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Asymptotic Behaviour of Solutions for Some Differential Equations in a Banach Space

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Abstract

We present some results of Eberlein-weakly almost periodic functions with values in a Banach space. Then, we apply these results to investigate the Eberlein-weakly almost periodic solutions of some differential equations in a Banach space.

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1 Introduction

The existence of periodic solutions or almost periodic solutions or Eberlein weak almost periodic solutions is very important in the qualitative studies of many problems. Among numerous results in this direction we mention the following result which is classical in the theory of ordinary differential equations. Let us consider the following system of differential equations in finite dimensional space

$$\frac{d}{dt}x(t) = Bx(t) + g(t), t \in \mathbb{R}, \tag{1.1}$$

where B is a constant $n \times n$-matrix and $g : \mathbb{R} \to R^n$ is a continuous and ω-periodic function. In [13], Massera studied the existence of periodic solutions of Equation (1.1) and he proved the equivalence between the existence of bounded solutions on $\mathbb{R}^+$ and the existence of ω-periodic solutions. As a generalization of this result, Bohr and Neugebauer, see [8], studied the existence of almost periodic solutions of Equation (1.1) in the case where the function g is almost periodic. More precisely, they proved that every bounded solution in the whole line is almost periodic solution. Here we propose to extend the Bohr-Neugebauer to the Equation (1.2).

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The introduction of Eberlein weakly almost periodic functions played a role in the theory of topological semigroups, where we only cite the book of Berglund, Junghenn and Milnes [14], and, more recently, in differential equations. The role of Eberlein weak almost periodicity in the latter context has been investigated by Ruess and Summers in [16] and [17]

In this paper, we study the asymptotic behavior of the solutions to some differential equations of the general form

$$\frac{d}{dt}x(t) \quad = \quad f(t,x(t)), \;\; t \in \mathbb{R}, \tag{1.2}$$

in a Banach space. The emphasis is on the weak almost periodicity in the sense of Eberlein properties of the solution, in particular on almost periodicity.

Main results specify conditions on the underlying Banach space and the map f, for which by using a result due to E. Hanebaly [10], and some dissipative type conditions for the term f leads to a Eberlein-weak almost periodic solution.

The goal of this work is to prove the existence of Eberlein weak almost periodic solutions of Equation (1.2) without the hyperbolicity condition. More precisely, we will show that the existence of an Eberlein weak almost periodic solution of Equation (1.2) is equivalent to the existence of a bounded solution on $\mathbb{R}^+$. Our approach is based on the Dissipativity property of f.

Our work is organized as follows. In section 2, we state some facts on Eberlein weak almost periodic functions with values in a Banach space; in section 3, we prove the main theorem on Eberlein weak almost periodic solution of the equation (1.2) under the essential property of dissipativity, we also consider the particular case of almost periodic situation. The last section is devoted to some examples to illustrat the work.

2 Notations and basic properties of Eberlein-weakly almost periodic functions

2.1 Semi - inner product

Throughout this paper, let $(E,\|\cdot\|)$ be a Banach space. For $x \in E$, $\|x\|$ be the norm of x. Given $x,y \in E$, we let $\langle x,y\rangle_+$ and $\langle x,y\rangle_-$ denote the upper and lower semi-inner product of x and y; i.e.,

$$\langle x,y\rangle_+ \quad = \quad \|y\| \lim_{t\to 0^+} \frac{1}{t}(\|y+tx\| - \|y\|)$$

and

$$\langle x,y\rangle_- \quad = \quad \|y\| \lim_{t\to 0^+} \frac{1}{t}(\|y\| - \|y-tx\|).$$

Remark 2.1. i) Both limits exist for every norm, and they coincide with the inner product, if E is a Hilbert space.
ii) In the case when E is a uniformly convex space we have $\langle x,y\rangle_+ = \langle x,y\rangle_-$.

The following lemma on the functionals $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_-$ is well know (see, for instance, [4]).

Lemma 2.2. *Let x, y and z be in E. Then, the functionals $\langle \cdot,\cdot \rangle_+$ and $\langle \cdot,\cdot \rangle_-$ have the following properties :*
(1) $\langle x,y\rangle_- = -\langle -x,y\rangle_+$;
(2) $\left|\langle x,y\rangle_{\pm}\right| \leq \|x\|\,\|y\|$;
(3) $\langle x+z,y\rangle_- \leq \langle x,y\rangle_- + \langle z,y\rangle_+$;
(4) $\langle x+z,y\rangle_+ \leq \langle x,y\rangle_+ + \langle z,y\rangle_+$;
(5) Let $x(.)$ be a function from a real interval J into E such that $x'(t_0)$ exists for an interior point t_0 of J. Then, $D^- \|x(t_0)\|$ exists and

$$D^- \|x(t_0)\| \quad = \quad \langle x'(t_0), x(t_0)\rangle_- ,$$

where $D^- \|x(t_0)\|$ denotes the left derivative of $\{t \to \|x(t)\|\}$ at t_0.

2.2 Eberlein-weakly almost periodic functions

Let (D,τ) be a topological space. The linear space of bounded continuous (respectively continuous) functions defined on the real line $\mathbb{R}$ (respectively on $\mathbb{R}\times D$) with values in E, will be denoted by $C_b(\mathbb{R},E)$ (respectively by $C(\mathbb{R}\times D,E)$), and for $s\in\mathbb{R}$ and $f\in C(\mathbb{R}\times D,E)$, if $T(s)f\in C(\mathbb{R}\times D,E)$ denotes the function defined by

$$(T(s)f)(t,x) = f(t+s,x), \text{ for } (t,x)\in\mathbb{R}\times D,$$

the set of all sequential cluster points of $\{T(s)f : s\in\mathbb{R}\}$, will be denoted by $H_T(f)$, that is,

$$H_T(f) := \left\{ \begin{array}{l} g\in C(\mathbb{R}\times D,E) \text{ such that for some sequence } (t_n)_{n\in IN}\subset\mathbb{R}, \\ ((T(t_n)f))_{n\in IN} \text{ converges to } g \text{ pointwise on } \mathbb{R}\times D, \text{ that is,} \\ \lim_{n\to\infty}(T(t_n)f)(t,x) = g(t,x) \text{ for any } (t,x)\in\mathbb{R}\times D. \end{array} \right\}.$$

The set $H_T(f)$ is considered as a topological space. Let $\{x_{t_n}(.)\}_{n\in IN}$ is a sequence of translations of $x(.)\in C_b(\mathbb{R},E)$. $x_{t_n}(.) \rightharpoonup y(.)$ denotes $\{x_{t_n}(.)\}_{n\in IN}$ converges weakly to y in $C_b(\mathbb{R},E)$.
The space $C_b(\mathbb{R},E)$ will always be supposed to be endowed with the sup-norm.

Definition 2.3. A function $f\in C_b(\mathbb{R},E)$ is almost periodic (a.p.), if the set of its translations

$$O(f) := \{f_s : s\in\mathbb{R}\}\subset C_b(\mathbb{R},E)$$

is relatively compact in $C_b(\mathbb{R},E)$. The set of all almost periodic functions is denoted by $AP(\mathbb{R},E)$.

The canonical wakening of the definition leads to the notion of weakly almost periodic functions, as done by Eberlein [7].

Definition 2.4. A function $f\in C_b(\mathbb{R},E)$ is Eberlein-weakly almost periodic (E-w.a.p.), if the set of its translations

$$O(f) := \{f_s : s\in\mathbb{R}\}\subset C_b(\mathbb{R},E)$$

is relatively compact in the weak topology of $C_b(\mathbb{R},E)$. The set of all Eberlein-weak almost periodic functions is denoted by $W(\mathbb{R},E)$.

In [5], Deleeuw and Glicksberg proved that the following decomposition holds :

$$W(\mathbb{R},E) \quad = \quad AP(\mathbb{R},E) \quad \oplus^t \quad W_0(\mathbb{R},E), \tag{2.1}$$

where $W_0(\mathbb{R},E)$ denotes the subspace of those Eberlein-weakly almost periodic functions, which contains zero in the weak closure of the orbit, i.e.,

$$W_0(\mathbb{R},E) \quad = \{f \in W(\mathbb{R},E) : \text{for a sequence } (s_n)_{n\in IN} \subseteq \mathbb{R},\, f_{s_n} \rightharpoonup 0\},$$

In order to prove the weak compactness of the translates, we need the following result :

Proposition 2.5. *[16] A subset $H \subset C_b(\mathbb{R},E)$, is relatively weakly compact, if and only if,*
(i) H is bounded in $C_b(\mathbb{R},E)$, and
(ii) for all $(h_m)_{m\in IN} \subset H$, $(t_n)_{n\in IN} \subset \mathbb{R}$ and $(x_n^*)_{n\in IN} \subset B_{E^*}$ the following double limits condition holds :

$$\lim_{n\to+\infty}\lim_{m\to+\infty} \langle h_m(t_n), x_n^*\rangle \quad = \quad \lim_{m\to+\infty}\lim_{n\to+\infty} \langle h_m(t_n), x_n^*\rangle\,,$$

whenever the iterated limits exist.
This criterion will be the main tool in verifying weak almost periodicity in the sense of Eberlein.

Corollary 2.6. *Let X be a Banach space. If $f \in C(J,X)$ is uniformly continuous with relatively compact range in X such that, for some $x \in X$, one has*

$$\Lambda_k(f) \quad \subseteq \quad C_0(J,X) + \{\alpha_x\}\,,$$

where α_x is the function defined on J with values in X by :

$$\alpha_x(t) \quad = \quad x,\ \forall t \in J.$$

Then, f is Eberlein weakly almost periodic.

Example 2.7. For any γ in $C_0(-1,1)$, let us consider the scalar function $\rho(\gamma)$ defined on $I\!R^+$ by:

$$\rho(\gamma)(t) \quad = \quad \begin{cases} \gamma(t-2^{2n}), & \text{if } t \in \left(2^{2n}-1, 2^{2n}+1\right) \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the function $\rho(\gamma)$ is an Eberlein weakly almost periodic function, which is almost periodic only when $\gamma \equiv 0$.

Proof. In deed : first one let us prove that if γ is not identically equal to zero, $\rho(\gamma)$ is not almost periodic.
Assume the contrary $\rho(\gamma)$ is almost periodic. Then, $\forall\varepsilon > 0$ there exists a real number $l = l(\varepsilon) > 0$, such that in every interval I of length l, there exists a real number τ for which one has:

$$\|\rho(\gamma)(t+\tau) - \rho(\gamma)(t)\| < \varepsilon, \text{ for all } t \in I\!R^+.$$

It follows that, there exists a real number $l > 0$, such that in every interval I of length l, there exists a real number τ for which one has:

$$\|\rho(\gamma)(t+\tau) - \rho(\gamma)(t)\| < \frac{\sup\limits_{0\le t\le 6} \|\rho(\gamma)(t)\|}{2}, \text{ for all } t \in I\!R^+.$$

For n_0 large enough, we can always make the possiblity such that the interval

$$\left[2^{2n_0} + 1, 2^{2(n_0+1)} - 1\right]$$

contains an interval $[a,b]$ with length l and $\left|(2^{2(n_0+1)} - 1) - b\right| > 6$, it follows that, it will be exist $\tau \in [a,b]$ such that :

$$\|\rho(\gamma)(t+\tau) - \rho(\gamma)(t)\| < \frac{\sup\limits_{0\le t\le 6} \|\rho(\gamma)(t)\|}{2}, \text{ for every } t \in I\!R^+. \tag{2.2}$$

But as we have ,

$$\left|(2^{2(n_0+1)} - 1) - \tau\right| > 6,$$

we will have $f_\tau = 0$ on $[0,6]$. So, the relation (2.2) becames

$$\sup_{0\le t\le 6} \|\rho(\gamma)(t)\| \quad \le \quad \|\rho(\gamma)(t+\tau) - \rho(\gamma)(t)\| \le \frac{\sup\limits_{0\le t\le 6} \|\rho(\gamma)(t)\|}{2},$$

which is absurd. Then, if $\gamma \neq 0$, $\rho(\gamma)$ is not almost periodic.
Let us prove that $\rho(\gamma)$ is Eberlein weakly almost periodic.
By the corollary (2.6), since $\rho(\gamma)$ is bounded and uniformly continuous, it suffices to prove that there exists $x \in I\!R^+$ such that

$$\Lambda_k(\rho(\gamma)) \quad \subseteq \quad C_0(I\!R^+) + \{\alpha_x\}.$$

We will prove more than this result, every element g of $\Lambda_k(\rho(\gamma))$ is with compact support and is included in some interval of length small than 2.
Let $g \in \Lambda_k(\rho(\gamma))$,
i) If $g \equiv 0$, the result is clear.
ii) If $g \neq 0$, then there exists a $t_0 \in I\!R^+$, such that $g(t_0) \neq 0$. But, $g \in \Lambda_k(\rho(\gamma))$, then there exists a sequence $(w_n)_n$ of elements of $I\!R^+$ satisfying $w_n \underset{n\to+\infty}{\to} +\infty$ and $\rho(\gamma)_{w_n} \underset{n\to+\infty}{\to} g$. It follows that, the sequence $(\rho(\gamma)(w_n + t_0))_n$ converges to $g(t_0)$ which gives the existence of an $n_{t_0} \in I\!N^*$ such that :

$$\|\rho(\gamma)(w_n + t_0)\| > \frac{\|g(t_0)\|}{2}, \ \forall n \ge n_{t_0}.$$

So, $\forall n \ge n_{t_0}$, $\rho(\gamma)(w_n + t_0) \neq 0$. Then, there exist $\delta \in (0,1)$ and $(k_n)_n \subset I\!N$ with $k_n \underset{n\to+\infty}{\to} +\infty$ such that :

$$w_n + t_0 \quad \in \quad \left(2^{2k_n} - \delta, 2^{2k_n} + \delta\right), \quad \forall n \ge n_{t_0}. \tag{2.3}$$

Let $\alpha \in [2,+\infty)$ and let us prove that $\rho(\gamma)(u := t_0+\alpha) = 0$ (i.e. the support of $\rho(\gamma)$ is included in some interval of type $[a,a+2]$ with $a \in I\!R^+$).
Assume that $\rho(\gamma)(u) \neq 0$. In the same manner, one has the existence of $n_u \in I\!N^*$, $\eta \in (0,1)$ and $(J_n)_n \subset I\!N$ with $J_n \underset{n\to+\infty}{\rightarrow} +\infty$ such that :

$$w_n + u \in \left(2^{2J_n} - \eta, 2^{2J_n} + \eta\right), \quad \forall n \geq n_u. \tag{2.4}$$

Consider $n \geq \max\{n_{t_0}, n_u\}$ verifying $2^{2k_n} > 2+\alpha$ (we can always prove the existence of a such n because $k_n \underset{n\to+\infty}{\rightarrow} +\infty$). From the relations (2.3) and (2.4), we have the following inequalities :

$$\begin{array}{lcccl} 2^{2k_n} - \delta + \alpha - w_n & \leq & u & \leq & 2^{2k_n} + \delta + \alpha - w_n \\ & & \text{and} & & \\ 2^{2J_n} - \eta - w_n & \leq & u & \leq & 2^{2J_n} + \eta - w_n. \end{array} \tag{2.5}$$

Assume that $k_n < J_n$, from (2.5), one has :

$$\begin{aligned} 0 \leq 2^{2k_n} + \delta + \alpha - w_n - (2^{2k_n} - \delta + \alpha - w_n) &= 2^{2k_n}(1 - 2^{2(J_n - k_n)}) + \alpha + \delta + \eta \\ &\leq \alpha + (\delta + \eta) - 2^{2k_n}, \end{aligned}$$

which is impossible since by hypothesis one has $2^{2k_n} > 2+\alpha$. It follows that, one has $J_n \leq k_n$ from (2.5), we have also :

$$\begin{aligned} 0 \leq 2^{2J_n} + \eta - w_n - (2^{2k_n} - \delta + \alpha - w_n) &= 2^{2J_n}(1 - 2^{2(k_n - J_n)}) - \alpha + \delta + \eta \\ &\leq (\delta + \eta) - \alpha, \end{aligned}$$

which contradicts the fact that $\alpha \geq 2$. hence we have the dezired result. □

2.2.1 Mappings leaving Eberlein-weak almost periodicity invariant

In this subsubsection, we start with the question for which $f : \mathbb{R} \times E \to E$ and given $g \in W(\mathbb{R}, E)$, the composition $t \mapsto f(t, g(t))$ is Eberlein-weakly almost periodic.

Definition 2.8. A function $f \in C(\mathbb{R} \times D, E)$ is said to be Eberlein-weak almost periodic in t uniformly for $x \in D$ (D.E-w.a.p), if $f(.,x) \in C_b(\mathbb{R}, E)$ is Eberlein -weakly almost periodic for each $x \in D$, for every compact subset $K \subset D$, $f/_{\mathbb{R}\times K}$ is bounded and the mapping $K \ni x \mapsto f(\cdot, x) \in W(\mathbb{R}, E)$ is continuous.

In the sequel, we denote these functions by :

$$W(\mathbb{R} \times D, E) := \{f \in C(\mathbb{R} \times D, E) \colon f \text{ is D.E-w.a.p}\}.$$

and by $WRC(\mathbb{R} \times D, E)$, the subset of those functions $f \in W(\mathbb{R} \times D, E)$, such that for all $x \in D$, $f(\cdot, x)$ has a relatively compact range.

Theorem 2.9. *[16] Suppose that $D = (E, weak)$, $f \in W(\mathbb{R} \times D, E)$ and $g \in W(\mathbb{R}, E)$, then*

$$t \mapsto f(t, g(t)) \in W(\mathbb{R}, E).$$

Corollary 2.10. *For a Banach space Y, let $D = (Y, \|\cdot\|)$ and $f \in W(\mathbb{R} \times D, E)$, then for every $g \in W(\mathbb{R}, E)$ with a relatively compact range,*

$$t \mapsto f(t, g(t)) \in W(\mathbb{R}, E).$$

Proof. The reader will have no difficulty to apply the previous theorem to

$$K := \overline{g(\mathbb{R})}$$

since $(K, weak) = (K, \|\cdot\|)$ hence one will obtain the result. □

In order to discuss our results in the case where $f(t,x)$ is almost periodic in t uniformly for $x \in D$, we introduce the projection operator on the almost periodic component

$$Q \;:\; W(\mathbb{R}, E) \;\to\; AP(\mathbb{R}, E).$$

By the fact that the almost periodic component can be obtained as the weak limit of a sequence of translates (2.1), it follows that $\|Q\| \le 1$.

Theorem 2.11. *[11] Let $D = (E, weak)$, $f \in W(\mathbb{R} \times D, E)$ and $g \in W(\mathbb{R}, E)$. Then, the identity*

$$Qf(.,g(.)) \;=\; f^a(., g^a(.))$$

holds, where $f^a(.,x)$ and $g^a(.)$ denote the almost periodic component of $f(.,x)$ and g respectively.

Corollary 2.12. *Let $D = (E, weak)$, $f \in W(\mathbb{R} \times D, E)$ and $y \in W(\mathbb{R}, E)$ the solution of the differential equation :*

$$\frac{d}{dt} y(t) \;=\; f(t, y(t)), \;\; t \in \mathbb{R}.$$

Then, the almost periodic component $y^a(.)$ of $y(.)$ satisfies the following differential equation :

$$\frac{d}{dt} y^a(t) \;=\; f^a(t, y^a(t)), \;\; t \in \mathbb{R},$$

where $f^a(.,x)$ denotes the almost periodic component of $f(.,x)$.

3 Differential Equations

In this section, we apply the above results to study the weak almost periodicity in the sense of Eberlein of solutions for the differential equation (1.2) :

$$\frac{d}{dt} x(t) \;=\; f(t, x(t)), \;\; t \in \mathbb{R},$$

where $f \in C(\mathbb{R} \times D, E)$.

Hypothesis :

(H_1) let $D = (E, weak)$, and $f \in WRC(\mathbb{R}\times D, E)$ dissipative, i.e., there exists a constant $c > 0$, such that :

$$\langle f(t,x) - f(t,y), x-y\rangle_{-} \le -c\,\|x-y\|^2, \text{ for all } (t,x,y)\in\mathbb{R}\times E\times E.$$

(H_2) $H_T(f)$, the set of all sequential cluster points of $\{T(s)f : s\in\mathbb{R}\}$, is simply sequentially compact, i.e.,
for every sequence $(t_n)_{n\in IN}\subset\mathbb{R}$, there exist a subsequence $(t_{n_k})_{k\in IN}$ and a function $g\in C(\mathbb{R}\times D, E)$ such that $(T(t_{n_k})f)_{k\in IN}$ converges to g pointwise on $\mathbb{R}\times D$, that is,

$$((T(t_{n_k})f)(t,x))_{k\in IN} \text{ converges to } g(t,x) \text{ on } E, \text{ for any } (t,x)\in\mathbb{R}\times D.$$

Remark 3.1. : Notice that the dissipativity property is essentially weaker than the classical Lipschitz condition:

$$\|f(x)-f(y)\| \le k\,\|x-y\|,$$

but it only guarantees existence of solutions to the right.

Theorem 3.2. *Assume that E is reflexive. Under the hypothesis (H_1) and (H_2), equation (1.2) has one and only one solution $x(\cdot)\in C_b(\mathbb{R},E)$ which is Eberlein-weakly almost periodic.*

For the proof of theorem (3.2), we shall need the following technical lemmas.

Lemma 3.3. *Assume that E is reflexive, and let $f\in W(\mathbb{R}\times D,E)$ be dissipative. For every $g\in H_T(f)$, g is dissipative, and for every bounded subset B of E, $g/_{\mathbb{R}\times B}\in C_b(\mathbb{R}\times B,E)$.*

Proof. Since $g\in H_T(f)$, then there exist a sequence $(t_n)_{n\in IN}\subset\mathbb{R}$ such that for every $(t,x)\in\mathbb{R}\times E$

$$((T(t_n)f)(t,x))_{n\in IN} \text{ converges in } E \text{ to } g(t,x).$$

Thus, for $(t,x,y)\in\mathbb{R}\times E\times E$, there exists $n_0 = n_0(t,x,y)\in IN$ such that

$$\begin{gathered}\|(T(t_{n_0})f)(t,x) - g(t,x)\|_E \le \varepsilon,\\ \text{and}\\ \|(T(t_{n_0})f)(t,y) - g(t,y)\|_E \le \varepsilon.\end{gathered} \tag{3.1}$$

By lemma (2.2), we obtain,

$$\begin{aligned}\langle g(t,x)-g(t,y), x-y\rangle_{-} \le\;& \langle g(t,x) - (T(t_{n_0})f)(t,x), x-y\rangle_{+}\\ &+\langle (T(t_{n_0})f)(t,x) - (T(t_{n_0})f)(t,y), x-y\rangle_{-}\\ &+\langle (T(t_{n_0})f)(t,y) - g(t,y), x-y\rangle_{+}.\end{aligned}$$

Thus, by the above estimate (3.1), we have

$$\langle g(t,x)-g(t,y), x-y\rangle_{-} \le 2\varepsilon + \langle (T(t_{n_0})f)(t,x) - (T(t_{n_0})f)(t,y), x-y\rangle_{-}.$$

Since f is dissipative, we conclude that g is dissipative.
Let B be a bounded subset of E. Since E is reflexive, then $\overline{B}^{\omega}$ is compact in D (where ω

denotes the weak topology), and since $f \in W(\mathbb{R} \times D, E)$, we have that $f/_{\mathbb{R}\times\overline{B}^{\omega}}$ is bounded. By the fact that for all $(t,x) \in \mathbb{R} \times B$,

$$\|g(t,x)\| \leq \|g(t,x) - (T(t_{n_0})f)(t,x)\| + \|(T(t_{n_0})f)(t,x)\|.$$

From (3.1), we derive that :
for every $\varepsilon > 0$,

$$\|g(t,x)\| \leq \varepsilon + \sup_{t\in\mathbb{R}, x\in\overline{B}^{\omega}} \|f(t,x)\|,$$

thus

$$\|g(t,x)\| \leq \sup_{t\in\mathbb{R},\, x\in\overline{B}^{\omega}} \|f(t,x)\|, \text{ for all } (t,x) \in \mathbb{R} \times B.$$

The lemma is proved. □

Lemma 3.4. *Assume that E is reflexive, and let $f \in W(\mathbb{R} \times D, E)$ such that $H_T(f)$ is simply sequentially compact. Then, for every compact subset $K \subset D$ and sequence $(t_n)_{n\in IN} \subset \mathbb{R}$, there exist a subsequence $(t^K_{n_k})_{k\in IN} \subset (t_n)_{n\in IN}$ and a function $g \in C(\mathbb{R} \times D, E)$ such that $((T(t^K_{n_k})f)(\cdot,x))_{k\in IN}$ converges weakly in $C_b(\mathbb{R}, E)$ to $g(\cdot,x)$ uniformly for x on K.*

Proof. Let $(t_n)_{n\in IN} \subset \mathbb{R}$. Since $H_T(f)$ is simply sequentially compact, then there exist a subsequence $(t_{n_k})_{k\in IN}$ and a function $g \in C(\mathbb{R} \times D, E)$ such that : for every $(t,x) \in \mathbb{R} \times E$,

$$((T(t_{n_k})f)(t,x))_{k\in IN} \text{ converges in } E \text{ to } g(t,x).$$

Let us prove that for every compact subset $K \subset D$, there exist a subsequence $(t^K_{n_{k_j}})_{j\in IN} \subset (t_{n_k})_{k\in IN}$ such that

$$((T(t^K_{n_{k_j}})f)(\cdot,x))_{j\in IN} \rightharpoonup g(\cdot,x), \text{ uniformly for } x \text{ on } K.$$

Let K be a compact subset of D. First, we show that

$$\Delta_K := \{(T(t)f)(\cdot,x) : t \in \mathbb{R} \text{ and } x \in K\}$$

is a subset of a closed and separable subspace Y_K of $C_b(\mathbb{R}, E)$. Given any $\varepsilon > 0$, since for a compact subset of D

$$K \ni x \mapsto f(\cdot,x) \in W(\mathbb{R}, E)$$

is continuous, we find an $n(\varepsilon)$ and $(z_i)_{i=1}^{i=n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{i=n(\varepsilon)} \{x : \|f(\cdot,x) \to f(\cdot,z_i)\| < \varepsilon\}.$$

Since $H_K := \{f(\cdot,x) : x \in K\}$ is compact,

$$\begin{array}{rccc} T: & \mathbb{R} \times K & \to & C_b(\mathbb{R}, E) \\ & (t,x) & \mapsto & (T(t)f)(.,x) \end{array}$$

is continuous, and

$$\Delta_K := \{(T(t)f)(\cdot,x) : t \in \mathbb{R} \text{ and } x \in K\} \subset \overline{\bigcup_{i=1}^{i=n(\varepsilon)} T([-n,n])H_K},$$

where $(T(t))_{t\in\mathbb{R}}$ denotes the group of translations.
Note that continuous images of separable spaces are separable, we obtain that Δ is a subset of the closed and separable subspace

$$Y_K := \overline{span}\left\{ \bigcup_{i=1}^{i=n(\varepsilon)} T([-n,n])H_K \right\}$$

of $C_b(\mathbb{R},E)$.
By the fact that

$$\Delta_K \subset \bigcup_{i=1}^{i=n(\varepsilon)} O(f(\cdot,x_i)) + \varepsilon B_{C_b(\mathbb{R},E)}$$

we obtain the relative weak compactness of Δ_K.
As from Dunford- Schwartz [6], we recall, that the weak topology on weak compact subsets in separable Banach spaces is a metric topology. Hence the weak topology on $\overline{\Delta_K}^{\omega}$ is metric. Thus, we may pass to subsequence $(t^K_{n_{k_j}})_{j\in IN} \subset \mathbb{R}$, such that the limit of $(T(t^K_{n_{k_j}})f)(\cdot,x))_{j\in IN}$ exist in the weak topology of $C_b(\mathbb{R},E)$, for all $x \in K$.
Using that for all $(t,x) \in \mathbb{R}\times E$,

$$\lim_{j\to+\infty} \left\| (T(t^K_{n_{k_j}})f)(t,x) - g(t,x) \right\|_E = 0,$$

a standard trick of topology gives

$$\omega - \lim_{j\to+\infty} (T(t^K_{n_{k_j}})f)(\cdot,x) = g(\cdot,x)$$

uniformly for $x \in K$. □

Lemma 3.5. *Assume that E is reflexive, let $x_0(\cdot) \in C_b(\mathbb{R},E)$ and $f \in W(\mathbb{R}\times D,E)$. Then, for every sequence $(t_n)_{n\in IN} \subset \mathbb{R}$, there exist a subsequence $(t_{n_k})_{k\in IN} \subset \mathbb{R}$ and a function $g \in C(\mathbb{R}\times D,E)$ such that $((T(t_{n_k})f)(\cdot,x_0(\cdot))_{k\in IN}$ converges weakly in $C_b(\mathbb{R},E)$ to $g(\cdot,x_0(\cdot))$.*

Proof. Let $K_0 := \overline{\bigcup_{t\in\mathbb{R}} \{x_0(t) : t \in \mathbb{R}\}}^{\omega}$. Since $x_0(.)$ is bounded and E is reflexive, K_0 is a weak compact subset of the closed separable subspace $Y := \overline{span}\{x_0(\mathbb{R})\}$ of E. Hence the weak topology on K_0 is metric. Let $d_0(.,.)$ denotes the metric on K_0. By Lemma (3.4), for $(t_n)_{n\in IN} \subset \mathbb{R}$, there exist a subsequence $(t^{K_0}_{n_k})_{k\in IN} \subset \mathbb{R}$ and $g \in C(\mathbb{R}\times D,E)$ such that $((T(t^{K_0}_{n_k})f)(\cdot,x))_{k\in IN}$ converges weakly in $C_b(\mathbb{R},E)$ to $g(\cdot,x)$ uniformly on $x \in K_0$.
Claim: $((T(t^{K_0}_{n_k})f)(\cdot,x_0(\cdot))_{k\in IN}$ converges weakly to $g(\cdot,x_0(\cdot))$ in $C_b(\mathbb{R},E)$. Since, for all $(t,x) \in \mathbb{R}\times E$

$$((T(t^{K_0}_{n_k})f)(t,x))_{k\in IN} \text{ converges in } E \text{ to } g(t,x),$$

it will suffice to show that $\left\{(T(t^{K_0}_{n_k})f)(.,x_0(.)) : k \in IN\right\}$ is weakly relatively compact in $C_b(\mathbb{R},E)$. Thus, for given sequences $(t^{K_0}_{n_{k_j}})_{j\in IN} \subset (t^{K_0}_{n_k})_{k\in IN}$ and $(s_m,x^*_m)_{m\in IN} \subset \mathbb{R}\times B_{E^*}$, we have to verify the following identity:

$$\lim_{j\to+\infty}\lim_{m\to+\infty} \left\langle (T(t^{K_0}_{n_{k_j}})f)(s_m,x_0(s_m)),x^*_m \right\rangle = \lim_{m\to+\infty}\lim_{j\to+\infty} \left\langle (T(t^{K_0}_{n_{k_j}})f)(s_m,x_0(s_m)),x^*_m \right\rangle,$$

whenever the iterated limits exist. By the weak compactly of K_0, without loss of generality $x_0(s_m) \rightharpoonup x_0 \in K_0$.
We define,

$$\begin{aligned} z_{j,m} &= \left\langle (T(t_{n_{k_j}}^{K_0})f)(s_m, x_0(s_m)), x_m^* \right\rangle \\ z &= \lim_{j\to+\infty}\lim_{m\to+\infty} z_{j,m} \\ y_{j,m} &= \left\langle (T(t_{n_{k_j}}^{K_0})f)(s_m, x_0), x_m^* \right\rangle. \end{aligned}$$

By our hypothesis, we have $\{t \mapsto f(t,x_0)\}$ is E-w.a.p., thus $\{y_{j,m}\}_{j,m\in IN}$ satisfies the double limit conditions, let $y \in \mathbb{R}$ be the double limit,
CLAIM : $y = z$.
Now,

$$|y-z| \leq |y - y_{j,m}| + |y_{j,m} - z_{j,m}| + |z_{j,m} - z|.$$

From the convergence of $\{y_{j,m}\}_{j,m\in IN}$ and $\{z_{j,m}\}_{j,m\in IN}$, we derive that for every $\varepsilon > 0$, there exists an $n_0 \in IN$, such that for $m \geq n_0$, there exists an $m_n \in IN$, such that

$$\begin{aligned} |y - y_{j,m}| &< \frac{\varepsilon}{3}, \text{ for all } m \geq m_n, \text{ and} \\ |z_{j,m} - z| &< \frac{\varepsilon}{3}, \text{ for all } m \geq m_n. \end{aligned}$$

Using that $x_0(s_m) \rightharpoonup x_0 \in K_0$, for a given $\delta > 0$, there exists $m_1 \in IN$, such that

$$d_0(x_0(s_m), x_0) < \delta, \text{ for all } m \geq m_1. \tag{3.2}$$

Applying the continuity of the map

$$K_0 \ni x \to f(.,x) \in W(\mathbb{R}, E)$$

on x_0, for $\varepsilon > 0$, we find a $\delta > 0$, and according to the previous estimate (3.2), there is an $m_1 = m(\varepsilon) \in IN$, such that

$$\|f(.,x_0(s_m)) - f(.,x_0)\| < \varepsilon, \text{ for all } m \geq m_1.$$

This yields, that

$$|y-z| \leq \varepsilon,$$

and hence $y = z$. By using the same arguments, we find that

$$\lim_{m\to+\infty}\lim_{j\to+\infty} \left\langle (T(t_{n_{k_j}}^{K_0})f)(s_m, x_0(s_m)), x_m^* \right\rangle = y,$$

the lemma is proved. □

Proof. **Proof of theorem** (2.9) **:**
The existence and uniqueness of a solution $x(.)$ in $C_b(\mathbb{R}, E)$ follows immediately from (H_1) (see [10, theorem 2, pp.140]).
Let us prove that $x(.)$ is E-w.a.p..
To this end, we choose sequences $(w_n)_{n\in IN} \subset \mathbb{R}$ and $(t_m, x_m^*)_{m\in IN} \subset \mathbb{R} \times B_{E^*}$ so that the limits

$$\alpha = \lim_{n\to+\infty}\lim_{m\to+\infty} \langle x(w_n + t_m), x_m^* \rangle \text{ and } \beta = \lim_{m\to+\infty}\lim_{n\to+\infty} \langle x(w_n + t_m), x_m^* \rangle$$

both exist. According to proposition (2.5), it will suffice to show that $\alpha = \beta$.
Since $H_T(f)$ is simply sequentially compact, there exist a subsequence $(w_{n_k})_{k\in IN} \subset \mathbb{R}$ and $g \in H_T(f)$, such that for all $(t,x) \in \mathbb{R}\times E$,

$$((T(w_{n_k})f)(t,x))_{k\in IN} \text{ converges in } E \text{ to } g(t,x).$$

From lemma (3.3), g satisfies the conditions of **theorem** 2 pp.140 [10], then equation

$$\frac{d}{dt}x(t) \quad = \quad g(t,x(t)),\ t\in\mathbb{R},$$

has a unique solution $x_0(.)$ in $C_b(\mathbb{R},E)$. By Lemma (2.2), without loss of generality, we can assume that

$$((T(t_{n_k})f)(\cdot,x_0(\cdot))_{k\in IN} \text{ converges weakly in } C_b(\mathbb{R},E) \text{ to } g(\cdot,x_0(\cdot)). \tag{3.3}$$

Consider the sequence $(\phi_k)_{k\in IN} \subset C(\mathbb{R})$ defined by

$$\phi_k(t) := \|x(t+w_{n_k}) - x_0(t)\|^2, \text{ for all } t\in\mathbb{R}.$$

For all $k \in IN$, we have

$$\begin{aligned}
&\frac{1}{2}D^-\phi_k(t) = \langle x'(w_{n_k}+t) - x_0'(t), x(w_{n_k}+t)-x_0(t)\rangle_- \\
&= \langle (T(w_{n_k})f)(t,x(w_{n_k}+t)) - g(t,x_0(t)), x(w_{n_k}+t)-x_0(t)\rangle_- \\
&= \langle (T(w_{n_k})f)(t,x(w_{n_k}+t)) - (T(w_{n_k})f)(t,x_0(t)) \\
&+ (T(w_{n_k})f)(t,x_0(t)) - g(t,x_0(t)), x(w_{n_k}+t)-x_0(t)\rangle_- .
\end{aligned}$$

Making use of the upper and lower semi-inner product properties (see Lemma (2.2)), and taking into account the dissipativity of f, we have

$$\frac{1}{2}D^-\phi_k(t) \quad \leq \quad -c\phi_k(t) + b_k(t)\sqrt{\phi_k(t)}, \text{ for all } t\in\mathbb{R},$$

where

$$b_k(t) := \|(T(w_{n_k})f)(\cdot,x_0(.))(t) - g(\cdot,x_0(\cdot))(t)\|_E, \text{ for all } t\in\mathbb{R}.$$

Hence by R. H. Martin result [12], for all $t_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\|x(t+w_{n_k}) - x_0(t)\|_E \quad &\leq \quad e^{-c(t-t_0)}\|x(t_0+w_{n_k}) - x_0(t_0)\|_E \\
&+ \quad \int_{t_0}^t e^{-c(t-s)} b_k(s)ds, \text{ for all } t\geq t_0.
\end{aligned}$$

Since $x(\cdot)$ and $x_0(\cdot)$ are bounded, by tending t_0 to $-\infty$, we obtain, for all $k\in IN$,

$$\|x(t+w_{n_k}) - x_0(t)\|_E \quad \leq \quad \int_{-\infty}^t e^{-c(t-s)} b_k(s)ds, \text{ for all } t\in\mathbb{R}. \tag{3.4}$$

As for all k, j and $m \in IN$,

$$\begin{aligned}
\left|\langle x(w_{n_k}+t_m), x_m^*\rangle - \left\langle x(w_{n_j}+t_m), x_m^*\right\rangle\right| \quad &\leq \quad \|x(t_m+w_{n_k}) - x_0(t_m)\|_E \\
&+ \quad \left\|x(t_m+w_{n_j}) - x_0(t_m)\right\|_E,
\end{aligned}$$

from the above estimate (3.4), we obtain

$$\begin{array}{rcl}\left|\langle x(w_{n_k}+t_m),x_m^*\rangle-\left\langle x(w_{n_j}+t_m),x_m^*\right\rangle\right| & \leq & \int_{-\infty}^{t_m} e^{-c(t_m-s)}b_k(s)ds \\ & + & \int_{-\infty}^{t_m} e^{-c(t_m-s)}b_j(s)ds.\end{array}$$

Thus,

$$\left|\langle x(w_{n_k}+t_m),x_m^*\rangle-\left\langle x(w_{n_j}+t_m),x_m^*\right\rangle\right| \leq A_{k,m}+A_{j,m}. \tag{3.5}$$

The uniform boundedness of the sequence of linear functionals,

$$\begin{array}{rccl}\Psi_m: & C_b(\mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \int_{-\infty}^{t_m} e^{-c(t_m-s)}h(s)ds,\end{array}$$

if we prove that

$$\|(T(w_{n_k})f)(\cdot,x_0(\cdot))-g(\cdot,x_0(\cdot))\| \rightharpoonup 0 \text{ in } C_b(\mathbb{R}).$$

by going over to appropriate subsequences, we can assume that the iterated double limits for $(A_{k,m})_{k,m\in IN}$ exist. Since they have to coincide, they have to be zero. Starting with $\lim\limits_{k\to+\infty}$, then $\lim\limits_{m\to+\infty}$, and at last $\lim\limits_{j\to+\infty}$ in (3.5), we obtain

$$|\alpha-\beta|\leq 0$$

which concludes the proof.
Thus, let us show that

$$\|(T(w_{n_k})f)(\cdot,x_0(\cdot))-g(\cdot,x_0(\cdot))\| \rightharpoonup 0 \text{ in } C_b(\mathbb{R}).$$

First we show that the set $\{\|(T(w_{n_k})f)(\cdot,x_0(\cdot))-g(\cdot,x_0(\cdot))\|\}_{k\in IN}$ is weakly relatively compact in $C_b(\mathbb{R})$. Thus, for given a subsequence $(w_{n_{k_j}})_{j\in IN}\subset(w_{n_k})_{k\in IN}$ and $(s_m)_{m\in IN}\subset\mathbb{R}$, we have to verify the following identity:

$$\begin{array}{l}\lim\limits_{j\to+\infty}\lim\limits_{m\to+\infty}\left\|(T(w_{n_{k_j}})f)(s_m,x_0(s_m))-g(s_m,x_0(s_m))\right\| \\ =\lim\limits_{m\to+\infty}\lim\limits_{j\to+\infty}\left\|(T(w_{n_{k_j}})f)(s_m,x_0(s_m))-g(s_m,x_0(s_m))\right\|,\end{array}$$

whenever the iterated limits exist. By the weak compactly of K_0, without loss of generality $x_0(s_m)\rightharpoonup x_0\in K_0$.
From the continuity of

$$K_0\ni x \;\to\; f(.,x)\in W(\mathbb{R},E), \text{ and}$$

$$K_0\ni x \;\to\; g(.,x)\in C_b(\mathbb{R},E)$$

on x_0, we obtain for $\varepsilon>0$, an $m_0=m(\varepsilon)\in IN$, such that

$$\begin{array}{rcl}\|f(.,x_0(s_m))-f(.,x_0)\| & < & \varepsilon, \text{ for all } m\geq m_0, \text{ and} \\ \|g(.,x_0(s_m))-g(.,x_0)\| & < & \varepsilon, \text{ for all } m\geq m_0.\end{array} \tag{3.6}$$

Since $g \in H_T(f)$,

$$\lim_{j\to+\infty} \left\| (T(w_{n_{k_j}})f)(s_m, x_0(s_m)) - g(s_m, x_0(s_m)) \right\| = 0, \text{ for all } m \in IN.$$

Thus, let us prove that

$$\lim_{j\to+\infty} \lim_{m\to+\infty} \left\| (T(w_{n_{k_j}})f)(s_m, x_0(s_m)) - g(s_m, x_0(s_m)) \right\|_E = 0.$$

Since for all $j,k \in IN$,

$$\begin{aligned} \left\| (T(w_{n_{k_j}})f)(s_m, x_0(s_m)) - g(s_m, x_0(s_m)) \right\|_E &\leq \left\| (T(w_{n_{k_j}})f)(s_m, x_0(s_m)) - (T(w_{n_{k_j}})f)(s_m, x_0) \right\|_E \\ + \left\| (T(w_{n_{k_j}})f)(s_m, x_0) - g(s_m, x_0(s_m)) \right\|_E &+ \| g(s_m, x_0(s_m)) - g(s_m, x_0) \|_E , \end{aligned}$$

by (3.6), we obtain that for all $m \geq m_0$,

$$\left\| (T(w_{n_{k_j}})f)(s_m, x_0(s_m)) - g(s_m, x_0(s_m)) \right\|_E \leq 2\varepsilon + \left\| (T(w_{n_{k_j}})f)(s_m, x_0) - g(s_m, x_0(s_m)) \right\|_E .$$

As by hypothesis, $\{t \mapsto f(t, x_0)\}$ has a weakly relatively compact range, a diagonalization argument gives us a subsequences, such that the iterated limits exist for the sequence $((T(w_{n_{k_{j_q}}})f)(s_{m_l}, x_0))_{q,l \in IN}$. By the fact that we only have to verify the equality of the iterated limits we may pass to these subsequences. In order to avoid subindices, we assume that the iterated limits exist for $((T(w_{n_{k_j}})f)(s_m, x_0))_{m,j \in IN}$. The characterization of weak compactness gives

$$\lim_{j\to+\infty} \lim_{m\to+\infty} (T(w_{n_{k_j}})f)(s_m, x_0) - g(s_m, x_0(s_m)) = \lim_{m\to+\infty} \lim_{j\to+\infty} (T(w_{n_{k_j}})f)(s_m, x_0) - g(s_m, x_0(s_m)).$$

Since the convergence holds in norm, by the estimate (3.3), we derive that

$$\{ \| (T(w_{n_k})f)(\cdot, x_0(\cdot)) - g(\cdot, x_0(\cdot)) \| \}_{k \in IN}$$

is weakly relatively compact in $C_b(\mathbb{R})$. Using that

$$\lim_{k\to+\infty} \| (T(w_{n_k})f)(t, x_0(t)) - g(t, x_0(t)) \| = 0, \text{ for all } t \in \mathbb{R},$$

a standard trick of topology gives

$$\| (T(w_{n_k})f)(\cdot, x_0(\cdot)) - g(\cdot, x_0(\cdot)) \| \rightharpoonup 0 \text{ in } C_b(\mathbb{R}),$$

which concludes the proof. □

3.1 Almost periodic solutions

In this section, we shall discuss an existence theorem for an almost periodic solution of equation (1.2).

A function $f \in C(\mathbb{R} \times D, E)$ is said to be almost periodic if $f(t,x)$ is almost periodic in t

uniformly with respect to x in bounded set of D.
In the sequel, we denote these functions by :

$$AP(\mathbb{R}\times D,E) \quad := \{f\in C(\mathbb{R}\times D,E) \colon f \text{ is almost periodic}\}.$$

Let $f\in AP(\mathbb{R}\times D,E)$. Bochner's theorem implies that $H_T(f)$ is a minimal set. Also,
(i) for any sequence $(t_n)_{n\in IN}\subset\mathbb{R}$, there exist a subsequence $(t_{n_k})_{k\in IN}\subset\mathbb{R}$ and a function $g\in C(\mathbb{R}\times D,E)$ such that $(T(t_{n_k})f)(t,x)\to g(t,x)$ as $n\to\infty$, uniformly in $s\in\mathbb{R}$ and x in bounded set of D; and
(ii) for any compact subset $K\subset D$ such that ω_K is metric (ω_K designs the weak topology on K), then $f(t,x)$ is uniformly continuous on $\mathbb{R}\times K$.
For a detailed account of other results on this direction we refer to [3].
Remark : By the fact that for any compact subset $K\subset D$ such that ω_K is metric, $f(t,x)$ is uniformly continuous on $\mathbb{R}\times K$, it's easy to derive that the map

$$K\ni x\mapsto f(.,x)\in AP(\mathbb{R},E)$$

is continuous.

3.1.1 Main result

Hypothesis :
(H_1') $f\in AP(\mathbb{R}\times D,E)$;
(H_2') f is dissipative.

Theorem 3.6. *Assume that E is reflexive. Under the hypothesis (H_1') and (H_2'), equation (1.2) has one and only one solution $x(\cdot)\in C_b(\mathbb{R},E)$ which is almost periodic.*

Proof. The reader will have no difficulty to apply theorem (3.2) to f under hypothesis (H_1') and (H_2'), since for any $f\in AP(\mathbb{R}\times D,E)$, f satisfies all properties of a D.E-w.a.p. function and $H_T(f)$ is simply sequentially compact, hence one will obtain that equation (1.2) has a unique bounded solution $x(.)\in W(\mathbb{R},E)$.
By corollary (2.12), the almost periodic component $x^a(.)$ of $x(.)$ satisfies the following differential equation

$$\frac{d}{dt}z(t) \quad = \quad f^a(t,z(t)),\ t\in\mathbb{R}, \tag{3.7}$$

where $f^a(.,x)$ denotes the almost periodic component of $f(.,x)$.
As for all $x\in D$, $f^a(.,x)=f(.,x)$. From the above differential equation (3.7), we deduce that $x^a(.)$ satisfies equation (1.2). By the uniqueness of bounded solution, we obtain

$$x(.)=x^a(.),$$

hence $x(.)$ is almost periodic. Which conclude the proof. □

4 Examples

$I)$

$$\frac{dx}{dt} = Ax + f(t) \tag{4.1}$$

where $f : \mathbb{R} \to H$, where H is a Hilbert space, $t \to f(t)$ is continuous, bounded, Eberlein weakly almost periodic, and $A : H \to H$, defined by $Ax = -x$

A verifies $(Ax, x) = -\|x\|^2$, so A is dissipative; using the theorem (3.2) we have that equation (4.1) has only and only one bounded solution which is Eberlein weakly almost periodic.

$II)$ We consider the following equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} & = & -\Delta u + f(t,x) \quad t \geq 0, \qquad x \in \mathbb{R}^n \\ u(0,x) & = & u_1(x) \qquad\qquad \dfrac{\partial u}{\partial t}(0,x) = u_2(x) \end{cases} \tag{4.2}$$

Where

$$u_1 \in H^1(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) / \frac{\partial v}{\partial x_i} \in L^2(\mathbb{R}^n) \right\}$$

where $\dfrac{\partial v}{\partial x_i}$ is a distribution derivative $u_2 \in L^2(\mathbb{R}^n)$

f:$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying that f is Eberlein weakly almost periodic in t uniformly with respect to x , $f(t,.) \in L^2(\mathbb{R}^n)$,

$$\begin{array}{rccc} F: & \mathbb{R} & \to & L^2(\mathbb{R}^n) \\ & t & \to & f(t,.) \end{array}$$

is Eberlein weakly almost periodic.

So the problem (4.2) is equivalent to the following

$$\left\{ \frac{\partial}{\partial t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix} \right. \tag{4.3}$$

with the variables change $u_2 = \dfrac{\partial u_1}{\partial t}$

Henceforth the phase space is $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$

Proposition 4.1. *[15] The operator*

$$A = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}$$

is the infinitesimal generator of a C_0 *semi group* $S(t)$ *on the space* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *satisfying the following inequality*

$$\begin{array}{rcccccc} \|S(t)\| & \leq & 4\exp(-2t), & t & \geq & 0 \\ D(A) & = & H^2(\mathbb{R}^n) & \times & L^2(\mathbb{R}^n). \end{array}$$

So if we put $V = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$*, then, the problem (4.3) becomes*

$$\begin{cases} \frac{d}{dt}V = AV + F(t), & t \geq 0 \\ V(0) = V_0 \end{cases} \tag{4.4}$$

Proposition 4.2. *If F is a continuously differentiable function such that*

$$\sup_{t\in\mathbb{R}} \left\| \frac{dF(t)}{dt} \right\| < \infty.$$

Then equation (4.4) has one and only one bounded solution W which is Eberlein weakly almost periodic solution.

Remark 4.3. In this case, the function that defined by $v(t,x) = W(t)(x)$ satisfies the problem (4.2) and is Eberlein weakly almost periodic in $L^2(\mathbb{R}^n)$ norm.

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