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Fractional Integro-differential Equations with State-Dependent Delay on an Unbounded Domain

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Abstract

We are concerned with the existence of solutions for fractional integro-differential equations with state-dependent delay on an infinite interval. Our results are based on Schauder's fixed point theorem combined with the diagonalization process.

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1 Introduction

In this paper, we study the existence of mild solutions, defined on the positive semi-infinite real interval $J := [0, +\infty)$, for semilinear integro-differential equations of fractional order

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds = f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J \tag{1.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.2}$$

where $1 < \alpha < 2$ and $A : D(A) \subset E \to E$ is the generator of an integral resolvent family defined on a complex Banach space $(E, |\cdot|)$, the convolution integral in the equation is known as the Riemann-Liouville fractional integral, $f : [0, +\infty) \times \mathcal{B} \to E$ and $\rho : [0, \infty) \times \mathcal{B} \to \mathbb{R}$ are appropriated functions. For any continuous function y defined on $(-\infty, +\infty)$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belongs to some abstract phase space $\mathcal{B}$, to be specified later.

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Fractional integro-differential equations arise in modeling processes in applied sciences (physics, engineering, finance, biology ...). Many problems in acoustics, electromagnetics, viscoelasticity, hydrology and other areas of application can be modeled by fractional differential equations; see the books [20, 22, 26].

The Cauchy problem for abstract differential equations involving Riemann-Liouville fractional integral in the linear part have been treated by Cuevas and Souza in [12, 13], where they studied S-asymptotically w-periodic solutions. Wang and Chen [28] considered a Cauchy problem for fractional integro-differential equations with time delay and nonlocal initial condition. Uniqueness and existence results of mild solution for fractional integro-differential equations with state-dependent delay on a semi-infinite interval have been established by Benchohra and Litimein [10] in Fréchet spaces.

Many properties of solutions for differential equations and inclusions, such as stability or oscillation, require global properties of solutions. This is the main motivation to look for sufficient conditions that ensure global existence of mild solutions for problem (1.1)-(1.2). There are two major approaches in the literature to establish existence of solutions to any problems on infinite intervals. The first approach is based on a diagonalization process [1, 2, 3] whereas the second is based on the recent nonlinear alternative of Leray Schauder type due to Frigon and Granas for contraction maps in Fréchet spaces [5, 6, 7, 8, 14].

In this paper, we study the existence of solutions for fractional integro-differential equations with state-dependent delay on an infinite interval. Our results are based on Schauder's fixed point theorem [15] combined with the diagonalization process.

2 Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([0,n];E)$, $n \in \mathbb{N}$ be the Banach space of all continuous functions from $J_n = [0,n]$ into E with the usual norm $\|y\|_n = \sup\{|y(t)| : 0 \leq t \leq n\}$.

$B(E)$ be the space of all bounded linear operators $N : E \to E$, with the usual supremum norm

$$\|N\|_{B(E)} = \sup \{ \, |N(y)| \; : \; |y| = 1 \, \}.$$

A measurable function $y : J_n \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [29]).

Let $L^1(J_n, E)$ denotes the Banach space of measurable functions $y : J_n \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^n |y(t)| \, dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [18] (see also [11]) and follow the terminology used in [21]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, n) \to E$ is continuous on J_n and $y_0 \in \mathcal{B}$, then for every $t \in J_n$ the following conditions hold :

(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{\,|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J_n.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_n = \sup\{K(t) : t \in J_n\}$ and $M_n = \sup\{M(t) : t \in J_n\}$.

Remark 2.1. 1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [21].

Example 2.2. Let:

BC the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;

$$C^\infty := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\right\};$$

$$C^0 := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. However, BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 2.3. The spaces C_g, UC_g, C_g^∞ and C_g^0.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \left\{\phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\right\};$$

$$C_g^0 := \left\{\phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0\right\}.$$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). We consider the following condition on the function g.

(g_1) For all $a > 0$, $\sup_{0\le t\le a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty.$

They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 2.4. The space C_γ.

For any real positive constant γ, we define the functional space C_γ by

$$C_\gamma := \left\{\phi \in C((-\infty,0],E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1)-(A_3)$ are satisfied.

Definition 2.5. A function $f : J \times \mathcal{B} \to E$ is said to be an Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t,.) : \mathcal{B} \to E$ is continuous;

(ii) for each $y \in \mathcal{B}$ the function $f(.,y) : J \to E$ is measurable.

The Laplace transformation of a function $f \in L^1_{loc}([0,\infty),E)$ is defined by

$$\mathcal{L}(f)(\lambda) :=: \widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re(\lambda) > \omega,$$

if the integral is absolutely convergent for $Re(\lambda) > \omega$. In order to defined the mild solution of the problems $(1.1)-(1.2)$ we recall the following definition

Definition 2.6. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space E. We call A the generator of an integral resolvent if there exists $\omega > 0$ and a strongly continuous function $S : [0,+\infty) \to B(E)$ such that

$$\left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t)x dt, \ Re\lambda > \omega, x \in E.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The following result is a direct consequence of ([23], Proposition 3.1 and Lemma 2.2).

Proposition 2.7. *Let $\{S(t)\}_{t\ge0} \subset B(E)$ be an integral resolvent family with generator A. Then the following conditions are satisfied:*

a) $S(t)$ is strongly continuous for $t \ge 0$ and $S(0) = I$;

b) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \ge 0$;

c) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds.$$

d) Let $x \in D(A)$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$ and

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)xds.$$

In particular, $S(0) = a(0)$.

Remark 2.8. The uniqueness of resolvent is well-know (see Prüss [27]).

If an operator A with domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function which satisfies $\lim_{t\to 0^+} \frac{\|S(t)\|_{B(E)}}{a(t)} < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - a(t)x}{(a*a)(t)},$$

see ([24], Theorem 2.1). For example, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup (see [9]) and $a(t) = t$ actually corresponds to the generator of a sine family (see [4]). A characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for C_0-semigroups, can be directly deduced from ([24], Theorem 3.4). More information on the C_0-semigroups and sine families can be found in [9, 16, 17, 25].

Definition 2.9. A resolvent family of bounded linear operators, $\{S(t)\}_{t>0}$, is called uniformly continuous if

$$\lim_{t\to s} \|S(t) - S(s)\|_{B(E)} = 0.$$

3 Main results

Now we define the mild solution for the initial value problem $(1.1)-(1.2)$.

Definition 3.1. We say that the function $y : (-\infty, +\infty) \to E$ is a mild solution of $(1.1)-(1.2)$ if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)\, f(s, y_{\rho(s,y_s)})\, ds \quad for\ each\ t \in J. \tag{3.1}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_φ), is frequently verified by continuous and bounded functions. For more details, see for instance [21].

Lemma 3.3. *([19], Lemma 2.4) If $y : \mathbb{R} \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \le (M_n + L^\phi)\|\phi\|_{\mathcal{B}} + K_n \sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J_n,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Theorem 3.4. *Assume that*

$(H1)$ *The operator solution $S(t)_{t\in J}$ is compact for $t > 0$.*

$(H2)$ *The function $f : J \times \mathcal{B} \to E$ is Carathéodory.*

$(H3)$ *There exists a function $p \in L^1(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that*

$$|f(t,u)| \le p(t)\,\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

$(H4)$ *For each $n \in \mathbb{N}$, there exists $r_n > 0$ such that*

$$r_n \ge M\,\psi(K_n r_n + c_n)\int_0^n p(s)ds$$

where

$$c_n := (M_n + L^\phi + K_n MH)\|\phi\|_{\mathcal{B}}$$

$(H5)$ *For each $t \in J$ and each bounded set $B \subset \mathcal{B}$, the set $\{f(t, y_{\rho(t,y_t)}), y \in B\}$ is relatively compact in E.*

Then the problem (1.1) – (1.2) *has a mild solution on* $(-\infty,+\infty)$.

Proof. The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)ds = f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J_n := [0,n], \tag{3.2}$$

$$y_0 = \phi \in \mathcal{B}, \tag{3.3}$$

Let

$$B_n = \{y : (-\infty, n] \to E : y|_{[0,n]} \text{ continuous and } y_0 \in \mathcal{B}\},$$

where $y|_{[0,n]}$ is the restriction of y to the real compact interval $[0,n]$.

Part I: We begin by showing that the problem $(3.2)-(3.3)$ has a solution $y_n \in B_n$. Consider the operator $N : B_n \to B_n$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t)\,\phi(0) + \displaystyle\int_0^t S(t-s)\, f(s, y_{\rho(s,y_s)})\, ds, & \text{if } t \in J_n. \end{cases} \tag{3.4}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(3.2)-(3.3)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ S(t)\,\phi(0), & \text{if } t \in J_n. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_n$ with $z_0 = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ z(t), & \text{if } t \in J_n. \end{cases}$$

If $y(\cdot)$ satisfies (3.1), we can decompose it as $y(t) = \bar{z}(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J_n$ and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t S(t-s)\, f(s, \bar{z}_{\rho(s,\bar{z}_s+x_s)} + x_{\rho(s,\bar{z}_s+x_s)})\, ds \quad \text{for } t \in J_n.$$

Let

$$B_n^0 = \{z \in B_n : z_0 = 0 \in \mathcal{B}\}.$$

For any $z \in B_n^0$ we have

$$\|z\|_n = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s \leq n\} = \sup\{|z(s)| : 0 \leq s \leq n\}.$$

Thus $(B_n^0, \|\cdot\|_n)$ is a Banach space. We define the operator $F : B_n^0 \to B_n^0$ by :

$$F(z)(t) = \int_0^t S(t-s)\, f(s, \bar{z}_{\rho(s,\bar{z}_s+x_s)} + x_{\rho(s,\bar{z}_s+x_s)})\, ds \quad \text{for } t \in J_n. \tag{3.5}$$

Obviously the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point. We shall show that the operators F satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.
Let

$$C_n = \{z \in B_n^0, \|z\|_n \leq r_n\},$$

where r_n is the constant from $(H4)$. It is clear that C_n is a closed, convex subset of B_n^0.
Step 1: F is continuous.

Let z_q be a sequence such that $z_q \longrightarrow z$ in B_n^0. Then for each $t \in J_n$

$$\begin{aligned}
|F(z_q)(t)-F(z)(t)| &= \Big|\int_0^t S(t-s)\,[f(s,\overline{z}_{q\rho(s,\overline{z}_{qs}+x_s)}+x_{\rho(s,\overline{z}_{qs}+x_s)}) \\
&- f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})]\,ds\Big| \\
&\leq M\int_0^t |f(s,\overline{z}_{q\rho(s,\overline{z}_{q_s}+x_s)}+x_{\rho(s,\overline{z}_{qs}+x_s)}) \\
&- f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds.
\end{aligned}$$

Since $f(s,.)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|F(z_q)-F(z)\|_n \to 0 \text{ as } q\to+\infty.$$

Thus F is continuous.

Step 2: $F(C_n)\subset C_n$.

Let $z\in C_n$, we show that $F(z)\in C_n$. For each $t\in J_n$ we have

$$\begin{aligned}
|Fz(t)| &\leq M\int_0^t |f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds \\
&\leq M\int_0^t p(s)\,\psi\Big(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}\Big)\,ds \\
&\leq M\int_0^t p(s)\,\psi\Big(K_nr_n+(M_n+L^\phi+K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}\Big)\,ds.
\end{aligned}$$

Set

$$c_n := (M_n+L^\phi+K_nMH)\|\phi\|_{\mathcal{B}}$$

Then, we have

$$\|Fz\|_n \leq M\,\psi(K_nr_n+c_n)\int_0^n p(s)ds.$$

By $(H4)$, we have

$$\|Fz\|_n \leq r_n.$$

Step 3: $F(C_n)$ is an equicontinuous set.

Let $\tau_1,\tau_2\in J_n$ with $\tau_2>\tau_1$, and $z\in C_n$. Then

$$
\begin{aligned}
&|F(z)(\tau_2)-F(z)(\tau_1)|\\
\leq\ & \Big|\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})\,ds\Big|\\
&+\Big|\int_{\tau_1}^{\tau_2}S(\tau_2-s)|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds\Big|\\
\leq\ & \int_0^{\tau_1}|S(\tau_2-s)-S(\tau_1-s)||f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds\\
&+\int_{\tau_1}^{\tau_2}|S(\tau_2-s)||f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds\\
\leq\ & \psi(r_nK_n+c_n)\int_0^{\tau_1}|S(\tau_2-s)-S(\tau_1-s)|p(s)ds\\
&+M\ \ \psi(r_nK_n+c_n)\int_{\tau_1}^{\tau_2}p(s)ds.
\end{aligned}
$$

The right-hand of the above inequality tends to zero as $\tau_2-\tau_1\to 0$, since $S(t)$ is uniformly continuous. As a consequence of Steps 1 to 3, $(H5)$ together with the Arzelá-Ascoli theorem, the operator F is completely continuous.

Therefore, we deduce from Schauder's fixed point theorem that F has a fixed point $z_n\in C_n$ which is a solution of problem (3.2) – (3.3) .

Part II: The diagonalization process

We now use the following diagonalization process. For $k\in\mathbb{N}$, let

$$
u_k(t)=\begin{cases} z_k(t), & t\in[0,n_k];\\ z_k(n_k), & t\in[n_k,\infty).\end{cases} \tag{3.6}
$$

Here $\{n_k\}_k\in\mathbb{N}^*$ is a sequence of numbers satisfying

$$
0<n_1<n_2<...<n_k<\ldots\uparrow\infty.
$$

Let $S=\{u_k\}_{k=1}^{\infty}$.

For $k\in\mathbb{N}$ and $t\in[0,n_1]$ we have

$$
u_{n_k}(t)=\int_0^t S(t-s)f(s,(\overline{u}_{n_k})_{\rho(s,(\overline{u}_{n_k})_s+x_s)}+x_{\rho(s,(\overline{u}_{n_k})_s+x_s)})ds.
$$

Thus, for $k\in\mathbb{N}$ and $t,h\in[0,n_1]$ we have

$$
u_{n_k}(t)-u_{n_k}(h)=\int_0^t[S(t-s)-S(h-s)]f(s,(\overline{u}_{n_k})_{\rho(s,(\overline{u}_{n_k})_s+x_s)}+x_{\rho(s,(\overline{u}_{n_k})_s+x_s)})ds
$$

and by $(H3)$, we have

$$
|u_{n_k}(t)-u_{n_k}(h)|\leq\psi(r_{n_1}K_{n_1}+c_{n_1})\int_0^t|S(t-s)-S(h-s)|p(s)ds.
$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence F_1^* of $\mathbb{N}$ and a function $v_1 \in B_{n_1}^0$ with $u_{n_k} \to v_1$ in $B_{n_1}^0$ as $k \to \infty$ through F_1^*. Let $N_1 = N_1^* \setminus \{1\}$.

Also for $k \in \mathbb{N}$ and $t, h \in [0, n_2]$ we have

$$|u_{n_k}(t) - u_{n_k}(x)| \le \psi(r_{n_2} K_{n_2} + c_{n_2}) \int_0^t |S(t-s) - S(h-s)| p(s) ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence F_2^* of F_1 and a function $v_2 \in B_{n_2}^0$ with $u_{n_k} \to v_2$ in $B_{n_2}^0$ as $k \to \infty$ through F_2^*. Note that $v_1 = v_2$ on $[0, n_1]$ since $F_2^* \subseteq F_1$. Let $F_2 = F_2^* \setminus \{2\}$. Proceed inductively to obtain for $m \in \{3, 4, \ldots\}$a subsequence F_m^* of F_{m-1} and a function $v_m \in B_{n_m}^0$ with $u_{n_k} \to v_m$ in $B_{n_m}^0$ as $k \to \infty$ through F_m^*. Let $F_m = F_m^* \setminus \{m\}$.

Define a function z as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Then define $z(t) = v_m(t)$. Then $z \in B_\infty$ and $z_0 = 0$.

Again fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Then for $n \in F_m$ we have

$$u_{n_k}(t) = \int_0^t S(t-s) f(s, (\overline{u}_{n_k})_{\rho(s, (\overline{u}_{n_k})_s + x_s)} + x_{\rho(s, (\overline{u}_{n_k})_s + x_s)}) ds.$$

Let $n_k \to \infty$ through F_m to obtain

$$v_m(t) = \int_0^t S(t-s) f(s, \overline{v}_{m_{\rho(s, \overline{v}_{m_s} + x_s)}} + x_{\rho(s, \overline{v}_{m_s} + x_s)}) ds.$$

i.e

$$z(t) = \int_0^t S(t-s) f(s, \overline{z}_{\rho(s, \overline{z}_s + x_s)} + x_{\rho(s, \overline{z}_s + x_s)}) ds.$$

We can use this method for each $h \in [0, n_m]$, and for each $m \in \mathbb{N}$. Thus the constructed function y is a mild solution of (1.1) – (1.2). This completes the proof of the theorem.

4 An Example

To apply our abstract results, we consider the fractional differential equation with state-dependent delay of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) - \dfrac{1}{\Gamma(\mu-1)} + \displaystyle\int_t^0 (t-s)^{\mu-2} L u_\xi u(s,\xi) ds \\ \quad = \dfrac{e^{-\gamma t + t} |u(t - \sigma(u(t,0)), \xi)|}{3(e^{-t} + e^t)(1 + |u(t - \sigma(u(t,0), \xi))|)}, & t \in [0,\infty),\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,\infty), \\ u(\theta,\xi) = u_0(\theta,\xi), & \theta \in (-\infty, 0],\ \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $1 < \mu < 2$, $\sigma \in C(\mathbb{R}, [0, \infty))$, $\gamma > 0$, L_ξ stands for the operator with respect to the spatial variable ξ which is given by:

$$L_\xi = \frac{\partial^2}{\partial \xi^2} - r, \quad \text{with} \quad r > 0.$$

Take $E = L^2([0,\pi],\mathbb{R})$ and the operator $A := L_\xi : D(A) \subset E \to E$ with domain

$$D(A) := \{\ u \in E\ : u'' \in E,\ u(0) = u(\pi) = 0\ \}.$$

Clearly A is densely defined in E and is sectorial. Hence A is a generator of a solution operator on E. For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma = \{\phi \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\}$$

with the norm

$$||\phi||_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|y(\theta)|.$$

Notice that the phase space $\mathcal{B}_\gamma$ satisfies axioms (A_1), (A_2) (see [21] for more details).

Set

$$y(t)(\xi) = u(t,\xi), \quad t \in [0,\infty),\ \xi \in [0,\pi].$$

$$\phi(\theta)(\xi) = u_0(\theta,\xi), \quad t \in [0,\infty),\ \theta \le 0.$$

$$f(t,\varphi)(\xi) = \frac{e^{-\gamma t+t}\varphi(0,\xi)}{3(e^{-t}+e^{t})(1+\varphi(0,\xi))}, \quad t \in [0,\infty),\ \xi \in [0,\pi].$$

$$\rho(t,\varphi) = t - \sigma(\varphi(0,0)).$$

Theorem 4.1. *Let $\varphi \in \mathcal{B}_\gamma$ be such that (H_φ) holds, and let $t \to \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of* (4.1).

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