WELL-POSEDNESS RESULT FOR A NONLINEAR ELLIPTIC PROBLEM INVOLVING VARIABLE EXPONENT AND ROBIN TYPE BOUNDARY CONDITION

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Abstract

In this work we study the following nonlinear elliptic boundary value problem,
\[ b(u) - \text{div} a(x, \nabla u) = f \text{ in } \Omega, \quad a(x, \nabla u) \cdot \eta = -|u|^{p(x)-2} u \text{ on } \partial \Omega, \]
where \( \Omega \) is a smooth bounded open domain in \( \mathbb{R}^N \), \( N \geq 1 \) with smooth boundary \( \partial \Omega \). We prove the existence and uniqueness of a weak solution for \( f \in L^{\infty}(\Omega) \), the existence and uniqueness of an entropy solution for \( L^1 \)-data \( f \). The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

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1 Introduction

This paper is motivated by phenomena which are described by Robin type boundary problem of the form

\[
\begin{align*}
&b(u) - \text{div} \ a(x, \nabla u) = f \text{ in } \Omega, \\
&a(x, \nabla u) \cdot \eta = -|u|^{p(x) - 2} u \text{ on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a smooth bounded open domain in \( \mathbb{R}^N, N \geq 3 \) with smooth boundary \( \partial \Omega \) and \( \eta \) the outer unit normal vector on \( \partial \Omega \). When \( p(\cdot) \equiv 2 \), we obtain an homogeneous Robin condition. Therefore, (1.1) includes a Robin boundary problem.

The study of problems involving variable exponent has received considerable attention in recent years (cf. [4,5,7-17,19-27, 29-34]) due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [4]), electrorheological fluids (see [11,22,29,30]) or image restauration (see [9]).

When the boundary value condition is a Neumann or Robin boundary condition in the context of variable exponent, we must work in general with the space \( W^{1,p(\cdot)}(\Omega) \) instead of the common space \( W^{1,p(\cdot)}_0(\Omega) \). The main difficulty which appears in this case of existence and also uniqueness of solutions is that the famous Poincar inequality does not apply (see [8]). We must use the Poincar-Wirtinger inequality instead of the Poincar inequality but due to the average number, it is not easy to use the Poincar-Wirtinger inequality to obtain appropriate properties for problem involving more general operator and data considered in this paper. We use in this paper a Poincar-Sobolev type inequality to get the main apriori estimate for the proof of the existence and uniqueness of entropy solution (see the proof of proposition 4.7 below). Recently, Ouaro (see [25]) studied the following problem

\[
\begin{align*}
&-\text{div} \ a(x, \nabla u) + |u|^{p(x) - 2} u = f \text{ in } \Omega, \\
&a(x, \nabla u) \cdot \eta = \varphi \text{ on } \partial \Omega,
\end{align*}
\]

under the following assumptions:

\[
\begin{align*}
&\{ \ p(\cdot) : \Omega \to \mathbb{R} \text{ is a measurable function such that } \\
&\quad 1 < p_- \leq p_+ < +\infty \},
\end{align*}
\]

where \( p_- := \text{ess inf}_{x \in \Omega} p(x) \) and \( p_+ := \text{ess sup}_{x \in \Omega} p(x) \).

For the vector fields \( a(\cdot, \cdot) \), we assume that \( a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is Carathodory and is the continuous derivative with respect to \( \xi \) of the mapping \( A : \Omega \times \mathbb{R}^N \to \mathbb{R}, A = A(x, \xi) \), i.e. \( a(x, \xi) = \nabla_{\xi} A(x, \xi) \) such that:

- The following equality holds

\[
A(x, 0) = 0,
\]

for almost every \( x \in \Omega \).
• There exists a positive constant $C_1$ such that
\begin{equation}
|a(x, \xi)| \leq C_1 \left( j(x) + |\xi|^{p(x)-1} \right)
\end{equation}
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where $j$ is a nonnegative function in $L^{p'(x)}(\Omega)$, with $1/p(x) + 1/p'(x) = 1$.

• There exists a positive constant $C_2$ such that for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$,
\begin{equation}
(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0.
\end{equation}

• The following inequalities hold
\begin{equation}
|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x) A(x, \xi)
\end{equation}
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

Under assumptions (1.3)-(1.7), Ouaro (see [25]) proved the existence and uniqueness of entropy solutions of problem (1.2) for $L^1$-data $f$ and $\varphi$. Assumption on the function $A$ and the use of the quantity $|u|^{p(x)-2} u$ allowed Ouaro, in particular, to exploit a minimization method for the proof of existence of a weak solution for (1.2) when the data $f$ and $\varphi$ are in $L^\infty(\Omega)$ and $L^\infty(\partial \Omega)$ respectively [25]. Note also that the uniqueness of weak and entropy solutions of (1.2) in [25] is due to the fact that $s \mapsto |s|^{|p(x)-2|} s$ is increasing.

In this paper, we improve the result in [25] by making less regularity on the data $a$ and $b$. More precisely:

\begin{equation}
\begin{cases}
p(\cdot) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that} \\
1 < p_- \leq p_+ < +\infty,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
b : \mathbb{R} \to \mathbb{R} \text{ is continuous, surjective, nondecreasing function} \\
such that $b(0) = 0$.
\end{cases}
\end{equation}

For the vector field $a(\cdot, \cdot)$, we assume that $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory such that:

• there exists a positive constant $C_2$ with
\begin{equation}
|a(x, \xi)| \leq C_2 \left( j(x) + |\xi|^{p(x)-1} \right)
\end{equation}
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where $j$ is a nonnegative function in $L^{p'(x)}(\Omega)$, with $
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
there exists a positive constant $C_3$ such that for every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, the following inequalities hold:

$$
(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0 \quad (1.11)
$$

and

$$
a(x, \xi) \cdot \xi \geq C_3 |\xi|^{p(x)} \quad (1.12)
$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

The remaining part of the paper is the following: in section 2, we introduce some notations/functional spaces. In section 3, we prove the existence and the uniqueness of weak solution of (1.1) when the data $f \in L^\infty(\Omega)$. Using the results of section 3, we study in section 4, the question of the existence and the uniqueness of entropy solution of (1.1) when the data $f \in L^1(\Omega)$.

## 2 Assumptions and preliminaries

As the exponent $p(.)$ appearing in (1.10) and (1.12) depends on the variable $x$, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$
\rho_{p(.)}(u) := \int_{\Omega} |u|^{p(x)}dx
$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$
|u|_{p(.)} = \inf \{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\}
$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where

$$
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1
$$

Finally, we have the Hölder type inequality:

$$
\left| \int_{\Omega} uvdx \right| \leq \left( \frac{1}{p_-} + \frac{1}{(p')_-} \right) |u|_{p(.)} |v|_{p'(.)},
$$

(2.1)

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.

Let

$$
W^{1,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \},
$$

which is a Banach space equipped with the following norm

$$
||u||_{1,p(.)} = |u|_{p(.)} + (|\nabla u|)_{p(.)}.
$$
The space \((W^{1,p,\cdot}(\Omega), ||.||_{1,p,\cdot})\) is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular \(\rho_{p,\cdot}\) of the space \(L^{p,\cdot}(\Omega)\). We have the following result (see [16]):

**Lemma 2.1** If \(u_n, u \in L^{p,\cdot}(\Omega)\) and \(p_+ < +\infty\), then the following properties hold:

(i) \(|u|_{p,\cdot} > 1 \Rightarrow |u|_{p,\cdot}^p \leq \rho_{p,\cdot}(u) \leq |u|_{p,\cdot}^p\);

(ii) \(|u|_{p,\cdot} < 1 \Rightarrow |u|_{p,\cdot}^p \leq \rho_{p,\cdot}(u) \leq |u|_{p,\cdot}^p\);

(iii) \(|u|_{p,\cdot} < 1\) (respectively \(= 1; > 1\)) \(\Leftrightarrow \rho_{p,\cdot}(u) < 1\) (respectively \(= 1; > 1\));

(iv) \(u_n \rightharpoonup 0\) (respectively \(\rightarrow +\infty\)) \(\Leftrightarrow \rho_{p,\cdot}(u_n) \rightharpoonup 0\) (respectively \(\rightarrow +\infty\));

(v) \(\rho_{p,\cdot}(u/|u|_{p,\cdot}) = 1\).

For a measurable function \(u : \Omega \rightarrow \mathbb{R}\), we introduce the following notation:

\[
\rho_{1,p,\cdot}(u) = \int_{\Omega} |u|_{p,\cdot}^{p(x)} \, dx + \int_{\Omega} |\nabla u|_{p,\cdot}^{p(x)} \, dx.
\]

We have the following lemma (see [32,34]):

**Lemma 2.2** If \(u \in W^{1,p,\cdot}(\Omega)\) then the following properties hold:

(i) \(|u|_{1,p,\cdot} < 1\) (respectively \(= 1; > 1\)) \(\Leftrightarrow \rho_{1,p,\cdot}(u) < 1\) (respectively \(= 1; > 1\));

(ii) \(|u|_{1,p,\cdot} < 1 \Leftrightarrow \|u\|_{p,\cdot}^{p_{1,\cdot}} \leq \rho_{1,p,\cdot}(u) \leq \|u\|_{p,\cdot}^{p_{1,\cdot}}\);

(iii) \(|u|_{1,p,\cdot} > 1 \Leftrightarrow \|u\|_{p,\cdot}^{p_{1,\cdot}} \leq \rho_{1,p,\cdot}(u) \leq \|u\|_{p,\cdot}^{p_{1,\cdot}}\);

(iv) \(u_n \rightharpoonup 0\) (respectively \(\rightarrow +\infty\)) \(\Leftrightarrow \rho_{1,p,\cdot}(u_n) \rightharpoonup 0\) (respectively \(\rightarrow +\infty\));

Put

\[
p^\gamma(x) := (p(x))^{\gamma} := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N \\ \frac{N}{N-p(x)}, & \text{if } p(x) \geq N; \end{cases}
\]

then we have the following embedding result:

**Proposition 2.3** Let \(p \in C(\partial \Omega)\) and \(p_+ > 1\). If \(q \in C(\partial \Omega)\) satisfies the condition

\[
1 \leq q(x) < p^\gamma(x), \ \forall \ x \in \partial \Omega,
\]

then, there is a compact embedding \(W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega)\). In particular, there is a compact embedding \(W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial \Omega)\).

Let us introduce the following notation: given two bounded measurable functions \(p(\cdot), q(\cdot) : \Omega \rightarrow \mathbb{R}\), we write

\[
q(\cdot) \ll p(\cdot) \ \text{if} \ \text{ess inf}_{\chi \in \Omega} (p(x) - q(x)) > 0.
\]

**Remark 2.4.** Observe that we use the same notation \(f\) for \(f\) and its trace when convenient.

### 3 Existence and uniqueness of weak solution

In this part, we study the existence and the uniqueness of a weak solution of (1.1) when the data \(f \in L^{p_{\cdot}}(\Omega)\).
Definition 3.1 A weak solution of (1.1) is a measurable function $u$ such that

$$u \in W^{1,p(\cdot)}(\Omega), \quad b(u) \in L^\infty(\Omega), \quad |u|^{p(\cdot)-2}u \in L^\infty(\partial\Omega)$$

and

$$\int_{\Omega} a(x, \nabla u) \nabla \varphi dx + \int_{\Omega} b(u) \varphi dx + \int_{\partial\Omega} |u|^{p(x)-2}u \varphi d\sigma = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in W^{1,p(\cdot)}(\Omega),$$ \hspace{1cm} (3.1)

where $d\sigma$ is the surface measure on $\partial\Omega$.

Notice that the integrals in (3.1) are well defined since for the third integral in the left-hand side, we can use the fact that the trace of $\varphi \in W^{1,p(\cdot)}(\Omega)$ on $\partial\Omega$ is well defined in $L^p(\partial\Omega)$, for $1 \leq p < +\infty$. The main result of this part is the following:

Theorem 3.2. Assume that (1.8)-(1.12) hold and $f \in L^\infty(\Omega)$. Then there exists a unique weak solution of (1.1).

Proof.

Part 1: Existence

For $k > 0$, we consider the following approximated problem:

$$\begin{cases}
T_k(b(u_k)) - \text{div} a(x, \nabla u_k) = f \quad \text{in } \Omega \\
a(x, \nabla u_k) \cdot \eta = T_k(-|u_k|^{p(x)-2}u_k) \quad \text{on } \partial\Omega,
\end{cases} \hspace{1cm} (3.2)$$

where for any $k > 0$, the truncation function $T_k$ is defined by $T_k(s) := \max\{-k, \min\{k,s\}\}$.

Note that as $T_k(b(u_k)) \in L^\infty(\Omega)$ and $T_k(|u_k|^{p(x)-2}u_k) \in L^\infty(\partial\Omega)$, thanks to [21, Theorem 3.1], there exists $u_k \in W^{1,p(\cdot)}(\Omega)$ which is a weak solution of (3.2).

We recall that for any $\varepsilon > 0$,

$$H_\varepsilon(s) = \min \left\{ \frac{s^+}{\varepsilon}, 1 \right\},$$

and if $\gamma$ is a maximal monotone operator defined on $\mathbb{R}$, we denote by $\gamma_0$ the main section of $\gamma$, i.e.

$$\gamma_0(s) = \begin{cases}
\text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \emptyset, \\
+\infty \text{ if } [s, +\infty) \cap D(\gamma) = \emptyset, \\
-\infty \text{ if } (-\infty, s] \cap D(\gamma) = \emptyset.
\end{cases}$$

We now show that $|b(u_k)| \leq \|f\|_{L^\infty(\Omega)}$ a.e. in $\Omega$ and $|u_k| \leq b_0^{-1} \left( \|f\|_{L^\infty(\Omega)} \right)$ a.e. in $\partial\Omega$ for all $k > 0$.

We take $\varphi = H_\varepsilon(u_k - M)$ as a test function in (3.1) for the weak solution $u_k$ and $M > 0$ a
We deduce that

\[
\int_{\Omega} a(x, \nabla u_k) \nabla H_\epsilon(u_k - M) \, dx + \int_{\Omega} T_k(b(u_k)) H_\epsilon(u_k - M) \, dx + \int_{\partial \Omega} T_k(|u_k|^{p(x)-2} u_k) H_\epsilon(u_k - M) \, d\sigma = \int_{\Omega} f H_\epsilon(u_k - M) \, dx.
\]

(3.3)

Let \( J := \int_{\Omega} a(x, \nabla u_k) \nabla H_\epsilon(u_k - M) \, dx \).

We deduce that \( J = \frac{1}{\epsilon} \int_{\{0 < u_k - M < \epsilon\}} a(x, \nabla u_k) \nabla H_\epsilon(u_k - M) \, dx \geq 0 \) then, according to (3.3), we obtain:

\[
\int_{\Omega} T_k(b(u_k)) H_\epsilon(u_k - M) \, dx + \int_{\partial \Omega} T_k(|u_k|^{p(x)-2} u_k) H_\epsilon(u_k - M) \, d\sigma \leq \int_{\Omega} f H_\epsilon(u_k - M) \, dx,
\]

(3.4)

which is equivalent to say

\[
\int_{\Omega} (T_k(b(u_k)) - T_k(b(M))) H_\epsilon(u_k - M) \, dx + \int_{\partial \Omega} T_k(|u_k|^{p(x)-2} u_k) H_\epsilon(u_k - M) \, d\sigma \leq \int_{\Omega} (f - T_k(b(M))) H_\epsilon(u_k - M) \, dx.
\]

(3.5)

As the two terms in the left-hand side in (3.5) are nonnegative then we deduce that

\[
\int_{\Omega} (T_k(b(u_k)) - T_k(b(M))) H_\epsilon(u_k - M) \, dx \leq \int_{\Omega} (f - T_k(b(M))) H_\epsilon(u_k - M) \, dx
\]

(3.6)

and

\[
\int_{\partial \Omega} T_k(|u_k|^{p(x)-2} u_k) H_\epsilon(u_k - M) \, d\sigma \leq \int_{\Omega} (f - T_k(b(M))) H_\epsilon(u_k - M) \, dx.
\]

(3.7)

We now let \( \epsilon \) goes to 0 in (3.6) and (3.7) to get:

\[
\int_{\Omega} (T_k(b(u_k)) - T_k(b(M)))^+ \, dx \leq \int_{\Omega} (f - T_k(b(M))) \text{sign}_0^+(u_k - M) \, dx
\]

(3.8)

and

\[
\int_{\partial \Omega} T_k(|u_k|^{p(x)-2} u_k) \text{sign}_0^+(u_k - M) \, d\sigma \leq \int_{\Omega} (f - T_k(b(M))) \text{sign}_0^+(u_k - M) \, dx.
\]

(3.9)

Choosing now \( M = b_0^{-1}(\|f\|_{L^p(\Omega)}) \) in (3.8) and (3.9)(M is a constant since \( b \) is onto) to obtain:

\[
\int_{\Omega} (T_k(b(u_k)) - T_k(\|f\|_{L^p(\Omega)}))^+ \, dx \leq \int_{\Omega} (f - T_k(\|f\|_{L^p(\Omega)})) \text{sign}_0^+(u_k - b_0^{-1}(\|f\|_{L^p(\Omega)})) \, dx,
\]

(3.10)
and
\[
\int_{\partial \Omega} T_k(|u_k|^{p(x)-2}u_k)\text{sign}_0(u_k - b_0^{-1} (\|f\|_{L^\infty(\Omega)})) \, d\sigma \\
\leq \int_{\Omega} (f - T_k(\|f\|_{L^\infty(\Omega)}))\text{sign}_0^+(u_k - b_0^{-1} (\|f\|_{L^\infty(\Omega)})) \, dx.
\] (3.11)

Hence, for all \( k > \|f\|_{L^\infty(\Omega)} \), it follows that
\[
T_k(b(u_k)) \leq \|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega
\] (3.12)
and
\[
u_k \leq b_0^{-1} (\|f\|_{L^\infty(\Omega)}) \text{ a.e. in } \partial \Omega.
\] (3.13)

It remains to prove that \( T_k(b(u_k)) \geq -\|f\|_{L^\infty(\Omega)} \) a.e. in \( \Omega \) and \( u_k \geq -b_0^{-1} (\|f\|_{L^\infty(\Omega)}) \) a.e. in \( \partial \Omega \) for all \( k > \|f\|_{L^\infty(\Omega)} \).

Let us remark that as \( u_k \) is a weak solution of (3.2), then \(-u_k\) is a weak solution of the following problem
\[
\begin{aligned}
\begin{cases}
T_k(\tilde{b}(u_k)) - \text{div} \tilde{a}(x, \nabla u_k) = \tilde{f} \text{ in } \Omega \\
\tilde{a}(x, \nabla u_k) \cdot \eta = T_k \left(-|u_k|^{p(x)-2}u_k\right) \text{ on } \partial \Omega,
\end{cases}
\end{aligned}
\] (3.14)

where \( \tilde{a}(x, \xi) = -a(x, -\xi), \tilde{b}(s) = -b(-s), \tilde{f} = -f \).

According to (3.12) and (3.13), we deduce that
\[
T_k(-b(u_k)) \leq \|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \text{ for all } k > \|f\|_{L^\infty(\Omega)}
\]
and
\[
u_k \leq b_0^{-1} (\|f\|_{L^\infty(\Omega)}) \text{ a.e. in } \partial \Omega.
\]

Therefore, we get
\[
T_k(b(u_k)) \geq -\|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \text{ for all } k > \|f\|_{L^\infty(\Omega)}
\] (3.15)
and
\[
u_k \geq -b_0^{-1} (\|f\|_{L^\infty(\Omega)}) \text{ a.e. in } \partial \Omega, \text{ for all } k > \|f\|_{L^\infty(\Omega)}.
\] (3.16)

It follows from (3.12), (3.13), (3.15) and (3.16) that for all \( k > \|f\|_{L^\infty(\Omega)} \),
\[
|b(u_k)| \leq \|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega
\] (3.17)
and
\[
|u_k| \leq b_0^{-1} (\|f\|_{L^\infty(\Omega)}) \text{ a.e. in } \partial \Omega.
\] (3.18)

We now fix \( k = \|f\|_{L^\infty(\Omega)} + (b_0^{-1} (\|f\|_{L^\infty(\Omega)}))^{p^*-1} + 2 \) in (3.2) to end the prove of the existence result.
Part 2: Uniqueness. Let \( u_1 \) and \( u_2 \) be two weak solutions of (1.1).

Let us take \( \varphi = u_1 - u_2 \) as test function in (3.1) for \( u_1 \) and also for \( u_2 \), to get

\[
\int_{\Omega} a(x, \nabla u_1) \cdot \nabla (u_1 - u_2) \, dx + \int_{\Omega} b(u_1)(u_1 - u_2) \, dx + \int_{\partial \Omega} |u_1|^{p(x)-2}u_1(u_1 - u_2) \, d\sigma = \int_{\Omega} f(u_1 - u_2) \, dx,
\]

and

\[
\int_{\Omega} a(x, \nabla u_2) \cdot \nabla (u_1 - u_2) \, dx + \int_{\Omega} b(u_2)(u_1 - u_2) \, dx + \int_{\partial \Omega} |u_2|^{p(x)-2}u_2(u_1 - u_2) \, d\sigma = \int_{\Omega} f(u_1 - u_2) \, dx.
\]

Substracting the two preceding relations, we obtain

\[
\int_{\Omega} (a(x, \nabla u_1) - a(x, \nabla u_2)) \cdot \nabla (u_1 - u_2) \, dx + \int_{\Omega} (b(u_1) - b(u_2))(u_1 - u_2) \, dx
\]

\[
+ \int_{\partial \Omega} (|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2)(u_1 - u_2) \, d\sigma = 0.
\]  

(3.19)

From (3.19) we deduce that

\[
\int_{\Omega} (a(x, \nabla u_1) - a(x, \nabla u_2)) \cdot \nabla (u_1 - u_2) \, dx = 0,
\]  

(3.20)

\[
\int_{\Omega} (b(u_1) - b(u_2))(u_1 - u_2) \, dx = 0
\]  

(3.21)

and

\[
\int_{\partial \Omega} (|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2)(u_1 - u_2) \, d\sigma = 0.
\]  

(3.22)

Since \( p_\rightarrow > 1 \), the following relation is true for any \( \xi, \eta \in \mathbb{R}, \xi \neq \eta \) (cf. [15])

\[
\left( |\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta \right) \langle \xi - \eta \rangle > 0.
\]  

(3.23)

Thanks to (3.20), (3.22), (3.23) and assumption (1.11), we get that there exists a constant \( c \) such that

\[
u_1 - u_2 = c \ a.e. \ in \ \Omega \ and \ u_1 - u_2 = 0 \ a.e. \ in \ \partial \Omega.
\]  

(3.24)

From (3.24), it follows that

\[
u_1 = u_2 \ a.e. \ in \ \Omega. \]

4 Entropy solutions

In this section, we study the existence and uniqueness of entropy solution to problem (1.1) when the right-hand side \( f \in L^1(\Omega) \). We first recall some notations.
For any $u \in W^{1,p(.)}(\Omega)$, we denote by $\tau(u)$ the trace of $u$ on $\partial \Omega$ in the usual sense. Set

$$T^{1,p(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{measurable such that } T_k(u) \in W^{1,p(.)}(\Omega), \text{ for any } k > 0 \right\}. $$

As $W^{1,p(.)}(\Omega) \subset W^{1,p-}(\Omega)$ and since $\Omega$ is bounded, then by [6, Lemma 2.1] (see also [1]), we have the following result:

**Proposition 4.1.** Let $u \in T^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v \chi_{\{|u|<k\}}$, for all $k > 0$. The function $v$ is denoted by $\nabla u$. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $v \in (L^p(.)(\Omega))^N$ and $v = \nabla u$ in the usual sense.

We define $T_{tr}^{1,p(.)}(\Omega)$ as the set of functions $u \in T^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

1. $u_n \to u$ a.e. in $\Omega$.
2. $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$.
3. There exists a measurable function $v$ on $\partial \Omega$, such that $u_n \to v$ a.e. in $\partial \Omega$.

The function $v$ is the trace of $u$ in the generalized sense. In the sequel the trace of $u \in T_{tr}^{1,p(.)}(\Omega)$ on $\partial \Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in T_{tr}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\phi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \phi) \in T_{tr}^{1,p(.)}(\Omega)$ and $tr(u - \phi) = tr(u) - tr(\phi)$ (see [2,3]).

We can now introduce the notion of entropy solution of (1.1).

**Definition 4.2.** A measurable function $u$ is an entropy solution to problem (1.1) if $u \in T_{tr}^{1,p(.)}(\Omega), b(u) \in L^1(\Omega), |u|^{p(.)-2} u \in L^1(\partial \Omega)$ and for every $k > 0$,

$$\int a(x,\nabla u).\nabla T_k(u - \phi)dx + \int b(u)T_k(u - \phi)dx + \int_{\partial \Omega} |u|^{p(.)-2} uT_k(u - \phi)d\sigma \leq \int fT_k(u - \phi)dx$$

(4.1)

for all $\phi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Notice that the integrals in (4.1) are well defined. Indeed, since $\phi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, then $(u - \phi) \in T_{tr}^{1,p(.)}(\Omega)$, hence $T_k(u - \phi) \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and consequently the first, the second and the fourth integral in (4.1) are well defined. Moreover, in the third integral, we can use the fact that the trace of $g \in W^{1,p(.)}(\Omega)$ on $\partial \Omega$ is well defined in $L^p(\partial \Omega)$.

Our main result in this section is the following:

**Theorem 4.3.** Assume (1.8)-(1.12) and $f \in L^1(\Omega)$, then there exists a unique entropy solution $u$ to problem (1.1).

In order to prove Theorem 4.3, we need the following propositions among which, some
can be proved following [7,26,27] with necessary changes in detail. But those which are new will be proved.

**Proposition 4.4.** Assume (1.8)-(1.12) and \( f \in L^1(\Omega) \). Let \( u \) be an entropy solution of (1.1). If there exists a positive constant \( M \) such that

\[
\int_{\{|u| > k\}} k^{q(\cdot)} dx \leq M \tag{4.2}
\]

then

\[
\int_{\{\nabla u|_{\alpha} > k\}} k^{q(\cdot)} dx \leq \|f\|_{L^1(\Omega)} + M, \quad \text{for all } k > 0,
\]

where \( \alpha(\cdot) = p(\cdot)/(q(\cdot) + 1) \) and \( q(\cdot) : \overline{\Omega} \to (0, +\infty) \) is measurable and such that \( q_- > 0 \).

**Proposition 4.5.** Assume (1.8)-(1.12) and \( f \in L^1(\Omega) \). Let \( u \) be an entropy solution of (1.1), then

\[
\int_{\Omega} |\nabla T_k(u)|^{p(\cdot)} dx \leq k\|f\|_{L^1(\Omega)} \quad \text{for all } k > 0,
\]

\[
\|b(u)\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}
\]

and

\[
\left\|u|^{p(\cdot)-2} u\right\|_{L^1(\partial\Omega)} = \left\|u|^{p(\cdot)-1}\right\|_{L^1(\partial\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{4.5}
\]

**Proof.** We will only prove relation (4.5) since the proof of relations (4.3) and (4.4) can be found in [7,26,27]. For this, we take \( \phi = 0 \) in relation (4.1) to get for all \( k > 0 \)

\[
\int_{\partial\Omega} |u|^{p(\cdot)-2} u T_k(u) d\sigma \leq k\|f\|_{L^1(\Omega)}. \tag{4.6}
\]

We deduce from (4.6) that

\[
\int_{\partial\Omega \cap \{u \geq k\}} |u|^{p(\cdot)-2} u T_k(u) d\sigma \leq k\|f\|_{L^1(\Omega)}
\]

which is equivalent to

\[
\int_{\partial\Omega \cap \{u \geq k\}} |u|^{p(\cdot)-2} u d\sigma - \int_{\partial\Omega \cap \{u \leq -k\}} |u|^{p(\cdot)-2} u d\sigma \leq \|f\|_{L^1(\Omega)}. \tag{4.7}
\]

It follows from (4.7) that

\[
\int_{\partial\Omega \cap \{u \geq k\}} |u|^{p(\cdot)-1} d\sigma \leq \|f\|_{L^1(\Omega)}. \tag{4.8}
\]

Finally, we let \( k \to 0 \) in (4.8) by using Fatou’s lemma to obtain relation (4.5). □

**Proposition 4.6.** Assume (1.8)-(1.12) and \( f \in L^1(\Omega) \). Let \( u \) be an entropy solution of (1.1), then

\[
\int_{\Omega} |\nabla T_k(u)|^p dx \leq \text{const}(\|f\|_{1,\Omega})(k + 1) \quad \text{for all } k > 0 \tag{4.9}
\]

\[
\int_{\partial\Omega} |u|^{p(\cdot)-2} u T_k(u) d\sigma \leq k\|f\|_{L^1(\Omega)} + M.
\]
Therefore, \( Z \)

Furthermore, for all \( k \)

The inequality (4.11) is equivalent to

Similarly, it follows that for all \( k \)

Adding relations (4.13) and (4.14) and using (4.5), we get (4.10).

**Proposition 4.7.** Assume (1.8)-(1.12) and \( f \in L^1(\Omega) \). Let \( u \) be an entropy solution of (1.1). Then

\[
\text{meas}\{ |u| > k \} \leq \frac{\text{const} \left( \|f\|_{L^1(\Omega)}, p, (p-)^{+}, \Omega \right)}{k^\alpha} \quad \text{for all } k \geq 1,
\]

and

\[
\text{meas}\{ |\nabla u| > k \} \leq \frac{\text{const} \left( \|f\|_{L^1(\Omega)}, p \right)}{k^{p-1}} \quad \text{for all } k \geq 1,
\]
where \((p_-)^* = \frac{1}{p_-} - \frac{1}{N}\) and \(\alpha = (p_-)^* \left(1 - \frac{1}{p_-}\right)\).

**Proof.** We only prove relation (4.15). The proof of (4.16) can be found in [7]. Using Proposition 4.6 (relation (4.9)), we obtain for all \(k \geq 1\) that

\[
\int_{\Omega} |\nabla T_k(u)|^{p_-} \, dx \leq K_1 k, \tag{4.17}
\]

where \(K_1\) is a positive real constant depending on \(\|f\|_1\) and meas(\(\Omega\)).

We now use a Poincar-Sobolev type inequality (see [28, Lemma in p. 308]) to get (since \(u \in T_r^{1, p_-} (\Omega)\)) that there exists a positive real constant \(K_2\) depending on \(\Omega\) such that

\[
\left(\int_{\Omega} |T_k(u)|^{(p_-)^*} \, dx \right)^{\frac{1}{(p_-)^*}} \leq K_2 \left( \left(\int_{\partial\Omega} |T_k(u)| d\sigma \right)^{p_-} + \int_{\Omega} |\nabla T_k(u)|^{p_-} \, dx \right), \tag{4.18}
\]

where \((p_-)^*\) is the Sobolev exponent with respect to \(p_-\). By Hölder inequality, we have the following

\[
\left(\int_{\partial\Omega} |T_k(u)| d\sigma \right)^{p_-} \leq \left( \|T_k(u)\|_{L^{p_-} (\partial\Omega)} \times \text{meas}_{N-1} (\partial\Omega)^{\frac{1}{p_-}} \right)^{p_-}. \tag{4.19}
\]

We deduce from (4.19) by using Proposition 4.6 (relation (4.10)) that for all \(k \geq 1\)

\[
\left(\int_{\partial\Omega} |T_k(u)| d\sigma \right)^{p_-} \leq K_3 k \tag{4.20}
\]

where \(K_3\) is a positive real constant which depends on \(\|f\|_1, p_-,\) meas(\(\Omega\)) and meas(\(\partial\Omega\)).

By (4.17), (4.18) and (4.20), we deduce that for all \(k \geq 1\),

\[
\left(\int_{\Omega} |T_k(u)|^{(p_-)^*} \, dx \right)^{\frac{1}{(p_-)^*}} \leq K_4 k, \tag{4.21}
\]

where \(K_4\) is a positive real constant depending only on \(\|f\|_1, p_-, (p_-)^*\), meas(\(\Omega\)) and meas(\(\partial\Omega\)).

It follows from (4.21) that

\[
\int_{\Omega} |T_k(u)|^{(p_-)^*} \, dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}, \tag{4.22}
\]

where \(K_5\) is a positive real constant depending only on \(\|f\|_1, p_-, (p_-)^*\), meas(\(\Omega\)) and meas(\(\partial\Omega\)).

Note that (4.22) implies that

\[
\int_{\{|u| > k\}} |T_k(u)|^{(p_-)^*} \, dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}. \tag{4.23}
\]

The inequality (4.23) is equivalent to the following

\[
\int_{\{|u| > k\}} k^{(p_-)^*} \, dx \leq K_5 k^{\frac{(p_-)^*}{p_-}},
\]
which in turn is also equivalent to
\[ k^{(p^*)}\meas(\{|u| > k\}) \leq K_k k^{(p^*)}. \]  
(4.24)
We deduce from (4.24), the following relation
\[ \meas(\{|u| > k\}) \leq K_k k^{(p^*)}(\frac{1}{p^*}-1). \]  
(4.25)
From (4.25), we deduce (4.15). □

We are now ready to give the proof of Theorem 4.3.

Proof of Theorem 4.3.

*Uniqueness of entropy solution.* Let \( h > 0 \) and \( u_1, u_2 \) be two entropy solutions of (1.1). We write the entropy inequality (4.1) corresponding to the solution \( u_1 \) with \( T_h(u_2) \) as a test function and to the solution \( u_2 \) with \( T_h(u_1) \) as a test function. Upon addition, we get
\[
\begin{align*}
\int_{\{|u_1 - T_h(u_2)| \leq k\}} a(x, \nabla u_1). \nabla (u_1 - T_h(u_2)) dx &+ \int_{\{|u_2 - T_h(u_1)| \leq k\}} a(x, \nabla u_2). \nabla (u_2 - T_h(u_1)) dx \\
+ \int_{\partial \Omega} |u_1|^{p(x)-2} u_1 T_h(u_1 - T_h(u_2)) d\sigma &+ \int_{\partial \Omega} |u_2|^{p(x)-2} u_2 T_h(u_2 - T_h(u_1)) d\sigma \\
+ \int_{\Omega} b(u_1) T_h(u_1 - T_h(u_2)) dx &+ \int_{\Omega} b(u_2) T_h(u_2 - T_h(u_1)) dx \\
\leq & \int_{\Omega} f(x) \left( T_h(u_1 - T_h(u_2)) + T_h(u_2 - T_h(u_1)) \right) dx.
\end{align*}
\]  
(4.26)
Now, define
\[ E_1 := \{|u_1 - u_2| \leq k, |u_2| \leq h\}, \quad E_2 := E_1 \cap \{|u_1| \leq h\}, \quad \text{and} \quad E_3 := E_1 \cap \{|u_1| > h\}. \]
We start with the first integral in (4.26). By (1.12), we have
\[
\begin{align*}
\int_{\{|u_1 - T_h(u_2)| \leq k\}} a(x, \nabla u_1). \nabla (u_1 - T_h(u_2)) dx &+ \int_{\{|u_2 - T_h(u_1)| \leq k\}} a(x, \nabla u_2). \nabla (u_2 - T_h(u_1)) dx \\
+ \int_{\partial \Omega} |u_1|^{p(x)-2} u_1 T_h(u_1 - T_h(u_2)) d\sigma &+ \int_{\partial \Omega} |u_2|^{p(x)-2} u_2 T_h(u_2 - T_h(u_1)) d\sigma \\
+ \int_{\Omega} b(u_1) T_h(u_1 - T_h(u_2)) dx &+ \int_{\Omega} b(u_2) T_h(u_2 - T_h(u_1)) dx \\
\leq & \int_{\Omega} f(x) \left( T_h(u_1 - T_h(u_2)) + T_h(u_2 - T_h(u_1)) \right) dx.
\end{align*}
\]  
(4.27)
Using (1.10) and (2.1), we estimate the last integral in (4.27) as follows:

\[
\begin{aligned}
&\int_{E_3} a(x, \nabla u_1) \cdot \nabla u_2 \, dx \\
&\leq C_1 \int_{E_3} \left( |j(x) + \|\nabla u_1\|_{L^{p(x)-1}}| \nabla u_2| \right) \, dx \\
&\leq C_1 \left( \|j\|_{L^{p(x)}} + \|\nabla u_1\|_{L^{p(x)-1}} \right) \|\nabla u_2\|_{L^{p(x)}} \\
\end{aligned}
\]  

(4.28)

where \( \|\nabla u_1\|_{L^{p(x)-1}} \). \( p'(\cdot, \{h < \|u_1\| \leq h + k\}) \) is an entropy solution to problem (1.1), by taking \( \phi = T_h(u_1) \) in the entropy inequality (4.1) we get (using (1.12)) that

\[
\int_{\{h < \|u_1\| \leq h + k\}} \|\nabla u_1\|_{L^{p(x)}} \, dx \leq k \|f\|_1.
\]

So, by Lemma 2.1, \( \|\nabla u_1\|_{L^{p(x)-1}} \|p'(\cdot, \{h < \|u_1\| \leq h + k\}) \leq C < +\infty \), where \( C \) is a constant which does not depend on \( h \).

Therefore,

\[
C_1 \left( \|j\|_{L^{p(x)}} + \|\nabla u_1\|_{L^{p(x)-1}} \right) \leq C_1 \left( \|j\|_{L^{p(x)}} + C \right) < +\infty.
\]

Since \( u_2 \) is an entropy solution to problem (1.1), by taking \( \phi = T_h(u_2) \) in the entropy inequality (4.1) we get (using (1.12)) that

\[
\int_{\{h < \|u_2\| \leq h + k\}} \|\nabla u_2\|_{L^{p(x)}} \, dx \leq k \int_{\{|u_2| > h\}} |f| \, dx.
\]

Using inequality (4.15) of Proposition 4.7, we have \( \text{meas}\{\|u_2\| > h\} \to 0 \) as \( h \to +\infty \). As \( f \in L^1(\Omega) \) we get

\[
k \int_{\{|u_2| > h\}} |f| \, dx \to 0 \text{ as } h \to +\infty \text{ for any fixed number } k > 0.
\]

From the above convergence we deduce that

\[
\lim_{h \to +\infty} \int_{\{h < \|u_2\| \leq h + k\}} \|\nabla u_2\|_{L^{p(x)}} \, dx = 0, \text{ for any fixed number } k > 0.
\]

Hence,

\[
\lim_{h \to +\infty} \int_{\{h - k < \|u_2\| \leq h\}} \|\nabla u_2\|_{L^{p(x)}} \, dx = \lim_{l \to +\infty} \int_{\{l < \|u_2\| \leq l + k\}} \|\nabla u_2\|_{L^{p(x)}} \, dx = 0,
\]

for any fixed number \( k > 0 \) with \( l = h - k \).

So by Lemma 2.1,

\[
\|\nabla u_2\|_{p'(\cdot, \{h - k < \|u_2\| \leq h\}) \to 0 \text{ as } h \to +\infty, \text{ for any fixed number } k > 0.
\]
Therefore, from (4.27) and (4.28), we obtain that
\[
\int_{\{|u_1 - T_h(u_2)| \leq k\}} a(x, \nabla u_1) \nabla (u_1 - T_h(u_2)) \, dx \geq I_h + \int_{E_2} a(x, \nabla u_1) \nabla (u_1 - u_2) \, dx,
\]  
(4.29)
where \( I_h \) converges to zero as \( h \to +\infty \).

We may adopt the same procedure to treat the second term in (4.26) to obtain
\[
\int_{\{|u_2 - T_h(u_1)| \leq k\}} a(x, \nabla u_2) \nabla (u_2 - T_h(u_1)) \, dx \geq J_h - \int_{E_2} a(x, \nabla u_2) \nabla (u_1 - u_2) \, dx,
\]  
(4.30)
where \( J_h \) converges to zero as \( h \to +\infty \).

Now, set for all \( h, k > 0 \),
\[
K_h = \int_\Omega b(u_1) T_k(u_1 - T_h(u_2)) \, dx + \int_\Omega b(u_2) T_k(u_2 - T_h(u_1)) \, dx
\]
and
\[
P_h = \int_{\partial \Omega} |u_1|^{p(x)-2} u_1 T_k(u_1 - T_h(u_2)) \, d\sigma + \int_{\partial \Omega} |u_2|^{p(x)-2} u_2 T_k(u_2 - T_h(u_1)) \, d\sigma.
\]
We have
\[
b(u_1) T_k(u_1 - T_h(u_2)) \to b(u_1) T_k(u_1 - u_2) \text{ a.e. in } \Omega \text{ as } h \to +\infty
\]
and
\[
|b(u_1) T_k(u_1 - T_h(u_2))| \leq k|b(u_1)| \in L^1(\Omega).
\]
Then by Lebesgue Theorem, we deduce that
\[
\lim_{h \to +\infty} \int_\Omega b(u_1) T_k(u_1 - T_h(u_2)) \, dx = \int_\Omega b(u_1) T_k(u_1 - u_2) \, dx.
\]  
(4.31)

Similarly, we have
\[
\lim_{h \to +\infty} \int_\Omega b(u_2) T_k(u_2 - T_h(u_1)) \, dx = \int_\Omega b(u_2) T_k(u_2 - u_1) \, dx.
\]  
(4.32)

Using (4.31) and (4.32), we get
\[
\lim_{h \to +\infty} K_h = \int_{\Omega} (b(u_1) - b(u_2)) T_k(u_1 - u_2) \, dx.
\]  
(4.33)

By the same procedure as above, we use the Lebesgue theorem to obtain
\[
\lim_{h \to +\infty} P_h = \int_{\partial \Omega} \left( |u_1|^{p(x)-2} u_1 - |u_2|^{p(x)-2} u_2 \right) \, T_k(u_1 - u_2) \, d\sigma.
\]  
(4.34)

We next examine the right-hand side of (4.26).
For all \( k > 0 \),
\[
f(x) \left( T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1)) \right) \to f(x) \left( T_k(u_1 - u_2) + T_k(u_2 - u_1) \right) = 0
\]
Lebesgue Theorem allows us to write
\[
\lim_{h \to +\infty} \int_{\Omega} f(x) \left( T_h(u_1 - T_h(u_2)) + T_h(u_2 - T_h(u_1)) \right) \, dx = 0. 
\] (4.35)

Using (4.29), (4.30), (4.33), (4.34) and (4.35), we get from (4.26) the following inequality:
\[
\begin{cases}
\int_{|u_1 - u_2| \leq k} \left( a(x, \nabla u_1) - a(x, \nabla u_2) \right) \left( \nabla u_1 - \nabla u_2 \right) \, dx + \\
\int_{\Omega} (b(u_1) - b(u_2)) T_h(u_1 - u_2) \, dx + \int_{\partial\Omega} \left| u_1 \right|^{p(x) - 2} u_1 - \left| u_2 \right|^{p(x) - 2} u_2 \right) T_h(u_1 - u_2) \, d\sigma \leq 0.
\end{cases}
\] (4.36)

It follows also from (4.36) that
\[
\begin{cases}
\int_{|u_1 - u_2| \leq k} \left( a(x, \nabla u_1) - a(x, \nabla u_2) \right) \left( \nabla u_1 - \nabla u_2 \right) \, dx = 0,
\end{cases}
\] (4.37)
\[
\int_{\Omega} (b(u_1) - b(u_2)) T_h(u_1 - u_2) \, dx = 0
\] (4.38)
and
\[
\int_{\partial\Omega} \left| u_1 \right|^{p(x) - 2} u_1 - \left| u_2 \right|^{p(x) - 2} u_2 \right) T_h(u_1 - u_2) \, d\sigma = 0,
\] (4.39)
for all \( k > 0 \).

From (4.37) and (1.11), it follows that
\[
u_1 - u_2 = c \text{ a.e. in } \Omega, \text{ where } c \text{ is a real constant.} \] (4.40)

By (4.39), we deduce that for all \( k \in \mathbb{N}^* \) there exists \( C_k \subset \partial\Omega \), \( \text{meas}(C_k) = 0 \) such that for all \( x \in \partial\Omega \setminus C_k \),
\[
\left( \left| u_1(x) \right|^{p(x) - 2} u_1(x) - \left| u_2(x) \right|^{p(x) - 2} u_2(x) \right) T_h(u_1(x) - u_2(x)) = 0.
\]

Therefore,
\[
\left( \left| u_1(x) \right|^{p(x) - 2} u_1(x) - \left| u_2(x) \right|^{p(x) - 2} u_2(x) \right) (u_1(x) - u_2(x)) = 0, \text{ for all } x \in \partial\Omega \setminus \bigcup_{k \in \mathbb{N}^*} C_k.
\] (4.41)

Now, we use (3.23) and (4.41) to get
\[
u_1 - u_2 = 0 \text{ a.e. on } \partial\Omega. \] (4.42)

Finally, (4.40) and (4.42) give
\[
u_1 = u_2 \text{ a.e. in } \Omega.
\]

*Existence of entropy solution.* Let \( f_n = T_n(f) \); then \( (f_n)_{n \in \mathbb{N}} \) is a sequence of bounded functions which strongly converges to \( f \) in \( L^1(\Omega) \) and such that
\[
\|f_n\|_1 \leq \|f\|_1, \text{ for all } n \in \mathbb{N}.
\] (4.43)
We consider the problem

\[
\begin{align*}
\begin{cases}
    b(u_n) - \text{div} a(x, \nabla u_n) &= f_n \text{ in } \Omega, \\
    a(x, \nabla u_n) \cdot n &= -|u_n|^{p(x)-2} u_n \text{ on } \partial \Omega.
\end{cases}
\end{align*}
\]  (4.44)

It follows from Theorem 3.2 that there exists a unique \( u_n \in W^{1,p(\cdot)}(\Omega) \) with \( b(u_n) \in L^\infty(\Omega) \) and \( |u_n|^{p(x)-2} u_n \in L^\infty(\partial \Omega) \) so that

\[
\int_{\Omega} a(x, \nabla u_n) \cdot \nabla \varphi dx + \int_{\Omega} b(u_n) \varphi dx + \int_{\partial \Omega} |u_n|^{p(x)-2} u_n \varphi d\sigma = \int_{\Omega} f_n \varphi dx,
\]  (4.45)

for all \( \varphi \in W^{1,p(\cdot)}(\Omega) \).

Our aim is to prove that these approximated solutions \( u_n \) tend to a measurable function \( u \) (as \( n \) goes to \(+\infty\)) which is an entropy solution to the limit problem (1.1). To start with, we first prove the following lemma:

**Lemma 4.8.** For any \( k > 0 \), \( \|T_k(u_n)\|_{1,p(\cdot)} \leq 1 + C \) where \( C = \text{const}(k, f, p_-, p_+, \text{meas}(\Omega)) \) is a positive constant.

**Proof.** By taking \( \varphi = T_k(u_n) \) in (4.45), we get

\[
\int_{\Omega} a(x, \nabla u_n) \cdot \nabla T_k(u_n) dx + \int_{\Omega} b(u_n) T_k(u_n) dx + \int_{\partial \Omega} |u_n|^{p(x)-2} u_n T_k(u_n) d\sigma = \int_{\Omega} f_n T_k(u_n) dx.
\]

Since all the terms in the left-hand side of the equality above are nonnegative and

\[
\int_{\Omega} f_n T_k(u_n) dx \leq k \|f_n\|_1 \leq k \|f\|_1,
\]

by using (1.12) we obtain

\[
\int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx \leq C k \|f\|_1. \]  (4.46)

We also have that

\[
\int_{\Omega} |T_k(u_n)|^{p(x)} dx = \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx + \int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx.
\]

Furthermore,

\[
\int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx \leq \begin{cases}
    k^{p_-} \text{meas}(\Omega) & \text{if } k \geq 1, \\
    \text{meas}(\Omega) & \text{if } k < 1
\end{cases}
\]

and

\[
\int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx \leq \begin{cases}
    k^{p_+} \text{meas}(\Omega) & \text{if } k \geq 1, \\
    \text{meas}(\Omega) & \text{if } k < 1.
\end{cases}
\]
This allows us to write
\[
\int_{\Omega} |T_k(u_n)|^{p(x)} \, dx \leq 2(1 + k^{p^+}) \text{meas}(\Omega). \tag{4.47}
\]
Hence, adding (4.46) and (4.47) yields
\[
\rho_{1, p(\cdot)}(T_k(u_n)) \leq Ck \|f\|_1 + (1 + k^{p^+}) \text{meas}(\Omega) = \text{const}(k, f, p_+, \text{meas}(\Omega)). \tag{4.48}
\]
For \(\|T_k(u_n)\|_{1,p(\cdot)} \geq 1\), we have according to Lemma 2.2 that
\[
\|T_k(u_n)\|_{1,p(\cdot)}^{p^+} \leq \rho_{1, p(\cdot)}(T_k(u_n)) \leq \text{const}(k, f, p_+, \text{meas}(\Omega)),
\]
which is equivalent to
\[
\|T_k(u_n)\|_{1,p(\cdot)} \leq \left(\text{const}(k, f, p_+, \text{meas}(\Omega))\right)^{\frac{1}{p^+}} = \text{const}(k, f, p_+, p_-, \text{meas}(\Omega)).
\]
The above inequality gives
\[
\|T_k(u_n)\|_{1,p(\cdot)} \leq 1 + \text{const}(k, f, p_+, p_-, \text{meas}(\Omega)).
\]
Then, the proof of Lemma 4.8 is complete.

From Lemma 4.8, we deduce that for any \(k > 0\), the sequence \((T_k(u_n))_{n \in \mathbb{N}}\) is uniformly bounded in \(W^{1,p(\cdot)}(\Omega)\) and so in \(W^{1,p(\cdot)}(\Omega)\). Then, up to a subsequence we can assume that for any \(k > 0\), \(T_k(u_n)\) converges weakly to \(\sigma_k\) in \(W^{1,p(\cdot)}(\Omega)\), and so \(T_k(u_n)\) strongly converges to \(\sigma_k\) in \(L^{p^+}(\Omega)\).

We next prove the following proposition:

**Proposition 4.9.** Assume that (1.8)-(1.12) hold and \(u_n \in W^{1,p(\cdot)}(\Omega)\) is the weak solution of problem (4.44), then the sequence \((u_n)_{n \in \mathbb{N}}\) is Cauchy in measure. In particular, there exists a measurable function \(u\) and a subsequence still denoted \((u_n)_{n \in \mathbb{N}}\) such that \(u_n \rightarrow u\) in measure.

**Proof.** Let \(s > 0\) and define
\[
E_n := \{|u_n| > k\}, \quad E_m := \{|u_m| > k\} \quad \text{and} \quad E_{n,m} := \{|T_k(u_n) - T_k(u_m)| > s\}
\]
where \(k > 0\) is to be fixed. We note that
\[
\{|u_n - u_m| > s\} \subset E_n \cup E_m \cup E_{n,m}
\]
and hence
\[
\text{meas}\{|u_n - u_m| > s\} \leq \text{meas}(E_n) + \text{meas}(E_m) + \text{meas}(E_{n,m}). \tag{4.49}
\]
Let \(\varepsilon > 0\). Using Proposition 4.7 (relation (4.15)), we choose \(k = k(\varepsilon)\) such that
\[
\text{meas}(E_n) \leq \varepsilon/3 \quad \text{and} \quad \text{meas}(E_m) \leq \varepsilon/3. \tag{4.50}
\]
Since \( T_k(u_n) \) strongly converges in \( L^p(\Omega) \), then it is a Cauchy sequence in \( L^p(\Omega) \).

Thus,

\[
\text{meas}(E_{n,m}) \leq \frac{1}{s} \int_{\Omega} |T_k(u_n) - T_k(u_m)|^p \, dx \leq \frac{\varepsilon}{3},
\]

for all \( n, m \geq n_0(s, \varepsilon) \).

Finally, from (4.49), (4.50) and (4.51), we obtain

\[
\text{meas}\{|u_n - u_m| > s\} \leq \varepsilon \text{ for all } n, m \geq n_0(s, \varepsilon).
\]

Relations (4.52) mean that the sequence \((u_n)_{n\in\mathbb{N}}\) is a Cauchy sequence in measure and the proof of Proposition 4.9 is complete.

Note that as \( u_n \to u \) in measure, up to a subsequence, we can assume that \( u_n \to u \) a.e. in \( \Omega \).

In the sequel, we need the following two technical lemmas (see [18,31]).

**Lemma 4.10.** Let \((v_n)_{n\in\mathbb{N}}\) be a sequence of measurable functions in \( \Omega \). If \( v_n \) converges in measure to \( v \) and is uniformly bounded in \( L^{p(\cdot)}(\Omega) \) for some \( 1 \leq p(\cdot) \in L^\infty(\Omega) \), then \( v_n \) strongly converges to \( v \) in \( L^1(\Omega) \).

The second technical lemma is a well known result in measure theory (see [18]):

**Lemma 4.11.** Let \((X, \mathcal{M}, \mu)\) be a measure space such that \( \mu(X) < +\infty \). Consider a measurable function \( \gamma : X \to [0, +\infty] \) such that

\[
\mu(\{x \in X : \gamma(x) = 0\}) = 0.
\]

Then, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\mu(A) < \varepsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma \, d\mu < \delta.
\]

We now set to prove that the function \( u \) in the Proposition 4.9 is an entropy solution of (1.1).

Let \( \varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \). For any \( k > 0 \), choose \( T_k(u_n - \varphi) \) as a test function in (4.45).

We get

\[
\int_{\Omega} a(x, \nabla u_n) \cdot \nabla T_k(u_n - \varphi) \, dx + \int_{\Omega} b(u_n) T_k(u_n - \varphi) \, dx + \int_{\partial\Omega} |u_n|^{p(\cdot)} - 2 u_n T_k(u_n - \varphi) \, d\sigma = \int_{\Omega} f_n(x) T_k(u_n - \varphi) \, dx.
\]

The following proposition is useful to pass to the limit in the first term of (4.53).

**Proposition 4.12.** Assume that (1.8) – (1.12) hold and \( u_n \in W^{1,p(\cdot)}(\Omega) \) be the weak solution of the problem (4.44), then

(i) \( \nabla u_n \) converges in measure to the weak gradient of \( u \);

(ii) for all \( k > 0 \), \( \nabla T_k(u_n) \) converges to \( \nabla T_k(u) \) in \( (L^1(\Omega))^N \).
(iii) for all $t > 0$, $a(x, \nabla T_t(u_n))$ strongly converges to $a(x, \nabla T_t(u))$ in $(L^1(\Omega))^N$ and weakly in $(L^{p'}(\Omega))^N$;
(iv) $u_n$ converges to some function $v$ a.e. on $\partial \Omega$.

Proof.

(i) We claim that the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ is Cauchy in measure.
Let $s > 0$ and consider

$$A_{n,m} := \{|\nabla u_n| > h\} \cup \{|\nabla u_m| > h\},$$
$$B_{n,m} := \{|u_n - u_m| > k\},$$
and

$$C_{n,m} := \{|\nabla u_n| \leq h, |\nabla u_m| \leq h, |u_n - u_m| \leq k, |\nabla u_n - \nabla u_m| > s\},$$

where $h$ and $k$ will be chosen later.

Note that

$$\{ |\nabla u_n - \nabla u_m| > s \} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}.$$  \hfill (4.54)

Let $\varepsilon > 0$. By Proposition 4.7 (relation (4.16)), we may choose $h = h(\varepsilon)$ large enough such that

$$\text{meas}(A_{n,m}) \leq \varepsilon / 3,$$  \hfill (4.55)

for all $n, m \geq 0$.

On the other hand, by Proposition 4.9

$$\text{meas}(B_{n,m}) \leq \varepsilon / 3,$$  \hfill (4.56)

for all $n, m \geq n_0(k, \varepsilon)$.

Moreover, since $a(x, \xi)$ is continuous with respect to $\xi$ for a.e. $x \in \Omega$, by assumption (1.11) there exists a real valued function $\gamma : \Omega \rightarrow [0, +\infty]$ such that $\text{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0$, and

$$(a(x, \xi) - a(x, \xi')).(\xi - \xi') \geq \gamma(x),$$  \hfill (4.57)

for all $\xi, \xi' \in \mathbb{R}^N$ such that $|\xi| \leq h$, $|\xi'| \leq h$, $|\xi - \xi'| \geq s$, for a.e $x \in \Omega$.

Let $\delta = \delta(\varepsilon)$ be given by Lemma 4.11, replacing $\varepsilon$ and $A$ by $\varepsilon / 3$ and $C_{n,m}$ respectively. As $u_n$ is a weak solution of (4.44), using $T_k(u_n - u_m)$ as a test function in (4.45), we get

$$\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - u_m)dx + \int_\Omega b(u_n)T_k(u_n - u_m)dx + \int_{\partial \Omega} |u_n|^{p(x)-2} u_n T_k(u_n - u_m)d\sigma = \int_\Omega f_n T_k(u_n - u_m)dx \leq k\|f\|_1.$$  

Similarly, we have for $u_m$ that

$$\int_\Omega a(x, \nabla u_m).\nabla T_k(u_m - u_n)dx + \int_\Omega b(u_m)T_k(u_m - u_n)dx + \int_{\partial \Omega} |u_m|^{p(x)-2} u_m T_k(u_m - u_n)d\sigma = \int_\Omega f_m T_k(u_m - u_n)dx \leq k\|f\|_1.$$
Adding the last two inequalities yields
\[
\int_{\{u_n - u_m \leq k\}} (a(x, \nabla u_n) - a(x, \nabla u_m))(\nabla u_n - \nabla u_m)dx + \int_{\Omega} \left( b(u_n) - b(u_m) \right) T_k(u_n - u_m)dx \\
+ \int_{\partial \Omega} \left( |u_n|^{p(x) - 2} u_n - |u_m|^{p(x) - 2} u_m \right) T_k(u_n - u_m)d\sigma \leq 2k\|f\|_1.
\]
Since the second and the third term of the above inequality are nonnegative, we obtain by using (4.57) that
\[
\int_{C_{n,m}} \gamma(x)dx \leq \int_{C_{n,m}} (a(x, \nabla u_n) - a(x, \nabla u_m))(\nabla u_n - \nabla u_m)dx \leq 2k\|f\|_1 < \delta,
\]
where \(k = \delta/4\|f\|_1\).

From Lemma 4.11, it follows that
\[
\text{meas}(C_{n,m}) \leq \varepsilon/3. \quad (4.58)
\]
Thus, using (4.54), (4.55), (4.56) and (4.58), we get
\[
\text{meas}(\{|\nabla u_n - \nabla u_m| > \delta\}) \leq \varepsilon, \text{ for all } n, m \geq n_0(s, \varepsilon) \quad (4.59)
\]
and then the claim is proved.

Consequently, \((\nabla u_n)_{n \in \mathbb{N}}\) converges in measure to some measurable function \(v\).

In order to end the proof of (i), we need the following lemma:

**Lemma 4.13**

(a) For a.e. \(t \in \mathbb{R}\), \(\nabla T_t(u_n)\) converges in measure to \(\nu \chi_{|u| < \langle t \rangle}\);

(b) for a.e. \(t \in \mathbb{R}\), \(\nabla T_t(u) = \nu \chi_{|u| < \langle t \rangle}\);

(c) \(\nabla T_t(u) = \nu \chi_{|u| < \langle t \rangle}\) holds for all \(t \in \mathbb{R}\).

**Proof.**

- **Proof of (a).**

We know that \(\nabla u_n \rightarrow v\) in measure. Thus, \(\chi_{|u| < \langle t \rangle} \nabla u_n \rightarrow \chi_{|u| < \langle t \rangle} v\) in measure.

Now, let us show that \((\chi_{|u_n| < \langle t \rangle} - \chi_{|u| < \langle t \rangle}) \nabla u_n \rightarrow 0\) in measure. For that, it is sufficient to show that \((\chi_{|u_n| < \langle t \rangle} - \chi_{|u| < \langle t \rangle}) \rightarrow 0\) in measure. Now, for all \(\delta > 0\),

\[
\{ |\chi_{|u_n| < \langle t \rangle} - \chi_{|u| < \langle t \rangle} | |\nabla u_n| > \delta \} \subset \{ |\chi_{|u_n| < \langle t \rangle} - \chi_{|u| < \langle t \rangle} | \neq 0 \}
\]

\[
\subset \{|u| = t\} \cup \{u_n < t < u\} \cup \{u < t < u_n\} \cup \{u_n < -t < u\} \cup \{u < -t < u_n\}.
\]

Thus,
\[
\text{meas} \left( \left| \chi_{|u_n| < \langle t \rangle} - \chi_{|u| < \langle t \rangle} \right| |\nabla u_n| > \delta \right) \leq \text{meas} \{|u| = t\} + \text{meas} \{u_n < t < u\} + \\
\text{meas} \{u < t < u_n\} + \text{meas} \{u_n < -t < u\} + \text{meas} \{u < -t < u_n\}.
\]

(4.60)
Note that
\[ \text{meas}\{|u| = t\} \leq \text{meas}\{t - h < u < t + h\} + \text{meas}\{t - h < u < -t + h\} \to 0 \text{ as } h \to 0 \]
for a.e. \( t \), since \( u \) is a fixed function. Next,
\[ \text{meas}\{u_n < t < u\} \leq \text{meas}\{t < u < t + h\} + \text{meas}\{|u - u_n| > h\}, \text{ for all } h > 0. \]

Due to Proposition 4.9, we have for all fixed \( h > 0 \), \( \text{meas}\{|u - u_n| > h\} \to 0 \text{ as } n \to +\infty.\)

Since \( \text{meas}\{t < u < t + h\} \to 0 \text{ as } h \to 0, \) for all \( \varepsilon > 0, \) one can find \( N \) such that for all \( n > N, \)
\[ \text{meas}\{u_n < t < u\} < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ by choosing } h \text{ and then } N. \]

Each of the other terms in the right-hand side of (4.60) can be treated in the same way as for \( \text{meas}\{u_n < t < u\} \).

Thus, \[ \text{meas}\left\{ \mathcal{X}_{\{|u| < t\}} - \mathcal{X}_{\{|u| < t\}} \left| \nabla u_n \right| > \delta \right\} \to 0 \text{ as } n \to +\infty. \]

Since \( \nabla T_i(u_n) = \nabla u_n \chi_{\{|u| < t\}}, \) the claim (a) follows.

• Proof of (b).

Let \( \psi_i \) be the weak \( W^{1,p(\cdot)} \)-limit of \( T_i(u_n) \), then it is also the strong \( L^1 \)-limit of \( T_i(u_n) \). But, as \( T_i \) is a Lipschitz function, the convergence in measure of \( u_n \) to \( u \) implies the convergence in measure of \( T_i(u_n) \) to \( T_i(u) \). Thus, by the uniqueness of the limit in measure, \( \psi_i \) is identified with \( T_i(u) \), we conclude that \( \nabla T_i(u_n) \to \nabla T_i(u) \) weakly in \( L^p(\cdot)(\Omega) \).

The previous convergence also ensures that \( \nabla T_i(u_n) \) converges to \( \nabla T_i(u) \) weakly in \( L^1(\Omega) \).

On the other hand, by (a), \( \nabla T_i(u_n) \) converges to \( v\chi_{\{|u| < t\}} \) in measure. By Lemma 4.10, since \( \nabla T_i(u_n) \) is uniformly bounded in \( L^p(\cdot)(\Omega) \), the convergence is actually strong in \( L^1(\Omega) \); thus it is also weak in \( L^1(\Omega) \). By the uniqueness of a weak \( L^1 \)-limit, \( v\chi_{\{|u| < t\}} \) coincides with \( \nabla T_i(u) \).

• Proof of (c)

Let \( 0 < t < s \), and \( s \) be such that \( v\chi_{\{|u| < s\}} \) coincides with \( \nabla T_i(u) \). Then
\[ \nabla T_i(u) = \nabla T_i(T_s(u)) = \nabla T_i(u) \mathcal{X}_{\{|T_i(u)| < t\}} = v\chi_{\{|u| < s\}} \mathcal{X}_{\{|u| < t\}} = v\chi_{\{|u| < t\}}. \]

Now, we can end the proof of (i). Indeed, combining Lemma 4.13-(c) and Proposition 4.1, (i) follows.

(ii) Let \( s > 0, k > 0 \) and consider
\[ F_{n,m} = \{|
abla u_n - \nabla u_m| > s, |u_n| \leq k, |u_m| \leq k\}, G_{n,m} = \{|
abla u_m| > s, |u_n| > k, |u_m| \leq k\}, \]
\[ H_{n,m} = \{|
abla u_n| > s, |u_n| > k, |u_m| \leq k\} \text{ and } I_{n,m} = \{0 > s, |u_m| > k, |u_n| > k\}. \]

Note that
\[ \{|
abla T_k(u_n) - \nabla T_k(u_m)| > s\} \subset F_{n,m} \cup G_{n,m} \cup H_{n,m} \cup I_{n,m}. \]  \hfill (4.61)

Let \( \varepsilon > 0. \) By Proposition 4.7, we may choose \( k(\varepsilon) \) such that
\[ \text{meas}(G_{n,m}) \leq \frac{\varepsilon}{4}, \text{meas}(H_{n,m}) \leq \frac{\varepsilon}{4} \text{ and } \text{meas}(I_{n,m}) \leq \frac{\varepsilon}{4}. \]  \hfill (4.62)

Therefore, using (4.59), (4.61) and (4.62) we get
\[ \text{meas}(\{|
abla T_k(u_n) - \nabla T_k(u_m)| > s\}) \leq \varepsilon, \text{ for all } n, m \geq n_1(s, \varepsilon). \]  \hfill (4.63)
Consequently, $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$.

Then, using lemmas 4.8 and 4.10, (ii) follows.

(iii) By lemmas 4.10 and 4.13, we have that for all $t > 0$, $a(x, \nabla T_t(u_n))$ strongly converges to $a(x, \nabla T_t(u))$ in $(L^1(\Omega))^N$ (as $n$ goes to $+\infty$) and $a(x, \nabla T_t(u))$ weakly converges to $\chi_t \in (L^{p'}(\Omega))^N$ (as $n$ goes to $+\infty$) in $(L^{p'}(\Omega))^N$. Since each of the convergences implies the weak $L^1$-convergence, $\chi_t$ can be identified with $a(x, \nabla T_t(u))$; thus, $a(x, \nabla T_t(u)) \in (L^{p'}(\Omega))^N$. The proof of (iii) is then complete.

(iv) As $u_n$ is a weak solution of (4.44), using $T_k(u_n)$ as a test function in (4.45), we get

$$\int_{\partial\Omega} |T_k(u_n)|^{p(x)} dx \leq \int_{\partial\Omega} |u_n|^{p(x)-2} u_n T_k(u_n) dx \leq k \|f\|_1,$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx \leq Ck \|f\|_1.$$

We deduce from the inequalities above that

$$\int_{\partial\Omega} |T_k(u_n)|^{p} dx \leq C(f, \Omega)k. \quad (4.64)$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|^{p} dx \leq C(C_3, f, \Omega)k, \quad (4.65)$$

for $k \geq 1$.

Note also that

$$\int_{\Omega} |T_k(u_n)|^{p} dx \leq 2(1+k^{p,x}) \text{meas}(\Omega) + \text{meas}(\Omega),$$

for $k \geq 1$.

Furthermore, $T_k(u_n)$ converges weakly to $T_k(u)$ in $W^{1,p}(\Omega)$ and since for every $1 \leq p \leq +\infty$,

$$\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega), u \mapsto \tau(u) = u|_{\partial\Omega}$$

is compact, we deduce that $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^p(\partial\Omega)$ and so, up to a subsequence, we can assume that $T_k(u_n)$ converges to $T_k(u)$ a.e. on $\partial\Omega$. In other words, there exists $A \subset \partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega \setminus A$ with $\mu(A) = 0$, where $\mu$ is the area measure on $\partial\Omega$.

Now, we use Hlder Inequality, (4.64) and (4.65) and the Poincar-Sobolev type inequality as in (4.18) to get

$$\int_{\Omega} |T_k(u_n)|^{p} dx \leq (\text{meas}(\Omega))^{\frac{1}{(p-\gamma)}} (Ck)^{\frac{1}{p-\gamma}} \quad (4.66)$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|^{p} dx \leq (\text{meas}(\Omega))^{\frac{1}{p-\gamma}} (Ck)^{\frac{1}{p-\gamma}}, \quad (4.67)$$

for $k \geq 1$.

By using Fatou’s Lemma in (4.66) and (4.67) we get as $n$ goes to $+\infty$ that

$$\int_{\Omega} |T_k(u)|^{p} dx \leq (\text{meas}(\Omega))^{\frac{1}{(p-\gamma)}} (Ck)^{\frac{1}{p-\gamma}} \quad (4.68)$$
and
\[ \int_{\Omega} |\nabla T_k(u)| \, dx \leq (\text{meas}(\Omega))^{\frac{1}{p+1}} (Ck)^{\frac{1}{p-1}}, \quad (4.69) \]
for \( k \geq 1 \).

For every \( k \geq 1 \), let \( A_k := \{ x \in \partial \Omega : |T_k(u(x))| < k \} \) and \( B = \partial \Omega \setminus \bigcup_{k \geq 1} A_k \).

We have that
\[ \mu(B) = \frac{1}{k} \int_B |T_k(u)| \, dx \leq \frac{1}{k} \int_{\partial \Omega} |T_k(u)| \, dx \leq \frac{C_1}{k} \| T_k(u) \|_{W^{1,1}(\Omega)} \leq \frac{C_1}{k} \| T_k(u) \|_{L^1(\Omega)} + \frac{C_1}{k} \| \nabla T_k(u) \|_{L^1(\Omega)}. \]

According to (4.68) and (4.69), we deduce by letting \( k \to +\infty \) that \( \mu(B) = 0 \).

Let us define in \( \partial \Omega \) the function \( v \) by
\[ v(x) := T_k(u(x)) \text{ if } x \in A_k. \]

We take \( x \in \partial \Omega \setminus (A \cup B) \); then there exists \( k > 0 \) such that \( x \in A_k \) and we have
\[ u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))). \]

Since \( x \in A_k \), we have \( |T_k(u(x))| < k \) and so \( |T_k(u_n(x))| < k \), from which we deduce that \( |u_n(x)| < k \).

Therefore,
\[ u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to +\infty. \]

This means that \( u_n \) converges to \( v \) a.e. on \( \partial \Omega \). The proof of the proposition 4.12 is then complete.

To complete the proof of existence of entropy solution it remains to show that
\[ |u_n|^{p(x)-2} u_n \to |u|^{p(x)-2} u \text{ in } L^1(\partial \Omega). \quad (4.70) \]

For this, let us see that \( \left( |u_n|^{p(x)-2} u_n \right)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^1(\partial \Omega) \). Indeed: As \( u_n \) is a weak solution of (4.44), using \( \frac{1}{k} T_k(u_n - u_m) \) as a test function in (4.45), we get
\[
\int_{\Omega} \frac{1}{k} a(x, \nabla u_n) \nabla T_k(u_n - u_m) \, dx + \int_{\Omega} b(u_n) \frac{1}{k} T_k(u_n - u_m) \, dx
+ \int_{\partial \Omega} |u_n|^{p(x)-2} u_n \frac{1}{k} T_k(u_n - u_m) \, d\sigma = \int_{\Omega} f_n \frac{1}{k} T_k(u_n - u_m) \, dx.
\]

Similarly for \( u_m \), with \( \frac{1}{k} T_k(u_m - u_n) \) as test function, we have
\[
\int_{\Omega} \frac{1}{k} a(x, \nabla u_m) \nabla T_k(u_m - u_n) \, dx + \int_{\Omega} b(u_m) \frac{1}{k} T_k(u_m - u_n) \, dx
+ \int_{\partial \Omega} |u_m|^{p(x)-2} u_m \frac{1}{k} T_k(u_m - u_n) \, d\sigma = \int_{\Omega} f_m \frac{1}{k} T_k(u_m - u_n) \, dx.
\]
Adding the last two identities yields
\[
\int_{(|u_n| - |u_m|)\leq k} \frac{1}{k} (a(x, \nabla u_n) - a(x, \nabla u_m)) \cdot (\nabla u_n - \nabla u_m) \, dx + \\
\frac{1}{k} \int_{\Omega} (b(u_n) - b(u_m)) \, T_k(u_n - u_m) \, dx \\
+ \int_{\partial \Omega} \left( |u_n|^{p(x)-2} u_n - |u_m|^{p(x)-2} u_m \right) \frac{1}{k} T_k(u_n - u_m) \, d\sigma \leq \int_{\Omega} |f_n - f_m| \, dx.
\] (4.71)

Letting \( k \to 0 \) and as the first and the second term in the left-hand side of inequality (4.71) are nonnegative, we get
\[
\int_{\partial \Omega} \left| u_n \right|^{p(x)-2} u_n - \left| u_m \right|^{p(x)-2} u_m \, d\sigma \leq \int_{\Omega} |f_n - f_m| \, dx.
\] (4.72)

Now, since \( \{f_n\}_{n \in \mathbb{N}} \) is convergent in \( L^1(\Omega) \), by (4.72) \( \{|u_n|^{p(x)-2} u_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^1(\partial \Omega) \). As \( L^1(\partial \Omega) \) is a Banach space and \( s \mapsto |s|^{p(x)-2} s \) is continuous and is a maximal monotone graph in \( \mathbb{R} \), then (see [3])
\[
|u_n|^{p(x)-2} u_n \rightharpoonup |u|^{p(x)-2} u \text{ in } L^1(\partial \Omega).
\] (4.73)

We are now able to pass to the limit in the identity (4.53).

For the right-hand side and the third term in the left-hand side of (4.53), the convergence is obvious since \( f_n \) strongly converges to \( f \) in \( L^1(\Omega) \), \( |u_n|^{p(x)-2} u_n \) strongly converges to \( |u|^{p(x)-2} u \) in \( L^1(\partial \Omega) \), \( T_k(u_n - \varphi) \) converges weakly-* to \( T_k(u - \varphi) \) in \( L^{\infty}(\Omega) \) and a.e in \( \Omega \), and \( T_k(u_n - \varphi) \) converges weakly-* to \( T_k(u - \varphi) \) in \( L^{\infty}(\partial \Omega) \) and a.e in \( \partial \Omega \).

For the second term of (4.53), we have
\[
\int_{\Omega} b(u_n) T_k(u_n - \varphi) \, dx = \int_{\Omega} (b(u_n) - b(\varphi)) T_k(u_n - \varphi) \, dx \\
+ \int_{\Omega} b(\varphi) T_k(u_n - \varphi) \, dx.
\]

The quantity \( (b(u_n) - b(\varphi)) T_k(u_n - \varphi) \) is nonnegative and since for all \( s \in \mathbb{R}, \ s \mapsto b(s) \) is continuous, we get
\[
(b(u_n) - b(\varphi)) T_k(u_n - \varphi) \rightharpoonup (b(u) - b(\varphi)) T_k(u - \varphi) \text{ a.e. in } \Omega.
\]

Then, it follows by Fatou’s Lemma that
\[
\liminf_{n \to +\infty} \int_{\Omega} (b(u_n) - b(\varphi)) T_k(u_n - \varphi) \, dx \geq \int_{\Omega} (b(u) - b(\varphi)) T_k(u - \varphi) \, dx.
\] (4.74)

We have \( b(\varphi) \in L^1(\Omega) \).

Since \( T_k(u_n - \varphi) \) converges weakly-* to \( T_k(u - \varphi) \) in \( L^{\infty}(\Omega) \) and \( b(\varphi) \in L^1(\Omega) \), it follows that
\[
\lim_{n \to +\infty} \int_{\Omega} b(\varphi) T_k(u_n - \varphi) \, dx = \int_{\Omega} b(\varphi) T_k(u - \varphi) \, dx.
\] (4.75)
Next, we write the first term in (4.53) in the following form
\[
\int_{\{u_n - \varphi \leq k\}} a(x, \nabla u_n) \cdot \nabla u_n \, dx - \int_{\{u_n - \varphi \leq k\}} a(x, \nabla u_n) \cdot \nabla \varphi \, dx.
\] (4.76)

Set \( l = k + \|\varphi\|_{\infty} \). The second integral in (4.76) is equal to
\[
\int_{\{u_n - \varphi \leq k\}} a(x, \nabla T_l(u_n)) \cdot \nabla \varphi \, dx.
\]

Since \( a(x, \nabla T_l(u_n)) \) is uniformly bounded in \( \left( L^{p'(\cdot)}(\Omega) \right)^N \) (by (1.10) and (4.46)), by Proposition 4.12-
(iii), it converges weakly to \( a(x, \nabla T_l(u)) \) in \( \left( L^{p'(\cdot)}(\Omega) \right)^N \). Therefore,
\[
\lim_{n \to +\infty} \int_{\{u_n - \varphi \leq k\}} a(x, \nabla T_l(u_n)) \cdot \nabla \varphi \, dx = \int_{\{u - \varphi \leq k\}} a(x, \nabla T_l(u)) \cdot \nabla \varphi \, dx.
\] (4.77)

Moreover, \( a(x, T_l(u_n)) \cdot \nabla u_n \) is nonnegative and converges a.e. in \( \Omega \) to \( a(x, \nabla u) \cdot \nabla u \).
Thanks to Fatou’s Lemma, we obtain
\[
\liminf_{n \to +\infty} \int_{\{u_n - \varphi \leq k\}} a(x, \nabla u_n) \cdot \nabla u_n \, dx \geq \int_{\{u - \varphi \leq k\}} a(x, \nabla u) \cdot \nabla u \, dx.
\] (4.78)

By (4.74), (4.75), (4.77) and (4.78), we get
\[
\int_{\Omega} a(x, \nabla u) \cdot \nabla T_k(u - \varphi) \, dx + \int_{\Omega} b(u)T_k(u - \varphi) \, dx + \int_{\partial \Omega} |u|^{p(x)-2} u T_k(u - \varphi) d\sigma \leq \int_{\Omega} f T_k(u - \varphi) \, dx.
\]

We conclude that \( u \) is an entropy solution of (1.1). □

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**References**


