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Optimization and Flow Invariance via First and Second Order Tangent Cones

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Abstract

This is a survey paper on optimization problems via the technique of first and second order tangent cones to a nonempty subset of a Banach space X. Such a technique is also used in the study of the flow invariance of a closed set with respect to a second order differential equation (motion on a given orbit in a force field). Many of the known results in these areas are included here.

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1 Introduction

The goal of this survey paper is twofold:

1) to present some of the main results obtained by the authors via Pavel and Ursescu tangent cones, results that concern constrained optimization problems, flow-invariance problems and some of their applications,

2) to point out the unifying effect of the theory of tangent cones in the areas of differential equations and optimization.

We are dealing with the following scalar set constrained minimization problem

$$F(x) = \text{Local Minimum } F(z), \quad z \in D, \tag{P}$$

where X is a Banach space of norm $\|\cdot\|$, $x \in D \subseteq U$, and $F : U \subseteq X \to \mathbb{R}$ is a function of class C^p on the open set U, p positive integer, or a locally Lipschitz function near x.

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Special interest is paid in the optimization problem with equality constraints obtained for $D = D_G = \{x \in X,\ G(x) = 0\}$, where $G : X \to \mathbb{R}^s$ is of class C^2.

Also we are concerned with conditions for a closed subset $D \subset U$ of a Banach space X to be flow-invariant with respect to the second order autonomous differential equation

$$u''(t) = F(u(t)), t \geq 0,$$

where $F : U \to X$ is a locally Lipschitz mapping on an open subset U of X.

Throughout the paper, if a function F is p-times differentiable at x, then $F^{'}(x)$, $F^{''}(x)$, $F^{'''}(x)$, $F^{(p)}(x)$, $p \geq 4$ denote its first, second, third and p-th order derivatives at x and $F^{(p)}(x)[y]^p = F^{(p)}(x)\underbrace{(y)\cdots(y)}_{p \text{ times}}$.

The paper is organized as follows. In Section 2 that is based on [9], [11], [22], [27], [29], we recall the definitions of the first and second order tangent cones and some of their characterizations. In Section 3 we present the optimality conditions given by the authors in [11], [12], [19], [31], [32] via the tangent cones and analyze illustrative examples. We devote Section 4 to some of the main results in the literature concerning flow invariance problems and their applications in Flight Mechanics ([29], [22]), results that are expressed by means of the tangent cones as well.

2 First and Second Order Tangent Cones

The tangent cones are the main tools for formulating the results of this paper.

Definition 2.1. Let D be a nonempty subset of a Banach space X and let $x \in D$ be a given point.

i) (Ursescu, [34]) An element $v \in X$ is called a tangent vector to D at x if

$$\lim_{t \to 0+} \frac{1}{t} d(x + tv; D) = 0. \tag{2.1}$$

ii) (Pavel and Ursescu, [29]) An element $w \in X$ is called a second order tangent vector to D at $x \in D$, if there is $v \in X$, such that

$$\lim_{t \to 0+} \frac{1}{t^2} d(x + tv + \frac{t^2}{2} w; D) = 0, \tag{2.2}$$

where $d(x; D) = \inf\{||x - y||;\ y \in D\}$.

The sets of all first and second order tangent vectors to D at $x \in D$ are denoted by $T_x D$ and $T_x^2 D$, respectively.

The cone $T_x D$ is also known as the cone of attainable directions [15] or the adjacent cone [3].

For a given $w \in T_x^2 D$, a vector v as in (1.3) is said to be an associate of w, or associated with w, or a correspondent vector of w.

The definition of $T_x^2 D$ was suggested by a formula of Pavel from 1975 (see [27] with $f(x)$ in place of w).

In the theory of motion on a given orbit on D_G in a force field F on D_G in Flight Mechanics, the vectors v and w above are the initial velocity and the acceleration at x, respectively.

It is obvious that if x is an interior point of D, then $T_xD = T_x^2D = X$.

Proposition 2.2. *i) (Lemma 3.1, [29]) The fact that v belongs to T_xD is equivalent to the existence of a function $\gamma_1 : (0,\infty) \to X$ with $\gamma_1(t) \to 0$ as $t \to 0+$, and*

$$x + t(v + \gamma_1(t)) \in D, \ \forall t > 0. \tag{2.3}$$

ii) (Lemma 3.2, [29]) The fact that w belongs to T_x^2D with the corespondent vector $v \in X$, as in (1.2) is equivalent to the existence of a function $\gamma_2 : (0,\infty) \to X$ with $\gamma_2(t) \to 0$ as $t \to 0+$, and

$$x + tv + \frac{t^2}{2}(w + \gamma_2(t)) \in D, \ \forall t > 0. \tag{2.4}$$

It can easily be seen that $0 \in T_xD$ (take $\gamma_1 \equiv 0$), and $0 \in T_x^2D$ (take $\gamma_2 \equiv 0,\ v = 0$).

Proposition 2.3. *(Proposition 1.8 i), [22]) If $w \in T_x^2D$ then its associated vector v belongs to T_xD.*

Proposition 2.4. *i) (Proposition 1.2, [22]) T_xS is a closed cone in X.*
ii) (Proposition 1.8 ii), [22]) T_x^2D is a cone in X.

There are known the following characterizations of the first and second order tangent cones to the null-set of a mapping G, i.e., $D_G = \{y \in X; G(y) = 0,\ G : X \to \mathbb{R}^s\}$.

Theorem 2.5. *(Corollary 3.1, [29]) Let $G : X \to \mathbb{R}^s$ be continuous in a neighborhood of x and Fréchet differentiable at x. If $G'(x)$ is onto, then*

$$T_xD_G = KerG'(x). \tag{2.5}$$

Here $KerG'(x)$ denotes the null space of $G'(x)$, i.e., $KerG'(x) = \{v \in X;\ G'(x)(v) = 0\}$.

Remark 2.6. If $G'(x)$ is not onto, then in general the strict inclusion $T_xD_G \subset KerG'(x)$ holds.

Theorem 2.7. *(Corollary 3.2, [29]) Let X be a normed linear space, $G : U \to \mathbb{R}^s$ twice Fréchet differentiable at x, with $G(x) = 0$, G continuous near x, and $G'(x) : X \to \mathbb{R}^s$ onto.*
Then $w \in T_x^2D_G$ with $v \in T_xD_G$ if and only if

$$G'(x)(v) = 0, \textit{ and } G''(x)(v)(v) + G'(x)(w) = 0. \tag{2.6}$$

Definition 2.8. ([35] Let X and Y be two normed spaces. The mapping $G : X \to Y$ is said to be p-regular at the point x if, given any $h \in KerG^{(p)}(x) = \{h \in X, G^{(p)}(x_0)[h]^p = 0\}$, $h \neq 0$, we have $G^{(p)}(x)[h]^{p-1}X = Y$, $p \geq 2$.

In the above two theorems, Pavel and Ursescu characterized the second order tangent vectors to D_G at $x \in D_G$ when $G : X \to \mathbb{R}^s$ is twice Fréchet differentiable at x and $G'(x)$ is onto. In [9], Constantin described the second order tangent cone to D_G at $x \in D_G$ when $G'(x) = 0$ and G has some additional properties.

Theorem 2.9. *(Theorem 1.3, [9]) Assume that $G : X \to \mathbb{R}^s$ is three times Fréchet differentiable at $x \in D_G$, $G^{'}(x) = 0$, and G is 2-regular at x and continuous near x.*

Then $w \in T_x^2 D_G$ with associated vector $v \in T_x D_G$, $v \neq 0$ if and only if

$$G^{''}(x)(v)(v) = 0,\ v \neq 0,\ \textit{and} \tag{2.7}$$

$$G^{'''}(x)(v)(v)(v) + 3G^{''}(x)(v)(w) = 0. \tag{2.8}$$

Remark 2.10. $T_x^2 D$ was extended to n-order $T_x^n D$ tangent cones, $n > 2$, by Pavel et al. [31]. Also $T_x^2 D$ was extended to Banach manifolds by Motreanu and Pavel [22].

3 Optimization via Tangent Cones

We are dealing with the following constrained minimization problem

$$F(x) = \text{Local Minimum } F(z), \quad z \in D, \tag{P}$$

where X is a Banach space of norm $\|\cdot\|$, $x \in D \subseteq U$, and $F : U \subseteq X \to \mathbb{R}$ is a function of class C^p on the open set U, p positive integer, or a locally Lipschitz function near x.

This section provides some necessary conditions and some sufficient conditions for a point $x \in D$ to be a local minimum of F on D using the theory of tangent cones in Pavel and Ursescu sense.

Recall that a point $x \in D$ is said to be a local minimum of a function $F : U \to \mathbb{R}$ on $D \subseteq X$, X Banach space, if there exists $\delta > 0$ such that $F(z) \geq F(x)$, for all $z \in U \cap D$ satisfying $0 < \|z - x\| < \delta$. If the defining inequality is strict, then x is said to be a strict local minimum of F on D.

3.1 Second Order Optimality Conditions via First and Second Order Tangent Cones

In this subsection we present second order necessary and sufficient optimality conditions for a point to be a local minimum for problem (P) with a smooth objective function. The necessary conditions are formulated for an arbitrary constraint set and the sufficient conditions are formulated for a constraint D_G given by a functional G.

It is well-known the method Lagrange developed for approaching such functional constrained optimization problems.

Theorem 3.1. *(Lagrange Multipliers Method) Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $G = (G_1, \ldots, G_s) : U \to \mathbb{R}^s$ be of class C^1 on the open set $U \subseteq \mathbb{R}^n$. If F has an extremum on $D_G = \{z \in U; G_1(z) = 0, G_2(z) = 0, \ldots, G_s(z) = 0\}$, at $x \in D_G$, and $G_1'(x), G_2'(x), \ldots, G_s'(x)$ are linearly independent, then there exists a unique vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, called a Lagrange multiplier vector, such that $F'(x) = \lambda G'(x) = \lambda_1 G_1'(x) + \lambda_2 G_2'(x) + \ldots + \lambda_s G_s'(x)$.*

Suppose in addition that F and G are on class C^2 on U.

If x is a local minimum of F on D_G, then $[F''(x) - \lambda G''(x)][y]^2 \geq 0$, for all y such that $G'(x)(y) = 0$.

Theorem 3.2. *(Proposition 3.2.1, [6]) Assume that $F : U \subseteq \mathbb{R}^n \to \mathbb{R}$, and $G = (G_1, \dots, G_s) : U \to \mathbb{R}^s$ are of class C^2 on the open set U, and let $x \in D_G = \{z \in U; G_1(z) = 0, G_2(z) = 0, \dots, G_s(z) = 0\}$ and $\lambda \in \mathbb{R}^s$ satisfy $F'(x) = \lambda G'(x)$. Then*

If $[F''(x) - \lambda G''(x)][y]^2 > 0$, for all $y \neq 0$ such that $G'(x)(y) = 0$, then $\bar{x}$ is a strict local minimum of F on D_G.

For a given $w \in T_x^2 D$, we set

$$S_w = \{v \in T_x D; \lim_{t \downarrow 0} t^{-2} dist(x + tv + \frac{1}{2!} t^2 w; D) = 0\}. \tag{3.1}$$

In words, S_w is the set of all "associates" v of w.

Note that in general S_w is strictly included in $T_x D$. For example, if D is the unit sphere of a real Hilbert space H of inner product $< \cdot >$ and norm $\|\cdot\|$, i.e., $D = B_1 = \{x \in H; \|x\| = 1\}$, then according to [22], [29]

$$T_x B_1 = \{v \in H; < x, v >= 0\}, \tag{3.2}$$

$$T_x^2 B_1 = \{w \in H; \text{there is } v \in H; < x, v >= 0, \|v\|^2 + < x, w >= 0\}, \tag{3.3}$$

which implies that

$$S_w = \{v \in H; < x, v >= 0, \|v\|^2 + < x, w >= 0\} \subset T_x B_1. \tag{3.4}$$

Theorem 3.3. *(Necessary Conditions) (Theorem 2.1, [12]) Let D be a nonempty subset of the Banach space X, and let x be a local minimum of $F : X \to \mathbb{R}$ on D.*

If F is of class C^2 on X, $w \in T_x^2 D$, and $F'(x)(v) = 0$ for all $v \in S_w$, then

$$F''(x)(v)(v) + F'(x)(w) \geq 0. \tag{3.5}$$

Proof By definition, F is said to be twice Fréchet differentiable at x if

$$F(x + z) - F(x) = F'(x)(z) + \frac{1}{2} F''(x)(z)(z) + \|z\|^2 r(z) \tag{3.6}$$

with some $r(z) \to 0$ as $z \to 0$.

Replacing $z = tv + \frac{1}{2} t^2 (w + r(t))$ into (3.6), we get

$0 \leq F(x + tv + \frac{1}{2} t^2 (w + r(t))) - F(x)$

$$= F'(x)(tv + \frac{1}{2} t^2 (w + r(t)) + \frac{1}{2} F''(x)(tv + \frac{1}{2} t^2 (w + r(t)))(tv + \frac{1}{2} t^2 (w + r(t))) + t^2 \alpha(t),$$

with some $\alpha(t) \to 0$, as $t \to 0$.

If $F'(x)(v) = 0$, dividing by t^2 and then letting $t \to 0$, one obtains (3.5), which completes the proof.

Clearly, the above theorem yields the following corollary.

Corollary 3.4. *Suppose that $F : X \to \mathbb{R}$ is a functional of class C^2 on a Banach space X and x is a local minimum of F on $D = D_G$, $G : X \to \mathbb{R}^s$.*

i) (Corollary 2.1, [12]) If $F'(x)(v) = 0$ for all $v \in T_x D_G$, then

$$F''(x)(v)(v) + F'(x)(w) \geq 0, \tag{3.7}$$

for all v and w satisfying

$$G(x + tv + \frac{1}{2!}t^2(w + r(t))) = 0, \text{ for some } r(t) \to 0, \text{as } t \to 0. \tag{3.8}$$

ii) (Theorem 3.1, [31]) Suppose in addition that G is of class C^2 and the Fréchet derivative $G'(x)$ of G at x is onto from X into $\mathbb{R}^s$. Then necessarily

$$F'(x)(v) = 0, \;\; F''(x)(v)(v) + F'(x)(w) \geq 0, \tag{3.9}$$

for all (v, w) satisfying

$$G'(x)(v) = 0, \;\; G''(x)(v)(v) + G'(x)(w) = 0. \tag{3.10}$$

Remark 3.5. The second order necessary optimality conditions of Corollary 3.4 ii) (Theorem 3.1, [31]) have been extended to higher order necessary conditions for sufficiently often Fréchet differentiable functions F and G by Constantin in Corollary 2.1, [9].

Theorem 3.6. *(Sufficient Conditions) (Theorem 2.2, [12]) Suppose that $G : X \to \mathbb{R}^s$ is of class C^2 on the finite dimensional normed space X, and the Fréchet derivative $G'(x)$ of G at x is onto from X into $\mathbb{R}^s$. Let $x \in D$ satisfy:*

$$F'(x)(v) = 0, \;\; F''(x)(v)(v) + F'(x)(w) > 0, \tag{3.11}$$

for all (v, w) different from zero, such that

$$G'(x)(v) = 0, \;\; G''(x)(v)(v) + G'(x)(w) = 0. \tag{3.12}$$

Then x is a strict local minimum of F on D_G.

Proof. It is easy to see that $F'(x)(v) = 0$, for all v in the null space of $G'(x)$, i.e., for all v with $G'(x)(v) = 0$, is equivalent to

$$F'(x)(w) = < \lambda, G'(x)(w) >, \tag{3.13}$$

for all $w \in X$, where $\lambda \in \mathbb{R}^s$ and $<,>$ is the inner product of $\mathbb{R}^s$.

This implies (by the elimination of w), that conditions (3.11) and (3.12) are equivalent to

$$F''(x)(v)(v) - \lambda G''(x)(v)(v) > 0, \tag{3.14}$$

for all $v \neq 0$, with $G'(x)(v) = 0$, i.e., the Hessian $F''(x) - \lambda G''(x)$ is strictly positive definite on the tangent cone of D_G at x. Indeed, in view of the hypothesis that $w \to G'(x)(w)$ is onto, for each v with $G'(x)(v) = 0$ there is w such that

$$G'(x)(w) = -G''(x)(v)(v). \tag{3.15}$$

By Theorem 3.2 (Proposition 3.2.1, [6]), relation (3.14) implies that x is a strict local minimum of F on D_G. The proof is complete.

Example 3.7. (Example 3, [13]) Consider the function $F(x_1,x_2) = x_2^6 + x_1^3 + 2x_1^2 - x_2^2 + 4x_1 + 4x_2$, subject to the constraint $G(x_1,x_2) = x_1^5 + x_2^4 + x_1 + x_2 = 0$, $F,G : \mathbb{R}^2 \to \mathbb{R}$.

It can be checked that $(0,0)$ is a constrained critical point with the Lagrange multiplier $\lambda = 4$.

The origin is a strict local minimum because the sufficient conditions of Theorem 3.2 are satisfied as $[F''(0,0) - 4G''(0,0)][y]^2 = 4y_1^2 - 2y_2^2 = 2y_2^2 > 0$, for all $y = (y_1,y_2) \neq 0$ such that $G'(0,0)(y) = y_1 + y_2 = 0$ as $y \neq 0$ implies $y_2 \neq 0$. Here $G''(0,0) = 0$.

We can draw the same conclusion by verifying the sufficient conditions given by (3.11) and (3.12). Indeed, $F^{'}(\bar{x})(w) + F^{''}(\bar{x})[v]^2 = 4v_1^2 - 2v_2^2 + 4(w_1 + w_1) = 2v_1^2 > 0$, for all $v \neq 0$ as $v_1 + v_2 = 0$ and $w_1 + w_2 = 0$.

Here we used the fact that $G'(0,0) = (1,1)$ is onto and thus the tangent cones can be characterized by means of Theorem 2.5 (Corollary 3.1, [29]) and of Theorem 2.7 (Corollary 3.2, [29]):

$T_{(0,0)}D_G = \{v = (v_1,v_2) \in \mathbb{R}^2; v_1 + v_2 = 0\}$,
$T^2_{(0,0)}D_G = \{w = (w_1,w_2) \in \mathbb{R}^2; w_1 + w_2 = 0\}$.

3.2 Second Order Necessary Conditions via First and Second Order Tangent Cones

In this subsection we present some second order necessary optimality conditions for a point to be a local minimum for problem (P) with a locally Lipschitz objective function. The main result of the subsection is due to Constantin [11] and extends the second order necessary conditions of Corollary 2.4 ii) (Theorem 3.1, [31]) given for smooth optimization problems to nonsmooth problems.

We make use of the concepts of Clarke's generalized derivative [8] and Páles and Zeidan's second order directional derivative [24].

Definition 3.8. If F is a real-valued locally Lipschitz mapping on an open set U of a Banach space X, and $x \in U$, then

i) ([8]) Clarke's generalized derivative of F at x is defined by

$$F^{\circ}(x;v) = \limsup_{(z,t)\to(x,0+)} \frac{F(z+tv) - F(z)}{t}, \; v \in X.$$

ii) ([24]) Páles and Zeidan's second order directional derivative of F at x is defined by

$$F^{\circ\circ}(x;v) = \limsup_{t\to 0+} 2\frac{F(x+tv) - F(x) - tF^{\circ}(x;v)}{t^2}, \; v \in X.$$

If F is Fréchet differentiable at x and locally Lipschitz near x then $F^{\circ}(x;v) = F'(x)(v)$. Furthermore, if F is two times differentiable at x and locally Lipschitz near x, then $F^{\circ\circ}(x;v) = F''(x)(v)(v)$, for all $v \in X$.

First order necessary conditions for problem (P) with locally Lipschitz data can be found in Ye [36].

Theorem 3.9. *(Lemma 3.1, [36]) Let $F : X \to \mathbb{R}$ be a locally Lipschitz functional on an open set U of a Banach space X. If x is a local minimum point of F on $D \subset U$, $D \neq \emptyset$, then*

$$F^{\diamondsuit}(x; v) \geq 0, \; for\; any\; v \in \Gamma^*(x, D).$$

Here $\Gamma^*(x, D)$ is Bouligand tangent cone or contingent cone to D at x in the closure of D ([7], [3]),

$$\Gamma^*(x, D) = \{v \in X;\ \liminf_{t\to 0+} \frac{1}{t} d(x + tv; D) = 0\}.$$

Also, according to [33], $F^{\diamondsuit}(x; v)$ is Michel-Penot directional derivative of F at x in the direction v defined by

$$F^{\diamondsuit}(x; v) = \sup_{w\in X} \limsup_{t\to 0+} \frac{F(x + t(v + w)) - F(x + tw)}{t},\ v \in X.$$

It follows directly from the definitions that $T_x D \subseteq \Gamma^*(x, D)$ and $F^{\circ}(x; v) \geq F^{\diamondsuit}(x; v)$ for any $v \in X$.

In particular, this implies that $F^{\circ}(x; v) \geq 0$, for all $v \in T_x D$.

The main result of this subsection gives some second order necessary condition derived by Constantin in [11] for a point x to be a local minimum for problem (P) with locally Lipschitz data.

Theorem 3.10. *(Theorem 3.1, [11]) Let $F : X \to \mathbb{R}$ be a locally Lipschitz functional on an open set U of a Banach space X.*

If x is a local minimum point of F on $D \subset U$, $D \neq \emptyset$, then

$$F^{\circ}(x; w) + F^{\circ\circ}(x; v) \geq 0, \tag{3.16}$$

for any $w \in T_x^2 D$ with correspondent vector $v \in T_x D$ such that $F^{\circ}(x; v) = 0$.

Proof. By Proposition 2.2 ii) (Lemma 3.2, [29]), for $w \in T_x^2 D$, there exists $\gamma_2(t) \to 0$ as $t \to 0+$,

$$x + tv + \frac{t^2}{2}(w + \gamma_2(t)) \in D,\ t > 0.$$

We use the following identity

$$F(x + tv + \frac{t^2}{2}(w + \gamma_2(t))) - F(x) = tF^{\circ}(x; v) + \frac{t^2}{2}[\frac{2}{t^2}(F(x + tv + \frac{t^2}{2}(w + \gamma_2(t))) - F(x + tv))]$$
$$+ \frac{t^2}{2}[\frac{2}{t^2}(F(x + tv) - F(x) - tF^{\circ}(x; v))].$$

If $F^{\circ}(x; v) = 0$, we divide both sides by $t^2/2$, and after passing to upper limit as $t \to 0+$, we obtain $F^{\circ}(x; w) + F^{\circ\circ}(x; v) \geq 0$, because the left-hand side of the above equality is nonnegative for t close enough to 0.

We give next an example in which Ye's first order necessary condition of extremum holds at $x = (0, 0)$ but our second order necessary condition classifies $(0, 0)$ as being nonoptimal. By providing additional information about the candidate $(0, 0)$ for a local minimum point, Theorem 3.10 (Theorem 3.1, [11]) helped us make explicit a situation left unclear by Ye's result.

Example 3.11. (Example 4.1, [11]) Let us consider the function $F(x_1,x_2) = |x_1| - x_2 - x_1^2 + x_2^2$, subject to $G(x_1,x_2) = x_1^3 + x_1x_2 + x_2^2 = 0,\ F, G : \mathbb{R}^2 \to \mathbb{R}$.

We notice that the point $(x_1,x_2) = (0,0)$ verifies the constraint.

The objective function F is locally Lipschitz on $\mathbb{R}^2$ but it is not differentiable at $(0,0)$.

The first order partial derivatives of G at $(0,0)$ are identically zero.

Since $G''(0,0)(v)$ is onto for any v with $G''(0,0)(v)(v) = 0,\ v \neq 0$, using the characterization of the second order tangent cones provided by Theorem 2.9 (Theorem 1.3, [9]), we obtain that $w \in T^2_{(0,0)}D_G$ with associated vector $v \in T_{(0,0)}D_G,\ v \neq 0$ if and only if $v_1v_2 + v_2^2 = 0$, $(v_1,v_2) \neq (0,0)$, and $2v_1^3 + v_1w_2 + v_2w_1 + 2v_2w_2 = 0$. If $w \in T^2_{(0,0)}D_G$ with correspondent vector $v = 0$ then $w_1w_2 + w_2^2 = 0$.

Next we find the explicit form of $F^\circ((0,0);v)$.

Using the inequality $|x_1 + tv_1| - |x_1| \le t|v_1|$ we obtain

$$F^\circ((0,0);v) = \limsup_{(x,t)\to((0,0),0+)} \frac{F(x+tv) - F(x)}{t} \le |v_1| - v_2.$$

Considering the sequence $(x_1^n, x_2^n) = (0,0) \to (0,0)$, we get

$$F^\circ((0,0);v) \ge \limsup_{t\to 0+} \frac{|tv_1| - tv_2 - t^2v_1^2 + t^2v_2^2}{t} = |v_1| - v_2.$$

Thus $F^\circ((0,0);v) = |v_1| - v_2,\ \forall\, v \in \mathbb{R}^2$.

In this example Ye's first order necessary condition of Theorem 3.9 (Lemma 3.1, [36])

$$F^\diamond((0,0);v) \ge 0,\ \forall\, v \in \Gamma^*((0,0);D_G)$$

is verified at the point $(0,0)$ as

$$F^\diamond((0,0);v) \ge \limsup_{t\to 0} \frac{F(tv) - F(0)}{t} = |v_1| - v_2 \ge 0,\ \forall\, v \in \Gamma^*((0,0);D_G),$$

because $\Gamma^*((0,0);D_G) \subseteq \left\{(v_1,v_2) \in \mathbb{R}^2,\ v_1v_2 + v_2^2 = 0\right\}$, and for all $v \in \Gamma^*((0,0);D_G)$ we have either $v_2 = 0$ and $F^\diamond((0,0);v) \ge |v_1| \ge 0$, or $v_1 + v_2 = 0$ and thus $F^\diamond((0,0);v) \ge |v_1| + v_1 \ge 0$. Hence $(0,0)$ is a candidate for being optimal.

We apply next Theorem 3.10 (Theorem 3.1, [11]).

First we find

$$\begin{aligned} F^{\circ\circ}((0,0);v) &= \limsup_{t\to 0+} 2\frac{F(tv) - F(0,0) - tF^\circ((0,0);v)}{t^2} \\ &= \limsup_{t\to 0+} 2\frac{|tv_1| - tv_2 - t^2v_1^2 + t^2v_2^2 - t(|v_1| - v_2)}{t^2} = 2(-v_1^2 + v_2^2),\ \forall\, v \in \mathbb{R}^2. \end{aligned}$$

For any $w \in T^2_{(0,0)}D_G$ with associated vector $v \in T_{(0,0)}D_G$, such that $F^\circ((0,0);v) = 0$, i.e. $v_2 = |v_1|$, we will check the sign of the expression

$$F^{\circ\circ}((0,0);v) + F^\circ((0,0);w) = -2v_1^2 + 2v_2^2 + |w_1| - w_2 = |w_1| - w_2.$$

We notice that in the case $v_1 + v_2 = 0,\ v \neq 0$ we get $w_2 = 2v_1^2 - w_1$, and

$$F^{\circ\circ}((0,0);v) + F^\circ((0,0);w) = |w_1| + w_1 - 2v_1^2,$$

which is not necessarily nonnegative since $v_1 \neq 0$. Therefore our second order necessary condition is violated and we can conclude that $(0,0)$ is not a local minimum point of F on D_G.

3.3 Higher Order Sufficient Optimality Conditions via First Order Tangent Cones

In this subsection we present some higher order sufficient optimality conditions for a sufficiently often Fréchet differentiable function F defined on a finite dimensional normed space X and subject to a convex set constraint.

Theorem 3.12. *(Theorem 3.1, [32]) Let D be a nonempty convex subset of a finite dimensional normed space X, $x \in D$ and $F : D \to \mathbb{R}$ of class C^2 such that:*
i) $F'(x)(v) \geq 0$, for all $v \in T_xD$,
ii) $F''(x)(v)(v) > 0$, for all $v \in T_xD$, with $v \neq 0$.
Then x is a strict local minimum of F on D.
Precisely, there are two positive numbers a and b such that:

$$F(z) - F(x) \geq a\|z - x\|^2, \text{ for all } z \in D, \|z - x\| < b.$$

Example 3.13. (Example 3.1, [32])
Let D be the triangle of vertices (0,0), (2,0), (2,1) in $\mathbb{R}^2$ and
$F(x_1,x_2) = \frac{1}{3}x_1^3 + x_1^2 + x_1 - x_2^2 + x_2$ with $x = (x_1,x_2)$.
Clearly $F'(x) = (x_1^2 + 2x_1 + 1, -2x_2 + 1)$, $F'(0,0) = (1,1)$ and the Hessian matrix is $F''(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$.
It is also easy to check that $T_{(0,0)}D = \{v = (v_1,v_2), v_1 \geq 2v_2 \geq 0\}$, so $v \neq 0$ implies $v_1 \neq 0$.
Therefore $F'(0,0)(v) = v_1 + v_2 \geq 0$, for all $v \in T_{(0,0)}D$.
We have $F''(0,0)(v)(v) = 2(v_1^2 - v_2^2) > 0$ for all $v \in T_{(0,0)}D$, $v \neq 0$.
According to Theorem 3.12, the point $x = (0,0)$ is a strict local minimum of F on D.

The above theorem has been generalized as follows. Note that the case $p = 4$ has been considered in Theorem 4.1, [32].

Theorem 3.14. *(Theorem 4.2, [19]) Let D be a nonempty convex subset of a finite dimensional normed space X, $x \in D$ and $F : D \to \mathbb{R}$ of class C^p, $p \geq 2$, such that:*
i) $F^{(i)}(x)(v) \geq 0$, for all $i = 1,\dots,p-1$ and for all $v \in T_xD$,
ii) $F^{(p)}(x)(v)^p > 0$, for all $v \in T_xD$, $v \neq 0$.
Then x is a strict local minimum of F on D.
Precisely, there are two positive numbers a and b such that:

$$F(z) - F(x) \geq a\|z - x\|^p, \text{ for all } z \in D, \|z - x\| < b.$$

Proof. Suppose by contradiction that the conclusion of this theorem is not true, i.e., that there is a sequence $z_i \in D$ with $z_i \to x$ such that

$$F(z_i) - F(x) < \frac{1}{i}\|z_p - x\|^p, \text{ for all integers } i \geq 1. \tag{3.17}$$

Set $d_i = \|z_i - x\|$ and $z_i - x = d_i v_i$ with $v_i = \dfrac{z_i - x}{\|z_i - x\|}$.

Since D is convex, $z_i - x = d_i v_i \in T_x D$, so $v_i \in T_x D$.

As $\|v_i\| = 1$, we may assume that v_i is convergent to an element v_0, so $v_0 \in T_x D$ as $T_x D$ is closed. Clearly $\|v_0\| = 1$. In view of the Taylor formula, (3.17) yields

$$\frac{1}{i}\|z_i - x\|^p > F(z_i) - F(x) = \sum_{k=1}^{p-1} d_i^k F^{(k)}(x)(v_i)^k + \frac{1}{p!} d_i^p F^{(p)}(x)(v_i)^p + d_i^p r_i,$$

with some $r_i \to 0$, for $i \to \infty$.

Then

$$\frac{1}{i}\|z_i - x\|^p > \frac{1}{p!} d_i^p F^{(p)}(x)(v_i)^p + d_i^p r_i,$$

with some $r_i \to 0$, for $i \to \infty$.

Dividing by d_i^p and then letting $r_i \to 0$, we get $0 \geq F^{(p)}(x)(v_0)(v_0)$, which is in conflict with the second hypothesis ii). This completes the proof.

In the proof of the above result we used the proposition below.

Proposition 3.15. *(Proposition 1.1, [12]) If D is convex, then for any $x \in D$,*

$$\{v = z - x, z \in D\} \subseteq T_x D.$$

Example 3.16. (Example 4.1, [32]) Let D be the triangle of vertices (0,0), (2,0), (2,1) in $\mathbb{R}^2$ and $F(x_1, x_2) = x_1^4 + x_1 - x_2^4 + x_2$ with $x = (x_1, x_2)$.

We have for $x = (0,0)$ that $F'(0,0) = (1,1)$, $T_{(0,0)}D = \{v = (v_1, v_2), v_1 \geq 2v_2 \geq 0\}$, so $v \neq 0$ implies $v_1 \neq 0$, $F'(0,0)(v) = v_1 + v_2 \geq 0$ for all $v \in T_{(0,0)}D$, $F''(0,0) = F'''(0,0) = 0$, and $F^{(4)}(0,0)(v)^4 = 4!(v_1^4 - v_2^4) > 0$ for all $v \in T_{(0,0)}D$, $v \neq 0$.

According to Theorem 3.14, the point $(0,0)$ is a strict local minimum of F on D.

4 Flow Invariance via Tangent Cones

In this paper we provide conditions for a closed subset $D \subset U$ of a Banach space X to be flow-invariant with respect to the second order autonomous differential equation

$$u''(t) = F(u(t)), t \geq 0, \tag{4.1}$$

where $F : U \to X$ is a locally Lipschitz mapping on an open subset U of a Banach space X.

The invariant sets for the first order differential equations were studied by Brézis [4], Crandall [14], Martin [20], Pavel and Iacob [26] and many other authors.

In [29], Pavel and Ursescu treated the problem of flow-invariance of a set with respect to the above second order differential equation using the theory of tangent cones.

Theorem 4.1. *(Definition 1.1, [22]) The nonempty set $D \subset U$ is said to be (right-hand) flow-invariant with respect to the second order differential equation (4.1) if the solution $u : [0,T) \to X$ to the Cauchy problem (4.1) determined by the initial conditions*

$$u(0) = x, u'(0) = v, \tag{4.2}$$

with $x \in D$, $v \in T_x D$, $F(x) \in T_x^2 D$ having correspondent vector v, satisfies

$$u(t) \in D, \forall\, t \in [0,T). \tag{4.3}$$

The constraints imposed to (x,v) are necessary conditions to have the invariance property (4.3).

In [29], Pavel and Ursescu introduced the following set

$$M_D = \{(x,v) \in D\times X : v_1 \in T_x D,\ F(x) \in T_x^2 D \text{ whose associated vector is } v\}. \tag{4.4}$$

The choice in (4.2) for the initial conditions was expresses in [22] by means of (4.4) as follows

$$(u(0),u'(0)) = (x,v) \in M_D.$$

This was justified by the following result of Pavel and Ursescu [29].

Proposition 4.2. *(Theorem 2.2 i), [29]) If $u : [0,T) \to X$ is a solution of (4.1) which satisfies the invariance property (4.3), then one has*

$$(u(t),u'(t)) \in M_D, \forall\, t \in [0,T). \tag{4.5}$$

Pavel and Ursescu ([29], [22]) reduced the problem of invariant sets for (4.1) to a similar problem for a first order differential equation, fact that allowed them to utilize a theorem proved by Nagumo [23] and, independently, by Brézis (Theorem 1, [4]), in order to obtain the following characterization of flow-invariant sets $D \subset U$ with respect to the second order differential equation (4.1).

Theorem 4.3. *(Theorem 2.4, [29]) Assume that M_D is a nonempty closed subset of $U\times X$, for a closed subset D of U. Then $D \subset U$ is a flow-invariant set with respect with the second order differential equation (4.1) if and only if $(v,F(x)) \in X^2$ is a tangent vector to M_D for all $(x,v) \in M_D$, i.e.,*

$$\lim_{t\downarrow 0} t^{-1} d((x,v)+t(v,F(x)); M_D) = 0.$$

Pavel and Ursescu gave a description of the sets $D_G = \{x \in U; G(x) = 0\}$ that are flow-invariant with respect to the second order differential equation $u''(t) = F(u(t))$, $t \geq 0$.

Theorem 4.4. *(Theorem 2.6, [29]) Assume that $G : U \to \mathbb{R}^s$, $s \geq 1$, is two times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^s$ is onto for each $x \in D_G$. Then M_{D_G} is given by*

$$M_{D_G} = \{(x,v) \in U\times X : G(x) = 0, G'(x)(v) = 0, G''(x)(v)(v) + G'(x)(F(x)) = 0\}. \tag{4.6}$$

Suppose further that G is three times Fréchet differentiable on U, the function $h : U \to \mathbb{R}^s$ given by

$$h(x) = G'(x)(F(x)), \forall\, x \in U,$$

is Fréchet differentiable, M_{D_G} is nonempty and the mapping
$(G'(x)(\cdot), G''(x)(v)(\cdot)) : X \to \mathbb{R}^s\times\mathbb{R}^s$ is onto for every $(x,v) \in M_{D_G}$.

Then D_G is flow-invariant with respect to the differential equation $u''(t) = F(u(t)), t \geq 0$ if and only if

$$G'''(x)(v)(v)(v) + 2G''(x)(v)(F(x)) + h'(x)(v) = 0,\ \forall\,(x,v) \in M_{D_G}. \tag{4.7}$$

Proof. Formula (4.6) follows directly from Pavel and Ursescu's description of the second order tangent cone, Theorem 2.7 (Corollary 3.2, [29]).

To prove the second part, we notice that, due to (4.6), the set M_{D_G} can be rewritten as

$$M_{D_G} = g^{-1}(0),$$

where $g : U \times X \to \mathbb{R}^{3s}$ is defined by

$$g(x,v) = (G(x), G'(x)(v), G''(x)(v)(v) + G'(x)(F(x)),$$

for all $(x,v) \in U \times X$.

It can be easily seen that under the hypothesis of the theorem, g is Fréchet differentiable and its Fréchet derivative is determined by the relation

$$g'(x,v)(u,y) = (G'(x)(u), G''(x)(u)(v) + G'(x)(y),$$

$$G'''(x)(u)(v)(v) + 2G''(x)(v)(y) + h'(x)(u)),\ \forall\,(x,v) \in U \times X,\ \forall\,(u,y) \in X \times X.$$

We now show that $g'(x,v) : X \times X \to \mathbb{R}^{3s}$ is onto for each $(x,v) \in M_{D_G}$, i.e., the equation $g'(x,v)(u,y) = (z_1, z_2, z_3)$ has a solution $(u,y) \in X \times X$ for any $(z_1,z_2,z_3) \in \mathbb{R}^{3s}$. Since $G'(x)$ is onto, there is $u \in X$ such that $G'(x)(u) = z_1$. Then the element $y \in X$ can be obtained using the fact that the mapping $(G'(x)(\cdot), G''(x)(v)(\cdot)) : X \to \mathbb{R}^s \times \mathbb{R}^s$ is onto.

Thus, $T_{(x,v)} M_{D_G} = g'(x,v)^{-1}(0)$.

Finally, Theorem 4.3 (Theorem 1.10, [22]) completes the proof.

Note that if $G : U \to \mathbb{R}^s$ is continuous, then $D_G = \{x \in U; G(x) = 0\}$ is closed in U.

Remark 4.5. The above result has been generalized by Constantin (Theorems 3 and 4, [10]) for the n-th order autonomous differential equation $u^{(n)}(t) = F(u(t))$, $t \geq 0$, $n \geq 3$.

Remark 4.6. (Remark 5.1, [29]) Recall that a function $F : U \to X$ can be regarded as a field of force on U, in the sense that to each vector position $x \in U$ is associated the vector force $F(x) \in X$.

The notion

$$D_G \textit{ is a flow-invariant set for the equation } u''(t) = F(u(t)),\ t \geq 0 \tag{4.8}$$

can be restated in terms of Flight Mechanics as follows

$$\textit{A mass particle projected from a point } x \in D_G \textit{ with velocity } v \in X \textit{ such that } (x,v) \in M_{D_G}$$

$$((\textit{given by } (4.6)),\ \textit{describes (under the action of the force field } F) \textit{ an orbit which lies in } D_G. \tag{4.9}$$

Under the hypothesis of Theorem 4.4 upon G, (4.9) happens if and only if (4.7) holds.

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