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SOME DISCRETE FRACTIONAL INEQUALITIES OF CHEBYSHEV TYPE

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Abstract

Using the discrete fractional sum operator, we establish some inequalities of Chebyshev type.

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1 Introduction

In 1882, Chebyshev proved the following result [3]:

> Let f and g be two integrable functions in $[0,1]$. If both functions are simultaneously increasing or decreasing for the same values of x in $[0,1]$, then
> $$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$
> If one function is increasing and the other decreasing for the same values of x in $[0,1]$, then
> $$\int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Since then, continuous and discrete generalizations and extensions of such inequalities have appeared in the literature (see [2, 8] and references therein). In 2009, Belarbi and Dahmani [1] proved that

$$(I^\alpha fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha}(I^\alpha f)(t)(I^\alpha g)(t), \quad t>0, \quad \alpha>0, \tag{1.1}$$

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where I^α is the Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ [6], and f and g are two synchronous functions (cf. Definition 2.5 below). Moreover, much more recently, a q-analogue of inequality (1.1) has appeared in the literature [7].

It is our aim with this paper to establish a discrete version of inequality (1.1) as well as some other related results. We will do this by using the discrete fractional sum operator defined by Miller and Ross [5] in 1989.

This paper is organized as follows: in Section 2 we provide the reader fundamental concepts and results needed throughout the paper. In Section 3 we state and prove our main achievements.

2 Preliminaries on Discrete Fractional Calculus

In this section we introduce the reader to basic concepts and results about discrete fractional calculus.

The power function is defined by

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}, \text{ for } x, x-y \in \mathbb{R}\backslash(\mathbb{Z}\backslash\mathbb{N}_0).$$

Remark 2.1. Using the properties of the Gamma function, it is easily seen that for $x \geq y \geq 0$, we get $x^{(y)} \geq 0$.

For $a \in \mathbb{R}$ and $0 < \alpha \leq 1$, we define the set $\mathbb{N}_a^\alpha = \{a+\alpha, a+\alpha+1, a+\alpha+2, \ldots\}$. Also, we use the notation $\sigma(s) = s+1$ for the shift operator and $(\Delta f)(t) = f(t+1) - f(t)$ for the forward difference operator.

For a function $f : \mathbb{N}_a^0 \to \mathbb{R}$, the *discrete fractional sum of order* $\alpha \geq 0$ is defined as

$$\begin{aligned}
({}_a\Delta^0 f)(t) &= f(t), \quad t \in \mathbb{N}_a^0, \\
({}_a\Delta^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \quad t \in \mathbb{N}_a^\alpha,\ \alpha > 0.
\end{aligned}$$

Remark 2.2. Note that the operator ${}_a\Delta^{-\alpha}$ with $\alpha > 0$ maps functions defined on $\mathbb{N}_a^0$ to functions defined on $\mathbb{N}_a^\alpha$. Also observe that if $\alpha = 1$, we get the summation operator

$$({}_a\Delta^{-1} f)(t) = \sum_{s=a}^{t-1} f(s).$$

The following result will be used in the sequel.

Lemma 2.3 (See [4, Corollary 10]). *If* $a \in \mathbb{R}$ *and* $\mu, \mu+\nu \in \mathbb{R}\backslash\{\ldots,-2,-1\}$*, then*

$$\left({}_a\Delta^{-\nu}(s-a+\mu)^{(\mu)}\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a+\mu)^{(\mu+\nu)}, \quad t \in \mathbb{N}_a^\nu.$$

Remark 2.4. The function $t \to (t-a)^{(\alpha)}$ defined on $\mathbb{N}_a^\alpha$, $a \in \mathbb{R}$ and $\alpha \geq 0$, is increasing. Indeed, we have that $\Delta(t-a)^{(\alpha)} = \alpha(t-a)^{(\alpha-1)}$ and $(t-a)^{(\alpha-1)} \geq 0$.

Definition 2.5. Two functions f and g are called synchronous, respectively asynchronous, on $\mathbb{N}_a^0$ if for all $\tau, s \in \mathbb{N}_a^0$, we have $(f(\tau)-f(s))(g(\tau)-g(s)) \geq 0$, respectively $(f(\tau)-f(s))(g(\tau)-g(s)) \leq 0$.

3 Discrete Fractional Inequalities

We start by proving the main result of this paper.

Theorem 3.1. *If* $\alpha > 0$ *and* f, g *are two synchronous functions on* $\mathbb{N}_a^0$, *then*

$$\left({}_a\Delta^{-\alpha} fg\right)(t) \geq \frac{\Gamma(\alpha+1)}{(t-a)^{(\alpha)}} \left({}_a\Delta^{-\alpha} f\right)(t) \left({}_a\Delta^{-\alpha} g\right)(t), \quad t \in \mathbb{N}_a^\alpha. \tag{3.1}$$

Proof. Since the functions f and g are synchronous on $\mathbb{N}_a^0$, then for all $\tau, s \in \mathbb{N}_a^0$, we have

$$(f(\tau) - f(s))(g(\tau) - g(s)) \geq 0,$$

i.e.,

$$f(\tau)g(\tau) + f(s)g(s) \geq f(\tau)g(s) + f(s)g(\tau). \tag{3.2}$$

Now, multiplying both sides of (3.2) by $\frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)}$, $t \in \mathbb{N}_a^\alpha$ and $\tau \in \{a, a+1, \ldots, t-\alpha\}$, we get

$$\begin{aligned} \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(\tau)g(\tau) + \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(s) \\ \geq \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(\tau)g(s) + \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(\tau). \end{aligned} \tag{3.3}$$

Now, summing both sides of (3.3) for $\tau \in \{a, a+1, \ldots, t-\alpha\}$, we obtain

$$\left({}_a\Delta^{-\alpha} fg\right)(t) + f(s)g(s)\left({}_a\Delta^{-\alpha} 1\right)(t) \geq g(s)\left({}_a\Delta^{-\alpha} f\right)(t) + f(s)\left({}_a\Delta^{-\alpha} g\right)(t). \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)}$, $t \in \mathbb{N}_a^\alpha$ and $s \in \{a, a+1, \ldots, t-\alpha\}$, we obtain

$$\begin{aligned} \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} \left({}_a\Delta^{-\alpha} fg\right)(t) + \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(s)\left({}_a\Delta^{-\alpha} 1\right)(t) \\ \geq \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} g(s)\left({}_a\Delta^{-\alpha} f\right)(t) + \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)\left({}_a\Delta^{-\alpha} g\right)(t), \end{aligned} \tag{3.5}$$

and again, summing both sides of (3.5) for $s \in \{a, a+1, \ldots, t-\alpha\}$, we get

$$\begin{aligned} \left({}_a\Delta^{-\alpha} 1\right)(t)\left({}_a\Delta^{-\alpha} fg\right)(t) + \left({}_a\Delta^{-\alpha} fg\right)(t)\left({}_a\Delta^{-\alpha} 1\right)(t) \\ \geq \left({}_a\Delta^{-\alpha} g\right)(t)\left({}_a\Delta^{-\alpha} f\right)(t) + \left({}_a\Delta^{-\alpha} f\right)(t)\left({}_a\Delta^{-\alpha} g\right)(t), \end{aligned}$$

i.e.,

$$\begin{aligned} \left({}_a\Delta^{-\alpha} f\right)(t)\left({}_a\Delta^{-\alpha} g\right)(t) &\leq \left({}_a\Delta^{-\alpha} 1\right)(t)\left({}_a\Delta^{-\alpha} fg\right)(t) \\ &= \frac{(t-a)^{(\alpha)}}{\Gamma(\alpha+1)} \left({}_a\Delta^{-\alpha} fg\right)(t), \end{aligned}$$

where we have used Lemma 2.3. This shows (3.1). □

Remark 3.2. The inequality sign in (3.1) is reversed if the functions are asynchronous on $\mathbb{N}_a^0$.

Example 3.3. Let $\beta \geq 0$ and consider the functions f_β defined by

$$f_\beta(t) = (t+\beta)^{(\beta)}, \quad t \in \mathbb{N}_0^0.$$

By Remark 2.4, it follows that f_β and f_γ are synchronous functions for $\beta, \gamma \geq 0$. Therefore, by Lemma 2.3 and Theorem 3.1, the inequality

$$\left({}_0\Delta^{-\alpha} f_\beta f_\gamma\right)(t) \geq \frac{\Gamma(\alpha+1)}{t^{(\alpha)}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t+\gamma)^{(\gamma+\alpha)} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t+\beta)^{(\beta+\alpha)}$$

holds for all $t \in \mathbb{N}_0^\alpha$.

Theorem 3.4. *If* $\alpha, \beta > 0$ *and* f, g *are two synchronous functions on* $\mathbb{N}_a^0$, *then*

$$\frac{(t-a)^{(\alpha)}}{\Gamma(\alpha+1)} \left({}_a\Delta^{-\beta} fg\right)(t) + \frac{(t-a)^{(\beta)}}{\Gamma(\beta+1)} \left({}_a\Delta^{-\alpha} fg\right)(t) \geq \left({}_a\Delta^{-\alpha} f\right)(t) \left({}_a\Delta^{-\beta} g\right)(t) + \left({}_a\Delta^{-\beta} f\right)(t) \left({}_a\Delta^{-\alpha} g\right)(t), \quad t \in \mathbb{N}_a^\alpha. \tag{3.6}$$

Proof. Proceeding as in the proof of Theorem 3.1 and using inequality (3.4), we can write

$$\frac{(t-\sigma(s))^{(\beta-1)}}{\Gamma(\beta)} \left({}_a\Delta^{-\alpha} fg\right)(t) + \frac{(t-\sigma(s))^{(\beta-1)}}{\Gamma(\beta)} f(s)g(s) \left({}_a\Delta^{-\alpha} 1\right)(t) \geq \frac{(t-\sigma(s))^{(\beta-1)}}{\Gamma(\beta)} g(s) \left({}_a\Delta^{-\alpha} f\right)(t) + \frac{(t-\sigma(s))^{(\beta-1)}}{\Gamma(\beta)} f(s) \left({}_a\Delta^{-\alpha} g\right)(t). \tag{3.7}$$

Now, summing both sides of (3.7) for $s \in \{a, a+1, \ldots, t-\beta\}$, we obtain the desired inequality (3.6). □

Remark 3.5. If we let $\alpha = \beta$ in Theorem 3.4, we obtain Theorem 3.1.

We end this manuscript with a generalization of Theorem 3.1.

Theorem 3.6. *Assume that* f_i, $1 \leq i \leq n$, *are* $n \in \mathbb{N}$ *functions on* $\mathbb{N}_a^0$ *satisfying*

$$\prod_{i=1}^{k-1} f_i \text{ and } f_k \text{ are synchronous for all } k \in \{2, \ldots, n\}, \tag{3.8}$$

$$f_i \geq 0 \text{ for } 3 \leq i \leq n. \tag{3.9}$$

Suppose that $\alpha > 0$. *Then, for all* $t \in \mathbb{N}_a^\alpha$, *we have*

$$\left({}_a\Delta^{-\alpha} \prod_{i=1}^{n} f_i\right)(t) \geq \left(\frac{\Gamma(\alpha+1)}{(t-a)^{(\alpha)}}\right)^{n-1} \prod_{i=1}^{n} \left({}_a\Delta^{-\alpha} f_i\right)(t). \tag{3.10}$$

Proof. In view of (3.8) and (3.9), we have

$$\begin{aligned}\left({}_a\Delta^{-\alpha}\prod_{i=1}^{n} f_i\right)(t) &\geq \frac{\Gamma(\alpha+1)}{(t-a)^{(\alpha)}}\left({}_a\Delta^{-\alpha}\prod_{i=1}^{n-1} f_i\right)(t)\left({}_a\Delta^{-\alpha}f_n\right)(t)\\ &\geq \left(\frac{\Gamma(\alpha+1)}{(t-a)^{(\alpha)}}\right)^2\left({}_a\Delta^{-\alpha}\prod_{i=1}^{n-2} f_i\right)(t)\prod_{i=n-1}^{n}\left({}_a\Delta^{-\alpha}f_k\right)(t)\\ &\vdots\\ &\geq \left(\frac{\Gamma(\alpha+1)}{(t-a)^{(\alpha)}}\right)^{n-1}\prod_{i=1}^{n}\left({}_a\Delta^{-\alpha}f_i\right)(t),\end{aligned}$$

where we repeatedly applied Theorem 3.1. □

Remark 3.7. If the functions f_i, $1 \leq i \leq n$, in Theorem 3.6 are either all nonnegative increasing or nonnegative decreasing, then both (3.8) and (3.9) are satisfied.

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