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AP_r–Almost Periodic Solutions to the Equation $\dot{x}(t) = Ax(t) + (k * x)(t) + f(t)$

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Abstract

This note is dedicated to the existence of almost periodic solutions of a certain class of functional equations, of the form (1) in the text, in spaces like $AP_r(R,C^n)$, $1 \le r \le 2$. Frequency domain conditions are involved in this study.

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This note is aimed at finding sufficient conditions, for the equation in the title, to possess AP_r-almost periodic solutions, for $f \in AP_r(R,C^n)$. The significance of the data involved in the functional differential equation

$$\dot{x}(t) = Ax(t) + (k * x)(t) + f(t)\ ,\ t \in R, \tag{1}$$

is the following:

(a) The vector valued functions $x, f : R \longrightarrow C^n$, i.e., the unknown element and the given term in (1), will belong to the space of almost periodic functions $AP_r(R,C^n)$, $1 \le r \le 2$, this space being defined and investigated in Corduneanu [1].

(b) $A \in \mathcal{L}(C^n,C^n)$, for given $n \in N$.

(c) $k * x$ is the convolution product (generalized), as defined in Corduneanu [1], for $k \in L^1(R,\mathcal{L}(C^n,C^n))$, and $x \in AP_r(R,C^n)$, by the formula

$$(k * x)(t) \simeq \sum_{j=1}^{\infty} \left(\int_R k(s)\, e^{-i\lambda_j s}\, ds \right) x_j\, e^{i\lambda_j t}, \tag{2}$$

where

$$x(t) \simeq \sum_{j=1}^{\infty} x_j\, e^{i\lambda_j t}\ ,\ t \in R. \tag{3}$$

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(d) The Fourier series of the given term $f(t)$ in (1) is

$$f(t) \simeq \sum_{j=1}^{\infty} f_j e^{i\lambda_j t} \ , \ t \in R. \tag{4}$$

Remark 1. As usual in the theory of almost periodic functions, in various senses, one always assumes that λ_j, $j \in N$, are real numbers, while f_j, $j \in N$, are such that $f_j \in C^n$, $j \in N$. The same remark is valid for $x(t)$, which is sought in the space $AP_r(R, C^n)$, $1 \leq r \leq 2$.

Remark 2. It has been shown in Corduneanu [1] that each $AP_r(R, C^n)$, $1 \leq r \leq 2$, is an invariant space for the convolution operator defined by formula (2), in which $\int_R k(s)e^{-i\lambda_j s}ds$, $j \in N$, represents the values of the Fourier transform of $k \in L^1$,

$$\tilde{k}(s) = \int_R k(t)e^{-its}dt \ , \ s \in R \tag{5}$$

for $s = \lambda_j$, $j \in N$.

Remark 3. In general, the Fourier series attached to an element $f \in AP_r(R, C^n)$, $1 \leq r \leq 2$, does not converge, in any usual sense, to that element. It is important to notice that it always converges in $AP_r(R, C^n)$!

The elements of $AP_r(R, C^n)$ are characterized by the condition

$$\sum_{j=1}^{\infty} |f_j|^r < +\infty. \tag{6}$$

For instance, for $r = 1$, one has absolute (and uniform) convergence, and in (2), (3) or (4), the sign $\simeq$ can be replaced by $=$. When $r = 2$, we deal with the space $B_2 = AP_2(R, C^n)$ of Besicovitch almost periodic functions. We notice the inclusion $AP_r \subset AP_2 = B_2$, for $r < 2$. In other words, all almost periodic function spaces $AP_r(R, C^n)$, with $r \in [1, 2]$, belong to the Besicovitch space B_2. These spaces have different norms than B_2-norm.

Remark 4. In Corduneanu [1], and Corduneanu, Mahdavi, and Li [2], it has been shown that in case $k \equiv 0$ on R, which implies that equation (1) reduces to an ordinary differential equation/system, the condition of (unique) solvability is

$$det[i\omega I - A] \neq 0 \ , \ \omega \in R. \tag{7}$$

Also, in case $A = O$, the condition for (unique) solvability is

$$det[i\omega I - \tilde{k}(\omega)] \neq 0 \ , \ \omega \in R. \tag{8}$$

The natural question, arising in regard to the (more complex) functional differential equation (1), is what kind of frequency type condition one must impose, in order to obtain solvability for (1)?

We shall provide an answer to this question, assuming a certain condition on the distribution of the frequencies occurring in the equation (1).

Theorem 1. *Let us consider the functional differential equation (1), under the following specified conditions/hypotheses:*

1. $A \in \mathcal{L}(C^n, C^n)$*, and (7) holds true;*
2. $k \in L^1(R, \mathcal{L}(C^n, C^n))$*;*
3. *The spectral/frequency condition*
$$|\lambda_j| \longrightarrow \infty \ , \ as \ \ j \longrightarrow \infty, \tag{9}$$
is verified;
4. $\tilde{k}$ *is such that*
$$det[i\omega I - A - \tilde{k}(\omega)] \neq 0 \ , \ \omega \in R. \tag{10}$$

Then, for each $f \in AP_r(R, C^n)$*,* $1 \leq r \leq 2$*, there exists a unique solution* $x(t) \in AP_r(R, C^n)$*, to equation (1).*

Proof. Let us substitute in the equation (1) the Fourier series for f and x, then identifying and equating the coefficients of the same exponentials $e^{i\lambda_j t}$, $j \in N$. We obtain,

$$\sum_{j=1}^{\infty} i\lambda_j x_j e^{i\lambda_j t} \simeq \sum_{j=1}^{\infty} \left[A + \left(\int_R k(s) e^{-i\lambda_j s} ds\right)\right] x_j e^{i\lambda_j t} + \sum_{j=1}^{\infty} f_j e^{i\lambda_j t}. \tag{11}$$

In order for (11) to take place, one must have

$$i\lambda_j x_j = \left[A + \left(\int_R k(s) e^{-i\lambda_j s} ds\right)\right] x_j + f_j \ , \ j \in N. \tag{12}$$

This infinite system in x_j, $j \in N$, has unique solution if and only if

$$i\lambda_j \neq A + \int_R k(s) e^{-i\lambda_j s} ds \ , \ j \in N. \tag{13}$$

Obviously, (13) is implied by the stronger condition (10). We shall keep it in the stronger form, in order to assume the validity of the approach for every $f \in AP_r(R, C^n)$. This happens because the set $\{\lambda_j;\ j \in N\}$ of Fourier exponents can change with the function f. Therefore, a λ_j could be any real number.

Since condition (10) assures the existence of all $x_j{}'s$, according to (13), it remains to prove the convergence of the series obtained for $x(t)$, in AP_r,

$$\sum_{j=1}^{\infty} |x_j|^r < \infty, \tag{14}$$

relying on our hypotheses, particularly on condition (6) for $f(t)$.

Let us notice that x_j, $j \in N$, are given by the formulas

$$x_j = \left(i\lambda_j I - A - \int_R k(s) e^{-i\lambda_j s} ds\right)^{-1} f_j \ , \ j \in N. \tag{15}$$

What is now needed to obtain (14), from (6), is a condition like

$$\left|det[i\omega I - A - \tilde{k}(\omega)]\right| \geq m > 0 \ , \ \omega \in R, \tag{16}$$

with $\tilde{k}(\omega)$ given by (5) — the Fourier transform of $k \in L^1$. Condition (16) will guarantee the invertibility of the matrix in (15).

We have assumed that A satisfies (7), which means that there exists $m_0 > 0$, such that

$$det[i\omega I - A] \geq m_0 > 0 \ , \ \omega \in R. \tag{17}$$

This is easily obtained because $|det[i\omega I - A]| \longrightarrow \infty$ as $|\omega| \longrightarrow \infty$, while it remains strictly positive on any interval $|\omega| \leq k$, due to (7).

But a determinant is a continuous function of its elements, and if we take into account the property

$$|\tilde{k}(\omega)| \longrightarrow 0 \ , \ as \ |\omega| \longrightarrow \infty, \tag{18}$$

there results that (17) implies the inequality

$$\left|det[i\omega I - A - \tilde{k}(\omega)]\right| \geq \frac{m_0}{2}, \tag{19}$$

provided we take ω, such that $|\omega| > M > 0$. But according to the assumption 3. in the Theorem 1, within the compact interval $|\omega| \leq M$, we have

$$\left|det[i\omega I - A - \tilde{k}(\omega)]\right| > m_1 > 0. \tag{20}$$

Hence, from (19) and (20) we obtain

$$\left|det[i\omega I - A - \tilde{k}(\omega)]\right| > min\left(m_1, \frac{m_0}{2}\right), \tag{21}$$

for all $\omega \in R$. Since $min(m_1, \frac{m_0}{2}) > 0$, the discussion carried above shows that a condition like (16) is verified, because of (7) and the property of the Fourier transform (18) (see, DeVito [3]).

Therefore, the system (12) is uniquely solvable, and taking into account (20), one finds the estimates

$$|x_j| \leq m^{-1}|f_j| \ , \ j \in N. \tag{22}$$

From (6) and (22) we obtain (14), which tells us that $x(t)$, whose Fourier series is indicated in (3), and which is a solution of the equation (1), is defining a solution of (1) in the space $AP_r(R,C^n)$. The uniqueness is guaranteed by its construction.

This ends the proof of Theorem 1.

Remark 5. It is easy to derive a similar result for the quasi-linear equation related to (1), namely

$$\dot{x}(t) = Ax(t) + (k * x)(t) + (fx)(t) \ , \ t \in R, \tag{23}$$

where $f : AP_r(R,C^n) \longrightarrow AP_r(R,C^n)$ is such that

$$|fx - fy|_r \leq \lambda |x - y|_r, \tag{24}$$

with λ a small constant.

The contraction mapping principle can be applied (see Corduneanu [1], Corduneanu, Mahdavi, and Li [2]).

References

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