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Gevrey Regularity for a Class of Solutions of the Linearized Spatially Homogeneous Boltzmann Equation Without Angular Cutoff*

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Abstract

In this paper, we study the Gevrey smoothing property for the non-negative solution of the linearized spatially homogeneous Boltzmann equation. Using pseudo-differential calculus and some techniques of mathematical analysis, we show that in the non-cutoff and non-Maxwellian case with the inverse power law potential, if the solution is Lipschitz continuous on the velocity variable, then it has the local Gevrey regularity.

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1 Introduction

In this paper, we consider the following Cauchy problem for the spatially homogeneous linear Boltzmann equation:

$$\begin{cases} \frac{\partial g}{\partial t} = Lg = Q(\mu, g) + Q(g, \mu), t > 0 \\ g_{|t=0} = g_0 \end{cases} \tag{1.1}$$

where $\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$ is the normalized Maxwellian distribution, and the function g depending only on the two variables $t \geq 0$(time) and $v \in \mathbb{R}^3$(velocity). The Boltzmann quadratic collision operator $Q(g, f)$ is of the following form:

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)\{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*$$

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where $\sigma \in \mathbb{S}^2$(unit sphere of $\mathbb{R}^3$), v' and v are the post- and pre-collision velocities respectively:

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

the Boltzmann collision kernel $B(z,\sigma)$ depends only on $|z|$ and the scalar product $< \frac{z}{|z|}, \sigma >$.

The early works for the Boltzmann equation are usually based on the Grad's cutoff assumption. However, the proof in [8] shows that the solution of (1.1) can not be smoother than the initial datum under this assumption. On the other hand, recently there are more and more research results without this assumption (see Refs. [2-7, 9, 10]). In [2], Alexandre et al. gave some estimates of the smooth property of the solutions to inhomogeneous Boltzmann equations. The authors in [3, 6, 9, 10] studied the Soblev regularity of the solutions. Although in the Maxwellian case, the Gevrey regularity of the solution of (1.1) has been proved in [10]. The methods in [10] seems can not be used in the non-Maxwellian case directly.

In this paper, we discuss the same issue with the non-Maxwellian case. Assume

$$B(|v-v_*|,\cos\theta) = \Phi(|v-v_*|)b(\cos\theta) = (1+|v-v_*|^2)^{\frac{\gamma}{2}}b(\cos\theta),\ \gamma\in(0,1]. \tag{1.2}$$

Note that in the form of $\Phi(|v-v_*|)$, the kinetic factor has been added a constant 1. As in [9], the corresponding potential is called the modified hard potential.

We consider the following angular non-cutoff case with the inverse power law potential

$$b(\cos\theta) \approx K\theta^{-2-2\alpha}\ when\ \theta\to 0,\ \cos\theta = < \frac{v-v_*}{|v-v_*|}, \sigma >,\ \theta\in[0,\frac{\pi}{2}],\ \alpha\in(0,\frac{1}{2}), \tag{1.3}$$

where K is a positive constant. The corresponding inverse power law potential $U(\rho) = \rho^{1-\kappa}$ for some $\kappa > 2$ and ρ being the distance between two particles.

Before state the main result, we recall the following definitions:

Definition 1.1 ([6, 10, 12]). *For $s \geq 1$, $u \in G^s(\mathbb{R}^3)$ which is the Gevrey class function space with index s, if there exists $C > 0$ such that for any $k \in \mathbb{N}$,*

$$\|D^k u\|_{L^2} \leq C^{k+1}(k!)^s,$$

or equivalently, there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0 <|D|>^{1/s}} u \in L^2(\mathbb{R}^3)$ where

$$\Lambda = <|D|> = (1+|D_v|^2)^{\frac{1}{2}},\ \|D^k u\|^2_{L^2} = \sum_{|\beta|=k} \|D^\beta u\|^2_{L^2}.$$

Definition 1.2 ([10]). *For a nonnegative initial datum $g_0(v) \in L^1_2(\mathbb{R}^3))$, $g(t,v)$ is called a weak solution of the Cauchy problem (1.1) if it satisfies:*

$$g(t,v) \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)) \cap L^2([0,T_0]; L^1_2(\mathbb{R}^3)) \cap L^\infty([0,T_0]; L^1(\mathbb{R}^3)); g(0,v) = g_0;$$

$$\int_{\mathbb{R}^3} g(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} g(0,v)\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} g(\tau,v)\partial_\tau\varphi(\tau,v)dv$$

$$= \int_0^t d\tau \int_{\mathbb{R}^3} L(g)(\tau,v)\varphi(\tau,v)dv,$$

for any test function $\varphi(t,v) \in L^\infty([0,T_0]; W^{2,\infty}(\mathbb{R}^3))$.

In the following discussion, we will use the notation of the weighted function space:

$$L_r^p(\mathbb{R}^3) = \{g; \|g\|_{L_r^p} < +\infty\}, H_r^p(\mathbb{R}^3) = \{g; \|g\|_{H_r^p} < +\infty\}$$

with the corresponding norm

$$\|g\|_{L_r^p} = \|g(v) < |v| >^r \|_{L^p},\ \|g\|_{H_r^p}^2 = \int_{\mathbb{R}^n} | < |D| >^p < |v| >^r g(v)|^2 dv, p, r \in \mathbb{R}.$$

Now we state the main result in this paper.

Theorem 1.1. *Assume g is a weak nonnegative solution of the Cauchy problem (1.1) with the corresponding collision cross section satisfies (1.2) and (1.3). If $g(t,\cdot)$ is a Lipschitz continuous function, then for any $0 < t \leq T_0$ and $\psi(v) \in C_0^\infty(\mathbb{R}^3)$, $\psi g(t,\cdot) \in G^{1/\alpha}(\mathbb{R}^3)$.*

2 The prior estimates

In this section, we first introduce the pseudo differential operator and then give some estimates on its commutation with some functions of the relative velocity.

For any $t \in [0, T_0]$ and $\delta \in [0,1]$, $d \geq 0$, we denote

$$G_{\delta,d}(t,\xi) = G_\delta(t,\xi) < |\xi| >^{-d} = \frac{1}{\delta + e^{-t<|\xi|>^\alpha}} < |\xi| >^{-d},$$

and

$$G(t,\xi) = G_{0,4}(t,\xi) = e^{t<|\xi|>^\alpha} < |\xi| >^{-4}.$$

The above pseudo differential operators have appeared and have been used in the proof of the Gevrey regularity in [10]. It is easy to see that for any fixed positive smooth function $\psi(v)$ which has a compact support, if we can prove $\psi g(t,\cdot) \in G^{1/\alpha}(\mathbb{R}^3)$, then the result of Theorem 1.1 can be obtained. So we can restrict $\psi(v) > 0$ in the following process.

We state the following theorem:

Theorem 2.1. *Suppose $I(\tau) = (Q(g,\mu), \psi G_\delta^2(\tau, D) < |D| >^{-8} \psi g)_{L^2}, \tau \in (0, T_0]$. Then for any fixed $s > 0$, there exists a constant $C = C(s) > 0$ independent of $\delta \in (0,1]$ satisfying*

$$I(\tau) \leq C\|g(\tau,\cdot)\|_{L^1_2} \|\psi g(\tau,\cdot)\|_{H^{-s}}.$$

In order to prove Theorem 2.1, we need to prove these following lemmas by Cauchy integral theorem:

Lemma 2.1. *Let $\Phi_1(|v-v_*|) = \Phi(|v-v_*|) < |v-v_*| >$. Assume the Fourier transform*

$$\mathcal{F}(\Phi_1(|v-v_*|)\mu(v_*))(\xi) = f_1(v,\xi)\hat{\mu}(\xi).$$

Then

$$f_1(v,\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\frac{|v_*|^2}{2}} [1 + |v-v_*|^2 - |\xi|^2 + 2i(v-v_*)\cdot\xi]^{\frac{\gamma+1}{2}} dv_*,$$

and

$$|f_1(v,\xi)| \leq C \cdot < |v| >^{\gamma+1} < |\xi| >^{\gamma+1}.$$

Proof. First we consider the case of $n = 1$.

$$\begin{aligned}\mathcal{F}(\Phi_1(|v-v_*|)\mu(v_*))(\xi) &= \int_{\mathbb{R}^1}(1+|v-v_*|^2)^{\frac{\gamma+1}{2}}(2\pi)^{-\frac{3}{2}}e^{-\frac{|v_*|^2}{2}-iv_*\cdot\xi}dv_*\\ &= (2\pi)^{-\frac{3}{2}}e^{-\frac{\xi^2}{2}}\int_{\mathbb{R}^1}[1+(v-v_*)^2]^{\frac{\gamma+1}{2}}e^{-\frac{(v_*+i\xi)^2}{2}}dv_*\\ &= (2\pi)^{-\frac{3}{2}}e^{-\frac{\xi^2}{2}}\int_C e^{-\frac{z^2}{2}}[1+(v-z+i\xi)^2]^{\frac{\gamma+1}{2}}dz,\end{aligned}$$

where $z = v_* + i\xi$, C denotes the curve: $v_* + i\xi, -\infty < v_* < \infty$. By Cauchy integral theorem ([11]), we obtain

$$\int_C e^{-\frac{z^2}{2}}[1+(v-z+i\xi)^2]^{\frac{\gamma+1}{2}}dz = \int_{\mathbb{R}^1}e^{-\frac{|v_*|^2}{2}}[1+(v-v_*+i\xi)^2]^{\frac{\gamma+1}{2}}dv_*.$$

Now we turn to consider the case of $n = 3$. Assume

$$v = (v_1, v_2, v_3), v_* = (v_{*1}, v_{*2}, v_{*3}), \xi = (\xi_1, \xi_2, \xi_3).$$

Using the above result, we have

$$\mathcal{F}(\Phi_1(|v-v_*|)\mu(v_*))(\xi) = \int_{\mathbb{R}^3}(1+|v-v_*|^2)^{\frac{\gamma+1}{2}}(2\pi)^{-\frac{3}{2}}e^{-\frac{|v_*|^2}{2}-iv_*\cdot\xi}dv_*$$

$$= (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^1}[1+(v_1-v_{*1})^2+(v_2-v_{*2})^2+(v_3-v_{*3})^2]^{\frac{\gamma+1}{2}}e^{-\frac{v_{*1}^2}{2}-iv_{*1}\xi_1}dv_{*1}\Big)$$
$$\times e^{-\frac{v_{*2}^2+v_{*3}^2}{2}-i(v_{*2}\xi_2+v_{*3}\xi_3)}dv_{*2}dv_{*3}$$

$$= (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^1}[1+(v_1-v_{*1}+i\xi_1)^2+(v_2-v_{*2})^2+(v_3-v_{*3})^2]^{\frac{\gamma+1}{2}}e^{-\frac{v_{*1}^2}{2}-\frac{\xi_1^2}{2}}dv_{*1}\Big)$$
$$\times e^{-\frac{v_{*2}^2+v_{*3}^2}{2}-i(v_{*2}\xi_2+v_{*3}\xi_3)}dv_{*2}dv_{*3}$$

$$= (2\pi)^{-\frac{3}{2}}e^{-\frac{|\xi|^2}{2}}\int_{\mathbb{R}^3}e^{-\frac{|v_*|^2}{2}}[1+\sum_{j=1}^{3}(v_j-v_{*j}+i\xi_j)^2]^{\frac{\gamma+1}{2}}dv_*$$

$$= (2\pi)^{-\frac{3}{2}}\hat{\mu}(\xi)\int_{\mathbb{R}^3}e^{-\frac{|v_*|^2}{2}}[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi]^{\frac{\gamma+1}{2}}dv_*.$$

Therefore, we conclude the result of Lemma 2.1. □

Lemma 2.2. *Let*

$$\xi^+ = \frac{\xi+|\xi|\sigma}{2}, \xi^- = \frac{\xi-|\xi|\sigma}{2}.$$

Assume the Fourier transform

$$\mathcal{F}(\Phi(|v-v_*|)\psi(v_*)\mu(v_*))(\xi) = f(v,\xi)\hat{\mu}(\xi).$$

Then

$$f(v,\xi) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}\psi(v_*-i\xi)e^{-\frac{|v_*|^2}{2}}[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi]^{\frac{\gamma}{2}}dv_*,$$

and

$$|f(v,\xi)| \le C\cdot <|v|>^{\gamma}<|\xi|>^{\gamma},$$
$$|f(v,\xi^+)-f(v,\xi)| \le C\cdot <|v|>^{\gamma}<|\xi|>^{\gamma+1}\sin\frac{\theta}{2}, \theta = \arccos<\frac{\xi}{|\xi|},\sigma>.$$

Proof. The above equality is obtained by the similar method as in Lemma 2.1. The first inequality is also easy to check. In fact, by the compact support function ψ and the above equality, there exists a positive constant C satisfying

$$\psi(v_* - i\xi) \equiv 0$$

when $|v_* - i\xi| = \sqrt{v_*^2 + \xi^2} > C$. Thus we conclude that f is also a compact support function on the variable ξ, that is,

$$f(v,\xi) = f(v,\xi)\mathcal{X}_E(\xi)$$

where the set $E = \{\xi; |\xi| \le C\}$. Therefore,

$$\begin{aligned}
|f(v,\xi)| &= |\hat{\mu}(\xi)^{-1}\mathcal{F}(\Phi(|v-v_*|)\psi(v_*)\mu(v_*))(\xi)|\mathcal{X}_E(\xi)\\
&= |\hat{\mu}(\xi)^{-1}\int_{\mathbb{R}^3}\Phi(|v-v_*|)\psi(v_*)\mu(v_*)e^{-iv_*\cdot\xi}dv_*|\mathcal{X}_E(\xi)\\
&\le \hat{\mu}(\xi)^{-1}\int_{\mathbb{R}^3}|\Phi(|v-v_*|)\psi(v_*)\mu(v_*)|dv_*\mathcal{X}_E(\xi)\\
&\le C <|v|>^{\gamma}\mathcal{X}_E(\xi)\\
&\le C\cdot <|v|>^{\gamma}<|\xi|>^{\gamma}.
\end{aligned}$$

The first inequality is obtained.
To prove the second inequality, by direct calculation, we have

$$\begin{aligned}
\partial_{\xi_i}(f(v,\xi)\hat{\mu}(\xi)) &= \hat{\mu}(\xi)[\partial_{\xi_i}f(v,\xi) - \xi_i f(v,\xi)]\\
&= \partial_{\xi_i}(\mathcal{F}(\Phi(|v-v_*|)\psi(v_*)\mu(v_*))(\xi))\\
&= \int_{\mathbb{R}^3}(1+|v-v_*|^2)^{\frac{\gamma}{2}}(2\pi)^{-\frac{3}{2}}\psi(v_*)e^{-\frac{|v_*|^2}{2}-iv_*\cdot\xi}(-iv_{*i})dv_*.
\end{aligned}$$

Using Cauchy integral theorem again, we obtain

$$\int_{\mathbb{R}^3}(1+|v-v_*|^2)^{\frac{\gamma}{2}}(2\pi)^{-\frac{3}{2}}\psi(v_*)e^{-\frac{|v_*|^2}{2}-iv_*\cdot\xi}(-iv_{*i})dv_* =$$

$$(2\pi)^{-\frac{3}{2}}\hat{\mu}(\xi)\int_{\mathbb{R}^3}\psi(v_*-i\xi)e^{-\frac{|v_*|^2}{2}}[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi]^{\frac{\gamma}{2}}\cdot(-\xi_i - iv_{*i})dv_*.$$

Therefore,

$$\begin{aligned}
\partial_{\xi_i}f(v,\xi) &= (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}\psi(v_*-i\xi)e^{-\frac{|v_*|^2}{2}}[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi]^{\frac{\gamma}{2}}\\
&\quad\cdot(-\xi_i - iv_{*i})dv_* + \xi_i f(v,\xi)\\
&= (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}\psi(v_*-i\xi)e^{-\frac{|v_*|^2}{2}}[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi]^{\frac{\gamma}{2}}\\
&\quad\cdot(-iv_{*i})dv_*.
\end{aligned}$$

Using the mean value theorem of differentials, we have

$$\begin{aligned}
|f(v,\xi^+) - f(v,\xi)| &\le C\cdot|\nabla_\xi f(v,\eta)|\cdot|\xi^+ - \xi|\\
&\le C'\cdot <|v|>^{\gamma}<|\eta|>^{\gamma}|\xi^+ - \xi|\\
&\le C''\cdot <|v|>^{\gamma}<|\xi|>^{1+\gamma}\sin\frac{\theta}{2},
\end{aligned}$$

here $\theta = \arccos < \frac{\xi}{|\xi|}, \sigma >$. This completes the proof of Lemma 2.2. □

Now we turn to prove Theorem 2.1, decomposing $I(\tau) = I_1(\tau) + I_2(\tau)$, where

$$I_1(\tau) = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma)g(v_*)\mu(v)[\psi(v') - \psi(v)]G^2_{\delta,4}(\tau,D)\psi g(v')d\sigma dv dv_*,$$

and

$$I_2(\tau) = (Q(g,\mu\psi), G^2_\delta(\tau,D) < |D| >^{-8} \psi g)_{L^2}.$$

Proof of Theorem 2.1. First, we estimate $I_1(\tau)$. Assume

$$b_1(\cos\theta) = b(\cos\theta)\sqrt{\frac{1-\cos\theta}{2}},$$

by (1.3), we have

$$\int_{\mathbb{S}^2} b_1(\cos\theta)d\sigma < +\infty.$$

Using the result of Lemma 2.1 and the means of Bobylev's formula (see Refs. [1, 10]),

$$e^{-\frac{|\xi|^2}{2}} = \hat{\mu}(\xi) \le \hat{\mu}(\xi^+) \le e^{-\frac{|\xi|^2}{4}},$$

we have

$$\begin{aligned}
|I_1(\tau)| &\le C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma)g(v_*)\mu(v)|v'-v|G^2_{\delta,4}(\tau,D)\psi g(v')d\sigma dv dv_* \\
&\le C\int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)|\overline{\widehat{\psi g}(\tau,\xi)}|\cdot|\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\Phi_1(|v-v_*|)b_1(\frac{\xi}{|\xi|}\cdot\sigma)\mu(v_*)g(v) \\
&\cdot e^{-i(v_*\cdot\xi^+ + v\cdot\xi^-)}d\sigma dv dv_*|d\xi \\
&= C\int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)|\overline{\widehat{\psi g}(\tau,\xi)}|\cdot|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\mathcal{F}(\Phi_1(|v-v_*|)\mu(v_*))(\xi^+)b_1(\frac{\xi}{|\xi|}\cdot\sigma) \\
&\cdot g(v)e^{-iv\cdot\xi^-}d\sigma dv|d\xi \\
&= C\int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)|\overline{\widehat{\psi g}(\tau,\xi)}|\cdot|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f_1(v,\xi^+)\hat{\mu}(\xi^+)b_1 g(v)e^{-iv\cdot\xi^-}d\sigma dv|d\xi \\
&\le C\|g(\tau,\cdot)\|_{L^1_2}\int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)|\overline{\widehat{\psi g}(\tau,\xi)}| < |\xi| >^{\gamma+1} e^{-\frac{|\xi|^2}{4}}d\xi \\
&\le C\|g(\tau,\cdot)\|_{L^1_2}\int_{\mathbb{R}^3}[< |\xi| >^{-s} |\overline{\widehat{\psi g}(\tau,\xi)}|][< |\xi| >^{s+\gamma-7} e^{2T_0< |\xi| >^\alpha - \frac{|\xi|^2}{4}}]d\xi \\
&\le C(\int_{\mathbb{R}^3}[< |\xi| >^{-s} |\overline{\widehat{\psi g}(\tau,\xi)}|]^2 d\xi)^{\frac12}(\int_{\mathbb{R}^3}[< |\xi| >^{s+\gamma-7} e^{2T_0< |\xi| >^\alpha - \frac{|\xi|^2}{4}}]^2 d\xi)^{\frac12} \\
&\cdot\|g(\tau,\cdot)\|_{L^1_2} \\
&\le C\|g(\tau,\cdot)\|_{L^1_2}(\int_{\mathbb{R}^3}[< |\xi| >^{-s} |\overline{\widehat{\psi g}(\tau,\xi)}|]^2 d\xi)^{\frac12} \\
&\le C\|g(\tau,\cdot)\|_{L^1_2}\|\psi g(\tau,\cdot)\|_{H^{-s}}.
\end{aligned}$$

Secondly, we estimate $I_2(\tau)$. Also using Bobylev identity, we have

$$\begin{aligned}
I_2(\tau) &= \int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)\overline{\widehat{\psi g}(\tau,\xi)}[\int_{\mathbb{R}^6}\int_{\mathbb{S}^2}\Phi(|v-v_*|)b(\frac{\xi}{|\xi|}\cdot\sigma)\mu(v_*)\psi(v_*)g(v) \\
&\times(e^{-i(v_*\cdot\xi^+ + v\cdot\xi^-)} - e^{-iv_*\cdot\xi})d\sigma dv dv_*]d\xi \\
&= \int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)\overline{\widehat{\psi g}(\tau,\xi)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[\mathcal{F}(\Phi(|v-v_*|)\mu(v_*)\psi(v_*))(\xi^+)bg(v)e^{-iv\cdot\xi^-} \\
&-\mathcal{F}(\Phi(|v-v_*|)\mu(v_*)\psi(v_*))(\xi)bg(v)]d\sigma dv d\xi \\
&= \int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)\overline{\widehat{\psi g}(\tau,\xi)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[f(v,\xi^+)-f(v,\xi)]\hat{\mu}(\xi^+)bg(v)e^{-iv\cdot\xi^-}d\sigma dv d\xi \\
&+ \int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)\overline{\widehat{\psi g}(\tau,\xi)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f(v,\xi)[\hat{\mu}(\xi^+)-\hat{\mu}(\xi)]bg(v)e^{-iv\cdot\xi^-}d\sigma dv d\xi \\
&+ \int_{\mathbb{R}^3} G^2_{\delta,4}(\tau,\xi)\overline{\widehat{\psi g}(\tau,\xi)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f(v,\xi)\hat{\mu}(\xi)bg(v)[e^{-iv\cdot\xi^-}-e^0]d\sigma dv d\xi \\
&= I_{21}+I_{22}+I_{23}.
\end{aligned}$$

Because (see Ref. [10])

$$|\hat{\mu}(\xi^+)-\hat{\mu}(\xi)| \le \hat{\mu}(\xi^+)|\xi|^2\sin^2\frac{\theta}{2},\ e^{-\frac{|\xi|^2}{2}} = \hat{\mu}(\xi) \le \hat{\mu}(\xi^+) \le e^{-\frac{|\xi|^2}{4}}$$

and

$$\begin{aligned}
|e^{-iv\cdot\xi^-}-e^0| &= |-2\sin\frac{v\cdot\xi^-}{2}(\sin\frac{v\cdot\xi^-}{2}+i\cos\frac{v\cdot\xi^-}{2})| \\
&\le C|v||\xi^-| \\
&\le C|v||\xi|\sin\frac{\theta}{2}.
\end{aligned}$$

Thus, by the result of Lemma 2.2 and the condition (1.3), for any $i\in\{1,2,3\}$, we have

$$\begin{aligned}
|I_{2i}| &\le C\cdot\|g(\tau,\cdot)\|_{L^1_2}\int_{\mathbb{R}^3} G^2_\delta(\tau,\xi) <|\xi|>^{-5} e^{-\frac{|\xi|^2}{4}}|\overline{\widehat{\psi g}(\tau,\xi)}|d\xi \\
&\le C\cdot\|g(\tau,\cdot)\|_{L^1_2}\int_{\mathbb{R}^3}[e^{2T_0<|\xi|>^\alpha} <|\xi|>^{s-5} e^{-\frac{|\xi|^2}{4}}][<|\xi|>^{-s}|\overline{\widehat{\psi g}(\tau,\xi)}|]d\xi \\
&\le C(s)\cdot\|g(\tau,\cdot)\|_{L^1_2}\{\int_{\mathbb{R}^3}(<|\xi|>^{-s}|\overline{\widehat{\psi g}(\tau,\xi)}|)^2 d\xi\}^{\frac{1}{2}} \\
&\le C(s)\cdot\|g(\tau,\cdot)\|_{L^1_2}\|\psi g(\tau,\cdot)\|_{H^{-s}}.
\end{aligned}$$

Combining with the estimate of $I_1(\tau)$, we obtain the result of Theorem 2.1. □

Suppose

$$\Omega_1 = \mathrm{supp}(\psi),\ \Omega_2 = \{v\in\Omega_1 : g(\tau,v)=0\}.$$

Then, $\Omega_2 \subset \Omega_1$.
Clearly there are two cases:
1. $\Omega_2 = \Omega_1$. In this case, $\psi g(v) \equiv 0$ for any $v \in \mathbb{R}^3$;

2. $\Omega_2 \neq \Omega_1$.
In case 2, assume $\Omega = \Omega_1/\Omega_2$. Then it is easy to see

$$\psi(\Omega) > 0 \text{ and } g(\Omega) > 0.$$

Because g satisfying the Lipschitz continuity, then for any $\epsilon > 0$, there exists $\Omega_\epsilon \subset \Omega$, satisfying

$$\inf_{v\in\Omega_\epsilon} \psi(v) = r_0 > 0, \ \inf_{v\in\Omega_\epsilon} g(v) = r_1 > 0 \ \text{ and } \int_{\Omega/\Omega_\epsilon} |G^2(\tau, D)\psi g| dv < \epsilon. \tag{2.1}$$

Lemma 2.3. *For any* $\epsilon > 0$,

$$\begin{aligned} A_1 &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} [(1+|v-v_*|^2)^{\frac{\gamma}{2}} - 1] b\mu(v_*)[g(v) - g(v')]\psi G^2_{\delta,4}\psi g(v') d\sigma dv_* dv \\ &\le O(1)||G_{\delta,4}\psi g||^2_{L^2} + \epsilon \cdot C_{g,\epsilon}. \end{aligned}$$

Proof. Since case 1 is evident, we only consider case 2. Using the Lipschitz continuity of the weak solution g,

$$\int_{\mathbb{R}^3} (1+|v_*|^{1+\gamma})\mu(v_*) dv_* < +\infty,$$

$$|\frac{dv}{dv'}| = \frac{4}{(cos\frac{\theta}{2})^2} \le C,$$

(2.1), and the fact that

$$\text{supp}(\psi) = \Omega_1, \ \ \psi \ge 0, \ \ \int_0^{\frac{\pi}{2}} b \sin\frac{\theta}{2} \sin\theta d\theta < +\infty,$$

we have

$$\begin{aligned} A_1 &\le C \cdot \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} |v-v_*|^\gamma b\mu(v_*)|v-v'|\psi G^2_{\delta,4}\psi g(v') d\sigma dv_* dv \\ &= C \cdot \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} \frac{|v'-v_*|^\gamma}{(\cos\frac{\theta}{2})^\gamma} b\mu(v_*)|v_* - v'|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \psi G^2_{\delta,4}\psi g(v') d\sigma dv_* dv \\ &= C \cdot \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} \frac{|v'-v_*|^{1+\gamma}}{(\cos\frac{\theta}{2})^{1+\gamma}} (b\sin\frac{\theta}{2})\mu(v_*)\psi G^2_{\delta,4}\psi g(v') d\sigma dv_* dv \\ &\le C \cdot \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} (1+|v'|^{1+\gamma})(1+|v_*|^{1+\gamma})(b\sin\frac{\theta}{2})\mu(v_*)\psi G^2_{\delta,4}\psi g(v') d\sigma dv_* dv \\ &\le C \cdot \int_{\mathbb{R}^3} \psi G^2_{\delta,4}\psi g(v') dv' \\ &= C \cdot (\int_{\Omega_\epsilon} \psi G^2_{\delta,4}\psi g(v) dv + \int_{\Omega/\Omega_\epsilon} \psi G^2_{\delta,4}\psi g(v) dv) \\ &\le \frac{C}{r_1}\int_{\Omega_\epsilon} \psi g G^2_{\delta,4}\psi g(v) dv + C' \cdot \int_{\Omega/\Omega_\epsilon} G^2_{\delta,4}\psi g(v) dv \\ &\le \frac{C}{r_1}\int_{\Omega_\epsilon} \psi g G^2_{\delta,4}\psi g(v) dv + C' \cdot \int_{\Omega/\Omega_\epsilon} |G^2\psi g(v)| dv \\ &\le \frac{C}{r_1}\int_{\mathbb{R}^3} \psi g G^2_{\delta,4}\psi g(v) dv + \epsilon \cdot C' \\ &= O(1)||G_{\delta,4}\psi g||^2_{L^2} + \epsilon \cdot C_{g,\epsilon}. \end{aligned}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *For any $\epsilon > 0$,*

$$A_2 = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)g(v)[\psi(v') - \psi(v)]G^2_{\delta,4}\psi g(v')d\sigma dv_* dv \leq O(1)\|G_{\delta,4}\psi g\|^2_{L^2} + \epsilon\cdot C'_{g,\epsilon}.$$

Proof. As in the proof of Lemma 2.3, by (2.1), we have

$$\begin{aligned}
\int_{\mathbb{R}^3} g(v)G^2_{\delta,4}\psi g(v)dv &= \int_{\Omega} g(v)G^2_{\delta,4}\psi g(v)dv \\
&= \int_{\Omega_\epsilon} g(v)G^2_{\delta,4}\psi g(v)dv + \int_{\Omega/\Omega_\epsilon} g(v)G^2_{\delta,4}\psi g(v)dv \\
&\leq \frac{1}{r_0}\int_{\Omega_\epsilon} \psi g(v)G^2_{\delta,4}\psi g(v)dv + \int_{\Omega/\Omega_\epsilon} g(v)G^2_{\delta,4}\psi g(v)dv \\
&\leq \frac{1}{r_0}\int_{\mathbb{R}^3} \psi g(v)G^2_{\delta,4}\psi g(v)dv + C\cdot\int_{\Omega/\Omega_\epsilon} G^2_{\delta,4}\psi g(v)dv \\
&\leq \frac{1}{r_0}\int_{\mathbb{R}^3} \psi g(v)G^2_{\delta,4}\psi g(v)dv + C\cdot\epsilon \\
&= O(1)\|G_{\delta,4}\psi g\|^2_{L^2} + C\cdot\epsilon.
\end{aligned}$$

Because of the Lipschitz continuity of g, there exists two positive constants C_1, C_2, satisfying

$$g(v) < C_1|v| + C_2, \quad \text{for } v\in\mathbb{R}^3.$$

So we have

$$\begin{aligned}
A_2 &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)g(v)[\psi(v') - \psi(v)]G^2_{\delta,4}\psi g(v')d\sigma dv_* dv \\
&\leq C\cdot\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)g(v)|v-v'|G^2_{\delta,4}\psi g(v')d\sigma dv_* dv \\
&\leq C\cdot\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)(C_1|v|+C_2)|v-v'|G^2_{\delta,4}\psi g(v')d\sigma dv_* dv \\
&\leq C\cdot\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)(C_1|v-v'|+C_2+C_1|v'|)|v-v'|G^2_{\delta,4}\psi g(v')d\sigma dv_* dv \\
&\leq C\cdot\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\mu(v_*)(C_1|v'-v_*|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+C_2+C_1|v'|)|v'-v_*|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}G^2_{\delta,4}\psi g(v')d\sigma dv_* dv.
\end{aligned}$$

By using the condition that ψ has a compact support and $|\frac{dv}{dv'}| < C$, we have

$$\begin{aligned}
A_2 &\leq C\cdot\int_{\mathbb{R}^3}\mu(v_*)(1+|v_*|)^2 dv_*\cdot\int_0^{\frac{\pi}{2}} b\sin\frac{\theta}{2}\sin\theta d\theta\cdot\int_{\mathbb{R}^3} G^2_{\delta,4}\psi g(v)dv \\
&\leq C\cdot\int_{\mathbb{R}^3} G^2_{\delta,4}\psi g(v)dv.
\end{aligned}$$

Thus

$$\begin{aligned}
A_2 &\le C\cdot\int_{\Omega} G_{\delta,4}^2\psi g(v)dv \\
&\le C\cdot\{\int_{\Omega_\epsilon} G_{\delta,4}^2\psi g(v)dv+\int_{\Omega/\Omega_\epsilon} G_{\delta,4}^2\psi g(v)dv\} \\
&\le \frac{C}{r_1}\int_{\Omega_\epsilon} gG_{\delta,4}^2\psi g(v)dv + C\cdot\int_{\Omega/\Omega_\epsilon}|G^2\psi g(v)|dv \\
&\le \frac{C}{r_1}(O(1)\|G_{\delta,4}\psi g\|_{L^2}^2 + C'\cdot\epsilon)+C\cdot\epsilon \\
&= O(1)\|G_{\delta,4}\psi g\|_{L^2}^2+\epsilon\cdot C'_{g,\epsilon}.
\end{aligned}$$

This concludes the proof of Lemma 2.4. □

Lemma 2.5. *For any $\theta\in[0,\frac{\pi}{2}],\gamma\in(0,1]$, there exists a independent constant C satisfying*

$$|\frac{1}{\cos^3\frac{\theta}{2}}[(1+\frac{|v-v_*|^2}{\cos^2\frac{\theta}{2}})^{\frac{\gamma}{2}}-1]-[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1]|\le C(\frac{1}{\cos^{\gamma+3}\frac{\theta}{2}}-1)(1+|v-v_*|^2)^{\frac{\gamma}{2}}.$$

Proof.

$$\begin{aligned}
J &= |\frac{1}{\cos^3\frac{\theta}{2}}[(1+\frac{|v-v_*|^2}{\cos^2\frac{\theta}{2}})^{\frac{\gamma}{2}}-1]-[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1]| \\
&\le |\frac{1}{\cos^{3+\gamma}\frac{\theta}{2}}[(\cos^2\frac{\theta}{2}+|v-v_*|^2)^{\frac{\gamma}{2}}-(1+|v-v_*|^2)^{\frac{\gamma}{2}}]|+|(1-\frac{1}{\cos^3\frac{\theta}{2}})| \\
&\quad +|(\frac{1}{\cos^{3+\gamma}\frac{\theta}{2}}-1)(1+|v-v_*|^2)^{\frac{\gamma}{2}}| \\
&\le (\frac{1}{\cos^2\frac{\theta}{2}}-1)\frac{1}{\cos^{1+\gamma}\frac{\theta}{2}}+|(1-\frac{1}{\cos^3\frac{\theta}{2}})|+|(\frac{1}{\cos^{3+\gamma}\frac{\theta}{2}}-1)(1+|v-v_*|^2)^{\frac{\gamma}{2}}| \\
&\le C(\frac{1}{\cos^{\gamma+3}\frac{\theta}{2}}-1)(1+|v-v_*|^2)^{\frac{\gamma}{2}}.
\end{aligned}$$

This completes the proof of Lemma 2.5. □

3 Proof of Theorem 1.1

In this section, we will prove the main result of this paper by using the estimates which have been proved in the previous section. Suppose

$$B(|v-v_*|,\sigma)=\Phi(|v-v_*|)b=[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1]b+b=B_1+B_2.$$

corresponding,

$$Q=Q_1+Q_2$$

$$
\begin{aligned}
(Q_1(\mu,g),\psi G^2_{\delta,4}\psi g)_{L^2} &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2}[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1](\psi G^2_{\delta,4}\psi g(v')-\psi G^2_{\delta,4}\psi g(v))\\
&\cdot b\mu(v_*)g(v)d\sigma dv dv_*\\
&= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2}[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1][g(v)-g(v')]\psi G^2_{\delta,4}\psi g(v')\\
&\cdot b\mu(v_*)d\sigma dv dv_* + \int_{\mathbb{R}^6}\int_{\mathbb{S}^2}[g(v')\psi G^2_{\delta,4}\psi g(v')-g(v)\psi G^2_{\delta,4}\psi g(v)]\\
&\cdot[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1]b\mu(v_*)d\sigma dv dv_*\\
&= A_1+(1).
\end{aligned}
$$

By Lemma 2.3,

$$A_1 \le O(1)\|G_{\delta,4}\psi g\|^2_{L^2}+\epsilon\cdot C_{g,\epsilon}.$$

Using cancellation lemma (see Ref. [1]), Lemma 2.5 and the fact that ψ has compact support set, we have

$$
\begin{aligned}
(1) &= |S^1|\int_{\mathbb{R}^6}\int_0^{\frac{\pi}{2}}\{\frac{1}{\cos^3\frac{\theta}{2}}[(1+\frac{|v-v_*|^2}{\cos^2\frac{\theta}{2}})^{\frac{\gamma}{2}}-1]-[(1+|v-v_*|^2)^{\frac{\gamma}{2}}-1]\}b(\cos\theta)\\
&\cdot\sin\theta\mu(v_*)g(v)\psi G^2_{\delta,4}\psi g(v)d\theta dv dv_*\\
&\le O(1)\int_{\mathbb{R}^6} <|v-v_*|>^{\gamma}\mu(v_*)g(v)\psi G^2_{\delta,4}\psi g(v)dv dv_*\\
&\cdot\int_0^{\frac{\pi}{2}}(\frac{1}{\cos^{\gamma+3}\frac{\theta}{2}}-1)b(\cos\theta)\sin\theta d\theta\\
&\le O(1)\int_{\mathbb{R}^3}(1+|v|^{\gamma})g(v)\psi G^2_{\delta,4}\psi g(v)dv\\
&\le O(1)\int_{\mathbb{R}^3}g(v)\psi G^2_{\delta,4}\psi g(v)dv\\
&= O(1)\|G_{\delta,4}\psi g\|^2_{L^2}.
\end{aligned}
$$

Therefore

$$(Q_1(\mu,g),\psi G^2_{\delta,4}\psi g)_{L^2}\le O(1)\|G_{\delta,4}\psi g\|^2_{L^2}+\epsilon\cdot C_{g,\epsilon}. \tag{3.1}$$

On the other hand, by direct calculation,

$$(Q_2(\mu,g),\psi G^2_{\delta,4}\psi g)_{L^2}=(Q_2(\mu,\psi g),G^2_{\delta,4}\psi g)_{L^2}+A_2.$$

Using the result in [10]:

$$C_\mu[\|G_{\delta,4}\psi g\|^2_{L^2}-(Q_2(\mu,\psi g),G^2_{\delta,4}\psi g)_{L^2}]\ge\|\Lambda^\alpha G_{\delta,4}\psi g\|^2_{L^2}.$$

Combining with Lemma 2.4, there exists three positive constants $C_i, i=\{1,2,3\}$ independent of δ satisfying

$$(Q_2(\mu,g),\psi G^2_{\delta,4}\psi g)_{L^2}\le C_1\|G_{\delta,4}\psi g\|^2_{L^2}-C_2\|\Lambda^\alpha G_{\delta,4}\psi g\|^2_{L^2}+C_3. \tag{3.2}$$

Because $G_\delta(\tau,\xi)\ge\frac{1}{2}$, we have

$$\|\psi g\|^2_{H^{-4}}\le C\|G_{\delta,4}\psi g\|^2_{L^2}.$$

By (3.1), (3.2) and Theorem 2.1, using Hölder inequality, we obtain

$$(Lg, \psi G_{\delta,4}^2 \psi g)_{L^2} \le C_1' \|G_{\delta,4}\psi g\|_{L^2}^2 - C_2' \|\Lambda^\alpha G_{\delta,4}\psi g\|_{L^2}^2 + C_3' + C_4' \|g\|_{L_2^1}^2 \tag{3.3}$$

where

$$C_i' > 0, i = 1, 2, 3, 4.$$

Finally, we give the proof of Theorem 1.1:

Proof of Theorem 1.1. Choosing $\psi G_{\delta,4}^2 \psi g$ to be the test function in Definition 1.3, we have

$$\frac{1}{2}\int_{\mathbb{R}^3} g(t)\psi G_{\delta,4}^2 \psi g(t) dv - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} g(\tau)\psi(\partial_t G_{\delta,4}^2(\tau))\psi g(\tau) dv d\tau,$$
$$= \frac{1}{2}\int_{\mathbb{R}^3} g_0 \psi G_{\delta,4}^2(0)\psi g_0 dv + \int_0^t (Lg(\tau), \psi G_{\delta,4}^2(\tau)\psi g(\tau))_{L^2} d\tau.$$

Since (see Ref. [10])

$$|\partial_t G_\delta(t,\xi)| \le G_\delta(t,\xi) < |\xi| >^\alpha \text{ and } \|G_{\delta,4}(0)\psi g_0\|_{L^2}^2 \le C\|\psi g_0\|_{L^1}^2,$$

we have

$$\left|\int_0^t \int_{\mathbb{R}^3} g(\tau)\psi(\partial_t G_{\delta,4}^2(\tau))\psi g(\tau) dv d\tau\right| \le 2\int_0^t \|\Lambda^\alpha G_{\delta,4}\psi g(\tau)\|_{L^2} \|G_{\delta,4}\psi g(\tau)\|_{L^2} d\tau.$$

Combining with (3.3), we obtain

$$\|G_{\delta,4}\psi g(t)\|_{L^2}^2 + C_2' \int_0^t \|\Lambda^\alpha G_{\delta,4}\psi g(\tau)\|_{L^2}^2 d\tau \le \|G_{\delta,4}\psi g_0\|_{L^2}^2 + C_1' \int_0^t \|G_{\delta,4}\psi g(\tau)\|_{L^2}^2 d\tau$$
$$+ 2\int_0^t \|\Lambda^\alpha G_{\delta,4}\psi g(\tau)\|_{L^2} \|G_{\delta,4}\psi g(\tau)\|_{L^2} d\tau + \int_0^t (C_3' + C_4' \|g(\tau)\|_{L_2^1}^2) d\tau$$
$$\le \|G_{\delta,4}\psi g_0\|_{L^2}^2 + \kappa \int_0^t \|\Lambda^\alpha G_{\delta,4}\psi g(\tau)\|_{L^2}^2 d\tau + (C_1' + C_\kappa)\int_0^t \|G_{\delta,4}\psi g(\tau)\|_{L^2}^2 d\tau + C_g.$$

Choosing $\kappa = C_2'$ and a constant R satisfying $(C_1' + C_\kappa)Rt \ge C_g$, we have

$$(\|G_{\delta,4}\psi g(t)\|_{L^2}^2 + R) \le (\|G_{\delta,4}\psi g_0\|_{L^2}^2 + R) + (C_1' + C_\kappa)\int_0^t (\|G_{\delta,4}\psi g(\tau)\|_{L^2}^2 + R) d\tau.$$

Then Gronwall inequality yields

$$(\|G_{\delta,4}\psi g(t)\|_{L^2}^2 + R) \le e^{(C_1'+C_\kappa)t}(\|G_{\delta,4}\psi g_0\|_{L^2}^2 + R) \le e^{C_3 t}(C_4\|\psi g_0\|_{L^1}^2 + R),$$

where C_3, C_4 are the positive constants independent of $\delta \in [0,1]$. Let $\delta \to 0$, we obtain for any given $t \in (0, T_0)$, $G\psi g(t) \in L^2$.

In addition, because there exists a positive constant $C_t < +\infty$ satisfying

$$\min_{\xi \in \mathbb{R}^3} e^{\frac{1}{2}t<|\xi|>^\alpha} < |\xi| >^{-4} \ge C_t$$

we have

$$e^{\frac{1}{2}t<|D_v|>^\alpha}\psi g(t,v) \in L^2,$$

and this completes the proof of Theorem 1.1. □

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