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Abstract

In this paper, consensus problem of the leader-following multi-agent system with a varying-velocity leader is analyzed. The system is considered with both time-varying input-delay and directed dynamic topologies, where the system delay is unknown and time-varying with a pre-specified upper bounded derivative. The stability analysis is performed with a proposed Lyapunov-Krosovskii functional. Sufficient delay-dependent condition in the form of Linear Matrix Inequalities (LMIs) is given to guarantee system consensus. Finally, numerical simulation verifies the theoretical results.

AMS Subject Classification: 93C23.

Keywords: Leader-following, consensus analysis, vary-velocity leader, time-varying delay, dynamic topology.

1 Introduction

Recently, with the significant improvements in computer science and technology, distributed coordinated control of the multi-agent system has attracted a lot of attention. The main idea of this control technology is using a group of mobile agents to accomplish a task, and its final goal is to make the system reach consensus. Although single agent has limited power of processing information, the interconnected system as a whole can perform complex tasks in a coordinated form. Thus, the multi-agent system has much more advantages than traditional control system, and has been applied to various fields such as formation control in robotic systems, unmanned aerial vehicle formation, biology, social-behavior, physics etc, [1-5].

An interesting topic is the consensus problem of the multi-agent system with a leader, where the leader is a special agent whose motion is independent of all the other agents. Such a problem is usually called leader-following consensus problem because the leader is followed by other agents in this system.

In the research of multi-agent systems, the critical challenge is to design an appropriate control protocol including neighbor-based rules for each agent as the information (position, velocity, acceleration, etc) exchanges among agents must be taken into consideration due

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to the fact that the behavior of every single agent is not only dependent on its own state but also on its neighbors'. Besides, another two questions must be considered in the consensus analysis of the system. On one hand, time-delay usually appears in control system and it is frequently a reason that leading system instability. Many results have been obtained in this area. For example, Ref.[6] proposes a leader-following consensus problem of autonomous agents with time-varying coupling delays. On the other hand, since the relative position is time-varying, the information interconnection topology among agents also changes with time. Hence, the topology is dynamic.

The main contribution of this work is obtaining the sufficient condition in the form of LMIs for multi-agent system to reach consensus with a varying-velocity leader and time-varying input-delay under dynamic topology. Here, the velocity of the leader is unknown while the acceleration of it is known. The simulations show that the system is able to reach to consensus if the proposed LMIs holds.

This paper is organized as follows: Section 2 introduces the preliminaries on algebra graph theory and the problem formulation. In section 3, an stability result is given for the system. Section 4 provides a simulation to verify the results.

2 Preliminaries

Generally speaking, the information exchange among agents in the multi-agent system can be described by the graph theory. So firstly we have an overview on the graph theory concepts as follows. A directed graph $G = \{V, E\}$ contains a set of nodes $V = \{v_1, v_2, \cdots, v_n\}$, and a set of edges $E \subseteq V \times V$, $E = \{(v_i, v_j) | v_i, v_j \in V, i \neq j\}$. Each agent is represented by a node and each $(v_i, v_j) \in E$ means agent v_i and v_j share the information with respect to their states, i.e., v_i and v_j are neighbors. $N_i(t)$ denotes the set of labels of those nodes which are neighbors of node v_i at time t $(i = 1, 2, \cdots, n)$. $A = [a_{ij}] \in R^{n\times n}$ is defined as the adjacency matrix of G, where $a_{ii} = 0$, and $a_{ij} \geq 0$, $a_{ij} > 0$ iff $(v_i, v_j) \in E$. The laplacian matrix of the weighted directed graph is defined as $L = D - A$, where $D = diag\{d_1, d_2, \ldots, d_n\} \in R^{n\times n}$ is a diagonal matrix and $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, 2, \ldots n$.

In this paper, the topology of the leader-following system is described by a directed graph $\tilde{G}$, which includes n followers and one leader.

In order to deal with the problem of dynamic topology, we need to consider a switching topology. Therefore, we define $\bar{G} = \{\tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_N\}$ as a set of graphs with all possible topologies including all possible interconnection graphs, and define $\beta = \{1, 2, \cdots, N\}$ as its index set. Besides, $B = diag\{b_1, b_2, \ldots, b_n\} \in R^{n\times n}$ denotes the adjacency matrix between leader and followers, where $b_i > 0$ means follower v_i is able to obtain the information from the leader, and $b_i = 0$, otherwise. A piecewise-constant switching signal $\sigma : [0, \infty) \to \beta$ is used to describe the variable information topology. Therefore, although $N_i(t)$ and $L_\sigma (\sigma \in \beta)$ are time-varying (switched at $t_i, i = 1, 2, ...$), they are time-invariant in any interval $[t_i, t_{i+1})$ (beginning at $t_0 = 0$), where $[t_i, t_{i+1})$ is an infinite, bounded, continuous time-intervals sequence, $i = 0, 1, \cdots, n$.

Lemma 2.1. *For any $a, b \in R^n$, and any appropriate positive definite matrix $\phi \in R^{n\times n}$, $2a^{\mathrm{T}} b \leq$*

$a^{\mathrm{T}}\phi^{-1}a + b^{\mathrm{T}}\phi\, b$ holds.

Lemma 2.2. *Suppose that a symmetric matrix is partitioned as $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, where S_{11} and S_{22} are phalanx. S is positive definite iff both S_{11} and $S_{22} - S_{21}S_{11}^{-1}S_{12}$ are positive definite or both S_{22} and $S_{11} - S_{12}S_{22}^{-1}S_{21}$ are positive definite.*

The multi-agent system is given as follows:

$$\dot{x}_i = u_i(t - \tau(t)),$$

where $x_i(t) \in R, u_i(t-\tau(t)) \in R$, indicates the state and the control input of the ith follower respectively ($i = 1, 2, \cdots, n$), and $\tau(t)$ represents the time-delay with $\tau(t) < d_1, \dot{\tau}(t) \leq d_2 < 1$.

The dynamic of the leader can be described as

$$\dot{x}_0 = v_0(t - \tau(t)), \tag{2.1}$$

$$\dot{v}_0 = a(t), \tag{2.2}$$

where $x_0(t) \in R$ is the position of the leader, $v_0(t) \in R$ is its velocity which is unknown, and $a(t)$ is the acceleration which is known.

In order to estimate $v_0(t)$, each follower has to use the information obtained from its neighbors in a decentralized way by $v_i(t)$. Therefore, the decentralized protocol can be expressed as follows:

$$u_i(t-\tau(t)) = -[\sum_{j\in N_i(t)} a_{ij}(t)(x_i(t-\tau(t)) - x_j(t-\tau(t))) + b_i(t)(x_i(t-\tau(t)) - x_0(t-\tau(t)))] + v_i(t-\tau(t)), \tag{2.3}$$

where $N_i(t)$ means the set including the neighbor agents of follower i at time t.

The protocol used to estimate $v_0(t)$ is

$$\dot{v}_i(t-\tau(t)) = a(t) - [\sum_{j\in N_i(t)} a_{ij}(t)(x_i(t-\tau(t)) - x_j(t-\tau(t))) + b_i(t)(x_i(t-\tau(t)) - x_0(t-\tau(t)))], \tag{2.4}$$

Let $\bar{x}_i = x_i - x_0, \bar{v}_i = v_i - v_0$, where $\bar{x}_i$ means the position error between agent i and leader, $\bar{v}_i$ means the velocity error between agent i and leader.

Then the system equation (4) can be rewritten as

$$u_i(t-\tau(t)) = -[\sum_{j\in N_i(t)} a_{ij}(t)(\bar{x}_i(t-\tau(t)) - \bar{x}_j(t-\tau(t))) + b_i(t)\bar{x}_i(t-\tau(t))] + v_i(t-\tau(t)), \tag{2.5}$$

and equation (5) can be rewritten as

$$\dot{v}_i(t-\tau(t)) = a(t) - [\sum_{j\in N_i(t)} a_{ij}(t)(\bar{x}_i(t-\tau(t)) - \bar{x}_j(t-\tau(t))) + b_i(t)\bar{x}_i(t-\tau(t))], \tag{2.6}$$

Bring the equation (6) and (7) into the equation (1-3), we can obtain the following equations

$$\begin{cases} \dot{\bar{x}}(t) = -(L_\sigma + B_\sigma)\bar{x}(t-\tau(t)) + \bar{v}(t-\tau(t)), \\ \dot{\bar{v}}(t) = -(L_\sigma + B_\sigma)\bar{x}(t-\tau(t)), \end{cases}$$

that is,

$$\begin{bmatrix} \dot{\bar{x}}(t) \\ \dot{\bar{v}}(t) \end{bmatrix} = \begin{bmatrix} -(L_\sigma + B_\sigma) & I_n \\ -(L_\sigma + B_\sigma) & 0_n \end{bmatrix} \begin{bmatrix} \bar{x}(t-\tau(t)) \\ \bar{v}(t-\tau(t)) \end{bmatrix},$$

where $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}, v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}}$, $\sigma \in \beta$, as been defined in section 2.

Define error vector $\varepsilon(t) = \begin{bmatrix} \bar{x}(t) \\ \bar{v}(t) \end{bmatrix} \in R^{2n}$, the error dynamics of the system is described as follows:

$$\dot{\varepsilon}(t) = F_\sigma \varepsilon(t-\tau(t)), \tag{2.7}$$

where $F_\sigma = \begin{bmatrix} -(L_\sigma + B_\sigma) & I_n \\ -(L_\sigma + B_\sigma) & 0_n \end{bmatrix}$, L_σ represents the Laplacian matrix of the graph $\tilde{G}_\sigma(t)$, B_σ represents the adjacency matrix between leader and followers of the graph $\tilde{G}_\sigma(t)$.

3 Main result

In this section, in order to make the system achieve consensus, the stability of system (8) is analyzed.

Theorem 3.1. *For any fixed $0 < d_2 < 1$, the system consensus can be realized if the following linear matrix inequality holds.*

$$\begin{pmatrix} PF_\sigma + F_\sigma^{\mathrm{T}}P + Q & PF_\sigma & 0 \\ F_\sigma^{\mathrm{T}}P & -(1-d_2)R/(d_1) & 0 \\ 0 & 0 & -(1-d_2)Q + d_1 F_\sigma^{\mathrm{T}} R F_\sigma \end{pmatrix} < 0$$

where P, Q, R are positive definite matrixes, d_1 is the upper bound of $\tau(t)$, and all of them can be obtained from the LMIs tool box of MatLab.

Proof. Define a Lyapunov-Krasovskii functional for system (8) as follows:

$$V = \varepsilon^{\mathrm{T}}(t)P\varepsilon(t) + \int_{t-\tau(t)}^{t} \varepsilon^{\mathrm{T}}(s)Q\varepsilon(s)ds + \int_{-\tau(t)}^{0}\int_{t+\theta}^{t} \dot{\varepsilon}^{\mathrm{T}}(s)R\dot{\varepsilon}(s)dsd\theta$$

where P, Q, R are positive definite matrixes.

Calculating $\dot{V}$, we get

$$\begin{aligned}\dot{V} = {} & 2\varepsilon^{\mathrm{T}}(t)PF_{\sigma}\varepsilon(t-\tau(t)) + \varepsilon^{\mathrm{T}}(t)Q\varepsilon(t) \\ & -(1-\dot{\tau}(t))\varepsilon^{\mathrm{T}}(t-\tau(t))Q\varepsilon(t-\tau(t)) + \tau(t)\dot{\varepsilon}^{\mathrm{T}}(t)R\dot{\varepsilon}(t) \\ & -(1-\dot{\tau}(t))\int_{t-\tau(t)}^{t} \dot{\varepsilon}^{\mathrm{T}}(s)R\dot{\varepsilon}(s)ds.\end{aligned}$$

Since

$$\varepsilon(t-\tau(t)) = \varepsilon(t) - \int_{t-\tau(t)}^{t} \dot{\varepsilon}(s)ds,$$

we have, by Lemma 1,

$$\begin{aligned}2\varepsilon^{\mathrm{T}}(t)PF_{\sigma}\varepsilon(t-\tau(t)) &= 2\varepsilon^{\mathrm{T}}(t)PF_{\sigma}\varepsilon(t) - \int_{t-\tau(t)}^{t} 2(F_{\sigma}P\varepsilon(t))^{\mathrm{T}}\dot{\varepsilon}(s)ds \\ &\le 2\varepsilon^{\mathrm{T}}(t)PF_{\sigma}\varepsilon(t) + \tau(t)/(1-d_2)\varepsilon^{\mathrm{T}}(t)PF_{\sigma}R^{-1}F_{\sigma}^{\mathrm{T}}P\varepsilon(t) \\ &\quad +(1-d_2)\int_{t-\tau(t)}^{t} \dot{\varepsilon}^{\mathrm{T}}(s)R\dot{\varepsilon}(s)ds \\ &\le 2\varepsilon^{\mathrm{T}}(t)PF_{\sigma}\varepsilon(t) + d_1/(1-d_2)\varepsilon^{\mathrm{T}}(t)PF_{\sigma}R^{-1}F_{\sigma}^{\mathrm{T}}P\varepsilon(t) \\ &\quad +(1-d_2)\int_{t-\tau(t)}^{t} \dot{\varepsilon}^{\mathrm{T}}(s)R\dot{\varepsilon}(s)ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{V}(t) \quad \le \quad & \varepsilon^{\mathrm{T}}(t)(F_{\sigma}^{\mathrm{T}}P + PF_{\sigma} + d_1/(1-d_2)PF_{\sigma}R^{-1}F_{\sigma}^{\mathrm{T}}P + Q)\varepsilon(t) \\ & +\varepsilon^{\mathrm{T}}(t-\tau(t))(-(1-d_2)Q + d_1F_{\sigma}^{\mathrm{T}}RF_{\sigma})\varepsilon(t-\tau(t))\end{aligned} \tag{3.1}$$

Thus, $\dot{V}(t) < 0$ holds if the following inequalities hold

$$F_{\sigma}^{\mathrm{T}}P + PF_{\sigma} + d_1/(1-d_2)PF_{\sigma}R^{-1}F_{\sigma}^{\mathrm{T}}P + Q < 0, \tag{3.2}$$

$$-(1-d_2)Q + d_1F_{\sigma}^{\mathrm{T}}RF_{\sigma} < 0. \tag{3.3}$$

According to Lemma 2,

$F_{\sigma}^{\mathrm{T}}P + PF_{\sigma} + d_1/(1-d_2)PF_{\sigma}R^{-1}F_{\sigma}^{\mathrm{T}}P + Q < 0 \Leftrightarrow$

$$\begin{pmatrix} PF_\sigma + F_\sigma^{\mathrm{T}}P + Q & PF_\sigma \\ F_\sigma^{\mathrm{T}}P & -(1-d_2)R/(d_1) \end{pmatrix} < 0$$

So (10) and (11) is equivalent to

$$\begin{pmatrix} PF_\sigma + F_\sigma^{\mathrm{T}}P + Q & PF_\sigma & 0 \\ F_\sigma^{\mathrm{T}}P & -(1-d_2)R/(d_1) & 0 \\ 0 & 0 & -(1-d_2)Q + d_1 F_\sigma^{\mathrm{T}} R F_\sigma \end{pmatrix} < 0$$

where P, Q, R, d_1 can be obtained by using the LMIs toolbox in MatLab. □

4 Simulations

In this section, we will provide a numerical simulation to illustrate the theoretical results derived in the above sections. Consider a system consists of four followers and one leader, we assume the topology of the system switches among three graphs for every 0.1 second. The topology begin switching at graph G_1. Suppose the adjacency weight among followers are 1.0, and that between leader and each follower is 2.0. Then the Laplacian matrixes of these graphs and the interconnection relationship between the leader and the followers are as following:

$$L_1 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$L_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Take $d_2 = 0.1$, by using the LMIs tool box, we obtain $d_1 = 0.0396$, and P, Q, R as follows

$$P = \begin{pmatrix} 11.3241 & -0.5472 & -0.5432 & -0.1907 & -2.5151 & 0.2396 & 0.3068 & 0.1204 \\ -0.5472 & 4.3226 & -0.7953 & -0.1871 & 0.2852 & -2.3153 & 0.5527 & -0.0089 \\ -0.5432 & -0.7953 & 4.6876 & 0.2303 & 0.2233 & 0.3901 & -2.5167 & -0.0164 \\ -0.1907 & -0.1871 & 0.2303 & 5.6034 & 0.3858 & 0.2758 & 0.2142 & -2.9490 \\ -2.5151 & 0.2852 & 0.2233 & 0.3858 & 4.8772 & 0.3858 & 0.4976 & 0.5785 \\ 0.2396 & -2.3153 & 0.3901 & 0.2758 & 0.3858 & 4.7762 & 0.3354 & 0.2393 \\ 0.3068 & 0.5527 & -2.5167 & 0.2142 & 0.4976 & 0.3354 & 4.8975 & 0.7525 \\ 0.1204 & -0.0089 & -0.0164 & -2.9490 & 0.5785 & 0.2393 & 0.7525 & 5.5026 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1.8500 & -0.4220 & -0.4636 & -0.5594 & -0.8526 & 0.0618 & 0.2402 & 0.1273 \\ -0.4220 & 1.6132 & -0.8022 & -0.0995 & 0.2764 & -0.7242 & 0.2378 & -0.0814 \\ -0.4636 & -0.8022 & 1.9297 & -0.1877 & 0.1320 & 0.2755 & -0.8308 & 0.1338 \\ -0.5594 & -0.0995 & -0.1877 & 2.6388 & 0.2196 & 0.2606 & 0.1387 & -0.8903 \\ -0.8526 & 0.2764 & 0.1320 & 0.2196 & 1.8244 & -0.2701 & -0.2922 & -0.2587 \\ 0.0618 & -0.7242 & 0.2755 & 0.2606 & -0.2701 & 1.6971 & -0.4853 & -0.0732 \\ 0.2402 & 0.2378 & -0.8308 & 0.1387 & -0.2922 & -0.4853 & 1.8461 & -0.1180 \\ 0.1273 & -0.0814 & 0.1338 & -0.8903 & -0.2587 & -0.0732 & -0.1180 & 2.3066 \end{pmatrix}$$

$$R = \begin{pmatrix} 0.3753 & -0.0149 & -0.0136 & 0.0004 & -0.0401 & 0.0106 & -0.0046 & -0.0080 \\ -0.0149 & 0.3573 & -0.0113 & -0.0034 & -0.0087 & -0.0406 & 0.0065 & 0.0069 \\ -0.0136 & -0.0113 & 0.3695 & 0.0147 & 0.0033 & -0.0087 & -0.0376 & -0.0038 \\ 0.0004 & -0.0034 & 0.0147 & 0.3655 & 0.0026 & -0.0052 & 0.0089 & -0.0392 \\ -0.0401 & -0.0087 & 0.0033 & 0.0026 & 0.3423 & 0.0179 & 0.0220 & 0.0289 \\ 0.0106 & -0.0406 & -0.0087 & -0.0052 & 0.0179 & 0.3433 & 0.0288 & 0.0108 \\ -0.0046 & 0.0065 & -0.0376 & 0.0089 & 0.0220 & 0.0288 & 0.3421 & 0.0325 \\ -0.0080 & 0.0069 & -0.0038 & -0.0392 & 0.0289 & 0.0108 & 0.0325 & 0.3341 \end{pmatrix}$$

Take $\tau(t) = 0.03\cos(|t|)$, Fig.1 shows the position tracking errors of followers. From Fig.1, we can see that position error converge to zero, i.e., the system reaches to consensus.

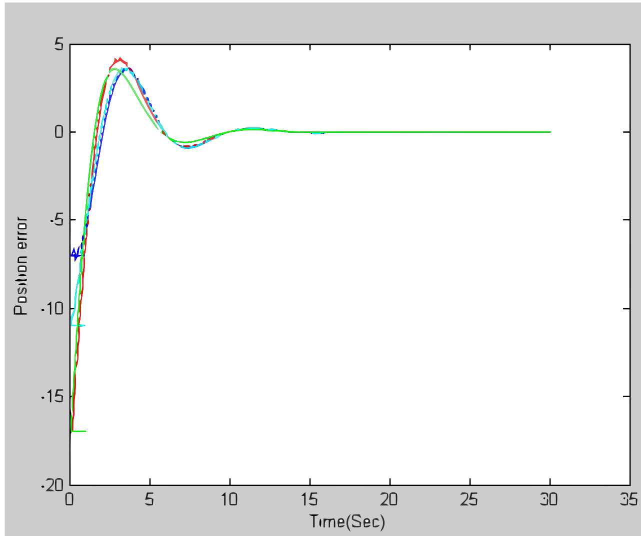

Figure 1. Position tracking errors of followers

References

[1] R M. Murray, Recent Research in Cooperative Control of Multivehi-cle Systems. *Dynamic Systems, Measurement, and Control* . **129** (2007), 571–583.

[2] W. Ni, D. Cheng, Leader-following consensus of multi-agent systems under fixed and switching topologies. *System and Control Letters*. **59** (2010), 209–217.

[3] P. Lin, Y. Jia, Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies.*Automatica.* **45** (2009), 2154–2158.

[4] Y. G. Sun, L. Wang, and G. Xie, Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays. *System and Control Letters*. **57** (2008), 175–183.

[5] W. Yu, G. Chen, and M. Cao, Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica,* doi:10.1016/j.automatica.2010.03.006.

[6] A. Jadbabaie, J. Lin, and A. S. Morse, Leader-follower coordination of multi-agent systems with coupling time delays. *Physical A* . **374**(9) (2007), 853–863.