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INTRODUCTION TO THE GROUP OF SYMPLECTOMORPHISMS

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Abstract

In these Lecture Notes of a mini-course delivered in the " Séminaire Itinérant de Géometrie et Physique Mathématique, " Geometry and Physics V" at the University Cheikh Anta Diop , Dakar in May 2007, we introduce the group of symplectic diffeomorphisms, the main results on its algebraic structure and on some of its local and global properties. This survey culminates with the most recent results on Hofer geometry, the definitions of the groups of symplectic and hamiltonian homeomorphisms, and the introduction to the C^0 symplectic topology.

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1 Preliminaries

1.1 Basic definitions

A *symplectic form* on a smooth manifold M is a non-degenerate closed 2-form ω. "Non-degenerate" means that the mapping $\tilde{\omega} : T(M) \to T^*(M)$, $X \mapsto \tilde{\omega}(X)$ where $\tilde{\omega}(X)(Y) = \omega(X,Y)$ is an isomorphism. We denote the 1-form $\tilde{\omega}(X)$ by $i(X)\omega$.

The couple (M,ω) of a smooth manifold M and a symplectic form ω is called a *symplectic manifold*. *Symplectic Geometry* is the study of geometric properties of symplectic manifolds. Three good references on symplectic geometry are [10], [32] and [19]. Symplectic manifolds are even dimensional and if $dim(M) = 2n$, ω^n is a volume-form, called the *Liouville volume*. Hence M is oriented.

A smooth function $f : M \to \mathbf{R}$ gives rise to a vector field X_f uniquely defined by the equation

$$i(X_f)\omega = df$$

called the *Hamiltonian vector field* with hamiltonian f.

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The *support* of a vector field X on a smooth manifold M, is the closure of the set $\{x \in M | X(x) \neq 0\}$. The support of a function f is defined likewise: it is the closure of the set $\{x \in M | f(x) \neq 0\}$.

If a function f has a compact support, so does X_f, hence X_f generates a flow ϕ_t^f of M such that $(\phi_t^f)^*\omega = \omega$. Indeed $L_{X_f}\omega = di(X_f)\omega + i(X_f)d\omega = d(df) = 0$. This shows that (ϕ_t^f) preserves ω.

A *symplectic diffeomorphism* or a *symplectomorphism* of a symplectic manifold (M, ω) is a C^∞ diffeomorphism $h : M \to M$ such that $h^*\omega = \omega$. The support of a diffeomorphism h is the closure of $\{x \in M | h(x) \neq x\}$. The set of all symplectomorphisms with compact support form a group, denoted $Symp(M, \omega)$ (with the law of composition of mappings).

Hence we see that any smooth function with compact support gives rise to a symplectomorphism with compact support.

1.2 Examples

1. $(\mathbf{R^{2n}}, \omega_0 = \sum_i^n \mathbf{dx_i} \wedge \mathbf{dy_i})$ is a symplectic manifold. If $f : \mathbf{R^{2n}} \to \mathbf{R}$ is a smooth function, then

$$X_f = \sum_i (\partial f/\partial y_i)\partial x_i - (\partial f/\partial x_i)\partial y_i.$$

The flow ϕ_t^f is the solution of Hamilton equations. (Symplectic Geometry started off as the setting of *Classical Mechanics*).

For instance, consider the function

$$f(x_1, .., x_{n+1}, y_1, .., y_{n+1}) = 1/2 \sum x_i^2 + y_i^2.$$

Then

$$X_f = \sum_i y_i \partial/\partial x_i - x_i \partial/\partial y_i.$$

Setting $z_k = x_k + iy_k$, we see that the Hamilton equations:

$$\dot{x}_k = y_k$$

$$\dot{y}_k = -x_k$$

become $\dot{z}_k = -iz_k$. Hence the flow ϕ_t^f is the family of diffeomorphisms

$$\phi_t^f(z) = z(0)e^{-it},$$

where $z = (z_1, .., z_n)$.

This flow induces an action of the circle S^1 on the sphere S^{2n+1}. The quotient of this action is the complex projective space CP^n.

2. The torus T^{2n}. The symplectic form on $\mathbf{R^{2n}}$ is invariant by translations and therefore induces a symplectic form ω on $T^{2n} = \mathbf{R^{2n}}/\mathbf{Z^{2n}}$. A translation on $\mathbf{R^{2n}}$ induces a symplectic diffeomorphism of T^{2n} called the rotation R_θ through $\theta \in T^{2n}$. This gives an inclusion $T^{2n} \subset Symp(T^{2n}, \omega)$.

Note that the symplectic form ω_0 of $\mathbf{R^{2n}}$ is exact. However the induced symplectic form on T^{2n} is not exact. It is a consequence of Stoke's theorem that on a compact symplectic manifold (M,ω) of dimension $2n$, then ω^k , k = 0,1,.., n, is not exact, since ω^n is a volume form. This remark shows that the spheres S^{2n} admit no symplectic form for any $n \geq 2$.

3. The cotangent space $M = T^*(N)$ of a smooth manifold N. The *Liouville 1-form* on $T^*(N)$, is the 1-form λ_N defined as follows: let $(q,\theta) \in T^*(N)$, $q \in N, \theta \in T_q^*(N)$ and $\xi \in T_{(q,\theta)}T^*(N)$, then

$$\lambda_N(q,\theta)(\xi) = \theta(\pi_*\xi)$$

where $\pi : T^*N \to N$ is the canonical projection. One can check that

$$\omega_N = d\lambda_N$$

is a symplectic form. The space $T^*(N)$ is the "phase space" in Classical Mechanics. On $T^*(N)$, the local coordinates $(q,p), q \in M, p \in T_q^*(N)$ are called respectively the position and impulsion of a particle.

4. An oriented surface, with its volume form, is a symplectic manifold since the notions of volume form coincides with the notion of symplectic form in dimension 2.

5. The cartesian product $M_1 \times M_2$ of 2 symplectic manifolds (M_i, ω_i) carries the following symplectic form $\omega_1 \ominus \omega_2 = \pi_1^*\omega_1 - \pi_2^*\omega_2$, where π_i are the projections of $M_1 \times M_2$ on each factor. For any non zero numbers λ_i, one can also consider the symplectic form $\lambda_1\omega_1 \oplus \lambda_2\omega_2$.

6. A *contact form* on a $2n+1$ dimensional manifold N is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form. $(N \times \mathbf{R}, \omega = d(e^t\alpha))$ is a symplectic manifold, called the symplectization of the contact manifold (N,α).

1.3 Exercises

1. Show that the symplectic form $\omega = \sum_i dx_i \wedge dy_i$ on $\mathbf{R^{2n+2}}$ induces a symplectic form on CP^n.
2. Let $\gamma : [0,1] \to \mathbf{R^{2n}}$ be a C^1 path, $\gamma(t) = (p(t), q(t))$ and $H : \mathbf{R^{2n}} \to \mathbf{R}$ a smooth function. Consider the "action-functional"

$$A(\gamma) = \int_0^1 (p(t).\dot{q}(t) - H((p(t),q(t))dt$$

where $p(t).\dot{q}(t)$ is the usual dot product and $\dot{q} = \frac{d}{dt}q(t)$. Show that critical loops of A correspond to periodic orbits of the Hamiltonian vector field X_H.
3. Show that if α is a 1-form on a smooth manifold N, then $\alpha^*\lambda_N = \alpha$ where λ_N is the Liouville 1-form on $T^*(N)$.
4. Show that a diffeomorphism $h : N \to N'$ induces a symplectic diffeomorphism $h_* : T^*N \to T^*N'$. In fact h_* maps the Liouville form of N' to the Liouville form of N.
5. Show that if θ is a closed 2-form on N, then $\omega_\theta = d\lambda_N + \pi^*\theta$ is a symplectic form on $T^*(N)$. Here $\pi : T^*N \to N$ is the canonical projection.

If θ is exact, show that there exists a diffeomorphism $\phi : T^*(N) \to T^*(N)$ such that $\phi^*(\omega_\theta) = \omega_N$.

6. Let $H(M,\omega)$ be the group generated by all ϕ_1^f, and their inverses, for all smooth functions f with compact support. This is an "infinite dimensional " group of symplectomorphisms. Show that the group $H(M,\omega)$ is a normal subgroup of $Symp(M,\omega)$.

7. Let (M,ω) be a symplectic manifold and $f : M \to M$ be a smooth map such that $f^*\omega = \lambda\omega$ for some function λ. Show that λ is a constant provided that the dimension of M is at least 4. Moreover if M is compact show that $\lambda = \pm 1$. (Liberman).

1.4 Two basic theorems [10]

The first important theorem in symplectic geometry is the:

Darboux theorem (see [10])

Let (M,ω) be a 2n dimensional symplectic manifold. For each point $x \in M$, there is a local chart (U,ϕ) where U is an open neigborhood of x, and a diffeomorphism $\phi : U \to \mathbf{R^{2n}}$ such that $\phi^*(\sum_i^n dx_i \wedge dy_i) = \omega_{|U}$.

This theorem says that all symplectic manifolds look alike locally. Therefore there are no symplectic local invariants. All symplectic invariants are of a global nature.

The global equivalence of two symplectic forms is an open question on compact manifolds:

Given two symplectic forms ω, ω' on a smooth manifold M, is there a diffeomorphism $h : M \to M$ such that $h^*\omega' = \omega$?

The answer in the case the symplectic manifold is open is given by Gromov's h-principle. In general this is a very difficult question for closed manifolds. For instance Taubes showed (using Gromov-Witten invariants) that any symplectic form cohomologous to the symplectic form above on CP^2 is equivalent to it.

The following (weak) global equivalence theorem of symplectic structures will be useful:

Moser theorem [20]

Let ω_t be a smooth family of symplectic forms on a compact manifold M such that the cohomology classes $[\omega_t] \in H^2(M,\mathbf{R})$ of ω_t are constant, then there exists a smooth family of diffeomorphisms ϕ_t such that $\phi_0 = id$ and $\phi_t^*\omega_t = \omega_0$.

Exercise 8 Show that if ω is a symplectic form on a smooth manifold M and if θ is a closed 2-form on M which is C^1 close to zero, then $\omega + \theta$ is again a symplectic form.

Moreover if θ is exact and M is compact, show that there exists a diffeomorphism ϕ such that $\phi^*\omega = \omega + \theta$.

2 Introducing $Symp(M,\omega)$ and $Ham(M,\omega)$

We will give $Symp(M,\omega)$ the C^∞-compact-open topology. This is the topology of uniform convergence over all compact subsets of $h \in Symp(M,\omega)$, and all its partial derivatives (in local charts). In section 2,2, we will see that $Symp(M,\omega)$ is locally connected by smooth arcs.

A diffeomorphism ϕ is said to be *isotopic to the identity* if there exists a smooth map $H : M \times [0,1] \to M$ such that if $h_t : M \to M$ is given by $h_t(x) = H(x,t)$, then h_t is a C^∞ diffeomorphism , $h_0 = id_M$ and $h_1 = \phi$. We say that h_t is an isotopy from ϕ to the identity.

A symplectomorphism $\phi \in Symp(M,\omega)$ is isotopic to the identity if in the definition above, for all t, h_t is a symplectomorphism with compact support. We will say that h_t is a symplectic isotopy (with compact support) form ϕ to the identity. We consider the space $Symp(M,\omega)_0$ of symplectomorphisms which are isotopic to the identity. One shows that this is a group, which coincides with the identity component in $Symp(M,\omega)$.

Exercise 9 Show that any symplectic diffeomorphism of $(\mathbf{R^{2n}},\omega)$ is isotopic to the identity (through non compactly supported isotopies). (Use the "Alexander trick").

Exercise 10 Let (M,ω) be a compact oriented surface with orientation (symplectic) form ω. Show that the inclusion $Symp(M,\omega)_0 \subset Diff(M)_0$, where $Diff(M)_0$ is the identity component in the group of all diffeomorphisms (with the C^∞ topology), is a homotopy equivalence. (Use Moser theorem).

The homotopy type of $Symp(M,\omega)$ is known for all oriented compact surfaces : the inclusions $S0(3) \subset Diff(S^2)_0$ [28], $T^2 \subset Diff(T^2)_0$ [11] are homotopy equivalence and if M is a compact surface of genus bigger than 1, then $Diff(M)_0$ is contractible [11]. Conclude now using exercise 10.

In higher dimensions, almost nothing is known. Let us just cite two known results:
(i) $Symp(\mathbf{R^4},\omega_\mathbf{0})$ is contractible (Gromov).

(ii) $Symp(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$ is homotopy equivalent to $SO(3) \times S0(3)$. (Gromov).

An isotopy h_t of a manifold gives rise to a family of vector fields $\dot{h}_t$ defined by

$$\dot{h}_t(x) = \frac{dh_t}{dt}(h_t^{-1}(x)).$$

Conversely a family of vector fields X_t with compact support gives rise to an isotopy ϕ_t, via the existence and uniqueness theorem of solutions of ODE:

$$\frac{d\phi_t}{dt}(x) = X_t(\phi_t(x)), \quad \phi_0(x) = x.$$

Exercise 11 Prove that if h_t, g_t are 2 isotopies, and if $u_t = h_t g_t$

$$\dot{u}_t = \dot{h}_t + (h_t)_* \dot{g}_t$$

(Use the chain rule from Calculus).
Deduce from the formula above the formula for $\dot{v}_t$ where $v_t = (h_t)^{-1}$.

Exercise 12 Let $h_{(s,t)}$ be a 2-parameter family of diffeomorphisms of a smooth manifold M such that $h_{(0,0)} = 1d_M$. Let $X_{(s,t)}, Y_{(s,t)}$ be the family of vector fields defined by:

$$X_{(s,t)}(x) = \frac{d}{dt} h_{(s,t)}(h^{-1}_{(s,t)}(x));$$

$$Y_{(s,t)}(x) = \frac{d}{ds} h_{(s,t)}(h^{-1}_{(s,t)}(x))$$

Show that

$$\partial X_{(s,t)}/\partial s = \partial Y_{(s,t)}/\partial t + [X_{(s,t)}, X_{(s,t)}]$$

(Use Frobenius theorem).

If h_t is a symplectic isotopy, then $h_t^*\omega = \omega$. Differentiating this equation gives

$$L_{\dot{h}_t}\omega = 0$$

where L_X is the Lie derivative in direction X. By Cartan formula $L_X\alpha = i(X)d\alpha + d(i(X)\alpha)$, and the fact that $d\omega = 0$, the equation above says that

$$i(\dot{h}_t)\omega$$

is a closed 1-form.

We say that h_t is a *hamiltonian isotopy* if there exists a smooth family of smooth functions H_t such that

$$i(\dot{h}_t)\omega = dH_t.$$

A symplectomorphism $\phi \in Symp(M,\omega)$ is said to be a *hamiltonian diffeomorphism* if there exists a hamiltonian isotopy h_t such that $\phi = h_1$.

Let $Ham(M,\omega)$ denote the set of all hamiltonian diffeomorphisms of (M,ω), i.e. the set of time one maps of hamiltonian isotopies. It follows from exercises 11 that $Ham(M,\omega)$ is a normal subgroup of $Symp(M,\omega)_0$.

Remark. The group $Ham(M,\omega)$ contains the group $H(M,\omega)$ as a normal subgroup. We will see that these two groups coincide.

The groups above $Symp(M,\omega)$ and $Ham(M,\omega)$ depend of course on ω. If there exists a diffeomorphism $h : (M,\omega) \to (M',\omega')$ between 2 symplectic manifolds such that $h^*\omega' = \lambda\omega$ for some constant λ, then

$$I_h : Symp(M,\omega) \to Symp(M',\omega') \qquad \phi \mapsto h\phi h^{-1}$$

is an isomorphism. The converse is a deep theorem [6].

Theorem 2.1. *Let $u : Symp(M,\omega)_0 \to Symp(M',\omega')_0$ or $u : Ham(M,\omega)_0 \to Ham(M',\omega')_0$ be group isomorphisms. Then there exists a diffeomorphism $h : M \to M'$ such that $h^*\omega' = \lambda\omega$ for some constant λ and such that $u = I_h$.*

This theorem means that the groups $Symp(M,\omega)$ or $Ham(M,\omega)$ determine the symplectic geometry. This fact is a generalization of a theorem of Filipkiewicz [12] asserting that the group of diffeomorphisms determines the smooth structure.

The proof of this theorem uses the difficult theorem 2.3 and several "dynamical system" type arguments.

The following important property (n-fold transitivity for all n) , due to Boothby (see [2]) is much easier to prove:

Theorem 2.2. *Given two sets $\{x_1,..,x_n\}$ and $\{y_1,..,y_n\}$ of disjoint points on a connected symplectic manifold (M,ω), there is $h \in Ham(M,\omega)$ such that $h(x_i) = y_i$.*

Hence a connected symplectic manifold is a "homogeneous space"

$$M \approx Ham(M,\omega)/Ham(M,\omega)_a$$

where $Ham(M,\omega)_a$ is the isotropy subgroup of some point a. Note that two isotropy subgroups $Ham(M,\omega)_a$ and $Ham(M,\omega)_b$ are conjugate.

2.1 The flux homomorphism

Two symplectic isotopies h_t, g_t with $g_1 = h_1 = \phi$ are said to be homotopic relatively to ends if there exists a 2-parameter family of symplectic isotopies $K_{(s,t)}$ such that $K_{0,0} = id_M$, $K_{(0,t)} = h_t$, $K_{(1,t)} = g_t$ and $K_{(s,1)} = \phi$ for all s.

This is an equivalence relation among symplectic isotopies from ϕ to the identity. The set of all equivalence classes $[h_t]$ of symplectic isotopies h_t in $Symp(M,\omega)_0$ is the universal cover $Symp(\tilde{M},\omega)_0$ of $Symp(M,\omega)_0$.

For any symplectic isotopy (ϕ_t), we consider the 1-form

$$\Sigma(\phi_t) = \int_0^1 i(\dot{\phi}_t)\omega dt$$

Using exercise 12, one shows that if two symplectic isotopies h_t, g_t are homotopic relatively to ends, then

$$\Sigma(h_t) - \Sigma(g_t) = d\theta$$

for some 1-form θ.

This means that we get a well defined map $[h_t] \mapsto [\Sigma(h_t)] \in H^1(M,\mathbf{R})$ from the universal covering of $Symp(M,\omega)_0$ into the first de Rham cohomology group of M.

Denote the map above by :

$$\tilde{S} : Symp(\tilde{M},\omega)_0 \to H^1(M,\mathbf{R})$$

Using exercise 11, one shows that $\tilde{S}$ is a group homomorphism [9], [1], [2]. To show that this homomorphism is surjective, consider a cohomology class $a \in H^1(M,\mathbf{R})$ with compact supports, and a closed 1-form α with compact support representing a. If $\gamma(t)$ is the flow of the vector field V defined by $i(V)\omega = \alpha$, then $\tilde{S}[\gamma(t)] = a$.

The homomorphism $\tilde{S} : Symp\tilde{(}M,\omega) \to H^1(M,\mathbf{R})$ is called the *flux homomorphism* or the Calabi homomorphism.

The group

$$\Gamma = \tilde{S}(\pi_1(Symp(M,\omega)_0)))$$

is called the *the flux group*.

We get an induced surjective homomorphism

$$S : Symp(M,\omega)_0 \to H^1(M,\mathbf{R})/\Gamma.$$

Remarks

1. The commutator subgroup $[Symp(M,\omega)_0, Symp(M,\omega)_0]$ of $Symp(M,\omega)_0$ is contained in the kernel, $KerS$, of S since the range of S is an abelian group.

2. The groups $H(M,\omega)$ and $Ham(M,\omega)$ are contained in $KerS$.

3. Suppose $h \in Symp(M,\omega)_0$ can be written as $h = h_1 \dots .h_N$ where each h_i has compact support in contractible open set, then $S(h_i) = 0$, and hence $S(h) = \sum S(h_i) = 0$. Let $H_0(M,\omega)$ be the group generated by symplectomorphisms with compact supports in contractible open sets. Then $H_0(M,\omega) \subset KerS$.

Exercise 13 Prove that $H_0(M,\omega) = KerS$.

This exercise is not easy. We refer to [2] for a proof. The statement of exercise 13 is called the " fragmentation property" of $Ham(M,\omega)$.

We have the following deep result [1]:

Theorem 2.3. *Let (M,ω) be a connected compact symplectic manifold. Then KerS is a simple group (i.e. it contains no non-trivial normal subgroup).*

The proof of this theorem is very delicate. It uses the Arnold-Kolmogoroff-Nash-Moser-Sergeraert implicit function theorem in Frechet spaces [14],[27] and a generalization of the Mather-Thurston theory relating the homology of diffeomorphism groups and characteristic classes of foliations (see [2]).

Corollary 2.4. *Let (M,ω) be a compact symplectic manifold. Then*

$$H(M,\omega) = H_0(M,\omega) = Ham(M,\omega) = KerS = [Symp(M,\omega)_0, Symp(M,\omega)_0].$$

Recently Ono [23] proved the following

Theorem 2.5. *The flux group Γ is discrete, i.e. $Ham(M,\omega)$ is a closed subgroup of $Symp(M,\omega)_0$ with the C^∞ topology.*

This is a deep theorem whose proof relies on the Floer-Novikov homology.

Exercise 14 Let (M,ω) be a symplectic manifold where ω is exact: i.e there is a 1-form α such that $\omega = d\alpha$ (ex. the cotangent bundle, the symplectisation of a contact manifold).

Show that for any $\phi \in Symp(M,\omega)_0$, the 1-form $\phi^*\alpha - \alpha$ is closed and its cohomology class $[\phi^*\alpha - \alpha]$ is independent of the choice of α. Show that $\phi \mapsto [\phi^*\alpha - \alpha]$ is a group homomorphism which coincides with the flux S. In that case, the flux group is trivial.

Exercise 15. Let $L_\omega(M)$ be the set of symplectic vector fields, i.e. vector fields X such that $i(X)\omega$ is a closed form. Denote by $[i(X)\omega]$ its cohomology class. Show that $L_\omega(M)$ is a Lie algebra. Show that the map : $s : L_\omega(M) \to H^1(M,\mathbf{R}), \quad \mathbf{X} \mapsto [\mathbf{i}(\mathbf{X})\omega]$ is a surjective Lie algebra homomorphism and its kernel is the derived Lie algebra $[L_\omega(M), L_\omega(M)]$, (generated by commutators). Moreover $Ker s = [L_\omega(M), L_\omega(M)]$ is a simple Lie algebra.
This "infinitesimal version" of theorem 2.3 was proved by Calabi [9].

2.2 The local structure of $Symp(M,\omega)$

A submanifold L of a symplectic manifold (M,ω) is called a *lagrangian* submanifold if $dimL = (1/2)dimM$ and $i^*\omega = 0$ where $i : L \to M$ is the inclusion.

Exercise 16 Show that the graph of a 1-form α on N is a lagrangian submanifold of T^*N iff α is closed. (Use exercise 3).

Exercise 17 Let $h : M \to M$ be a symplectomorphism of (M,ω). Show that its graph is a lagrangian submanifold of $(M \times M, \omega \ominus \omega)$. For instance the diagonal $\Delta \subset (M \times M, \omega \ominus \omega)$ is a lagrangian submanifold (the graph of the identity).

The following result due to Kostant-Sternberg-Weinstein) [13] describes a neigborhood of a lagrangian submanifold inside the ambiant manifold:

Theorem 2.6. *There exists a diffeomorphism k from a neigborhood* $\mathbf{U}$ *of* Δ *in* $M \times M$ *onto* $T^*(\Delta) \approx T^*(M)$ *such that* $k_{|\Delta \approx M}$ *is the identity and* $k^*\omega_M = \omega \ominus \omega$

.

If h is a symplectomorphism C^1 close enough to the identity, and its graph $\Gamma(h)$ fits inside the neighborhood $\mathbf{U}$, then $k(\Gamma(h))$ is a lagrangian submanifold of $T^*(M)$, which is C^1 close to the diagonal; it is then the graph of a closed 1-form $W(h)$. The correspondence

$$h \mapsto W(h)$$

is a smooth chart of a neighborhood $\mathbf{U}$ of the identity in $Symp(M,\omega)_0$, into a neighborhood $\mathbf{W}$ of zero in the space $Z^1(M)$ of closed i-forms, called the Weinstein chart.

If $h \in \mathbf{U}$, we get a "canonical" isotopy $h_t = W^{-1}(tW(h))$ from h to the identity. Hence $Symp(M,\omega)_0$ is smoothly locally contractible, and locally connected by smooth arcs.

Exercise 18 Show that $\tilde{S}([h_t]) = [W(h)]$, where $[W(h)]$ is the cohomology class of the form $W(h)$[5].

Therefore the Weinstein chart takes a small neigborhood of the identity in $Ham(M,\omega)$ to the space $B^1(M)$ of exact 1-forms. The space $B^1(M)$ is isomorphic to the space $\mathbf{A} = \mathbf{C^\infty_n(M)}$ of "normalized functions on M. A function f is normalized if $\int_M f\omega^n = 0$ when M compact or it has compact support when M is not compact.

(Heuristically), one says that the Lie algebra of $Ham(M,\omega)$ is the space $\mathbf{A} = \mathbf{C^\infty_n(M)}$.

Let $h \in U$, the zeros of $W(h)$ correspond to intersection points of the graph of h, i.e. to the fixed points of h. From exercise 18, we deduce the following observation of Weinstein [32]:

Theorem 2.7. *A C^1 small hamiltonian diffeomorphism h on compact manifold has as many fixed points as can have a smooth function on a compact manifold.*

3 Some global properties

3.1 The Arnold Conjecture

Theorem 2.7 is a particular case of a general conjecture made by Arnold in the 60's. It says that *if h is a hamiltonian diffeomorphism of a compact symplectic manifold (M,ω) such that its graph $\Gamma(h)$ intersects the diagonal transversally, then the number of its fixed points is no smaller than the number of critical points a smooth function is allowed to have.*

Another formulation is that the number of fixed points is bounded from below by the sum of the Betti numbers of M.

This conjecture is being solved nowadays. It has been a driving force which led to tremendous developpement in "Symplectic Topology".

One first observe that the set of fixed points of a hamiltonian diffeomorphism are in 1-1 correspondence with critical points of a functional the *action-functional* on the infinite dimensional space " of contractible loops on M (see exercise 2). *Floer homology* is the homology whose chains are generated by these critical points. The main result is that *Floer Homology* is isomorphic to the singular homology. The Arnold conjecture then follows. Floer homology is a very rich and very complicated machinery. The first steps of the theory can be read in [8], [26]. We refer to [21] for more advanced readings.

3.2 Symplectic and hamiltonian rigidities

Let (M,ω) be a compact symplectic manifold. Let $Diff(M)$ be the group of all C^∞ diffeomorphisms of M endowed with the C^r topologies, $0 \leq r \leq \infty$. We have the following "rigidity" theorem due to Eliashberg-Gromov [16]:

Theorem 3.1. *Let (M,ω) be a compact symplectic manifold. Then $Symp(M,\omega)$ is C^0 closed in $Diff(M)$.*

This theorem says that the symplectic character of a diffeomorphism "survives" topological limits. This is an indication that there exists a " symplectic topology".

Lalonde, Polterovich and McDuff [18] have also discovered a **hamiltonian rigidity** phenomenon:

Theorem 3.2. *Let ω and ω' be two symplectic forms on a closed manifold M. Suppose a loop l_t in $Symp(M,\omega)$ is homotopic to a loop $l'_t \in Ham(M,\omega')$ through $Diff^\infty(M)$ then l_t is homotopic to a loop l^*_t in $Ham(M,\omega)$ through $Symp(M,\omega)$ regardless of the relation between the two symplectic forms.*

This shows that being " hamiltonian" is topological in nature.

The symplectic and hamiltonian rigidities above reveal that symplectic geometry underlines a topology, one may call the C^0**-symplectic topology**. This is a new mathematical discipline in which almost nothing is known. We will say a few words about it at the end of these lectures.

Eliashberg-Gromov's theorem is an immediate consequence of the following

Theorem 3.3. *Let $\phi_j : B(1) \to (\mathbf{R^{2n}}, \omega)$ be a sequence of symplectic embeddings converging locally uniformly to a map $\phi : B(1) \to \mathbf{R^{2n}}$. If ϕ is differentiable at 0, then $\phi'(0)$ is a symplectic map.*

This is an easy consequence of the existence of *symplectic capacities* (Ekeland-Hofer) [16].

Symplectic capacities [16]

Let (M,ω) be a 2n dimensional symplectic manifold. It is obvious that any symplectomorphism h preserve the Liouville volume $\Omega = \omega^n$. **Symplectic Topology** emerged from the search of invariants which distinguish "volume preserving properties" from symplectic properties of symplectic manifolds of dimension $2n \geq 4$.

Darboux' theorem says that near every point in (M,ω), there is a local diffeomorphism of a small ball of radius r, centered at the origin $B(0,r)$ in $\mathbf{R^{2n}}$ into M such that $\phi^*(\omega) = (\omega_0)_{|}\phi^{-1}(B(0,r)$ where $\omega_0 = \sum dx_i \wedge dy_i$.

Let us look at the largest ball that can be embedded in M, i.e. we define:

$$D(M,\omega) = sup\{\pi r^2\}$$

where r runs over all the radii of balls $B(0,r)$ that can be embedded symplectically in (M,ω). This is a symplectic invariant called the *Gromov width*.

Definition 3.4. Let $\mathbf{S(2n)}$ be the class of all 2n dimensional symplectic manifolds.

A *capacity* is a map $c : \mathbf{S(2n)} \to \mathbf{R} \cup \infty$ such that

(1) $c(M,\omega) \leq c(N,\tau)$ if there exists a symplectic emdebbing $\phi : (M,\omega) \to (N,\tau)$. (monotonicity)

(2) $c(M,\alpha\omega) = |\alpha| c(M,\omega)$ for all non-zero number α. (conformality)

(3) $c(B(0,1),\omega_0) = c(Z(1),\omega_0) = \pi$
where
$Z(r) = \{x_1,..,x_n,y_1,...y_n)|x_1^2 + y_1^2 \leq r^2]$

If n= 1, then $c(M,\omega) = |\int_M \omega|$ is a capacity. If $n \geq 2$, the condition (3) excludes this example.

In [16] one can find the proof of the following theorem of Gromov:

Theorem 3.5. *The Gromov width is a symplectic capacity.*

The existence of a symplectic capacity implies immediately the following famous Gromov non-sqeezing theorem:

Theorem 3.6. *If there is a symplectic embedding from $Z(r) \subset \mathbf{R^{2n}}$ into the ball $B(0,R)$, then $r \leq R$.*

This theorem has been popularized as " the Gromov camel".

The proof of Theorem 3.2 uses Floer homology and the notion of " hamiltonian fibrations", i.e. fibrations with $Ham(M,\omega)$ as structural groups. A classical construction , called the "clutching" associates to a loop in $Ham(M,\omega)$ a hamitonian fibration over S^2. The flux homomorphism appears as the " boundary homomorphism" in the Wang exact sequence of the associated hamiltonian fibration over S^2 [18].

4 Metrics on $Symp(M,\omega)$ and $Ham(M,\omega)$

A *Finsler structure* on a smooth manifold Z is a norm in each tangent space T_zZ which varies smoothly with $z \in Z$. (In general these norms may not come from a scalar product: hence this may not be a riemannian structure). We may define the length of a curve $z : [a,b] \to Z$ as:

$$length(z) = \int_a^b norm(\dot{z}(t))dt.$$

We saw that the Lie algebra of $Ham(M,\omega)$ is the space $\mathbf{A}$ of smooth normalized functions on M. Hence any norm $||.||$ which is invariant by $Ham(M,\omega)$ defines a Finsler structure on $Ham(M,\omega)$. For any smooth path $\phi_t \in Ham(M,\omega)$, and any such norm $||,||$ on $\mathbf{A}$, we define the length as

$$length(\phi_t) = \int_0^1 ||F_t||dt$$

where F_t is the family of functions such that $i_{\dot{\phi}_t}\omega = dH_t$.

One defines the "distance" between two hamiltonian diffeomorphisms ϕ, ψ as

$$d(\phi,\psi) = inf(length(f_t))$$

where the infimum is taken over all hamiltonian paths $\{f_t\}$ with $f_0 = id$ and $f_1 = \psi\phi^{-1}$.

It is easy to verify that the function d is a pseudo-distance: it satisfies the properties of a distance but may be degenerate: $d(\phi,\psi) = 0$ may not imply that $\phi = \psi$.

The choice of the metric $||,||$ on $\mathbf{A}$ is very important. For instance if we take the L_p norm

$$||H||_p = (\int_M |H|^p(\omega)^n)^{1/p}$$

$p \geq 1$ then the corresponding pseudo metric is degenerate[26].

However, the L_∞-norm

$$||H||_\infty = osc(H) = maxH - minH$$

gives a non-degenerate function, i.e. a genuine metric, called the Hofer metric.

For $\phi \in Ham(M,\omega)$, choose a hamiltonian isotopy $\Phi = (\phi_t)$ from ϕ to the identity. Hofer defined the length of this isotopy

$$l_H(\Phi) = \int_0^1 osc(F_t)dt$$

where $i(\dot{\phi}_t) = dF_t$.

Exercise 19 The length function satisfies :
(i) $l_H(\Phi) \geq 0$
(ii) $l_H(\Phi.\Phi') \leq l_H(\Phi) + l_H(\Phi')$ where $(\Phi.\Phi')(t) = (\phi_t\phi'_t)$,
(iii) $l_H(\Phi) = l_H(\Phi^{-1})$
(iv) $l_H(h.\Phi.h^{-1}) = l_H(\Phi)$ for all sympectomorphism h.
(Hint : use exercise 11).

Consider

$$\nu(\phi) = inf(l_H(\Phi))$$

where the infimum is taken over all hamiltonian isotopies from ϕ to the identity.

Theorem 4.1. *The function $\nu(\phi)$ is a bi-invariant metric on $Ham(M,\omega)$.*

Therefore the function $d(\phi,\psi) = \nu(\phi.\psi^{-1})$ on $Ham(M,\omega)$ is a bi-invariant distance. We call $\nu(\phi) = ||\phi||$ the Hofer norm of ϕ and $d(\phi,\psi)$ the Hofer distance from ϕ to ψ.

In Theorem 4.1, only the non-degeneracy is difficult to prove. We give below an outline of its proof. The other properties come straight form exercise 19.

This theorem was proved first by Hofer for $M = \mathbf{R^{2n}}$ with its natural symplectic form, using infinite dimensional variational methods [15], then got improved by Viterbo using generating functions [30] and Polterovich using Gromov's J-holomorphic curves [25], and has been proved in its full generality by Lalonde-McDuff using J-holomorphic curves [17].

Definition 4.2. The *displacement energy* $e(A)$ of a bounded subset A of M is defined as follows:

$$e(A) = inf\{||\phi||, \phi \in Ham(M,\omega), \phi(A) \cap A = \emptyset\}$$

We have the following fact due to Eliashberg-Polterovich) [24]

Theorem 4.3. *For any non-empty open set A, $e(A)$ is strictly positive.*

The connection between the displacement energy and symplectic capacities is given by this result of Hofer, Lalonde-Mc Duff (see [16]):

Theorem 4.4.

$$sup\{c(U) | U \text{ open and } \phi(U) \cap U = \emptyset\} \leq ||\phi||.$$

4.1 Hofer geometry

This is the geometry of $Ham(M,\omega)$ equipped with the Hofer metric. The *Hofer topology* is the topology induced by the Hofer distance. This geometry and topology are not well understood.

For instance the following question does not have a complete answer: Let $(Ham(M,\omega),C^r)$ and $(Ham(M,\omega),\mathbf{H})$ be the group $Ham(M,\omega)$ endowed respectively with the C^r topology and the Hofer topology $\mathbf{H}$, is the identity map

$$(Ham(M,\omega),C^r) \to (Ham(M,\omega),\mathbf{H})$$

continuous?

Clearly the answer is yes if $r \geq 1$. When $< \pi_2(M),\omega >= 0$, Ostrover [24] constructed an example of a sequence in $Ham(M,\omega)$ converging C^0 to the identity, but whose Hofer norm goes to infinity. Therefore the map above is not continuous in this case. This example also proves the following

Theorem 4.5. [29], [24] *Let (M,ω) be a compact symplectic manifold satisfying $< \pi_2(M),\omega >= 0$, then the diameter of $(Ham(M,\omega,\mathbf{H})$ is infinite.*

4.2 Generalization to $Symp(M,\omega)$

A natural question is to try to extend the Hofer metric from the group $Ham(M,\omega)$ to the whole group $Symp(M,\omega)_0$. There are at least two ways to obtain a bi-invariant metric on $Symp(M,\omega)$ using the Hofer metric $||.||$ on $Ham(M,\omega)$.

1. Choose a positive number K and for $\phi \in Symp(M,\omega)$ define $||\phi||_K$ to be $min(||\phi||,K)$ if $\phi \in Ham(M,\omega)$, and $||\phi||_K = K$ otherwise (Han).

2. Fix a positive real number a and define for all $\phi \in Symp(M,\omega)$, $||\phi||_a = sup\{||\phi f \phi^{-1} f^{-1}||$ where $f \in Ham(M,\omega), ||f|| \leq a\}$. (Lalonde-Polterovich).

However, these metrics don't restrict to the Hofer norm on $Ham(M,\omega)$. Moreover $Ham(M,\omega)$ has a finite diameter in the restriction of these metrics to $Ham(M,\omega)$.

The question of extending the Hofer metric was studied in [4]. For instance one observes that the Hofer metric on $Ham(T^{2n},\omega)$ extends to the whole $Symp(T^{2n},\omega)$. They prove the following result [4]:

Theorem 4.6. *Let (M,ω) be a symplectic manifold such that the homomorphism S admits a continuous homomorphic right inverse, then the Hofer distance extends to a distance on $Symp(M,\omega)_0$ which is right invariant but not left invariant.*

It is not easy to characterize symplectic manifolds satisfying the conditions of theorem 15.

In the rest of this paragraph, we show how to use the Hofer metric to get a "Hofer-like " metric on the group $Symp(M,\omega)_0$.

An intrinsic topology on the space $symp(\omega,M)$ of symplectic vector fields

Let (M,ω) be a compact symplectic manifold. The "Lie algebra" of $Symp(M,\omega)$ is the space $sym(M,\omega)$ of symplectic vector fields, i.e the set of vector fields X such that $i_X\omega$ is a closed form. We give a norm $|.|$ on $symp(M,\omega)$ this way. First we fix a riemannian metric g, and a basis $\mathbf{B} = \{\mathbf{h_1},..,\mathbf{h_k}\}$ of harmonic 1-forms.

On the set $Harm^1(M,g)$ of harmonic 1-forms, we put the following "Euclidean" norm. If $H \in Harm^1(M,g)$ and $H = \sum \lambda_i h_i$, define

$$|H|_{\mathbf{B}} =: \sum |\lambda_i|.$$

Given $X \in sym(M,\omega)$, we consider the Hodge decomposition of $i_X\omega$ [31] : there is a unique harmonic 1-form H_X and a unique function u_X such that

$$i_X\omega = H_X + du_X$$

Now we define a norm $||.||^g_{\mathbf{B}}$ on the space $symp(M,\omega)$ by:

$$||X||^g_{\mathbf{B}} = |H_X|_{\mathbf{B}} + osc(u_X)$$

It is easy to see that this is indeed a norm.

The metric topology on $symp(M,\omega)$ defined by $||.|^g_{\mathbf{B}}$ is independent of the choice of the riemannian metric g and of the basis $\mathbf{B}$ of harmonic 1-forms:

Theorem 4.7. *For all riemannian metric g and all basis* $\mathbf{B}$ *of harmonic 1-forms, the norms* $||.|^g_{\mathbf{B}}$ *are all equivalent.*

In the sequel, we will fix one riemannian metric g and one basis $\mathbf{B}$ of harmonic 1-forms and simply denote by $||.||$ the norm $||.|^g_{\mathbf{B}}$.

The norm $||.||$ is not invariant by $Symp(M,\omega)$. Hence it does not define a Finsler metric on $Symp(M,\omega)$. But we still can define the length of a symplectic isotopy $\Phi = \phi_t$.

Consider the Hodge decomposition of the closed 1-form $i(\dot\phi_t)\omega$:

$$i(\dot\phi_t)\omega = H^{\Phi}_t + du^{\Phi}_t.$$

Define now the "length" of Φ to be

$$length(\Phi) = \int_0^1 (|H^{\Phi}_t| + osc(u^{\Phi}_t))dt = \int_0^1 ||\dot\phi_t)||dt$$

Given $\phi \in Symp(M,\omega)_0$, we let

$$e_0(\phi) = inf(length(\Phi))$$

where the infimum is taken over all symplectic isotopies Φ from ϕ to the identity.

We have the following generalization of the Hofer metric [3]:

Theorem 4.8. *The function* $\phi \mapsto e(\phi) = ((e_0(\phi) + e_0(\phi^{-1}))/2$ *defines a metric on* $Symp(M,\omega)$ *whose restriction to* $Ham(M,\omega)$ *is bounded from above by the Hofer metric, i.e.*

$$e(\phi \leq ||\phi||.$$

Moreover $Ham(M,\omega)$ *with the induced topology is a closed subgroup.*

Unlike the Hofer distance, the distance $d(\phi,\psi) = e(\phi.(\psi^{-1}))$ is not bi-invariant.

5 The C^0 symplectic topology

The C^0 *symplectic topology* is the topology underlying Symplectic Geometry, which manifests itself in various rigidity properties (ex. Eliashberg-Gromov symplectic rigidity, Lalonde-Mc Duff-Polterovich hamiltonian rigidity,...).

According to Oh-Muller ([22]) the automorphism group of the C^0 topology is the group

$$Sympeo(M,\omega) =: \overline{Symp(M,\omega)}$$

which is the closure of $Symp(M,\omega)$ in the group $Homeo(M)$ of homeomorphisms of M endowed with the C^0 topology.

The C^0 topology on $Homeo(M)$ coincides with the metric topology coming from the metric

$$\overline{d}(g,h) = max(sup_{x\in M} d_0(g(x),h(x)), sup_{x\in M} d_0(g^{-1}(x),h^{-1}(x))$$

where d_0 is a distance on M induced by some riemannian metric.

On the space $PHomeo(M)$ of continuous paths $\gamma : [0,1] \to Homeo(M)$, one has the distance

$$\overline{d}(\gamma,\mu) = sup_{t\in[0,1]} \overline{d}(\gamma(t),\mu(t))$$

Consider the space $PHam(M)$ of all isotopies $\Phi_H = [t \mapsto \Phi_H^t]$ where Φ_H^t is the family of hamiltonian diffeomorphisms obtained by integration of the family of vector fields X_H for a smooth family $H(x,t)$ of real functions on M.

Definition 5.1. *The hamiltonian topology* [22] on $PHam(M)$ is the metric topology defined by the distance

$$d_{ham}(\Phi_H,\Phi_{H'}) = ||H-H'|| + \overline{d}(\Phi_H,\Phi_{H'})$$

where $||H-H'|| = \int_0^1 osc(H-H')dt$.

Let $Hameo(M,\omega)$ denote the space of all homeomorphisms h such that there exists a continuous path $\lambda \in PHomeo(M)$ such that $\lambda(1) = h$ and there exists a Cauchy sequence (for the d_{ham} norm) of hamiltonian isotopies Φ_{H^n}, which C^0 converges to λ (in the $\overline{d}$ metric). The following is the first important theorem in the C^0 symplectic topology [22]:

Theorem 5.2. *The set $Hameo(M,\omega)$ is a topological group. It is a normal subgroup of $Sympeo(M,\omega)$. If $H^1(M,\mathbf{R}) \neq \mathbf{0}$, then $Hameo(M,\omega)$ is strictly contained in $Sympeo(M,\omega)$.*

This group is the topological analogue of the group $Ham(M,\omega)$.

On the space $Iso(M,\omega)$ of symplectic isotopies of (M,ω) we define a distance D_0 as follows: if $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are symplectic isotopies:

$$D_0(\Phi,\Psi) = |||\dot{\phi}_t - \dot{\psi}_t||| =: \int_0^1 (|H_t^\Phi - H_t^\Psi| + osc(u^{\Phi_t} - u^{\Psi_t}))dt.$$

Denote by $\Phi^{-1} = (\phi_t^{-1})$ and by $\Psi^{-1} = (\psi_t^{-1})$ the inverse isotopies.

We define a distance D by:

$$D(\Phi,\Psi) = (D_0(\Phi,\Psi) + D_0(\Phi^{-1},\Psi^{-1}))/2$$

Following [22], we define the *symplectic distance* on $Iso(M,\omega)$ by:

$$d_{symp}(\Phi,\Psi) = \bar{d}(\Phi,\Psi) + D(\Phi,\Psi).$$

One obtains the following generalization of the group $Hameo(M,\omega)$ [7]:

Theorem 5.3. *Let (M,ω) be a compact symplectic manifold. The set $SSympeo(M,\omega)$ of all homeomorphisms of M such that there exists a path $\lambda \in PHomeo(M)$ with $\lambda(1) = h$ and such that there exists a Cauchy sequence (for the distance d_{symp}) of symplectic isotopies Φ_n , which converges in the C^0 topology (induced by the metric $\bar{d}$) to λ, is a subgroup of $Sympeo(M,\omega)$, which contains $Hameo(M,\omega)$ as a normal subgroup. It is arcwise connected and is contained in the identity component of $Sympeo(M,\omega)$. Moreover its commutator subgroup $[SSympeo(M,\omega), SSympeo(M,\omega)]$ is contained in $Hameo(M,\omega)$.*

The group $SSympeo(M,\omega)$ is called the group of strong symplectic homeomorphisms. This group is probably strictly smaller than $Sympeo(M,\omega)$: its topology is more involved, combining the C^0 topology and the Hofer topology.

The groups just mentioned above $Sympeo(M,\omega)$, $Hameo(M,\omega)$, $SSympeo(M,\omega)$ are largely unknown. They will be the focus of intense research in the next future.

Final remark

We observed that most of the results surveyed in these lectures concern *compact* manifolds. In fact it is hard to deal with non compact manifolds and non compactly supported diffeomorphisms. The topologies are bad, and the one-to-one relation between isotopies and family of smooth vector fields breaks down.

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