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Weighted Ostrowski–Grüss Inequalities on Time Scales

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Abstract

In this paper, we study Ostrowski–Grüss and Ostrowski-like inequalities on time scales and thus unify and extend corresponding continuous and discrete versions from the literature. We present corresponding inequalities by using the time scales L^∞-norm and also by using the time scales L^p-norm. Several interesting inequalities representing special cases of our general results are supplied.

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1 Introduction

In 1938, A. Ostrowski (see [15, Formula (2)]) presented the following interesting integral inequality.

Theorem 1.1. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f' \in L^\infty((a,b))$, i.e.,*

$$\left\|f'\right\|_\infty := \sup_{s\in(a,b)} \left|f'(s)\right| < \infty,$$

then for all $t \in [a,b]$, we have

$$\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\mathrm{d}s\right| \le \left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty. \tag{1.1}$$

In 2007, B. Pachpatte (see [17, Theorem 1 and Theorem 2]) established new generalizations of Ostrowski-type inequalities involving two functions, whose derivatives belong to L^p-spaces.

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Theorem 1.2. *Let $p > 1$ and $q := p/(p-1)$. If $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous such that $f', g' \in L^p([a,b])$, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} < \infty \quad \text{and} \quad \|g'\|_p = \left(\int_a^b |g'(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} < \infty,$$

then for all $t \in [a,b]$, we have

$$\left| f(t)g(t) - \frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\mathrm{d}s + f(t)\int_a^b g(s)\mathrm{d}s\right]\right| \leq \frac{(B(t))^{\frac{1}{q}}}{b-a} \frac{|g(t)|\|f'\|_p + |f(t)|\|g'\|_p}{2} \tag{1.2}$$

and

$$\left| f(t)g(t) - \frac{1}{b-a}\left[g(t)\int_a^b f(s)\mathrm{d}s + f(t)\int_a^b g(s)\mathrm{d}s\right] + \left(\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s\right)\left(\frac{1}{b-a}\int_a^b g(s)\mathrm{d}s\right)\right| \leq \left(\frac{(B(t))^{\frac{1}{q}}}{b-a}\right)^2 \|f'\|_p \|g'\|_p, \tag{1.3}$$

where

$$B(t) := \frac{1}{q+1}\left[(t-a)^{q+1} + (b-t)^{q+1}\right].$$

In 1988, S. Hilger [10] introduced the time scales theory to unify continuous and discrete analysis. Since then, many authors have studied certain integral inequalities on time scales, see, e.g., [1–6, 11, 14, 18, 19]. In [3], M. Bohner and T. Matthews established the time scales version of Ostrowski's inequality, hence unifying discrete, continuous and other versions of Theorem 1.1.

This work is organized as follows: In Section 2, we briefly present the general definitions and theorems connected to the time scales calculus. Next, in Section 3 and Section 4, we obtain time scales versions of weighted Ostrowski–Grüss and Ostrowski-like inequalities using the L^∞-norm and the L^p-norm, respectively. Our proofs utilize generalizations of so-called Montgomery inequalities, see [12, page 565] and [13, page 261].

2 General Definitions

Now we introduce some necessary time scales elements and refer the reader to the books [5, 6] for further details.

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, *left-dense*, and *left-scattered* provided $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. The set $\mathbb{T}^\kappa$ is defined to be equal to the set $\mathbb{T}$ without its left-scattered maximum (if it exists). A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* and we write $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ if it is continuous at all right-dense points and its left-sided limits exist and are finite at all left-dense points, and f is called *delta differentiable* at $t \in \mathbb{T}^\kappa$, with *delta derivative* $f^\Delta(t) \in \mathbb{R}$, provided given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

If f is differentiable such that f^Δ is rd-continuous, then we write $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the *delta integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where} \quad a, b \in \mathbb{T}.$$

Example 2.2. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $f^\Delta(t) = f'(t)$ for all $t \in \mathbb{R}$ and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t \quad \text{for all} \quad a, b \in \mathbb{R},$$

and if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $f^\Delta(t) = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$ and

$$\int_0^n f(t)\Delta t = \sum_{t=0}^{n-1} f(t) \quad \text{for all} \quad n \in \mathbb{N}.$$

Some results about integrals that will be used in this paper are contained in [5, Section 1.4] and collected as follows.

Theorem 2.3. *If a function is rd-continuous, then it possesses a delta antiderivative. For* $f, g \in \mathrm{C}_{\mathrm{rd}}([a,b], \mathbb{R})$ *and* $a, b, c \in \mathbb{T}$, *we have*

$$\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$$
$$\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$$
$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$
$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b |f(t)|\,\Delta t,$$

and, if additionally $f, g \in \mathrm{C}^1_{\mathrm{rd}}([a,b], \mathbb{R})$,

$$\int_a^b f(t) g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t) g(\sigma(t))\Delta t.$$

We also need the time scales monomials (see [5, Section 1.6]) defined as follows.

Definition 2.4. Let $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$g_0(t,s) := h_0(t,s) := 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$g_{k+1}(t,s) := \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}$$

and

$$h_{k+1}(t,s) := \int_s^t h_k(\tau,s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}.$$

Assumption (H). *From now on, until the end of this paper, we assume that $\mathbb{T}$ is a time scale and that $a,b \in \mathbb{T}$ such that $a < b$. By writing $[a,b]$, we mean $[a,b] \cap \mathbb{T}$. Moreover, $w \in \mathrm{C}_{\mathrm{rd}}([a,b],[0,\infty))$ is such that*

$$m(a,b) := \int_a^b w(t)\Delta t < \infty,$$

and we also define

$$p_w(t,s) := \begin{cases} \displaystyle\int_a^s w(\tau)\Delta\tau & \text{for} \quad a \le s < t \\ \displaystyle\int_b^s w(\tau)\Delta\tau & \text{for} \quad t \le s \le b. \end{cases}$$

3 Weighted Ostrowski–Grüss Inequalities in L^∞-Norm

Theorem 3.1. *Assume* (H). *If $f,g \in \mathrm{C}^1_{\mathrm{rd}}([a,b],\mathbb{R})$ such that $f^\Delta, g^\Delta \in L^\infty((a,b))$, i.e.,*

$$\left\|f^\Delta\right\|_\infty := \sup_{s\in(a,b)} \left|f^\Delta(s)\right| < \infty \quad \text{and} \quad \left\|g^\Delta\right\|_\infty = \sup_{s\in(a,b)} \left|g^\Delta(s)\right| < \infty, \tag{3.1}$$

then for all $t \in [a,b]$, we have

$$\begin{aligned} &\left| f(t)g(t) - \frac{1}{2m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s + f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right]\right| \\ &\qquad \le \left(\frac{1}{m(a,b)}\int_a^b (\sigma(s)-t)w(s)\operatorname{sgn}(s-t)\Delta s\right)\frac{|g(t)|\left\|f^\Delta\right\|_\infty + |f(t)|\left\|g^\Delta\right\|_\infty}{2} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\left| f(t)g(t) - \frac{1}{m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s + f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right]\right. \\ &\qquad \left. + \left(\frac{1}{m(a,b)}\int_a^b w(s)f(\sigma(s))\Delta s\right)\left(\frac{1}{m(a,b)}\int_a^b w(s)g(\sigma(s))\Delta s\right)\right| \\ &\qquad\qquad \le \left(\frac{1}{m(a,b)}\int_a^b (\sigma(s)-t)w(s)\operatorname{sgn}(s-t)\Delta s\right)^2 \left\|f^\Delta\right\|_\infty \left\|g^\Delta\right\|_\infty. \end{aligned} \tag{3.3}$$

Proof. Using integration by parts from Theorem 2.3 twice, we have

$$\begin{aligned}
\int_a^b p_w(t,s)f^\Delta(s)\Delta s &= \int_a^t\left(\int_a^s w(\tau)\Delta\tau\right)f^\Delta(s)\Delta s + \int_t^b\left(\int_b^s w(\tau)\Delta\tau\right)f^\Delta(s)\Delta s \\
&= f(t)\int_a^t w(\tau)\Delta\tau - \int_a^t w(s)f(\sigma(s))\Delta s - f(t)\int_b^t w(\tau)\Delta\tau - \int_t^b w(s)f(\sigma(s))\Delta s \\
&= m(a,b)f(t) - \int_a^b w(s)f(\sigma(s))\Delta s
\end{aligned}$$

and thus

$$f(t) - \frac{1}{m(a,b)}\int_a^b w(s)f(\sigma(s))\Delta s = \frac{1}{m(a,b)}\int_a^b p_w(t,s)f^\Delta(s)\Delta s. \tag{3.4}$$

Replacing f by g in (3.4), we obtain

$$g(t) - \frac{1}{m(a,b)}\int_a^b w(s)g(\sigma(s))\Delta s = \frac{1}{m(a,b)}\int_a^b p_w(t,s)g^\Delta(s)\Delta s. \tag{3.5}$$

Using a similar calculation, we find

$$\begin{aligned}
\int_a^b |p_w(t,s)|\Delta s &= \int_a^t\left(\int_a^s w(\tau)\Delta\tau\right)\Delta s - \int_t^b\left(\int_b^s w(\tau)\Delta\tau\right)\Delta s \\
&= t\int_a^t w(\tau)\Delta\tau - \int_a^t w(s)\sigma(s)\Delta s + t\int_b^t w(\tau)\Delta\tau + \int_t^b w(s)\sigma(s)\Delta s \\
&= \int_a^b \sigma(s)w(s)\operatorname{sgn}(s-t)\Delta s - t\int_a^b w(s)\operatorname{sgn}(s-t)\Delta s \\
&= \int_a^b (\sigma(s)-t)w(s)\operatorname{sgn}(s-t)\Delta s.
\end{aligned} \tag{3.6}$$

Now multiplying (3.4) by $g(t)$ and (3.5) by $f(t)$, adding the resulting identities, rewriting, and taking absolute values, we have

$$\begin{aligned}
&\left|f(t)g(t) - \frac{1}{2m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s + f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right]\right| \\
&= \frac{1}{2m(a,b)}\left|g(t)\int_a^b p_w(t,s)f^\Delta(s)\Delta s + f(t)\int_a^b p_w(t,s)g^\Delta(s)\Delta s\right| \\
&\le \frac{1}{2m(a,b)}\left[|g(t)|\int_a^b |p_w(t,s)|\left|f^\Delta(s)\right|\Delta s + |f(t)|\int_a^b |p_w(t,s)|\left|g^\Delta(s)\right|\Delta s\right].
\end{aligned} \tag{3.7}$$

Using now (3.1) and (3.6) in (3.7), we obtain (3.2).

Next, multiplying the left and right sides of (3.4) and (3.5) and taking absolute values, we get

$$\begin{aligned}
&\Bigg|f(t)g(t)-\frac{1}{m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s+f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right]\\
&\qquad+\left(\frac{1}{m(a,b)}\int_a^b w(s)f(\sigma(s))\Delta s\right)\left(\frac{1}{m(a,b)}\int_a^b w(s)g(\sigma(s))\Delta s\right)\Bigg|\\
&=\frac{1}{m^2(a,b)}\left|\left(\int_a^b p_w(t,s)f^\Delta(s)\Delta s\right)\left(\int_a^b p_w(t,s)g^\Delta(s)\Delta s\right)\right|\\
&\le\frac{1}{m^2(a,b)}\left(\int_a^b |p_w(t,s)|\left|f^\Delta(s)\right|\Delta s\right)\left(\int_a^b |p_w(t,s)|\left|g^\Delta(s)\right|\Delta s\right).
\end{aligned}\tag{3.8}$$

Using now (3.1) and (3.6) in (3.8), we obtain (3.3). □

Corollary 3.2. *In addition to the assumptions of Theorem 3.1, let* $w(t)=1$ *for all* $t\in[a,b]$*. Then for all* $t\in[a,b]$*, we have*

$$\begin{aligned}
&\left|f(t)g(t)-\frac{1}{2(b-a)}\left[g(t)\int_a^b f(\sigma(s))\Delta s+f(t)\int_a^b g(\sigma(s))\Delta s\right]\right|\\
&\qquad\le\frac{h_2(t,a)+g_2(b,t)}{b-a}\,\frac{|g(t)|\left\|f^\Delta\right\|_\infty+|f(t)|\left\|g^\Delta\right\|_\infty}{2}
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\Bigg|f(t)g(t)-\frac{1}{b-a}\left[g(t)\int_a^b f(\sigma(s))\Delta s+f(t)\int_a^b g(\sigma(s))\Delta s\right]\\
&\qquad+\left(\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(\sigma(s))\Delta s\right)\Bigg|\\
&\qquad\le\left(\frac{h_2(t,a)+g_2(b,t)}{b-a}\right)^2\left\|f^\Delta\right\|_\infty\left\|g^\Delta\right\|_\infty.
\end{aligned}\tag{3.10}$$

Proof. We just have to use Theorem 3.1 and

$$\begin{aligned}
\int_a^b(\sigma(s)-t)\operatorname{sgn}(s-t)\Delta s&=-\int_a^t(\sigma(s)-t)\Delta s+\int_t^b(\sigma(s)-t)\Delta s\\
&=\int_t^a(\sigma(s)-t)\Delta s+\int_t^b(\sigma(s)-t)\Delta s\\
&=g_2(a,t)+g_2(b,t)=h_2(t,a)+g_2(b,t),
\end{aligned}$$

where we also applied Theorem 2.3, Definition 2.4, and [5, Theorem 1.112]. □

Example 3.3. If we let $g(t)=1$ for all $t\in[a,b]$, then (3.9) becomes

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right|\le\frac{h_2(t,a)+g_2(b,t)}{b-a}\left\|f^\Delta\right\|_\infty,\tag{3.11}$$

which is the Ostrowski inequality on time scales as given in [3, Theorem 3.5]. If $\mathbb{T}=\mathbb{R}$ in (3.11), then we obtain (1.1) in Theorem 1.1. If $\mathbb{T}=\mathbb{Z}$, $a=0$, and $b=n\in\mathbb{N}$ in (3.11), then we obtain

$$\left|f(t)-\frac{1}{n}\sum_{s=1}^{n}f(s)\right|\le\frac{1}{n}\left[\frac{n^2-1}{4}+\left(t-\frac{n+1}{2}\right)^2\right]\|\Delta f\|_\infty,$$

an inequality that is given by S. Dragomir in [8, Theorem 3.1].

Example 3.4. If we let $\mathbb{T}=\mathbb{R}$, then (3.9) and (3.10) become

$$\left|f(t)g(t)-\frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\mathrm{d}s+f(t)\int_a^b g(s)\mathrm{d}s\right]\right|$$
$$\le\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\frac{|g(t)|\|f'\|_\infty+|f(t)|\|g'\|_\infty}{2}$$

and

$$\left|f(t)g(t)-\frac{1}{b-a}\left[g(t)\int_a^b f(s)\mathrm{d}s+f(t)\int_a^b g(s)\mathrm{d}s\right]\right.$$
$$\left.+\left(\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s\right)\left(\frac{1}{b-a}\int_a^b g(s)\mathrm{d}s\right)\right|$$
$$\le\left(\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\right)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty,$$

respectively.

Example 3.5. If we let $\mathbb{T}=\mathbb{Z}$, $a=0$, and $b=n\in\mathbb{N}$, then (3.9) and (3.10) become

$$\left|f(t)g(t)-\frac{1}{2n}\left[g(t)\sum_{s=1}^{n}f(s)+f(t)\sum_{s=1}^{n}g(s)\right]\right|$$
$$\le\frac{1}{n}\left[\frac{n^2-1}{4}+\left(t-\frac{n+1}{2}\right)^2\right]\frac{|g(t)|\|\Delta f\|_\infty+|f(t)|\|\Delta g\|_\infty}{2}$$

and

$$\left|f(t)g(t)-\frac{1}{n}\left[g(t)\sum_{s=1}^{n}f(s)+f(t)\sum_{s=1}^{n}g(s)\right]+\left(\frac{1}{n}\sum_{s=1}^{n}f(s)\right)\left(\frac{1}{n}\sum_{s=1}^{n}g(s)\right)\right|$$
$$\le\left(\frac{1}{n}\left[\frac{n^2-1}{4}+\left(t-\frac{n+1}{2}\right)^2\right]\right)^2\|\Delta f\|_\infty\|\Delta g\|_\infty,$$

respectively. This is the discrete Ostrowski–Grüss inequality, which can be found in [16, Theorem 2.1].

4 Weighted Ostrowski–Grüss Inequalities in L^p-Norm

Theorem 4.1. *Assume* (H). *Let* $p > 1$ *and* $q := p/(p-1)$. *If* $f, g \in \mathrm{C}^1_{\mathrm{rd}}([a,b],\mathbb{R})$ *such that* $f^\Delta, g^\Delta \in L^p([a,b])$*, i.e.,*

$$\left\|f^\Delta\right\|_p := \left(\int_a^b \left|f^\Delta(s)\right|^p \Delta s\right)^{\frac{1}{p}} < \infty \quad \text{and} \quad \left\|g^\Delta\right\|_p = \left(\int_a^b \left|g^\Delta(s)\right|^p \Delta s\right)^{\frac{1}{p}} < \infty,$$

then for all $t \in [a,b]$*, we have*

$$\left|f(t)g(t) - \frac{1}{2m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s + f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right]\right| \leq \left\|\frac{p_w(t,\cdot)}{m(a,b)}\right\|_q \frac{|g(t)|\left\|f^\Delta\right\|_p + |f(t)|\left\|g^\Delta\right\|_p}{2} \tag{4.1}$$

and

$$\left|f(t)g(t) - \frac{1}{m(a,b)}\left[g(t)\int_a^b w(s)f(\sigma(s))\Delta s + f(t)\int_a^b w(s)g(\sigma(s))\Delta s\right] + \left(\frac{1}{m(a,b)}\int_a^b w(s)f(\sigma(s))\Delta s\right)\left(\frac{1}{m(a,b)}\int_a^b w(s)g(\sigma(s))\Delta s\right)\right| \leq \left\|\frac{p_w(t,\cdot)}{m(a,b)}\right\|_q^2 \left\|f^\Delta\right\|_p \left\|g^\Delta\right\|_p. \tag{4.2}$$

Proof. As in the proof of Theorem 3.1, we obtain (3.7) and (3.8). From (3.7) and (3.8), using Hölder's inequality on time scales (see [5, Theorem 6.13]), we obtain (4.1) and (4.2), respectively. □

Corollary 4.2. *In addition to the assumptions of Theorem 4.1, let* $w(t) = 1$ *for all* $t \in [a,b]$*. Then for all* $t \in [a,b]$*, we have*

$$\left|f(t)g(t) - \frac{1}{2(b-a)}\left[g(t)\int_a^b f(\sigma(s))\Delta s + f(t)\int_a^b g(\sigma(s))\Delta s\right]\right| \leq \left(\int_a^t \left(\frac{s-a}{b-a}\right)^q \Delta s + \int_t^b \left(\frac{b-s}{b-a}\right)^q \Delta s\right)^{\frac{1}{q}} \frac{|g(t)|\left\|f^\Delta\right\|_p + |f(t)|\left\|g^\Delta\right\|_p}{2} \tag{4.3}$$

and

$$\left|f(t)g(t) - \frac{1}{b-a}\left[g(t)\int_a^b f(\sigma(s))\Delta s + f(t)\int_a^b g(\sigma(s))\Delta s\right] + \left(\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right)\left(\frac{1}{b-a}\int_a^b g(\sigma(s))\Delta s\right)\right| \leq \left(\int_a^t \left(\frac{s-a}{b-a}\right)^q \Delta s + \int_t^b \left(\frac{b-s}{b-a}\right)^q \Delta s\right)^{\frac{2}{q}} \left\|f^\Delta\right\|_p \left\|g^\Delta\right\|_p. \tag{4.4}$$

Proof. We just have to use Theorem 4.1. □

Example 4.3. If we let $g(t)=1$ for all $t\in[a,b]$, then (4.3) becomes

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right| \le \left(\int_a^t \left(\frac{s-a}{b-a}\right)^q \Delta s+\int_t^b \left(\frac{b-s}{b-a}\right)^q \Delta s\right)^{\frac{1}{q}} \left\|f^\Delta\right\|_p, \tag{4.5}$$

which is a new time scales Ostrowski inequality. If $\mathbb{T}=\mathbb{R}$ in (4.5), then we obtain

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s\right| \le \left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

an inequality that is given by S. Dragomir and S. Wang in [9], see also [7, Theorem 2]. If $\mathbb{T}=\mathbb{Z}$, $a=0$, and $b=n\in\mathbb{N}$ in (4.5), then we obtain

$$\left|f(t)-\frac{1}{n}\sum_{s=1}^n f(s)\right| \le \frac{1}{n}\left(\sum_{s=1}^{t-1} s^q+\sum_{s=1}^{n-t} s^q\right)^{\frac{1}{q}} \|\Delta f\|_p,$$

which turns into, e.g., when $p=q=2$,

$$\left|f(t)-\frac{1}{n}\sum_{s=1}^n f(s)\right| \le \frac{1}{n}\sqrt{\frac{(t-1)t(2t-1)+(n-t)(n-t+1)(2n-2t+1)}{6}}\,\|\Delta f\|_2.$$

Example 4.4. If we let $\mathbb{T}=\mathbb{R}$, then (4.1) and (4.2) become

$$\left|f(t)g(t)-\frac{1}{2m(a,b)}\left[g(t)\int_a^b w(s)f(s)\mathrm{d}s+f(t)\int_a^b w(s)g(s)\mathrm{d}s\right]\right| \le \left\|\frac{p_w(t,\cdot)}{m(a,b)}\right\|_q \frac{|g(t)|\,\|f'\|_p+|f(t)|\,\|g'\|_p}{2}$$

and

$$\left|f(t)g(t)-\frac{1}{m(a,b)}\left[g(t)\int_a^b w(s)f(s)\mathrm{d}s+f(t)\int_a^b w(s)g(s)\mathrm{d}s\right] + \left(\frac{1}{m(a,b)}\int_a^b w(s)f(s)\mathrm{d}s\right)\left(\frac{1}{m(a,b)}\int_a^b w(s)g(s)\mathrm{d}s\right)\right| \le \left\|\frac{p_w(t,\cdot)}{m(a,b)}\right\|_q^2 \left\|f'\right\|_p \left\|g'\right\|_p,$$

respectively, and (4.3) and (4.4) become (1.2) and (1.3), respectively, in Theorem 1.2, and by choosing $t=(a+b)/2$ in these inequalities, we obtain the inequalities given in [2, Remark 2].

Example 4.5. If we let $\mathbb{T} = \mathbb{Z}$, $a = 0$, and $b = n \in \mathbb{N}$, then (4.3) and (4.4) become

$$\left| f(t)g(t) - \frac{1}{2n}\left[g(t)\sum_{s=1}^{n} f(s) + f(t)\sum_{s=1}^{n} g(s) \right] \right| \leq \frac{1}{n}\left(\sum_{s=1}^{t-1} s^q + \sum_{s=1}^{n-t} s^q \right)^{\frac{1}{q}} \frac{|g(t)|\,\|\Delta f\|_p + |f(t)|\,\|\Delta g\|_p}{2}$$

and

$$\left| f(t)g(t) - \frac{1}{n}\left[g(t)\sum_{s=1}^{n} f(s) + f(t)\sum_{s=1}^{n} g(s) \right] + \left(\frac{1}{n}\sum_{s=1}^{n} f(s) \right)\left(\frac{1}{n}\sum_{s=1}^{n} g(s) \right) \right| \leq \left(\frac{1}{n}\left(\sum_{s=1}^{t-1} s^q + \sum_{s=1}^{n-t} s^q \right)^{\frac{1}{q}} \right)^2 \|\Delta f\|_p\,\|\Delta g\|_p,$$

respectively, which are new discrete Ostrowski–Grüss inequalities.

References

[1] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002. Dynamic equations on time scales.

[2] Martin Bohner and Thomas Matthews. The Grüss inequality on time scales. *Commun. Math. Anal.*, 3(1):1–8 (electronic), 2007.

[3] Martin Bohner and Thomas Matthews. Ostrowski inequalities on time scales. *JIPAM. J. Inequal. Pure Appl. Math.*, 9(1):Article 6, 8, 2008.

[4] Martin Bohner, Thomas Matthews, and Adnan Tuna. Diamond-alpha Grüss type inequality on time scales. *Int. J. Dyn. Syst. Differ. Equ.*, 3(1/2):234–247, 2011.

[5] Martin Bohner and Allan Peterson. *Dynamic equations on time scales*. Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[6] Martin Bohner and Allan Peterson. *Advances in dynamic equations on time scales*. Birkhäuser Boston Inc., Boston, MA, 2003.

[7] Sever S. Dragomir. A generalization of the Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a,b]$ and applications in numerical integration. *J. Math. Anal. Appl.*, 255(2):605–626, 2001.

[8] Sever S. Dragomir. The discrete version of Ostrowski's inequality in normed linear spaces. *JIPAM. J. Inequal. Pure Appl. Math.*, 3(1):Article 2, 16 pp. (electronic), 2002.

[9] Sever S. Dragomir and Song Wang. A new inequality of Ostrowski's type in L_p-norm. *Indian J. Math.*, 40(3):299–304, 1998.

[10] Stefan Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*. PhD thesis, Universität Würzburg, 1988.

[11] Wenjun Liu and Quốc-Anh Ngô. An Ostrowski-Grüss type inequality on time scales. *Comput. Math. Appl.*, 58(6):1207–1210, 2009.

[12] Dragoslav S. Mitrinović, Josip E. Pečarić, and Arlington M. Fink. *Inequalities involving functions and their integrals and derivatives*, volume 53 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[13] Dragoslav S. Mitrinović, Josip E. Pečarić, and Arlington M. Fink. *Classical and new inequalities in analysis*, volume 61 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[14] Quốc Anh Ngô and Wenjun Liu. A sharp Grüss type inequality on time scales and application to the sharp Ostrowski-Grüss inequality. *Commun. Math. Anal.*, 6(2):33–41, 2009.

[15] Alexander Ostrowski. Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. *Comment. Math. Helv.*, 10(1):226–227, 1937.

[16] Baburao G. Pachpatte. New discrete Ostrowski-Grüss like inequalities. *Facta Univ. Ser. Math. Inform.*, 22(1):15–20, 2007.

[17] Baburao G. Pachpatte. On a new generalization of Ostrowski type inequality. *Tamkang J. Math.*, 38(4):335–339, 2007.

[18] Mehmet Zeki Sarikaya. New weighted Ostrowski and Čebyšev type inequalities on time scales. *Comput. Math. Appl.*, 60(5):1510–1514, 2010.

[19] Adnan Tuna and Durmus Daghan. Generalization of Ostrowski and Ostrowski-Grüss type inequalities on time scales. *Comput. Math. Appl.*, 60(3):803–811, 2010.