in Theorem 6.6 we introduce an auxiliary norm $\omega = \sup_{x,y}\{|g(x,y)|, x \in y+I\}$ and prove that

$$|\omega - ||f||_\infty| \le \epsilon|I|||g||_\infty \tag{6.7}$$

together with

$$||g||_\infty \le C\omega \tag{6.8}$$

Then a standard bootstrap yields the required estimate if $\epsilon|I| < 1/C$.

Let us prove (6.8). Since the problem is translation invariant, (6.3) holds for $I + y$ repacing I. Therefore for every y (6.3) can be rewritten

$$\sup_x |g(x,y)| \le C \sup_{x\in y+I} |g(x,y)| \tag{6.9}$$

It then suffices to compute the supremum in y to deduce (6.8) from (6.9). We now prove (6.7). Let us evaluate $g(x,y) - g(x,z)$. For every fixed x the spectrum of the function $g(x,y)$ of the y variable is contained in V. Then Bernstein theorem applies and for every fixed x yields

$$|g(x,y) - g(x,z)| \le \epsilon|y-z| \sup_y |g(x,y)| \le \epsilon|y-z|||g||_\infty \tag{6.10}$$

Therefore

$$|g(x,y)| \le |g(x,z)| + \epsilon|y-z|||g||_\infty \tag{6.11}$$

Replacing z by x in the RHS of (6.11) and assuming $x \in y+I$ we obtain

$$|g(x,y)| \le |f(x)| + \epsilon|I|||g||_\infty \tag{6.12}$$

Taking the supremum in $x \in y+I$ we obtain $\omega \le ||f||_\infty + \epsilon|I|||g||_\infty$ and $||f||_\infty \le \omega + \epsilon|I|||g||_\infty$ is proved similarly. This ends the proof of the first half of Theorem 6.6.

The proof of the second implication is easier. It suffices to use the following lemma

Lemma 6.7. *If there exists a continuous function θ tending to 0 at infinity and a constant C such that for every $f \in C_\Lambda$ we have*

$$||f||_\infty \le C||\theta f||_\infty \tag{6.13}$$

then Λ is a coherent set for frequencies.

The proof is obtained by a standard bootstrap argument. We let I be large enough such that $\sup_{x\notin I} |\theta(x)| \le \frac{1}{2C}$ and split the RHS of (6.13) in the sum
$\sup_{x\in I} |\theta(x)f(x)| + \sup_{x\notin I} |\theta(x)f(x)| \le ||\theta||_\infty \sup_{x\in I} |f(x)| + \frac{1}{2}||f||_\infty$.

The proof we gave does not yield the optimal value of ϵ. Here is a sharp result ([18], Theorem 12).

Theorem 6.8. *Let Λ be a model set defined by a window K and a lattice $\Gamma \subset \mathbb{R}\times\mathbb{R}^n$. For a positive ω let us assume the following three properties*

(a) $p_1(\Gamma^*)$ *is dense in* $\mathbb{R}$

(b) $p_2(\Gamma^*)$ *is dense in* $\mathbb{R}^n$

(c) two distinct points of the compact set $U = [-\omega,\omega]\times K$ *are never congruent modulo* Γ.

Then for every sequence a_λ, $\lambda\in\Lambda$, of functions whose spectra are contained in $[-\omega,\omega]$ we have

$$\|g(x,y)\|_{L^\infty(\mathbb{R}^2)} \le C\|f(x)\|_{L^\infty(\mathbb{R})}$$

where $f = \sum_{\lambda\in\Lambda} a_\lambda(x)\exp(i\lambda x)$ and $g(x,y) = \sum_{\lambda\in\Lambda} a_\lambda(y)\exp(i\lambda x)$.

This result gives the optimal value of ω ([18], Theorem 12) while Theorem 6.6 was only showing that such an $\omega > 0$ exists. Before giving more details let us provide the reader with the geometrical intuition which paves the way to the proof. We consider the mapping $\Theta : \mathbb{R} \mapsto \Delta$ which is defined in Lemma 3.2. Then we have $\Lambda = \{x\in\mathbb{R}^n; \Theta(x)\in M\}$ where M is a compact subset which is $n-1$ dimensional and transverse to $\Theta(\mathbb{R})$. Then $\Theta(\Lambda+V)$ is contained in a compact set N which is the product $M\times\Theta(V)$. It implies that the slow function and the quick functions are now decoupled. The proof will follow this guide.

The functions a_λ can be assumed to be finite trigonometric sums and the series (6.4) to be a finite sum. The general case will follow by a limiting argument where the topology on $L^\infty(\mathbb{R})$ is $\sigma(L^\infty, L^1)$. Let us be more explicit. Let us consider a function $f\in L^\infty$ whose spectrum is contained in $\Lambda + [-\omega,\omega]$. The first step in the approximation is to replace f by a sequence f_j whose Fourier transforms have compact supports. We have $\|f_j\|_\infty \le \|f\|_\infty$ and f is the limit of f_j with respect to the $\sigma(L^\infty, L^1)$ topology. The second step is to replace each f_j by a sequence $P_{j,k}$ of trigonometric sums which converge to f_j as k tends to infinity. Here the topology is the uniform convergence on compact sets. Moreover we have $\|P_{j,k}\|_\infty \le \|f_j\|_\infty \le \|f\|_\infty$. This part of the approximation is standard since the Fourier transform of f_j is supported by a finite union of intervals. By an abuse of notations we write again f instead of $P_{j,k}$. The estimate which is proved below yields $\|g_{j,k}\|_\infty \le C\|P_{j,k}\|_\infty$ and it suffices to pass to the limit to prove our claim.

We let $\sigma = \hat{f}$ be the Fourier transform of f. Then σ is a finite sum of Dirac masses. We have $\sigma = \sum_{\{\lambda\in\Lambda, |t|\le\omega\}} c(\lambda,t)\delta_{(\lambda+t)}$. We now forget to write the subscripts of this summation. Let $\mathcal{T} : \Lambda + [-\omega,\omega] \mapsto \mathbb{R}\times\mathbb{R}^n$ be defined by $\mathcal{T}(p_1(\gamma)+t) = (-t, p_2(\gamma))$, $\gamma\in\Gamma, |t|\le\omega$. We move σ on $\mathbb{R}\times\mathbb{R}^n$ by $\mathcal{T}$ and write $\tau = \sum c(\lambda,t)\delta_{(-t,p_2(\gamma))}$. We aim to compare τ to σ and more precisely to estimate the $L^\infty(\mathbb{R}^{n+1})$ norm of $\hat\tau$ by the $L^\infty(\mathbb{R})$ norm of $\hat\sigma(-x) = 2\pi f(x)$.

We first compute the restriction of the Fourier transform $\hat\tau$ of τ to the dual lattice Γ^*. It is given by

$$\hat\tau[p_1(\gamma^*), p_2(\gamma^*)] = \sum c(\lambda,t)\exp(i[tp_1(\gamma^*) - p_2(\gamma)\cdot p_2(\gamma^*)]) \tag{6.14}$$

But for every $\gamma\in\Gamma$ and every $\gamma^*\in\Gamma^*$ we have

$$p_1(\gamma)p_1(\gamma^*) + p_2(\gamma)\cdot p_2(\gamma^*) \in 2\pi\mathbb{Z} \tag{6.15}$$

which implies

$$\hat{\tau}[p_1(\gamma^*), p_2(\gamma^*)] = \sum c(\lambda, t)\exp(i[t + p_1(\gamma)]p_1(\gamma^*)) \tag{6.16}$$

We recognize the value of the one dimensional Fourier transform $\hat{\sigma}(\xi)$ of σ at $\xi = -p_1(\gamma^*)$. Since $p_1(\Gamma^*)$ is dense in $\mathbb{R}$ we have

Lemma 6.9. *The $L^\infty(\mathbb{R})$ norm of the Fourier transform of σ is equal to the $l^\infty(\Gamma^*)$ norm of the Fourier transform of τ*

$$||\hat{\sigma}||_\infty = ||\hat{\tau}||_{l^\infty(\Gamma^*)} \tag{6.17}$$

The second ingredient in the proof is Shannon's theorem which says the following

Lemma 6.10. *If two distinct points of a compact set U are never congruent modulo Γ then there exists a constant C such that for every measure τ supported by U the following holds*

$$||\hat{\tau}||_\infty \le C||\hat{\tau}||_{l^\infty(\Gamma^*)} \tag{6.18}$$

Lemma 6.10 is well known when $U = [-a, a]$ and when the lattice $\Gamma^* = h\mathbb{Z}$ with $0 < h < \frac{\pi}{h}$ but the same proof works as well in the general case. Lemma 6.9 and Lemma 6.10 imply Theorem 6.8.

Harmonious sets and coherent sets of frequencies are related by another connection, as indicated in the following theorem [19].

Theorem 6.11. *For any subset $\Lambda \subset \mathbb{R}$ the following two properties are equivalent*

(a) For each positive ε there exists a Delone set T_ε such that

$$\tau \in T_\varepsilon \Rightarrow ||f(x-\tau) - f(x)||_\infty \le \varepsilon ||f||_\infty \quad , \quad f \in C_\Lambda . \tag{6.19}$$

(b) Λ is harmonious.

If it is the case, then every $f \in C_\Lambda$ is an almost periodic function in the sense of Bohr.

If $f(x) = e^{i\lambda x}$, then (6.19) implies $|e^{i\lambda\tau} - 1| \le \varepsilon$ for each $\tau \in T_\varepsilon$. Therefore Λ is harmonious. Conversely let us assume that Λ is harmonious. Let Λ_ε^* be the ε-dual of Λ. The following lemma (N. Varopoulos, oral communication) will be used in the proof of Theorem 6.11.

Lemma 6.12. *Let $\eta \in (0, \pi/2]$ and $\theta(x)$ be the $2\pi-$periodic odd function of the real variable x defined by*

(a) $\theta(x) = \sin x$ if $|x - k\pi| \le \eta$, $k \in \mathbb{Z}$,

(b) $\theta(x) = \sin\eta$ if $\eta \le x \le \pi - \eta$.

Then the Fourier coefficients γ_k, $k \in \mathbb{Z}$, of $\theta(x)$ satisfy

$$\sum_{k\in\mathbb{Z}} |\gamma_k| \le C\eta\log(1/\eta) \tag{6.20}$$

where C is a numerical constant.

The proof is straightforward and will be omitted.

Corollary 6.13. *For every real number x one has*

$$|\sin x| \le \sin\eta \Rightarrow \sin x = \sum_{k\in\mathbb{Z}} \gamma_k \exp(ikx)$$

where the Fourier coefficients γ_k have a small l^1 norm as indicated in (6.20).

This immediately implies the following:

Corollary 6.14. *There exists a constant C such that for every $x \in \mathbb{R}$ and every $\epsilon > 0$,*

$$|\exp(ix) - 1| \le \epsilon \Rightarrow \exp(ix) - 1 = 2i\exp(ix/2)\sum_{-\infty}^{\infty} \gamma_k \exp(ikx/2) \tag{6.21}$$

where the Fourier coefficients γ_k satisfy (6.20) *with* $2\sin(\eta/2) = \epsilon$.

One writes $\exp(ix) - 1 = 2i\exp(ix/2)\sin(x/2)$ and uses Lemma 6.12. In what follows the factor $2i\exp(ix/2)$ in front of the RHS of (6.21) will be incorporated in the Fourier series expansion which changes the meaning of the coefficients γ_k without any modification in the claim.

Corollary 6.15. *For every $\varepsilon > 0$, $f \in C_\Lambda$ and $\tau \in \Lambda_\varepsilon^*$*

$$\|f(x+\tau) - f(x)\|_\infty \le C\epsilon\log(1/\epsilon)\|f\|_\infty \tag{6.22}$$

The proof is simple. We assume that f is a finite trigonometric sum and write $f(x) = \sum c(\lambda)\exp(i\lambda x)$. The definition of Λ_ε^* and Corollary 6.14 imply

$$\begin{aligned} f(x+\tau) - f(x) &= \sum c(\lambda)\exp(i\lambda x)[\exp(i\lambda\tau) - 1] = \sum\sum c(\lambda)\gamma_k \exp(i\lambda(x + k\tau/2)) \\ &= \sum \gamma_k f(x + k\tau/2). \end{aligned}$$

Finally $\|f(x+\tau) - f(x)\|_\infty \le \|f\|_\infty \sum |\gamma_k|$ which ends the proof of Corollary 6.15.

Keeping the same notations we have

Corollary 6.16. *Let us assume that $\kappa = C\epsilon\log(1/\epsilon) < 1$ and that Λ_ε^* is a Delone set. Then Λ is a coherent set of frequencies.*

Indeed let $T > 0$ be defined by $[0, T] + \Lambda_\varepsilon^* = \mathbb{R}$. Therefore every $x \in \mathbb{R}$ can be written $x = y + \tau,\ \tau \in \Lambda_\varepsilon^*,\ y \in [0, T]$. Corollary 6.15 implies

$$|f(y+\tau) - f(y)| \le \kappa\|f\|_\infty$$

for every $f \in C_\Lambda$. This yields

$$|f(x)| \le \sup_{y\in[0,T]} |f(y)| + \kappa\|f\|_\infty.$$

Finally $\|f\|_\infty \le \sup_{y\in[0,T]} |f(y)| + \kappa\|f\|_\infty$ and a simple bootstrap yields the required result since $\kappa < 1$. This proves Theorem 6.11.

This argument can be improved. We follow [20] and prove that Λ is a coherent set of frequencies under a weaker hypothesis than the one used in Corollary 6.16. An improved version of Lemma 6.12 will be used for achieving this goal. We first consider an auxiliary function $\omega(x)$ defined by the following properties

(a) $\omega(x)$ is $2\pi/5$-periodic

(b) $\omega(x)$ is an even function

(c) $\omega(x) = \sin(\pi/10 - x)$ on $[0, \pi/5]$.

Then a brute force calculation yields

Lemma 6.17. *The Fourier series expansion of $\omega(x)$ is given by*

$$\omega(x) = \sum_{-\infty}^{\infty} \alpha_k \exp(i5kx)$$

where the Fourier coefficients α_k are non-negative. We have

$$\sum_{-\infty}^{\infty} \alpha_k = \sin(\pi/10) \tag{6.23}$$

We then consider the function $\phi(x) = \omega(x - \pi/10)$ and observe that $\phi(x) = \sin x$ on every interval $[k\pi - \pi/10, k\pi + \pi/10]$, $k \in \mathbb{Z}$. It yields

$$|\sin x| \le \sin(\pi/10) \Rightarrow \sin x = \phi(x) \tag{6.24}$$

which is the *raison d'être* of the construction of ϕ. It implies

$$|\exp(ix) - 1| \le 2\sin(\pi/10) \Rightarrow \exp(ix) - 1 = 2i \exp(ix/2) \sum_{-\infty}^{\infty} \beta_k \exp(i5kx/2) \tag{6.25}$$

where β_k are the Fourier coefficients of ϕ. We have

$$\phi(x) = \sum_{-\infty}^{\infty} \beta_k \exp(i5kx) \ \text{ with } \ \sum_{-\infty}^{\infty} |\beta_k| = \sin(\pi/10) \tag{6.26}$$

We now repeat the proof of Corollary 6.15. Using the definition of Λ_ε^* we have $|\exp(i\lambda\tau) - 1| \le \epsilon$, $\lambda \in \Lambda$, and we continue as above to obtain

$$\begin{aligned} f(x+\tau) - f(x) &= \sum c(\lambda) \exp(i\lambda x)[\exp(i\lambda\tau) - 1] = 2i \sum\sum c(\lambda)\beta_k \exp(i\lambda(x + (5k+1)\tau/2)) \\ &= 2i \sum \beta_k f(x + (5k+1)\tau/2). \end{aligned}$$

Finally $\|f(x+\tau) - f(x)\|_\infty \le 2(\sum |\beta_k|)\|f\|_\infty$. The definition of β_k and Lemma 6.17 yield $\kappa = 2\sum|\beta_k| = 2\sum \alpha_k = 2\sin(\pi/10) = \frac{\sqrt{5}-1}{2} = 0.6180339\ldots < 1$.

We just proved the following :

Theorem 6.18. *Let Λ be a set of real numbers. Let us assume that Λ_ε^* is a Delone set when $\varepsilon = \frac{\sqrt{5}-1}{2}$. Then Λ is a coherent set of frequencies.*

Such sets Λ are not harmonious in general. It is likely that the critical value $\varepsilon = \frac{\sqrt{5}-1}{2}$ can be replaced by a larger one. If instead of ω one used the function ω_0 which is even and $2\pi/3$ periodic with $\omega_0 = \sin(\pi/6 - x)$ on $[0, \pi/3]$, the Fourier coefficients $\tilde{\alpha}_k$ of ω_0 would still be non-negative. But we have $\kappa = \sum \tilde{\alpha}_k = 2\sin(\pi/6) = 1$ which is forbidden by the bootstrap argument.

Let $\theta > 1$ and let Λ_θ be the set of all finite sums $\lambda = \sum_{k\geq 0} \epsilon_k \theta^k$; $\epsilon_k \in \{0, 1\}$. This set is playing a key role in elucidating the problem of spectral synthesis for the Cantor type set constructed with the dissection ratio $1/\theta$. Details are given in Section 8.

Theorem 6.19. *The following two properties are equivalent ones*

(a) every $f \in C_{\Lambda_\theta}$ is an almost periodic function in the sense of Bohr

(b) θ is a Pisot number.

The implication (a)$\Rightarrow$ (b) is not difficult. We use the following lemma.

Lemma 6.20. *Let us assume that θ is not a Pisot number. Then $P_n(x) = \Pi_0^{n-1} \cos(\theta^k x/2)$ tends to 0 uniformly on compact sets not containing 0.*

Indeed the sequence $|P_n(x)|$ is decreasing and it suffices to prove that $P_n(x_0) \to 0$ for every $x_0 \neq 0$ for obtaining the required uniform convergence. We use the simple observation that an infinite product $\Pi_0^\infty (1 - \epsilon_k)$ converges to 0 when $0 \leq \epsilon_k < 1$ and $\sum_0^\infty \epsilon_k = +\infty$. Then the required pointwise convergence is implied by Theorem 2.35. We have $|P_n| = |Q_n|$ where $Q_n(x) = \Pi_0^{n-1}\left(\frac{1+\exp(i\theta^k x)}{2}\right)$. Therefore $Q_n \in C_{\Lambda_\theta}$ converges uniformly to 0 on any compact set not containing 0 while $Q_n(0) = 1$. Piling up some translates of Q_n it is quite easy to construct an unbounded $f \in C_{\Lambda_\theta}$ of the form $f(x) = \sum_{k\geq 0} a_k Q_{n_k}(x - x_k)$.

For proving (b)$\Rightarrow$(a) in Theorem 6.19, we prove a stronger statement. In fact Λ_θ is a harmonious set when θ is a Pisot number (Lemma 2.38) and every harmonious set is a coherent set of frequencies, as Theorem 6.11 indicates.

7 Model sets and irregular sampling

If Λ is a coherent set of frequencies there exist a compact subset $K \subset \mathbb{R}^n$ and a constant C such that every $f \in C_\Lambda$ satisfies

$$\sup_{x\in\mathbb{R}^n} |f(x)| \leq C \sup_{x\in K} |f(x)| \tag{7.1}$$

If (7.1) holds for K it will hold for any set L containing K. The challenge is to find an optimal K for which (7.1) holds. If Λ is a lattice, then (7.1) holds for K if and only if

$$\Lambda^* + K = \mathbb{R}^n \tag{7.2}$$

where Λ^* is the dual lattice of Λ. Therefore the smallest K fulfilling (7.2) is a (compact) fundamental domain of Λ^*. We have $|K| = (2\pi)^n \operatorname{dens}\Lambda$ but this necessary condition is far from being sufficient. Surprisingly a condition on the measure of K suffices when Λ is a simple model set (which is defined below).

Lemma 7.1. *Let us assume that*

(a) K is Riemann integrable

(b) Λ is a simple model set

(c) the Lebesgue measure $|K|$ of K satisfies $|K| > (2\pi)^n \operatorname{dens}\Lambda$

Then (7.1) *holds.*

A proof is given in [16]. This result can be rephrased as a theorem of interpolation. Let $A(\mathbb{R}^n)$ denote the Banach algebra consisting of Fourier transforms $\hat{f}$ of functions $f \in L^1(\mathbb{R}^n)$. If Λ is any closed subset of $\mathbb{R}^n$ the Wiener algebra $A(\Lambda)$ consists of the restrictions to Λ of the functions $g \in A(\mathbb{R}^n)$ and the norm in $A(\Lambda)$ is the corresponding quotient norm. Two functions $g_1, g_2 \in A(\mathbb{R}^n)$ are identified if they have the same restriction to Λ.

Definition 7.2. *We say that K is a set of interpolation for the Wiener algebra $A(\Lambda)$ if for every sequence $c(\lambda) \in A(\Lambda)$ there exists an integrable function f supported by K such that*

$$\hat{f}(\lambda) = c(\lambda),\ \lambda \in \Lambda \tag{7.3}$$

We now consider the L^2 theory. Let $K \subset \mathbb{R}^n$ be a compact set and $E_K \subset L^2(\mathbb{R}^n)$ be the translation invariant subspace of $L^2(\mathbb{R}^n)$ consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}(\xi) = \int \exp(-ix\cdot\xi) f(x)\, dx$ is supported by K.

We now follow H. J. Landau [15].

Definition 7.3. *A set $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for E_K if there exists a constant C such that*

$$f \in E_K \Rightarrow \|f\|_2^2 \le C \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \tag{7.4}$$

A set $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for E_K if every sequence $c(\lambda) \in l^2(\Lambda)$ is the restriction to Λ of a function $f \in E_K$.

In full generality H.J. Landau proved that the sampling property (7.4) implies

$$2\pi \overline{\operatorname{dens}}\,\Lambda \ge |K|$$

and that the interpolation property implies

$$2\pi \underline{\operatorname{dens}}\,\Lambda \le |K|.$$

The upper and lower densities are defined below (Definition 8.1). These necessary conditions are not sufficient. Indeed $|K| < 2\pi \underline{\operatorname{dens}}\,\Lambda$ does not imply the sampling property even when $\Lambda = \mathbb{Z}$.

We also need the following

Definition 7.4. *A model set is simple if* $m = 1$ *and* $K = I$ *is an interval in Definition* 18.

A lattice cannot be a simple model set. This is fortunate since a lattice cannot be a "universal sampling set". A set Λ is a universal sampling set if for every compact set K, the property $|K| < 2\pi \underline{\operatorname{dens}}\, \Lambda$ implies (7.4). The following theorem is quite surprising since it says that simple model sets are "universal sampling sets" and should be preferred to lattices who are not "universal sampling sets".

Theorem 7.5. *Let* $\Lambda \subset \mathbb{R}^n$ *be a simple model set and* $K \subset \mathbb{R}^n$ *be a compact set. Then*

(a) If $|K| < (2\pi)^n \operatorname{dens} \Lambda$, *then* Λ *is a set of stable sampling for* E_K.

(b) If K *is Riemann integrable and if* $|K| > (2\pi)^n \operatorname{dens} \Lambda$, *then* Λ *is a set of stable interpolation for* E_K.

There are no reasons to believe that the assumption that $\Lambda \subset \mathbb{R}^n$ is a simple model set is necessary in Theorem 7.5.

Let us prove Theorem 7.5. It combines a beautiful theorem by A. Beurling [1] to some transference arguments similar to the ones introduced by Coifman and Weiss in [3].

We begin with the definition of the upper and lower density of a point set.

Definition 7.6. *Let* Λ *be an increasing sequence* $\lambda_j,\ j \in \mathbb{Z}$, *of real numbers such that* $\lambda_{j+1} - \lambda_j \geq \beta > 0$. *Let*

$$\overline{\operatorname{dens}}\, \Lambda = \limsup_{R \to \infty} R^{-1} \sup_{x \in \mathbb{R}} \operatorname{card}\{\Lambda \cap [x, x+R]\} \tag{7.5}$$

be the upper density of Λ. *The lower density is defined by replacing everywhere upper bounds by lower bounds.*

A. Beurling proved the following [1]

Proposition 7.7. *For any interval* J, *the condition*

$$|J| < 2\pi \underline{\operatorname{dens}}\, \Lambda \tag{7.6}$$

implies that Λ *is a set of stable sampling for* L^2_J *and similarly*

$$|J| > 2\pi \overline{\operatorname{dens}}\, \Lambda \tag{7.7}$$

implies that Λ *is a set of stable interpolation for* L^2_J.

The proof of (a) in Theorem 7.5 relies on the property of the auxiliary model set defined by

$$M_K = \{p_2(\gamma^*); \gamma^* \in \Gamma^*,\ p_1(\gamma^*) \in K\} \tag{7.8}$$

We know from Section 4 (see also [19]) that the density of the model set Λ_I is uniform and is given by $|I|/\operatorname{vol}(\Gamma)$ and similarly the density of M_K is $|K|/\operatorname{vol}(\Gamma^*)$. But $\operatorname{vol}(\Gamma^*) = (2\pi)^{n+1}/\operatorname{vol}(\Gamma)$. Therefore

$$|K| < (2\pi)^n \operatorname{dens} \Lambda_I \Rightarrow 2\pi \operatorname{dens} M_K < |I| \tag{7.9}$$

This remark is pivotal in the proof.

We sort the elements of M_K in increasing order and denote the corresponding sequence by $m_k;\, k \in \mathbb{Z}$. Then we have (Corollary 4.5, Section 4)

Lemma 7.8. *The sequence $\tilde{m}_k;\, k \in \mathbb{Z}$, is equidistributed on K.*

We now prove (a) of Theorem 7.5. We replace K by a larger compact set still denoted by K which is Riemann integrable and satisifies $|K| < (2\pi)^n \operatorname{dens} \Lambda_I$. By a standard density argument we can assume $\hat{f} \in C_0^\infty(K)$. As a consequence of Lemma 7.8 we get

$$\frac{1}{|K|}\|\hat{f}\|_2^2 = \lim_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T} |\hat{f}(\tilde{m}_k)|^2 \tag{7.10}$$

The right-hand side of (7.10) is given by

$$\lim_{\epsilon\downarrow 0} \epsilon \sum_{k\in\mathbb{Z}} |\phi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 \tag{7.11}$$

where $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi(0) = 1$.

At this stage we use the auxiliary function of the real variable x defined as

$$F_\epsilon(x) = \sqrt{\epsilon} \sum_{k\in\mathbb{Z}} \phi(\epsilon m_k) \hat{f}(\tilde{m}_k) \exp(i m_k x) \tag{7.12}$$

The Fourier transform of ϕ is denoted by ω. It will be assumed that $\omega \in C_0^\infty([-1,1])$ is a positive and even function. But $(2\pi)^n|K| < \operatorname{dens}\Lambda_I$ implies $|I| > 2\pi \operatorname{dens} M_K$. Therefore Beurling's theorem applies to the interval I, to the set of frequencies M_K and to the trigonometric sum defined in (7.10). Then one has

$$\epsilon \sum_{k\in\mathbb{Z}} |\phi(\epsilon m_k)|^2 |\hat{f}(\tilde{m}_k)|^2 \le C \int_I |F_\epsilon(x)|^2 dx \tag{7.13}$$

Let us compute the limit as $\epsilon \to 0$ of the term in the right-hand side of (7.13). To this aim, we use the definition of M_K and write

$$F_\epsilon(x) = \sqrt{\epsilon} \sum_{\gamma^*\in\Gamma^*} \phi(\epsilon p_2(\gamma^*)) \hat{f}(p_1(\gamma^*)) \exp(i p_2(\gamma^*) x) \tag{7.14}$$

The Poisson identity reads $\sum_{\gamma\in\Gamma} u(\gamma) = c(\Gamma) \sum_{\gamma^*\in\Gamma^*} \hat{u}(\gamma^*)$ where $u \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and yields

$$F_\epsilon(x) = c(\Gamma) \frac{1}{\sqrt{\epsilon}} \sum_{\gamma\in\Gamma} \omega\left(\frac{x - p_2(\gamma)}{\epsilon}\right) f(p_1(\gamma)) \tag{7.15}$$

It remains to calculate

$$\lim_{\epsilon\downarrow 0} \int_I |F_\epsilon(y)|^2 dy \tag{7.16}$$

where F_ϵ is given by (7.15). To this end, we notice that all terms in the right-hand side of (7.15) such that $|p_1(\gamma)| \ge \alpha + \epsilon$ vanish on $I = [-\alpha, \alpha]$. Indeed the support of ω is contained in $[-1,1]$. Therefore we can restrict the summation to the set $\Lambda_{I,\epsilon} = \{p_1(\gamma); \gamma \in \Gamma, |p_2(\gamma)| \le \alpha + \epsilon\}$. For $0 \le \epsilon \le 1$ we have

$$\lim_{\epsilon \to 0} \Lambda_{I,\epsilon} = \Lambda_I \text{ and } \Lambda_{I,\epsilon} \subset \Lambda_{I,1} \tag{7.17}$$

We split F_ϵ into a sum $F_\epsilon = F_\epsilon^N + R_N$. Here

$$F_\epsilon^N(x) = \frac{1}{\sqrt{\epsilon}} \sum_{\gamma \in A} \omega(\frac{x - p_2(\gamma)}{\epsilon}) f(p_1(\gamma)) \tag{7.18}$$

where

$$A = \{\gamma \in \Gamma, |p_1(\gamma)| \le N, |p_2(\gamma)| \le \alpha + \epsilon\} \tag{7.19}$$

and

$$R_N(x) = \frac{1}{\sqrt{\epsilon}} \sum_{\gamma \in B} \omega(\frac{x - p_2(\gamma)}{\epsilon}) f(p_1(\gamma)) \tag{7.20}$$

where

$$B = \{\gamma \in \Gamma, |p_1(\gamma)| > N, |p_2(\gamma)| \le \alpha + \epsilon\} \tag{7.21}$$

The triangle inequality yields $||R_N||_2 \le \epsilon_N ||\omega||_2$ with

$$\epsilon_N = \sum_{\gamma \in B} |f(p_1(\gamma))| \tag{7.22}$$

Since f belongs to the Schwartz class and since the collection of all $p_1(\gamma)$ such that $\gamma \in \Gamma$ and $|p_2(\gamma)| \le \alpha + 1$ is a model set, ϵ_N tends to 0 as N tends to infinity.

Estimating (7.18) is more involved. Since $|p_1(\gamma)| \le N$, the points $p_2(\gamma)$ appearing in (7.19) are separated by a distance larger than a positive constant β_N. If $0 < \epsilon < \beta_N$ the different terms in (7.18) have disjoint supports which implies

$$||F_\epsilon^N||_{L^2(I)} \le \sigma(N, \epsilon) ||\omega||_2 \tag{7.23}$$

where

$$\sigma(N, \epsilon) = \sum_{\gamma \in A} |f(p_1(\gamma))|^2 \tag{7.24}$$

If ϵ is small enough we have

$$\{\gamma \in \Gamma, |p_1(\gamma)| \le N, |p_2(\gamma)| \le \alpha + \epsilon\} = \{\gamma \in \Gamma, |p_1(\gamma)| \le N, |p_2(\gamma)| \le \alpha\}$$

and $\sigma(N, \epsilon) = \sigma_{N,0}$. Therefore

$$\lim_{\epsilon \to 0} \int_I |F_\epsilon(y)|^2 dy \le \sum_{\lambda \in \Lambda_I} |f(\lambda)|^2 + \epsilon_N ||\omega||_2 \tag{7.25}$$

and letting $N \to \infty$ we obtain the first claim. The proof of the second claim uses the same notations and strategy together with the first assertion of Beurling's theorem.

8 Spectral synthesis

In a beautiful paper [9] Carl Herz proved that the standard Cantor set E_3 is a set of spectral synthesis. Spectral synthesis will be defined below. In his proof Herz used a geometrical property of E_3. We have $E_3 \subset 3^{-j}\mathbb{Z} + 3^{-j}E_3$. In other words the Cantor set E_3 can be analyzed by the sequence of embedded grids $\Gamma_j = 3^{-j}\mathbb{Z}$. Let $F_j \subset E_3$ be the finite set consisting of the 2^j sums $\sum_0^{j-1} \epsilon_k 3^{-k}$, $\epsilon_k \in \{0, 1\}$. The property of spectral synthesis reduces to approximation schemes which are standard in numerical analysis. A "pseudo-measure" is a distribution S whose Fourier transform belongs to L^∞. Every pseudo-measure S supported by E_3 is the limit of a sequence S_j of sums of Dirac masses supported by F_j and this approximation of S by S_j is controlled as follows

$$\|\hat{S}_j\|_\infty \le C\|\hat{S}\|_\infty \tag{8.1}$$

where $\hat{S}$ is the Fourier transform of S. This approximation of pseudo-measures carried by a compact set E by atomic measures carried by E with a control on the L^∞ norms is the property of spectral synthesis.

Raphaël Salem conjectured that Herz theorem could be generalized to the Cantor set E_θ constructed with a dissection ratio which is the inverse of a Pisot number θ. One has $E_\theta \subset \theta^{-j}\Lambda_\theta + \theta^{-j}E_\theta$ where Λ_θ is the set of all finite sums $\sum_{k\ge 0} \epsilon_k \theta^k$, $\epsilon_k \in \{0, 1\}$. Whas it possible to prove (8.1) in that context ? That issue paved the way to my program on model sets.

When I began to study this problem the property of uniqueness for trigonometric expansion had been proved by R. Salem and A. Zygmund already. But let us define a set of uniqueness. For a compact set $E \subset \mathbb{R}$ we write $f \in \mathcal{F}_E$ if $f \in L^\infty(\mathbb{R})$ and if the spectrum of f is contained in E. Here the spectrum is the closed support of the distributional Fourier transform of f. If $E = E_\theta$ we write $\mathcal{F}_{E_\theta} = \mathcal{F}_\theta$.

Definition 8.1. *A compact set E is a set of uniqueness if and only if $f \in \mathcal{F}_E$ cannot tend to 0 at infinity unless $f = 0$ identically.*

Let us assume that E is a compact subset of the circle group $\mathbb{R}/2\pi\mathbb{Z}$. Then the property of uniqueness is equivalent to the following: if a formal trigonometric series $\sum_{-\infty}^{+\infty} c_k \exp(ikx)$ converges to 0 for every $x \notin K$ then $c_k = 0$, $k \in \mathbb{Z}$. This problem has a long history since it has already been raised by Riemann in his thesis.

A theorem by R. Salem and A. Zygmund [28] settles this issue for Cantor like sets.

Theorem 8.2. *The following two properties of a real number $\theta > 2$ are equivalent*

(a) θ is a Pisot number

(b) E_θ is a set of uniqueness.

A simple proof of this achievement will be given now (see also [20]). It paves the road to the solution of the problem of spectral synthesis. One way is almost obvious in the proof of Theorem 8.2. If θ is not a Pisot number, the Fourier transform of the Lebesgue measure

μ on E_θ tends to 0 at infinity. This measure μ is the limit as m tends to infinity of the sum μ_m of the Dirac masses 2^{-m} on each of the 2^m points of the finite set F_m. This finite set consists of all $x = \sum_0^{m-1} \alpha_k \theta^{-k}$, $\alpha_k \in \{0,1\}$. Let us observe that $F_m \subset E_\theta$ since E_θ is the compact set consisting of all real numbers $x = \sum_0^\infty \alpha_k \theta^{-k}$ where $\alpha_k \in \{0,1\}$. An obvious computation yields $|\hat{\mu}(x)| = \Pi_0^\infty |\cos(\theta^{-k} x)|$ and our claim follows from Theorem 2.35.

A set of uniqueness cannot, by definition, be the support of a distribution whose Fourier transform tends to 0 at infinity. Therefore if θ is not a Pisot number, E_θ cannot be a set of uniqueness.

We now assume that θ is a Pisot number and we prove that E_θ is a set of uniqueness. This is the difficult implication in Theorem 8.2. Let Λ_θ be the set of all real numbers $\lambda = \sum_0^\infty \alpha_k \theta^k$ where $\alpha_k \in \{0,1\}$ and where the sum is finite. Then Λ_θ is the *skeleton* of E_θ in the following sense: for every integer m we have

$$E_\theta \subset F_m + \theta^{-m} E_\theta \subset \theta^{-m} \Lambda_\theta + \theta^{-m} E_\theta \tag{8.2}$$

This geometrical property has far reaching consequences. The arithmetical property of Λ_θ which is needed in the proof of Theorem 8.2 is given by Lemma 8.3 which is coming now. Keeping the notations of Theorem 2.35 and Lemma 2.38, we denote by Ω_θ the ring of all algebraic integers of the field of θ and by Γ_1 the dense subgroup $\sigma_1(\Omega_\theta) \subset \mathbb{R}$. Let $G \simeq \mathbb{T}^n$ be the compact abelian group which is the dual of Γ_1. Let $J : \mathbb{R} \mapsto G$ the embedding of $\mathbb{R}$ into G which is the dual (in terms of Pontryagin duality) of the canonical embedding $\sigma_1(\Omega_\theta) \subset \mathbb{R}$.

Lemma 8.3. *These notations being kept $J(\Lambda_\theta + E_\theta)$ is contained in a compact subset K of G with $K \neq G$.*

The proof of Lemma 8.3 is immediate. We know that Λ_θ is contained in a model set. Using Lemma 3.2 we obtain $J(\Lambda_\theta) \subset \tilde{V}$ where $\tilde{V}$ is compact and contained in a dense subgroup of dimension $n-1$. It implies that $J(\Lambda_\theta + E_\theta)$ is contained in the direct sum $K = \tilde{V} + J(E_\theta)$. But E_θ is a set of measure 0 and the same holds for K since $J(E_\theta)$ is transverse to $\tilde{V}$.

We now return to Theorem 8.2. Let us argue by contradiction and assume that $F \in \mathcal{F}_\theta$ tends to 0 at infinity. Replacing F by $\beta F(x + x_0)$ if needed (β being a suitable constant factor), we can assume $F(0) = 1$. We then form $F_m(x) = F(\theta^m x)$ which tends to 0 everywhere on $\mathbb{R} \setminus \{0\}$. Let S be the distributional Fourier transform of F. Then S is supported by E_θ. We denote by S_m the Fourier transform of F_m. Then S_m is supported by $\theta^m E_\theta$. Using the embedding J we move S_m to G and write $\sigma_m = J_\#(S_m)$. The Fourier coefficients of σ_m are given by

$$\hat{\sigma}_m(x) = F(\theta^m x),\ x \in \Gamma_1 \tag{8.3}$$

It implies that the sequence σ_m tends weakly to the Haar measure on G. The sequence σ_m is bounded in $\mathcal{F}l^\infty(\Gamma_1)$ and the weak convergence refers to the duality between $\mathcal{F}l^1$ and $\mathcal{F}l^\infty$. But the supports of σ_m are included in the compact set $K \neq G$. We have reached a contradiction since the Haar measure on G is not supported by K.

In the late sixties I wished to prove something similar for the property of spectral synthesis. The property of spectral synthesis holds for a compact set E if every $f \in \mathcal{F}_E$ can be approximated in the topology $\sigma(L^\infty, L^1)$ by trigonometric sums whose frequencies belong to E.

Theorem 8.4. *If θ is a Pisot number, E_θ is a set of spectral synthesis.*

The full proof can be found in [20]. Here is a sketch of the proof. We know that $E_\theta \subset F_m + \theta^{-m} E_\theta$. Therefore for every $f \in \mathcal{F}_{E_\theta}$ and every integer m we have

$$f(x) = \sum_{y \in F_m} a_{y,m}(\theta^{-m} x) \exp(iyx) \tag{8.4}$$

where the spectra of the functions $a_{y,m}$ are contained in E_θ. These functions $a_{y,m}(\theta^{-m} x)$ are flat when compared to $\exp(iyx)$, $y \in F_m$ which are rapidly oscillating. It is then natural to approximate $f(x)$ by the trigonometric polynomials $f_m(x) = \sum_{y \in F_m} a_{y,m}(0) \exp(iyx)$. For some Pisot numbers which are characterized in [18], Theorem 6.8 implies the following. There exists of a constant C such that for every f whose spectrum is contained in E_θ we have

$$\|f_m\|_\infty \leq C \|f\|_\infty \tag{8.5}$$

and

$$f(x) = \lim_{m \to \infty} f_m(x) \tag{8.6}$$

where the convergence is uniform on each compact interval (see [18]).

The proof is a simple corollary of Theorem 6.8 if $\frac{1}{\theta - 1} < \omega$. This condition may not be satisfied for a Pisot number θ but will hold if θ is replaced by a suitable power θ^m. A more involved argument is needed in the general case and one can prove the property of spectral synthesis for all Pisot numbers [20].

9 Conclusion

The problems we raised and partially solved are white stones on a track. Each white stone leads to the next one. Model sets are harmonious sets. Model sets are the right tool to be used in the proof of the Salem & Zygmund theorem. Harmonious sets are coherent sets of frequencies. Coherent sets of frequencies play a seminal role in the problem of spectral synthesis. Finally simple model sets are sets of universal sampling.

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