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INTERIOR CONTROLLABILITY OF THE THERMOELASTIC PLATE EQUATION

HUGO LEIVA*
Department of Mathematics, Universidad de Los Andes,
Merida 5101- Venezuela

NELSON MERENTES†
Department of Mathematics, Universidad Central de Venezuela,
Caracas, Venezuela

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Abstract

In this paper we prove the interior controllability of the Thermoelastic Plate Equation

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta w = 1_\omega u_1(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = 1_\omega u_2(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ \theta = w = \Delta w = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \end{cases}$$

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed control $u_i \in L^2([0,\tau];L^2(\Omega)), i = 1,2$. Specifically, we prove the following statement: For all $\tau > 0$ the system is approximately controllable on $[0,\tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time $\tau > 0$.

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1 Introduction

This paper has been motivated by the works in [2], [8], [9], [10] and [12], where a new technique is used to prove the approximate controllability of some diffusion process.

Following [2] and [9], in this paper we study the interior approximate controllability of Thermoelastic Plate Equation

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta w = 1_\omega u_1(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = 1_\omega u_2(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ \theta = w = \Delta w = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \end{cases} \tag{1.1}$$

*E-mail address: hleiva@ula.ve
†E-mail address: nmerucv@gmail.com

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u_i \in L^2([0,\tau];L^2(\Omega)), i = 1,2.$ and and w, θ denote the vertical deflection and the temperature of the plate respectively. The derivation of the uncontrolled($u_i = 0, i = 1,2$) thermoelastic plate equation

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0, \ t \geq 0, \ x \in \Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = 0, \ t \geq 0, \ x \in \Omega, \\ \theta = w = \Delta w = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1.2}$$

can be found in J. Lagnese [7], where the author discussed stability of various plate models. J.U. Kim [6](1992) studied the system (1.2) with the following homogeneous Dirichlet boundary condition

$$\theta = \frac{\partial w}{\partial \eta} = w = 0, \ \text{on} \ \partial\Omega,$$

and he proved the exponential decay of the energy. Also, the stability of system (1.2) has been studied in [13].

Also, the controllability of system (1.2) with the controls acting in the whole set Ω was studied in [12]; more precise, the author study the approximate controllability of the following thermoelastic plate equation with Dirichlet boundary condition

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = a_1(x)u_1 + \cdots + a_m(x)u_m, \ t \geq 0, \ x \in \Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = b_1(x)u_1 + \cdots + b_m(x)u_m, \ t \geq 0, \ x \in \Omega, \\ \theta = w = \Delta w = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1.3}$$

where the controls $u_i \in L^2(0,t_1;\mathbb{R}); i = 1,2,\ldots,m.$

Moreover, the approximate controllability of the following thermoelastic plate equation with the controls acting in the whole set Ω is proved in [8]

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta w = u_1(t,x), \quad t > 0, \quad x \in \Omega \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = u_2(t,x), \quad t > 0, \quad x \in \Omega \\ \theta = w = \Delta w = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $u_i \in L^2([0,\tau];L^2(\Omega)), i = 1,2..$

In this paper,we are interested in the interior approximate controllability of the thermoelastic equation, which is more interesting problem from the applications point of view since the control is acting only in a subset or part of the plate Ω. Roughly speaking we prove the following statement: For all $\tau > 0$ the system is approximately controllable on $[0,\tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 3.7).

Our technique is simple and rests on the shoulders of the following fundamental results:

Theorem 1.1. *[10] The eigenfunctions of $-\Delta$ with Dirichlet boundary condition on Ω are real analytic functions.*

Theorem 1.2. *[1] Suppose $\Omega \subset \mathbb{R}^n$ is an open, non-empty and connected set, and f is a real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω. Then, $f = 0$ in Ω.*

2 Abstract Formulation of the Problem.

Let $Z = L^2(\Omega)$ and consider the linear unbounded operator
$A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{2.1}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with multiplicity γ_n equal to the dimension of the corresponding eigenspace.
a) There exists a complete orthonormal set $\{\phi_n\}$ of eigenvectors of A.

b) For all $z \in D(A)$ we have

$$Az = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n z, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n z = \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k}. \tag{2.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in z and

$$z = \sum_{n=1}^{\infty} E_n z, \quad z \in Z. \tag{2.4}$$

c) $-A$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n z. \tag{2.5}$$

d) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{j=1}^{\infty} \lambda_j^r E_j x. \tag{2.6}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X \times X$, which is a Hilbert Space with norm given by

$$\left\| \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2 + \|\theta\|.$$

Hence, (1.1) can be written as an abstract system of ordinary differential equations in the Hilbert space $Z = X^1 \times X \times X$ as follows:

$$\begin{cases} w' = v \\ v' = -A^2 w + \alpha A w + 1_\omega u_1 \\ \theta' = -\beta A\theta - \alpha A v + 1_\omega u_2 \end{cases} \tag{2.7}$$

Finally, system (1.1) can be rewritten as a first order system of ordinary differential equations in the Hilbert space $Z = X^1 \times X \times X$ as follows:

$$z' = \mathcal{A}z + B_\omega u, \quad z \in Z \quad t \geq 0, \tag{2.8}$$

where $u \in L^2([0,\tau];U)$, $U = L^2(\Omega) \times L^2(\Omega)$,

$$\mathcal{A} = \begin{bmatrix} 0 & I_X & 0 \\ -A^2 & 0 & -\alpha A \\ 0 & \alpha A & -\beta A \end{bmatrix}, \tag{2.9}$$

is an unbounded linear operator with domain

$$D(\mathcal{A}) = \{w \in H^4(\Omega) : w = \Delta w = 0\} \times D(A) \times D(A),$$

and $B : U \longrightarrow Z$, $B_\omega = \begin{bmatrix} 0 & 0 \\ 1_\omega & 0 \\ 0 & 1_\omega \end{bmatrix}$ is a bounded linear operator.

Proposition 2.1. *The adjoint of operators B_Ω and B_ω are given by*

$$B_\Omega^* = \begin{bmatrix} 0 & I_X & 0 \\ 0 & 0 & I_X \end{bmatrix}, \quad B_\omega^* = \begin{bmatrix} 0 & 1_\omega & 0 \\ 0 & 0 & 1_\omega \end{bmatrix}$$

Now, we shall prove that the linear unbounded operator $\mathcal{A}$ given by the linear thermoelastic plate equation (2.9) generates a strongly continuous semigroup which decays exponentially to zero. To this end, we will use the following Lemma from [11].

Lemma 2.2. *Let Z be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in Z with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_n P_n = P_n A_n, \ \ n = 1,2,3,\ldots \tag{2.10}$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad t \geq 0. \tag{2.11}$$

Then:
(a) $T(t)$ is a linear bounded operator if

$$\|e^{A_n t}\| \leq g(t), \ \ n = 1,2,3,\ldots \tag{2.12}$$

for some continuous real-valued function $g(t)$.
(b) Under the condition (2.12) $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup in the Hilbert space Z whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \quad z \in D(\mathcal{A}) \tag{2.13}$$

with

$$D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\} \tag{2.14}$$

(c) The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{2.15}$$

where $\bar{A}_n = A_n P_n$.

Theorem 2.3. *The operator $\mathcal{A}$, given by (2.9), is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ represented by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z_1, \quad t \geq 0 \tag{2.16}$$

where $\{P_j\}_{j\geq 0}$ is a complete family of orthogonal projections in the Hilbert space Z_1 given by

$$P_j = \begin{bmatrix} E_j & 0 & 0 \\ 0 & E_j & 0 \\ 0 & 0 & E_j \end{bmatrix}, \quad j = 1, 2, \ldots, \infty, \tag{2.17}$$

and

$$A_j = B_j P_j, \ \ B_j = \begin{bmatrix} 0 & 1 & 0 \\ -\lambda_j^2 & 0 & \alpha\lambda_j \\ 0 & -\alpha\lambda_j & -\beta\lambda_j. \end{bmatrix}, \ j \geq 1. \tag{2.18}$$

Moreover, the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$, $\sigma_3(j)$ of the matrix B_j are simple and given by:

$$\sigma_1(j) = -\lambda_j\rho_1, \ \ \sigma_2(j) = -\lambda_j\rho_2, \ \ \sigma_3(j) = -\lambda_j\rho_3$$

where $\rho_i > 0, i = 1, 2, 3$ are the roots of the characteristic equation

$$\rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0,$$

and this semigroup decays exponentially to zero

$$\|T(t)\| \leq Me^{-\mu t}, \ \ t \geq 0, \tag{2.19}$$

where

$$\mu = \lambda_1 min\{Re(\rho) : \ \ \rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0\}$$

Proof Let us compute $\mathcal{A}z$:

$$
\begin{aligned}
\mathcal{A}z &= \begin{bmatrix} 0 & I & 0 \\ -A^2 & 0 & \alpha A \\ 0 & -\alpha A & -\beta A \end{bmatrix} \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \\
&= \begin{bmatrix} v \\ -A^2 w + \alpha A\theta \\ -\alpha A v - \beta A\theta \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^{\infty} E_j v \\ -\sum_{j=1}^{\infty} \lambda_j^2 E_j w + \alpha \sum_{j=1}^{\infty} \lambda_j E_j \theta \\ -\alpha \sum_{j=1}^{\infty} \lambda_j E_j v - \beta \sum_{j=1}^{\infty} \lambda_j E_j \theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} \begin{bmatrix} E_j v \\ -\lambda_j^2 E_j w + \alpha\lambda_j E_j \theta \\ -\alpha\lambda_j E_j v - \beta\lambda_j E_j \theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} \begin{bmatrix} 0 & 1 & 0 \\ -\lambda_j^2 & 0 & \alpha\lambda_j \\ 0 & -\alpha\lambda_j & -\beta\lambda_j \end{bmatrix} \begin{bmatrix} E_j & 0 & 0 \\ 0 & E_j & 0 \\ 0 & 0 & E_j \end{bmatrix} \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} A_j P_j z.
\end{aligned}
$$

It is clear that $A_j P_j = P_j A_j$. Now, we need to check condition (2.12) from Lemma 2.2. To this end, we have to compute the spectrum of the matrix B_j. The characteristic equation of B_j is given by

$$\lambda^3 + \beta\lambda_j\lambda^2 + \lambda_j^2(1+\alpha^2)\lambda + \beta\lambda_j^3 = 0.$$

Then,

$$\left(\frac{\lambda}{\lambda_j}\right)^3 + \beta\left(\frac{\lambda}{\lambda_j}\right)^2 + \lambda_j^2(1+\alpha^2)\left(\frac{\lambda}{\lambda_j}\right) + \beta = 0.$$

Letting $\frac{\lambda}{\lambda_j} = -\rho$ we obtain the equation

$$\rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0. \tag{2.20}$$

From Routh Hurwitz Theorem we obtain that the real part of the roots ρ_1, ρ_2, ρ_3 of equation (2.20) are positive. Therefore, the eigenvalues $\sigma_1(j), \sigma_2(j)$, $\sigma_3(j)$ of B_j are given by

$$\sigma_1(j) = -\lambda_j\rho_1, \;\; \sigma_2(j) = -\lambda_j\rho_2, \;\; \sigma_3(j) = -\lambda_j\rho_3. \tag{2.21}$$

Since the eigenvalues of B_j are simple, there exists a complete family of complementaries projections $\{q_i(j)\}_{i=1}^3$ in $\mathbb{R}^3$ such that

$$\begin{cases} B_j &= \sigma_1(j)q_1(j) + \sigma_1(j)q_2(j) + \sigma_1(j)q_3(j) \\ e^{B_j t} &= e^{-\lambda_j\rho_1 t}q_1(j) + e^{-\lambda_j\rho_2 t}q_2(j) + e^{-\lambda_j\rho_3 t}q_3(j), \end{cases}$$

where $q_i(j),\ \ i=1,2,3$ are given by:

$$
\begin{aligned}
q_1(j) &= \frac{1}{(\rho_1-\rho_2)(\rho_1-\rho_3)}\begin{bmatrix} \rho_2\rho_3-1 & \frac{\rho_2+\rho_3}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_3-\rho_2) & \rho_2\rho_3-1-\alpha^2 & \alpha(\rho_2+\rho_3-\beta) \\ \lambda_j\alpha & -\alpha(\rho_2+\rho_3-\beta) & (\rho_3-\beta)^2-\alpha^2, \end{bmatrix} \\
q_2(j) &= \frac{1}{(\rho_2-\rho_1)(\rho_2-\rho_3)}\begin{bmatrix} \rho_1\rho_3-1 & \frac{\rho_1+\rho_3}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_3-\rho_1) & \rho_1\rho_3-1-\alpha^2 & \alpha(\rho_1+\rho_3-\beta) \\ \lambda_j\alpha & -\alpha(\rho_1+\rho_3-\beta) & (\rho_3-\beta)^2-\alpha^2, \end{bmatrix} \\
q_3(j) &= \frac{1}{(\rho_3-\rho_1)(\rho_3-\rho_2)}\begin{bmatrix} \rho_1\rho_2-1 & \frac{\rho_1+\rho_2}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_2-\rho_1) & \rho_1\rho_2-1-\alpha^2 & \alpha(\rho_1+\rho_2-\beta) \\ \lambda_j\alpha & -\alpha(\rho_1+\rho_2-\beta) & (\rho_2-\beta)^2-\alpha^2. \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{cases} A_j & = \sigma_1(j)P_{j1}+\sigma_1(j)P_{j2}+\sigma_1(j)P_{j3} \\ e^{A_jt} & = e^{-\lambda_j\rho_1 t}P_{j1}+e^{-\lambda_j\rho_2 t}P_{j2}+e^{-\lambda_j\rho_3 t}P_{j3}, \end{cases}
$$

and

$$
\mathcal{A}z=\sum_{j=1}^{\infty}\left\{\sigma_1(j)P_{j1}z+\sigma_2(j)P_{j2}z+\sigma_3(j)P_{j3}z\right\}, \tag{2.22}
$$

where, $P_{ji}=q_i(j)P_j$ is a complete family of orthogonal projections in Z_1.

To prove that $e^{A_nt}P_n : Z_1 \to Z_1$ satisfies condition (2.12) from Lemma 2.2, it will be enough to prove for example that $e^{-\lambda_n\rho_2 t}q_2(n)P_n, n=1,2,3,\dots$ satisfies the condition. In fact, consider $z=(z_1,z_2,z_3)^T \in Z_1$ such that $\|z\|=1$. Then,

$$
\|z_1\|_1^2=\sum_{j=1}^{\infty}\lambda_j^2\|E_jz_1\|^2\le 1,\ \ \|z_2\|_X^2=\sum_{j=1}^{\infty}\|E_jz_2\|^2\le 1\ \text{ and }\ \|z_3\|_X^2=\sum_{j=1}^{\infty}\|E_jz_3\|^2\le 1.
$$

Therefore, $\lambda_j\|E_jz_1\|\le 1,\ \ \|E_jz_2\|\le 1,\ \ \|E_jz_3\|\le 1\ \ j=1,2,\dots$. Then,

$$
|e^{-\lambda_j\rho_2 t}q_2(n)P_nz\|_{Z_1}^2=
$$

$$
\frac{e^{-2\lambda\rho_2 t}}{(\rho_2-\rho_1)^2(\rho_2-\rho_3)^2}\left\|\begin{matrix} (\rho_1\rho_3-1)E_nz_1+\frac{\rho_1+\rho_3}{\lambda_n}E_nz_2+\frac{\alpha}{\lambda_n}E_nz_3 \\ \lambda_n(\rho_3-\rho_1)E_nz_1+(\rho_1\rho_3-1-\alpha^2)E_nz_2+\alpha(\rho_1+\rho_3-\beta)E_nz_3 \\ \lambda_n\alpha E_nz_1+-\alpha(\rho_1+\rho_3-\beta)E_nz_2+[(\rho_3-\beta)^2-\alpha^2]E_nz_3 \end{matrix}\right\|_{Z_1}^2
$$

$$
\begin{aligned}
&= e^{-2\lambda_n\rho_2 t}\sum_{j=1}^{\infty}\lambda_j^2\|E_j\left((\rho_1\rho_3-1)E_nz_1+\frac{\rho_1+\rho_3}{\lambda_j}E_nz_2+\frac{\alpha}{\lambda_j}E_nz_3\right)\|^2\\
&+ e^{-2\lambda_n\rho_2}\sum_{j=1}^{\infty}\|E_j\left(\lambda_n(\rho_3-\rho_1)E_nz_1+(\rho_1\rho_3-1-\alpha^2)Ez_2+\alpha(\rho_1+\rho_3-\beta)Enz_3\right)\|^2\\
&+ e^{-2\lambda_n\rho_2 t}\sum_{j=1}^{\infty}\|E_j\left(\lambda_n\alpha E_nz_1+-\alpha(\rho_1+\rho_3-\beta)E_nz_2+[(\rho_3-\beta)^2-\alpha^2]E_nz_3\right)\|^2\\
&= e^{-2\lambda_n\rho_2 t}\lambda_n^2\|(\rho_1\rho_3-1)E_nz_1+\frac{\rho_1+\rho_3}{\lambda_n}E_nz_2+\frac{\alpha}{\lambda_n}E_nz_3\|^2\\
&+ e^{-2\lambda_n\rho_2 t}\|\lambda_n(\rho_3-\rho_1)E_nz_1+(\rho_1\rho_3-1-\alpha^2)E_nz_2\alpha(\rho_1+\rho_3-\beta)E_nz_3\|^2\\
&+ e^{-2\lambda_n\rho_2 t}\|\lambda_\alpha E_nz_1+-\alpha(\rho_1+\rho_3-\beta)E_nz_2+[(\rho_3-\beta)^2-\alpha^2]E_nz_3\|^2\\
&\leq e^{-2\lambda_n\rho_2 t}\left[|\rho_1\rho_3-1|+\rho_1+\rho_3+\alpha\right]^2\\
&+ e^{-2\lambda_n\rho_2 t}\left[|\rho_3-\rho_1|+|\rho_1\rho_3-1-\alpha^2|+\alpha|\rho_1+\rho_3-\beta|\right]^2\\
&+ e^{-2\lambda_n\rho_2 t}\left[\alpha+\alpha|\rho_1+\rho_3-\beta|+|(\rho_3-\beta)^2-\alpha^2|\right]^2\\
&\leq M^2e^{-2\lambda_n\rho_2 t}.
\end{aligned}
$$

where $M=M(\alpha,\beta)\geq 1$ depending on α and β. Then we have,

$$\|e^{-\lambda_n\rho_2 t}q_2(n)P_n\|_{Z_1}\leq M(\alpha,\beta)e^{-\lambda_n\rho_2 t},\ \ t\geq 0\ \ n=1,2,\ldots.$$

In the same way e obtain that

$$
\begin{aligned}
\|e^{-\lambda_n\rho_1 t}q_1(n)P_n\|_{Z_1} &\leq M(\alpha,\beta)e^{-\lambda_n\rho_1 t},\ \ t\geq 0\ \ n=1,2,\ldots,\\
\|e^{-\lambda_j\rho_3 t}q_3(n)P_n\|_{Z_1} &\leq M(\alpha,\beta)e^{-\lambda_n\rho_3 t},\ \ t\geq 0\ \ n=1,2,\ldots.
\end{aligned}
$$

Therefore,

$$\|e^{A_nt}P_n\|_{Z_1}\leq M(\alpha,\beta)e^{-\mu t},\ \ t\geq 0\ \ n=1,2,\ldots,$$

were

$$\mu=\lambda_1\min\{\mathrm{Re}(\rho):\ \ \rho^3-\beta\rho^2+(1+\alpha^2)\rho-\beta=0\}.$$

Hene, applying Lemma 2.2 we obtain that $\mathcal{A}$ generates a strongly contnuous semigroup given by (2.16). Next, we prove this semigroup decays exponentially to zero. In fact,

$$
\begin{aligned}
\|T(t)z\|^2 &= \sum_{j=1}^{\infty}\|e^{A_jt}P_jz\|^2\\
&\leq \sum_{j=1}^{\infty}\|e^{A_jt}\|^2\|P_jz\|^2\\
&\leq M^2(\alpha,\beta)e^{-2\mu t}\sum_{j=1}^{\infty}\|P_jz\|^2\\
&= M^2(\alpha,\beta)e^{-2\mu}\|z\|^2.
\end{aligned}
$$

Therefore,

$$\|T(t)\|\leq M(\alpha,\beta)e^{-\mu t},\ \ t\geq 0.$$

□

The following gap condition plays an important role in this paper

$$0 < \rho_1 < \rho_2 < \rho_3 \ \text{and} \ \frac{\lambda_{j+1}}{\lambda_j} > \frac{\rho_3}{\rho_1}, \quad j = 1, 2, 3, \dots. \tag{2.23}$$

Proposition 2.4. *The operator $P_j : Z_r \to Z_r, \quad j \geq 0$, defined by*

$$P_j = \begin{bmatrix} E_j & 0 & 0 \\ 0 & E_j & 0 \\ 0 & 0 & E_j \end{bmatrix}, \ j \geq 1, \tag{2.24}$$

is a continuous(bounded) orthogonal projections in the Hilbert space Z_r.

Proof First we shall show that $P_j(Z_r) \subset Z_r$, which is equivalent to show that $E_j(X^r) \subset X^r$. In fact, let x be in X^r and consider $E_j x$. Then

$$\sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n E_j x\|^2 = \lambda_j^{2r} \|E_j x\|^2 < \infty$$

Therefore, $E_j x \in X^r, \forall x \in X^r$.
Now, we shall prove that this projection is bounded. In fact, from the continuous inclusion $X^r \subset X$, there exists a constant $k > 0$ such that

$$\|x\| \leq k\|x\|_r, \ \ \forall x \in X^r.$$

Then, for all $x \in X^r$ we have the following estimate

$$\begin{aligned} \|E_j x\|_r^2 &= \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n E_j x\|^2 = \lambda_j^{2r} \|E_j x\|^2 \\ &\leq \lambda_j^{2r} \|x\|^2 \leq \lambda_j^{2r} k^2 \|x\|_r^2 \end{aligned}$$

Hence $\|E_j x\| \leq \lambda_j^r k \|x\|_r$, which implies the continuity of $E_j : X^r \to X^r$. So, P_j is a continuous projection on Z_r.

□

3 Proof of the Main Theorem

In this section we shall prove the main result of this paper on the controllability of the linear system (2.8). But, before we shall give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the the initial value problem

$$\begin{cases} z' = \mathcal{A}z + B_\omega u(t), z \in Z, \\ z(0) = z_0, \end{cases} \tag{3.1}$$

where the control function u belong to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s) B_\omega u(s) ds, \ \ t \in [0, \tau]. \tag{3.2}$$

Definition 3.1. (**Approximate Controllability**) The system (2.8) is said to be approximately controllable on $[0,\tau]$ if for every z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0,\tau;U)$ such that the solution $z(t)$ of (3.2) corresponding to u verifies:

$$z(0) = z_0 \text{ and } \|z(\tau) - z_1\| < \varepsilon.$$

Consider the following bounded linear operator:

$$G : L^2(0,\tau;Z) \to Z, \;\; Gu = \int_0^\tau T(\tau - s)B_\omega u(s)ds, \tag{3.3}$$

whose adjoint operator $G^* : Z \longrightarrow L^2(0,\tau;Z)$ is given by

$$(G^* z)(s) = B^*_\omega T^*(\tau - s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z. \tag{3.4}$$

The following lemma is trivial:

Lemma 3.2. *The equation (2.8) is approximately controllable on* $[0,\tau]$ *if, and only if,* $\overline{\text{Rang}(G)} = Z$.

The following result is well known from linear operator theory:

Lemma 3.3. *Let W and Z be Hilbert spaces and* $G^* \in L(Z,W)$ *the adjoint operator of the linear operator* $G \in L(W,Z)$. *Then*

$$\overline{\text{Rang}(G)} = Z \iff Ker(G^*) = \{0\}.$$

As a consequence of the foregoing Lemma one can prove the following result:

Lemma 3.4. *Let W and Z be Hilbert spaces and* $G^* \in L(Z,W)$ *the adjoint operator of the linear operator* $G \in L(W,Z)$. *Then* $\overline{\text{Rang}(G)} = Z$ *if, and only if, one of the following statements holds:*

a) $Ker(G^*) = \{0\}$.

b) $\langle GG^* z, z\rangle > 0$, $z \neq 0$ *in Z.*

c) $\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1} z = 0$.

d) $\sup_{\alpha > 0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1$.

The following theorem follows directly from (3.4), lemma 3.2 and lemma 3.4.

Theorem 3.5. *(2.8) is approximately controllable on* $[0,\tau]$ *iff*

$$B^*_\omega T^*(t)z = 0, \quad \forall t \in [0,\tau], \quad \Rightarrow z = 0. \tag{3.5}$$

For the proof of the main theorem of this paper we shall use the following version of Lemma 3.14 from [3] and Lemma 4.4 from [2].

Lemma 3.6. *Let* $\{\alpha_1(j)\}_{j\geq 1}$, $\{\beta_{1j}\}_{j\geq 1}$, $\{\alpha_2(j)\}_{j\geq 1}$,$\{\beta_{2j}\}_{j\geq 1}$ *and* $\{\alpha_3(j)\}_{j\geq 1}$, $\{\beta_{3j}\}_{j\geq 1}$, *be sequences of real numbers such that* $\alpha_3(j) < \alpha_2(j) < \alpha_1(j)$ *and*

$$\alpha_s(j+1) < \alpha_s(j), \quad \alpha_1(j+1) < \alpha_2(j), \quad \alpha_1(j+1) < \alpha_3(j), \quad \alpha_2(j+1) < \alpha_3(j). \tag{3.6}$$

for $s = 1,2,3; j = 1,2,3,\ldots$. *Then, for any* $\tau > 0$ *we have that*

$$\sum_{j=1}^{\infty} \left(e^{\alpha_1(j)t}\beta_{1j} + e^{\alpha_2(j)t}\beta_{2j} + e^{\alpha_3(j)t}\beta_{3j}\right) = 0, \quad \forall t \in [0,\tau] \tag{3.7}$$

if, and only if,

$$\beta_{1j} = \beta_{2j} = \beta_{3j} = 0, \forall j \geq 1. \tag{3.8}$$

Proof By analytic extension we obtain

$$\sum_{j=1}^{\infty} \left(e^{\alpha_1(j)t}\beta_{1j} + e^{\alpha_2(j)t}\beta_{2j} + e^{\alpha_3(j)t}\beta_{3j}\right) = 0, \quad \forall t \in [0,\infty).$$

Now, dividing this expression by $e^{\alpha_1(1)t}$ we get

$$\beta_{11} + \sum_{j=2}^{\infty} e^{(\alpha_1(j)-\alpha_1(1))t}\beta_{1j} + \sum_{j=1}^{\infty} e^{(\alpha_2(j)-\alpha_1(1))t}\beta_{2j} + \sum_{j=1}^{\infty} e^{(\alpha_3(j)-\alpha_1(1))t}\beta_{3j} = 0, \quad \forall t \in [0,\infty).$$

Since $\alpha_1(j)-\alpha_1(1)) < 0$ for $j > 1$ and $\alpha_2(j)-\alpha_1(1) < 0$, $\alpha_3(j)-\alpha_1(1) < 0$ for $j \geq 1$, then passing to the limit when $t \to \infty$ we obtain that $\beta_{11} = 0$
Then, we have that

$$\sum_{j=2}^{\infty} e^{\alpha_1(j)t}\beta_{1j} + \sum_{j=1}^{\infty} e^{\alpha_2(j)t}\beta_{2j} + \sum_{j=1}^{\infty} e^{\alpha_3(j)t}\beta_{3j} = 0, \quad \forall t \in [0,\infty).$$

Now, dividing this expression by $e^{\alpha_2(1)t}$ we get

$$\beta_{21} + \sum_{j=2}^{\infty} e^{(\alpha_1(j)-\alpha_2(1))t}\beta_{1j} + \sum_{j=2}^{\infty} e^{(\alpha_2(j)-\alpha_2(1))t}\beta_{2j} + \sum_{j=1}^{\infty} e^{(\alpha_3(j)-\alpha_2(1))t}\beta_{3j} = 0, \quad \forall t \in [0,\infty).$$

From (3.6) we have that $\alpha_1(j)-\alpha_2(1)) < 0$ and $\alpha_2(j)-\alpha_2(1) < 0$ for $j \geq 2$ and $\alpha_3(j)-\alpha_2(1) < 0$ for $j \geq 1$. Then passing to the limit when $t \to \infty$ we obtain that $\beta_{21} = 0$
Then, we have that

$$\sum_{j=2}^{\infty} e^{\alpha_1(j)t}\beta_{1j} + \sum_{j=2}^{\infty} e^{\alpha_2(j)t}\beta_{2j} + \sum_{j=1}^{\infty} e^{\alpha_3(j)t}\beta_{3j} = 0, \quad \forall t \in [0,\infty).$$

Now, dividing this expression by $e^{\alpha_3(1)t}$ we get

$$\beta_{31} + \sum_{j=2}^{\infty} e^{(\alpha_1(j)-\alpha_3(1))t}\beta_{1j} + \sum_{j=2}^{\infty} e^{(\alpha_2(j)-\alpha_3(1))t}\beta_{2j} + \sum_{j=2}^{\infty} e^{(\alpha_3(j)-\alpha_3(1))t}\beta_{3j} = 0, \quad \forall t \in [0,\infty).$$

From (3.6) we have that $\alpha_1(j)-\alpha_3(1))<0$, $\alpha_2(j)-\alpha_3(1)<0$ and $\alpha_3(j)-\alpha_3(1)<0$ for $j\geq 2$. Then passing to the limit when $t\to\infty$ we obtain that $\beta_{31}=0$.
Then, we have that

$$\sum_{j=2}^{\infty} e^{\alpha_1(j)t}\beta_{1j}+\sum_{j=2}^{\infty} e^{\alpha_2(j)t}\beta_{2j}+\sum_{j=2}^{\infty} e^{\alpha_3(j)t}\beta_{3j}=0, \quad \forall t\in[0,\infty).$$

Repeating this procedure from here, we would obtain that $\beta_{12}=\beta_{22}=\beta_{32}=0$, and continuing this way we get $\beta_{1j}=\beta_{2j}=\beta_{3j}=0, \forall j\geq 1$.

□

Now, we are ready to formulate and prove the main theorem of this work.

Theorem 3.7. *(Main Result) Under condition (2.23), for all nonempty open subset* ω *of* Ω *and* $\tau>0$ *the system (2.8) is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system (2.8) from initial state* z_0 *to an* ε *neighborhood of the final state* z_1 *at time* $\tau>0$ *is given by*

$$u_\alpha(t)=B_\omega^* T(\tau-t)(\alpha I+GG^*)^{-1}(z_1-T(\tau)z_0), \quad \alpha>0,$$

and the error of this approximation E_α *is given by*

$$E_\alpha=\alpha(\alpha I+GG^*)^{-1}(z_1-T(\tau)z_0), \quad \alpha>0.$$

Proof. We shall apply Theorem 3.5 to prove the controllability of system (2.8). To this end, we observe that

$$T^*(t)z=\sum_{j=1}^{\infty} e^{A_j^* t}P_j^* z, \ \ z\in Z, \ \ t\geq 0,$$

and, since the eigenvalues of the matrix A_j are simple, there exists a family of complete complementary projections $\{q_1(j),q_2(j),q_3(j)\}$ on $\mathbb{R}^3$ such that

$$e^{A_j^* t}=e^{\sigma_1(j)t}q_1^*(j)P_j^*+e^{\sigma_2(j)t}q_2^*(j)P_j^*+e^{\sigma_3(j)t}q_3^*(j)P_j^*.$$

Therefore,

$$B_\omega^* T^*(t)z=\sum_{j=1}^{\infty} B_\omega^* e^{A_j^* t}P_j^* z=\sum_{j=1}^{\infty}\sum_{s=1}^{3} e^{\sigma_s(j)t}B_\omega^* P_{s,j}^* z,$$

where $P_{s,j}=q_s(j)P_j=P_jq_s(j)$.

Now, suppose that $B_\omega^* T^*(t)z=0, \ \forall t\in[0,\tau]$. Then,

$$\begin{aligned} B_\omega^* T^*(t)z &= \sum_{j=1}^{\infty} B_\omega^* e^{A_j^* t}P_j^* z=\sum_{j=1}^{\infty}\sum_{s=1}^{3} e^{\sigma_s(j)t}B_\omega^* P_{s,j}^* z=0. \\ \iff & \sum_{j=1}^{\infty}\sum_{s=1}^{3} e^{\sigma_s(j)t}(B_\omega^* P_{s,j}^* z)(x)=0, \quad \forall x\in\Omega. \end{aligned}$$

The assumption (2.23) implies that the sequence $\{\alpha_s(j) = -\lambda_j \rho_s : s = 1,2,3; j = 1,2,\dots\}$ satisfies the conditions on Lemma 3.6. In fact, we have trivially that
$\alpha_3(j) < \alpha_2(j) < \alpha_1(j)$ and from (2.23) we obtain:

$$\frac{\lambda_{j+1}}{\lambda_j} > \frac{\rho_3}{\rho_1} > \frac{\rho_3}{\rho_2} \quad \text{and} \quad \frac{\lambda_{j+1}}{\lambda_j} > \frac{\rho_2}{\rho_1}.$$

Therefore,

$$-\lambda_{j+1}\rho_1 < -\lambda_j\rho_3, \quad -\lambda_{j+1}\rho_2 < -\lambda_j\rho_3, \quad -\lambda_{j+1}\rho_1 < -\lambda_j\rho_2$$

i.e.,

$$\alpha_1(j+1) < \alpha_2(j), \quad \alpha_1(j+1) < \alpha_3(j), \quad \alpha_2(j+1) < \alpha_3(j).$$

Then, from Lemma 3.6 we obtain for all x $\in \Omega$ that

$$(B_\omega^* P_{s,j}^* z)(x) = 0, \quad \forall x \in \Omega, \quad s = 1,2,3; \quad j = 1,2,3,\dots.$$

Since

$$q_i^*(j) = \begin{bmatrix} a_{11}^{ij} & a_{12}^{ij} & a_{13}^{ij} \\ a_{21}^{ij} & a_{22}^{ij} & a_{23}^{ij} \\ a_{31}^{ij} & a_{32}^{ij} & a_{33}^{ij} \end{bmatrix}, i = 1,2,3; \;\; j = 1,2,3,4,\dots,$$

we get $\forall x \in \Omega, \quad i = 1,2,3; \;\; j = 1,2,3,4,\dots$ that

$$(B_\omega^* P_{s,j}^* z)(x) = \begin{bmatrix} 1_\omega[a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) + a_{23}^{ij} E_j z_3(x)] \\ 1_\omega[a_{31}^{ij} E_j z_1(x) + a_{32}^{ij} E_j z_2(x) + a_{33}^{ij} E_j z_3(x)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is to say,

$$(B_\omega^* P_{s,j}^* z)(x) = \begin{bmatrix} a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) + a_{23}^{ij} E_j z_3(x) \\ a_{31}^{ij} E_j z_1(x) + a_{32}^{ij} E_j z_2(x) + a_{33}^{ij} E_j z_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall x \in \omega.$$

On the other hand, we know that $\phi_{n,k}$ are analytic functions, which implies the analyticity of $E_j z_i$. Then, from Theorem 1.1 we get for $i = 1,2,3; \;\; j = 1,2,3,4,\dots$ that

$$(B_\Omega^* P_{s,j}^* z)(x) = \begin{bmatrix} a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) + a_{23}^{ij} E_j z_3(x) \\ a_{31}^{ij} E_j z_1(x) + a_{32}^{ij} E_j z_2(x) + a_{33}^{ij} E_j z_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall x \in \Omega.$$

Hence

$$B_\Omega^* T^*(t) z = \sum_{j=1}^{\infty} B_\Omega^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^{3} e^{\sigma_s(j)t} B_\Omega^* P_{s,j}^* z = 0, \;\; \forall t \in [0,\tau].$$

Since system (1.4)(see [8]) is approximately controllable, then from Theorem 3.5 we get that $z = 0$.

□

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