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LOCATION OF SPECTRUM AND STABILITY OF SOLUTIONS FOR MONOTONE PARABOLIC SYSTEM

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1. Introduction. We consider the parabolic system of equations

$$\frac{\partial u}{\partial t} = a\Delta u + F(u, x'), \tag{1.1}$$

with the boundary condition

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{1.2}$$

where $u = (u_1, \dots, u_n)$, $x = (x_1, \dots, x_m) \in \Omega \subset R^m$, Ω is an infinite cylinder with the axis in the x_1-direction and with sufficiently smooth boundary $\partial\Omega$. The coordinates in the section of the cylinder are denoted by $x' = (x_2, \dots, x_m)$. We suppose that a is a constant diagonal matrix with positive diagonal elements and function $F = (F_1, \dots, F_n)$ satisfies the condition

$$\frac{\partial F_i}{\partial u_j} \geq 0, \quad i \neq j. \tag{1.3}$$

In this work we study local and global stability of travelling waves described by the problem (1.1), (1.2). We recall that a travelling wave solution is a solution of the form $u(x,t) = w(x_1 - ct, x_2, \dots, x_m)$. Here c is a constant, the wave velocity. The function $w(x)$ is a stationary solution of the problem

$$\frac{\partial v}{\partial t} = a\Delta v + c\frac{\partial v}{\partial x_1} + F(v, x'), \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{1.4}$$

As is known, local stability of travelling waves is determined by the location of the spectrum of the operator obtained by linearization of the right-hand side of (1.4) about the travelling wave $w(x)$,

$$Mu = a\Delta u + c\frac{\partial u}{\partial x_1} + F'(w(x), x')u, \ \ \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{1.5}$$

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Location of the continuous spectrum of this operator is known (see [1], [2]) and we will impose the conditions under which it lies in the left half plane of the complex plane. In this case the wave is asymptotically stable if all eigenvalues of the operator M except for 0, lie in the left half plane. The zero eigenvalue should be simple.

We note that the zero eigenvalue appears because of invariance of solutions with respect to translation in x_1. The eigenfunction corresponding to the zero eigenvalue is the derivative of the wave, $\partial w/\partial x_1$. Thus the stability conditions will be satisfied if the zero eigenvalue is principal, i.e., with the maximal real part, and simple. We show that these conditions are satisfied if the eigenfunction corresponding to the zero eigenvalue is positive or, in other words, if the travelling wave solution is monotone in x_1.

Taking into account other possible applications we consider the elliptic problem of more general form

$$Lu = a(x)\Delta u + \sum_{i=1}^{m} c_i(x)\frac{\partial u}{\partial x_i} + b(x)u, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{1.6}$$

We suppose that $a(x)$, $c_j(x)$, and $b(x)$ are sufficiently smooth matrices having limits as $x_1 \to \pm\infty$, $a(x)$ and $c_j(x)$ are diagonal matrices with diagonal elements $a_i(x)$ and $c_{ji}(x)$, $i = 1, \ldots, n$, respectively, $a_i(x) \geq a_0 > 0$, where a_0 is a constant. We assume also that the matrix $b(x)$ has nonnegative off-diagonal elements, $b_{ij}(x) \geq 0$, $i \neq j,\ i, j = 1, \ldots, n$.

We show that under some conditions there exists a real eigenvalue of the operator L and that the corresponding eigenfunction is positive. This real eigenvalue appears to be the principal eigenvalue of the operator L and it is simple. We obtain also a minimax representation of the principal eigenvalue. The Krein-Rutman properties of the elliptic operators under consideration generalize some known results for the scalar case ($n = 1$) (see [3]–[6], [20] and references there).

These results are applicable for the operator M and we obtain local stability of travelling waves. We prove also their global stability or convergence of solutions of the problem (1.1), (1.2) with the initial condition $u(x, 0) = f(x)$ to the wave solution. We assume here that the initial condition $f(x)$ is monotone in x_1 and generalize the technique developed in our previous works for the one-dimensional case ($m = 1, n \geq 1$) ([1], [7]).

We note that in the one-dimensional scalar case $n = m = 1$ there is a complete theory including global stability of travelling waves (see [1] and references there). For the multidimensional scalar equation ($m \geq 1,\ n = 1$) stability of travelling waves is studied in [8], [9]. In this work we present the first results for the multidimensional systems of equations ($m \geq 1,\ n \geq 1$).

The contents of the paper are as follows. Sections 2 and 3 are devoted to location of the spectrum of the operator L. In Section 4 we obtain a minimax representation of the principal eigenvalue. In Section 5 we prove stability of travelling waves. Global stability of travelling waves allows us to obtain a minimax representation of the wave velocity (Section 6).

2. Existence of a real eigenvalue and positive eigenfunction. To prove existence of a real eigenvalue with the corresponding positive eigenfunction we use the theory of analytical semigroups, which is possible because the operator L considered in $L_2(\Omega)$ is sectorial (see, for example, [10]–[12]).

Without loss of generality we can assume that its eigenvalues with the maximal real parts are located on the imaginary axis. We assume that there is a finite number of complex conjugate eigenvalues is_j, $j = 1, \dots, m$ and the remaining spectrum lies in the half plane $Re\,\lambda < -\epsilon$, where $\epsilon > 0$.

Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = Lu, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u(x,0) = f(x), \tag{2.1}$$

where $f(x)$ is a sufficiently smooth function. Its solution can be represented in the form

$$u(x,t) = \sum_{k=1}^{m} u_k(x,t)e^{is_k t} + u_0(x,t), \tag{2.2}$$

where $u_k(x,t)$ is a polynomial in t, $u_0(x,t) \to 0$ as $t \to \infty$ in $L_2(\Omega)$ (see, for example, [10]).

Let $\phi(x)$ be a smooth vector-valued function with a finite support. Denote

$$v(t) = \int_\Omega u(x,t)\phi(x)\,dx,$$
$$p_k(t) = \int_\Omega u_k(x,t)\phi(x)\,dx,$$
$$v_0(t) = \int_\Omega u_0(x,t)\phi(x)\,dx.$$

Here $u\phi$, $u_k\phi$, and $u_0\phi$ are inner products in R^n. Then from (2.2) we have

$$v(t) = \sum_{k=1}^{m} p_k(t)e^{is_k t} + v_0(t), \tag{2.3}$$

where

$$|v_0(t)| \le \tau e^{-\epsilon t} \tag{2.4}$$

and τ is a positive constant.

Lemma 2.1. *The operator L has a zero eigenvalue.*

Proof. Suppose that the initial condition $f(x)$ of the problem (2.1) is nonnegative and has a nonzero projection on the invariant subspace corresponding to the eigenvalues on the imaginary axis. Then the solution is also nonnegative and in the representation (2.2) some of the functions $u_k(x,t)$ are different from zero. Then a nonnegative function $\phi(x)$ can be chosen such that at least one of the functions $p_k(t)$ is not identically zero.

We can rewrite (2.3) in the form

$$v(t) = g(t) + v_0(t).$$

If there is no zero among the numbers s_j, then the function $g(t)$ is a combination of *sines* and *cosines* with polynomial coefficients:

$$g(t) = \sum_{j=1}^{m/2} a_j(t)\sin(s_j t) + b_j(t)\cos(s_j t).$$

We show that $g_0(t) = g(t) + \tau e^{-\epsilon t}$ cannot be nonnegative. Together with (2.4) it will give a contradiction of the nonnegativeness of $v(t)$.

Suppose that $g_0(t) \geq 0$, $g_0(t) \not\equiv 0$. We begin with the case where the eigenvalues of the operator L on the imaginary axis are simple. In this case the coefficients a_j and b_j are constant. Consider the Cauchy problem

$$v'(t) = g_0(t), \quad v(0) = 1.$$

Its solution is

$$v(t) = \text{const} + \sum_{j=0}^{m/2} \Big(-\frac{a_j}{s_j}\cos(s_j t) + \frac{b_j}{s_j}\sin(s_j t)\Big) - \frac{\tau}{\epsilon}e^{-\epsilon t}. \tag{2.5}$$

Obviously it is a bounded function. Since $g_0(t)$ is supposed to be nonnegative, $v(t)$ is nondecreasing and has a limit v_0 as $t \to \infty$.

Suppose for definiteness that b_j is not zero. We multiply (2.5) by $\sin(s_j t)$ and take an average value. We show that the average of the left-hand side tends to zero while the average of the right-hand side tends to a nonzero constant. This contradiction will prove the lemma.

We have

$$\begin{aligned}\frac{1}{t}\int_0^t v(\tau)\sin(s_j\tau)\,d\tau &= \int_0^1 v(ty)\sin(s_j ty)\,dy \\ &= \int_0^1 (v(ty) - v_0)\sin(s_j ty)dy + v_0\int_0^1 \sin(s_j ty)\,dy.\end{aligned}$$

It is easy to verify that both integrals in the right-hand side of the last equality tend to zero as $t \to \infty$.

We consider now the right-hand side of (2.5). All terms

$$\frac{1}{t}\int_0^t \sin(s_k\tau)\sin(s_j\tau)\,d\tau, \quad k \neq j$$

and

$$\frac{1}{t}\int_0^t \cos(s_k\tau)\sin(s_j\tau)\,d\tau, \quad \frac{1}{t}\int_0^t e^{-\epsilon t}\sin(s_j\tau)\,d\tau$$

tend to zero as $t \to \infty$. It remains to note that

$$\frac{1}{t}\int_0^t \sin^2(s_j\tau)\,d\tau \to \frac{1}{2}.$$

If not all eigenvalues of the operator L on the imaginary axis are simple, then the coefficients a_j and b_j can be polynomials. Denote by r the highest degree of the polynomials. We divide equality (2.5) by t^r and obtain the same contradiction as above. The lemma is proved.

Remarks. 1. Since the coefficients of the operator L are smooth and bounded, and we suppose also that the boundary of the domain is also sufficiently smooth, then the eigenfunction of the operator L corresponding to zero eigenvalue is a classical solution of the equation $Lu = 0$.

2. In the proof of existence of a real eigenvalue we have not used the assumption that the domain is cylindrical. The particular form of the boundary conditions is not important either.

Lemma 2.2. *There is a nonnegative eigenfunction corresponding to a zero eigenvalue.*

Proof. We consider again the representation (2.2) and denote

$$v(x,t) = \frac{1}{t^{r+1}}\int_0^t u(x,s)\,ds,$$

where r is the highest degree of the polynomials. Then there exists a limit function $v_0(x;f)$,

$$v(x,t) \to v_0(x;f), \quad t \to \infty$$

and

$$Lv_0 = 0.$$

The notation $v_0(x;f)$ shows the dependence of the limit function on the initial condition. Consider a continuous nonnegative initial condition $f(x) \in L^2(\Omega)$. It can be taken such that $v_0(x;f)$ is nonnegative and not identically zero. Indeed, let $z(x)$ be the eigenfunction corresponding to the zero eigenvalue. We take $f(x) \geq z(x)$. If $v_0(x;f) \equiv 0$, then $v_0(x, f-z) = -z(x) \not\equiv 0$. Since the solution with nonnegative initial condition is also nonnegative, then $-z(x) \geq 0$. The lemma is proved.

Remark. Suppose that the matrix $b(x)$ is functionally irreducible; i.e., the numerical matrix with the elements $\tilde{b}_{ij} = \sup_x |b_{ij}|$ is irreducible. Then the nonnegative eigenfunction is positive in $\bar{\Omega}$.

Thus we have proved the following theorem.

Theorem 2.1. *Suppose that there is a finite number of eigenvalues of the operator L with the real part $\operatorname{Re}\lambda = \lambda_0$ and all other spectrum lies in the half-plane $\operatorname{Re}\lambda < \lambda_0 - \epsilon$ with some positive ϵ. Then there is a real eigenvalue λ_0 of the operator L and a nonnegative corresponding eigenfunction.*

3. Extremal properties of the real eigenvalue with positive eigenfunction. In the previous section we showed existence of a real eigenvalue with nonnegative corresponding eigenfunction. In this section we show that if there exists a real eigenvalue λ_0 and the corresponding eigenfunction is positive, then all other eigenvalues have real parts less than λ_0.

We consider the operator

$$Lu = a(x)\Delta u + \sum_{j=1}^{m} c_j(x)\frac{\partial u}{\partial x_j} + b(x)u,$$

where a, b, and c_j are smooth matrices, a and c_j are diagonal, a with positive diagonal elements, $b(x)$ with nonnegative off-diagonal elements. We suppose that the matrices a, b, and c_j have limits as $x_1 \to \infty$, and that the matrices

$$b_{\pm} = \lim_{x_1\to\pm\infty} b(x) \tag{3.1}$$

have all eigenvalues in the left half plane. The limit (3.1) is supposed to be uniform with respect to x'. As above we suppose that the matrix $b(x)$ is functionally irreducible.

We denote by a_k, c_{jk}, $k = 1, \ldots, n$, the diagonal elements of the matrices a and c_j, respectively, and by b_{kl} the elements of the matrix b.

We begin with some auxiliary lemmas.

Lemma 3.1. *If $u(x) \geq 0$, $x \in \bar{\Omega}$ is a nonzero solution of the problem*

$$Lu \leq 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{3.2}$$

then $u(x) > 0$ in $\bar{\Omega}$.

Proof. Let $u(x) \geq 0$ be a nonzero solution of (3.2). Suppose that $u(x)$ is not strictly positive. Denote by u_k, $k = 1, \ldots, n$, the elements of the vector u. Obviously, $u_k(x)$ satisfies the inequality

$$a_k(x)\Delta u_k + \sum_{j=1}^{m} c_{jk}(x)\frac{\partial u_k}{\partial x_j} + b_{kk}(x)u_k \leq 0. \tag{3.3}$$

Hence $u_k(x)$ is strictly positive in Ω or it is identically zero. If $u_k(x_0) = 0$, $x_0 \in \partial\Omega$ and $u_k(x) \not\equiv 0$, then

$$\frac{\partial u_k}{\partial \nu}\Big|_{x=x_0} < 0$$

(see [13]). This contradicts the boundary condition.

Since the matrix $b(x)$ is functionally irreducible, if one of the functions $u_k(x)$ is not identically zero, then all other functions are not identically zero also. The lemma is proved.

Lemma 3.2. *Let $q_\pm$ be positive vectors such that $b_\pm q_\pm < 0$. Suppose that a number r is such that*

$$b(x)q_+ < 0 \quad \textit{for} \quad x_1 > r, \quad b(x)q_- < 0 \quad \textit{for} \quad x_1 < -r.$$

If the vector-valued function $u(x)$ satisfies the problem

$$Lu \le \gamma u \quad (\gamma \ge 0), \quad \frac{\partial u}{\partial \nu} = 0 \tag{3.4}$$

for $|x_1| \ge r$ and the conditions

$$u|_{x_1=\pm r} \ge 0, \quad \lim_{N\to\pm\infty} \inf_{x_1 \ge N(x_1 \le N)} u(x) \ge 0,$$

then

$$u(x) \ge 0 \quad \textit{for } |x_1| \ge r.$$

Proof. We consider only the case $x_1 \ge r$. The second case $x_1 \le -r$ is similar.

Suppose that for $x_1 > r$ the vector-valued function $u(x)$ is not nonnegative. Consider the function $v(x) = u(x) + \tau q_+$. We can choose $\tau > 0$ such that $v(x) \ge 0$, but it is not strictly positive. Indeed, if the function $u_k(x)$ has negative values, then they are reached at finite x_1. Without loss of generality we can assume that $v_1(x^0) = 0$, $x_1^0 > r$. The function $v_1(x)$ satisfies the inequality

$$a_1 \Delta v_1 + \sum_{j=1}^m c_{1j} \frac{\partial v_1}{\partial x_j} + \sum_{k=1}^n b_{1k} v_k < \gamma v_1.$$

If $x^0 \in \Omega$, then we have a contradiction in signs.

If $x^0 \in \partial\Omega$, then from the inequality

$$a_1 \Delta v_1 + \sum_{j=1}^m c_{1j} \frac{\partial v_1}{\partial x_j} + (b_{11}(x) - \gamma) v_1 < 0$$

follows

$$\left. \frac{\partial v_1}{\partial \nu} \right|_{x=x_0} < 0.$$

This contradicts the boundary condition in (3.4). The lemma is proved.

Lemma 3.3. *In the conditions of the previous lemma, a bounded solution u of the problem*

$$Lu = \gamma u, \quad u|_{x_1=r} = 0, \ \frac{\partial u}{\partial \nu}\Big|_{x\in\partial\Omega} = 0$$

for $x_1 \geq r$ is identically zero.

The proof is similar to that in [1], [7].

Theorem 3.1. *Suppose that the problem*

$$Lw = 0, \quad \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{3.5}$$

has a positive solution.

If

$$\lim_{x_1\to\pm\infty} w(x) = 0, \tag{3.6}$$

uniformly in x', then the following assertions hold:

1. *The problem*

$$Lu = \lambda u, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{3.7}$$

where $u(x) \to 0$ as $x_1 \to \pm\infty$ uniformly in x', does not have nontrivial solutions for $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$.

2. *Each solution of the problem*

$$Lu = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{3.8}$$

has the form $u(x) = kw(x)$, where k is a constant.

3. *There are no nonzero solutions of the problem*

$$L^s u = 0, \ L^{s-1}u \neq 0, \ \frac{\partial(L^k u)}{\partial \nu} = 0, \ k = 0, \dots, s-1$$

for an integer $s > 1$.

Proof. 1. Consider first the case where in (3.7), $\lambda = \alpha + i\beta$, $\alpha \geq 0$, $\beta \neq 0$. Suppose that there exists a nonzero solution $u(x) = u^1(x) + iu^2(x)$ of this problem.

We consider also the problem

$$\frac{\partial v}{\partial t} = Lv - \alpha v, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad v(x,0) = u^1(x). \tag{3.9}$$

Its solution is

$$v(x,t) = u^1(x)\cos(\beta t) - u^2(x)\sin(\beta t).$$

Denote $\hat{u} = (|u_1|, \dots, |u_n|)$, where $u = (u_1, \dots, u_n)$. We can choose a number $r > 0$ which satisfies the conditions of Lemma 3.2 (see Lemma 5.1, Chapter 4, [1]). Further, we can choose a positive τ such that

$$\hat{u}(x) \le \tau w(x) \quad \text{for } |x_1| \le r \tag{3.10}$$

and there are k and $x^0 \in \bar{\Omega}$, $|x_1^0| \le r$ for which

$$|u_k(x^0)| = \tau w_k(x^0). \tag{3.11}$$

In the domain $\Omega_r = \Omega \cap \{x_1 \ge r\}$ consider the problem

$$\frac{\partial y}{\partial t} = Ly - \alpha y, \;\; y(x,t)|_{x_1=r} = \hat{u}(x)|_{x_1=r}, \tag{3.12}$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{x\in\partial\Omega} = 0, \tag{3.13}$$

$$y(x,0) = \hat{u}(x) \tag{3.14}$$

and the corresponding stationary problem

$$L\bar{y} - \alpha\bar{y} = 0, \; \bar{y}(x)|_{x_1=r} = \hat{u}(x)|_{x_1=r}, \; \bar{y}(x)|_{x_1=\infty} = 0, \; \left.\frac{\partial \bar{y}}{\partial \nu}\right|_{x\in\partial\Omega} = 0. \tag{3.15}$$

It follows from Lemma 3.3 that solution of this problem is unique.

We show that the solution of the problem (3.12)–(3.14) converges to it. Denote $\tilde{y} = y - \bar{y}$. The function $\tilde{y}$ is a solution of the problem

$$\frac{\partial \tilde{y}}{\partial t} = L\tilde{y} - \alpha\tilde{y}, \;\; \tilde{y}(x,t)|_{x_1=r} = 0, \tag{3.16}$$

$$\left.\frac{\partial \tilde{y}}{\partial \nu}\right|_{x\in\partial\Omega} = 0, \tag{3.17}$$

$$\tilde{y}(x,0) = \hat{u}(x) - \hat{y}(x). \tag{3.18}$$

We should show that $\tilde{y}$ converges to zero as t increases. To do this we estimate the solution of the problem (3.16)–(3.18) by the solution $y^*(x,t)$ of the problem

$$\frac{\partial y^*}{\partial t} = Ly^* - \alpha y^*, \;\; y^*(x,t)|_{x_1=r} = e^{-t}\tau q_+, \tag{3.19}$$

$$\left.\frac{\partial y^*}{\partial \nu}\right|_{x\in\partial\Omega} = 0, \tag{3.20}$$

$$y^*(x,0) = \tau q_+, \tag{3.21}$$

for positive and sufficiently large τ. By virtue of the inequality $b(x)q_+ < 0$ for $x_1 > r$, $y^*(x,t)$ decreases in time for every x and converges to some stationary solution. From Lemma 3.3 follows that it is identically zero.

We have

$$y_*(x,t) \le \tilde{y}(x,t) \le y^*(x,t),$$

where $y_*(x,t)$ is a solution of the similar problem with a negative and sufficiently large in the absolute value τ. Since $y_*(x,t)$ and $y^*(x,t)$ tend to zero as $t \to \infty$, we finally obtain

$$\lim_{t\to\infty} y(x,t) = \bar{y}(x).$$

Since $v(x,t) \le \hat{u}(x)$ for $x_1 \ge r$, then

$$v(x,t) \le y(x,t) \text{ for } x_1 \ge r, \ t \ge 0.$$

From this

$$v(x,t) = v(x, t + \frac{2\pi n}{\beta}) \le y(x, t + \frac{2\pi n}{\beta}).$$

Passing to the limit as $n \to \infty$, we have

$$v(x,t) \le \bar{y}(x) \text{ for } x_1 \ge r, \ t \ge 0.$$

From Lemma 3.2 applied to the function $\tau w(x) - \bar{y}(x)$ it follows that $\bar{y}(x) \le \tau w(x)$ for $x_1 \ge r$. Hence

$$v(x,t) \le \tau w(x) \tag{3.22}$$

for $x_1 \ge r, \ t \ge 0$. Similarly this inequality can be proved for $x_1 \le -r$. From this and (3.10) it follows that (3.22) is valid for all $x \in \Omega$.

The function $z = \tau w - v \ge 0$ is a solution of the equation

$$\frac{\partial z}{\partial t} = Lz - \alpha z + \alpha \tau w.$$

So its kth element satisfies the inequality

$$\frac{\partial z_k}{\partial t} \ge a_k \Delta z_k + \sum_{j=1}^{m} c_{kj} \frac{\partial z_k}{\partial x_j} + b_{kk} z_k - \alpha z_k.$$

Since $z_k(x,t) \ge 0$, it is not identically zero, and it is periodic in t, then from the positiveness theorem it follows that $z_k(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. But this contradicts (3.11). Indeed, we can choose t_0 such that

$$z_k(x_0, t) = \tau w_k(x_0) - |u_k(x_0)| = 0.$$

This contradiction proves the theorem for nonreal λ.

Let now λ be real and positive, and $u(x)$ be a nonzero solution of (3.7). Without loss of generality we can assume that at least one of the components of the vector-valued function $u(x)$ has negative values. Otherwise we can change its sign.

Consider the function $v = u + \tau w$ and choose a positive τ such that $v(x) \geq 0$ for $|x_1| \leq r$, but it is not strictly positive, $v_k(x^0) = 0$ for some k and x^0. We have

$$Lv = \lambda v - \lambda \tau w, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{3.23}$$

By virtue of Lemma 3.2, $v(x) \geq 0$ in the whole of Ω. If $\lambda > 0$ and x^0 is an interior point, then the kth equation in (3.23) gives a contradiction. So $v_k(x) > 0$ for $x \in \Omega$ and, consequently, $x_0 \in \partial\Omega$. From the inequality

$$a_k \Delta v_k + \sum_{j=1}^{m} c_{kj} \frac{\partial v_k}{\partial x_j} + b_{kk} v_k - \lambda v_k \leq 0$$

it follows that at $x = x_0$, $\partial v_k / \partial \nu < 0$. This contradicts the boundary condition in (3.23).

2. If $\lambda = 0$, we obtain as above that $v(x)$ is a nonnegative and not strictly positive solution of the equation $Lv = 0$. Then $v(x) \equiv 0$.

3. Let now $s > 1$. Since the only solution of the equation $Lu = 0$ up to a constant factor is $w(x)$, then

$$L^{s-1} u = k w(x).$$

We put $k = -1$ and denote $v = L^{s-2} u$. Then $Lv = -w(x)$. By virtue of Lemma 3.2 we can choose τ such that $z = v + \tau w \geq 0$ but not strictly positive, $z(x_0) = 0$. The τ here is not necessarily positive.

We have $w(x) > 0$ in $\bar{\Omega}$ and $Lz < 0$. We obtain a contradiction in signs for $x_0 \in \Omega$ and a contradiction with the boundary condition for $x_0 \in \partial\Omega$. The theorem is proved.

4. Minimax representation. In this section we obtain a minimax representation for the principal eigenvalue, i.e., for the eigenvalue with the maximal real part. We consider the eigenvalue problem

$$Lv = \lambda v, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

with the same operator L as above. We suppose that the principal eigenvalue λ_0 is real and that the corresponding eigenfunction $w(x)$ is positive in $\bar{\Omega}$.

Everywhere below the limit as $x_1 \to \infty$ is supposed to be uniform in x'.

Theorem 4.1. *Let λ_0 be the principal eigenvalue of the problem*

$$Lv = \lambda v, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad v|_{x_1 = \pm\infty} = 0$$

and suppose all other spectrum lies in the half-plane $\mathrm{Re}\,\lambda < \lambda_0 - \epsilon$ *with some positive* ϵ*. Then the following inequality holds:*

$$\inf_{x\in\Omega,i} \frac{(Lu)_i}{u_i} \le \lambda_0 \le \sup_{x\in\Omega,i} \frac{(Lu)_i}{u_i}. \tag{4.1}$$

Here u *is an arbitrary sufficiently smooth function positive in* $\bar{\Omega}$ *and satisfying the conditions*

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \; u|_{x_1=\pm\infty} = 0.$$

Proof. Suppose that for some function $u(x)$ the right inequality in (4.1) is not satisfied. Then $Lu < \lambda_0 u$. We can choose τ not necessarily positive such that the function $v(x) = u(x) + \tau w(x)$ is nonnegative for $|x_1| \le r$, but not strictly positive. Here r is from Lemma 3.2. Since $Lv < \lambda_0 v$, then by virtue of this lemma $v(x) \ge 0$ in Ω. We obtain a contradiction with the maximum principle.

Suppose now that the left inequality in (4.1) is not satisfied for some function $u(x)$. Then there exists $\lambda_1 > \lambda_0$ such that

$$Lu > \lambda_1 u. \tag{4.2}$$

Consider the problem

$$\frac{\partial v}{\partial t} = Lv - \lambda_1 v, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad v(x,0) = u(x).$$

From (4.2) follows that $v(x,t)$ increases in t for every x. Then

$$v(x,t) \ge u(x), \; x \in \bar{\Omega}, \; t \ge 0.$$

On the other hand, all spectrum of the operator $L - \lambda_1 I$ is in the left half plane. Hence $v(x,t) \to 0$ as $t \to \infty$. This contradiction proves the theorem.

Corollary. *The following minimax representation of the principal eigenvalue holds:*

$$\lambda_0 = \inf_{u\in K} \sup_{x\in\Omega,i} \frac{(Lu)_i}{u_i} = \sup_{u\in K} \inf_{x\in\Omega,i} \frac{(Lu)_i}{u_i}. \tag{4.3}$$

Here K *is a class of sufficiently smooth functions satisfying the boundary conditions and positive in* $\bar{\Omega}$*.*

For the proof it is sufficient to note that $w(x) \in K$.

The next theorem determines the behavior of the principal eigenvalue under increase of the matrix $b(x)$.

Theorem 4.2. *Let the principal eigenvalues* λ *and* λ_1 *of the operatorsL and* $L_1 = L + d(x)$, $d(x) \geq 0, d(x) \not\equiv 0$ *be real and the corresponding eigenfunctions be positive. If the matrix* $b(x)$ *is functionally irreducible, then* $\lambda_1 > \lambda$.

Here $d(x)$ is the operator of multiplication by a matrix.

Proof. From the minimax representation it follows that $\lambda_1 \geq \lambda$. We show that the inequality is strict.

Denote by $w(x)$ and $w_1(x)$, respectively, the eigenfunctions corresponding to the principal eigenvalues of the operators L and L_1. Suppose that $\lambda_1 = \lambda$. Then $Lw_1 \leq \lambda w_1$. Applying the usual construction we find a value of τ such that the function $z = w_1 - \tau w$ is nonnegative but not strictly positive in $\bar{\Omega}$. It satisfies the equation

$$Lz - \lambda z + d(x)w_1 = 0.$$

From the maximum principle we obtain that $z \equiv 0$ and $d(x) \equiv 0$. This contradiction proves the theorem.

5. Stability of travlling waves. We consider the parabolic system of equations

$$\frac{\partial u}{\partial t} = a\Delta u + F(u, x'), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{5.1}$$

in the infinite cylinder Ω with the axis in the x_1-direction and a smooth boundary. Here a is a diagonal matrix with positive diagonal elements, x' is the variable in the section Ω' of the cylinder, F is a smooth vector-valued function satisfying the conditions

$$\frac{\partial F_i}{\partial u_j} \geq 0, \quad i \neq j. \tag{5.2}$$

We suppose that there exist constant vectors w_+ and w_- such that

$$F(w_\pm, x') = 0, \quad x' \in \Omega',$$

and the matrices $b_\pm = F'(w_\pm, x')$ are constant and have all eigenvalues in the left half plane.

The travelling wave solution of the problem (5.1) is a solution of the form

$$u(x,t) = w(x_1 - ct, x'),$$

where c is the wave velocity. Existence of travelling waves for one-dimensional monotone systems, i.e., for the systems with condition (5.2), is proved in [1], [14]. For the multidimensional scalar equation it is considered in [15]–[18]. For the multidimensional systems it will be published elsewhere [19]. Here we assume that there exists a travelling wave solution $w(x)$ of the problem (5.1) with the limits

$$\lim_{x_1 \to \pm\infty} w(x) = w_\pm.$$

Moreover we suppose that $w_+ < w_-$ and that the function $w(x)$ is monotone in x_1.

The function $w(x)$ is a solution of the problem

$$a\Delta w + c\frac{\partial w}{\partial x_1} + F(w, x') = 0, \quad \frac{\partial w}{\partial \nu} = 0. \tag{5.3}$$

We consider the linearized problem

$$Mu = 0, \quad \frac{\partial u}{\partial \nu} = 0,$$

where

$$Mu \equiv a\Delta u + c\frac{\partial u}{\partial x_1} + F'(w, x')u.$$

The function $u = -\partial w/\partial x_1$ is a positive solution of this problem. We suppose also that the matrix $F'(w(x), x')$ is functionally irreducible and the convergence

$$F'(w(x), x') \to b_\pm, \;\; x_1 \to \pm\infty$$

is uniform in x'. The matrices b_+ and b_- are supposed to have all eigenvalues in the left half plane.

Theorem 5.1. Stability to small perturbations. *If the initial condition of the Cauchy problem*

$$\frac{\partial u}{\partial t} = a\Delta u + c\frac{\partial u}{\partial x_1} + F(u, x'), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \tag{5.4}$$

is sufficiently close to a wave $w(x)$ in the norm $L_2(\Omega)$,

$$\|u(x, 0) - w(x)\|_{L_2} \le \epsilon, \tag{5.5}$$

then the solution converges exponentially to a shifted wave $w(x + h)$ with some h,

$$\|u(x, t) - w(x + h)\|_{L_2} \le Ke^{-st}. \tag{5.6}$$

Here the constants K and s are positive and do not depend on the initial condition.

Proof. By virtue of the connection between location of the spectrum of a linearized operator and stability of travelling waves ([1]), it is sufficient to show that the whole spectrum of the operator M except for 0 lies in the left half plane. $\lambda = 0$ should be a simple eigenvalue.

We recall that the spectrum of the operator M consists of the continuous spectrum and eigenvalues. The continuous spectrum is given by the eigenvalues of the problem

$$a\Delta'\tilde{u} - a\xi^2\tilde{u} + ci\tilde{u} + b_\pm\tilde{u} = \lambda\tilde{u}, \quad \frac{\partial \tilde{u}}{\partial \nu}\Big|_{\partial\Omega'} = 0 \tag{5.7}$$

in the section Ω' of the cylinder, for all real ξ. Since all eigenvalues of the matrices $b_\pm$ lie in the left half plane, then all eigenvalues of the problem (5.7) are also in the left half plane. The operators under consideration are Fredholm with index 0 in L_2 and in weighted Hölder spaces. So the eigenfunctions corresponding to eigenvalues with zero real part coincide in these two spaces and we can use results of Section 3. From them it follows that zero is a simple eigenvalue of the operator M and all other eigenvalues lie in the left half plane. The theorem is proved.

We show now that from the stability with respect to small perturbations and comparison theorems follows the global stability. We begin with some auxiliary estimations.

Lemma 5.1. *Let $u(x)$ be a smooth function monotone in x_1, $\lim_{x_1\to\pm\infty} u = w_\pm$. Then*

$$\|u_h(x) - u(x)\|_{L_2(\Omega)} \le h\text{const}\Big(\sup_x \Big|\frac{\partial u}{\partial x_1}\Big|\Big)^{1/2}, \tag{5.8}$$

where $u_h(x) = u(x_1 + h, x_2, \ldots, x_m)$ and the constant does not depend on the function $u(x)$.

Proof. We have

$$\int_{-\infty}^{\infty} |u(x_1+h,x') - u(x_1,x')|^2\, dx_1 = \int_{-\infty}^{\infty} \Big|h\int_0^1 \frac{\partial}{\partial x_1} u(x_1 + h\xi, x')d\xi\Big|^2 dx_1$$
$$\le h^2 \int_0^1 d\xi \int_{-\infty}^{\infty} \Big|\frac{\partial}{\partial x_1} u(x_1+h\xi,x')\Big|^2 dx_1 = h^2 \int_{-\infty}^{\infty} \Big|\frac{\partial}{\partial x_1} u(x_1,x')\Big|^2 dx_1.$$

Integrating this inequality in x', we obtain

$$\|u_h(x) - u(x)\|_{L_2(\Omega)} \le h\|\frac{\partial u}{\partial x_1}\|_{L_2(\Omega)}. \tag{5.9}$$

The next step is to estimate the norm

$$\|\frac{\partial u(x)}{\partial x_1}\|_{L_2(\Omega)}.$$

We have

$$\int_\Omega \Big|\frac{\partial u(x)}{\partial x_1}\Big|^2 dx \le \sup_x \Big|\frac{\partial u(x)}{\partial x_1}\Big| \int_\Omega \Big|\frac{\partial u(x)}{\partial x_1}\Big| dx$$
$$= \sup_x \Big|\frac{\partial u(x)}{\partial x_1}\Big| \Big|\int_\Omega \frac{\partial u(x)}{\partial x_1}\Big| dx = |w_- - w_+| \text{mes}\, \Omega' \sup_x \Big|\frac{\partial u(x)}{\partial x_1}\Big|.$$

The lemma is proved.

Thus we obtain that for a monotone-in-x_1 solution $u(x,t)$ of the problem (5.4), the shifted solution $u_h(x,t)$ is close in the $L_2(\Omega)$ norm.

We recall that the comparison theorem is valid for the monotone systems and it plays an important role. It can be formulated as follows. Let $u^1(x,0)$ and $u^2(x,0)$ be initial conditions of the Cauchy problem (5.4). If

$$u^1(x,0) \leq u^2(x,0), \quad x \in \Omega,$$

then

$$u^1(x,t) \leq u^2(x,t), \quad x \in \Omega, \quad t \geq 0.$$

We formulate now the main result of this section.

Theorem 5.2. (Global stability.) *Let $f(x)$ be a continuous initial condition of the problem* (5.4)*, monotone in x_1 and such that the norm $\|f(x) - w(x)\|_{L_2}$ is finite. Then the solution $u(x,t)$ of this problem converges to the travelling wave; i.e., the estimation* (5.6) *holds with some h, $K > 0$, which can depend on the initial condition and $s > 0$ which does not depend on the initial condition.*

Proof. We assume first that the initial condition has the form $f(x) = w(x) + \phi(x)$, where $\phi(x)$ is continuous nonnegative function with a finite support. Consider the function

$$f^1(x) = \begin{cases} f(x), \ w(x) \leq f(x) \leq w(x-h) \\ w(x-h), \ w(x-h) \leq f(x), \end{cases}$$

where $x - h$ denotes $(x_1 - h, x_2, \ldots, x_m)$. Here h is positive and such that

$$\|w(x-h) - w(x)\|_{L_2} \leq \frac{\epsilon}{2},$$

where ϵ is the same as in (5.5),

$$w(x) \leq f^1(x) \leq w(x-h) \quad x \in \Omega.$$

Then the solution $u^1(x,t)$ of the Cauchy problem with the initial condition $f^1(x)$ converges to a wave $w(x+h_1)$ for some h_1. The shifted solution $u^1(x-h,t)$ corresponds to the shifted initial condition $f^1(x-h)$ and converges to the shifted wave $w(x-h+h_1)$. Consider now the next initial condition

$$f^2(x) = \begin{cases} f(x), \ f^1(x) \leq f(x) \leq f^1(x-h) \\ f^1(x-h), \ f^1(x-h) \leq f(x), \end{cases}$$
$$f^1(x) \leq f^2(x) \leq f^1(x-h), \ x \in \Omega.$$

Then

$$u^1(x,t) \leq u^2(x,t) \leq u^1(x-h,t).$$

Since $u^1(x,t)$ converges to the wave $w(x+h_1)$, then after some time the solution $u^2(x,t)$ will be in an ϵ-neighbourhood of this wave and, consequently, will also converge to another shifted wave.

We continue this process and in a finite number of steps we have $f^n(x) = f(x)$. It gives convergence of the solution $u(x,t)$ to a shifted wave.

We have fulfilled this construction for a finite positive perturbation. If $\phi(x)$ still has a finite support but it is not necessarily nonnegative, then we consider first the initial condition

$$f_+(x) = w(x) + \max(\phi(x), 0)$$

and show that the solution converges to a wave $w(x+h)$ with some h. After that we apply a similar construction to the initial condition

$$f_-(x) = f_+(x) + \min(\phi(x), 0)$$

and show that the solution converges to a wave. It remains to note that $f_-(x) = f(x)$.

We make finally some remarks about the case where the support of the perturbation $\phi(x)$ is not finite. In this case the construction described above may not be finished in a finite number of steps. In this case it should be changed. For example, if $f(x) \geq w(x)$, we put

$$g_h(x) = \max(w(x-h), f(x)).$$

We note that $g_h(x)$ is monotone in x_1 and the norm $\|g_h(x) - w(x-h)\|_{L_2}$ is small for large h. Then the solution with the initial condition $g_h(x)$ converges to a wave. We fulfill now the construction described above beginning from the function $g_h(x)$. Since

$$g_h(x+h) \leq f(x) \leq g_h(x),$$

it will be finished in a finite number of steps. The theorem is proved.

6. Wave velocity. In this section we apply the result on the global stability of waves to obtain a minimax representation of the wave velocity.

The following representation takes place:

$$c = \inf_{\rho \in K} \sup_{x,i} \frac{a_i \Delta \rho_i + F_i(\rho, x')}{-\frac{\partial \rho}{\partial x_1}} = \sup_{\rho \in K} \inf_{x,i} \frac{a_i \Delta \rho_i + F_i(\rho, x')}{-\frac{\partial \rho}{\partial x_1}}. \tag{6.1}$$

Here a_i are the diagonal elements of the matrix a, F_i and ρ_i are the elements of the vectors F and ρ, respectively, K is the class of functions continuous with second derivatives, decreasing in x_1, satisfying the boundary conditions, and such that

$$\|\rho(x) - w(x)\|_{L_2} < \infty. \tag{6.2}$$

In the one-dimensional case a similar representation was obtained in [1], [7].

Since the travelling wave belongs to the class K, to prove (6.1) it is sufficient to show that for any function $\rho(x) \in K$ the following inequality holds:

$$\inf_{x,i} \frac{a_i \Delta \rho_i + F_i(\rho, x')}{-\frac{\partial \rho_i}{\partial x_1}} \leq c \leq \sup_{x,i} \frac{a_i \Delta \rho_i + F_i(\rho, x')}{-\frac{\partial \rho_i}{\partial x_1}}. \tag{6.3}$$

We verify the left inequality in (6.3). The right inequality can be shown in the similar way.

Suppose that it is not valid for some function $\rho \in K$. Then there exists $c_1 > c$ such that

$$c_1 \leq \inf_{x,i} \frac{a_i \Delta \rho_i + F_i(\rho, x')}{-\frac{\partial \rho_i}{\partial x_1}} \quad \text{or} \quad a\Delta\rho + c_1 \frac{\partial \rho}{\partial x_1} + F(\rho, x') \geq 0.$$

Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = a\Delta u + c_1 \frac{\partial u}{\partial x_1} + F(u, x'), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u(x, 0) = \rho(x).$$

From the previous inequality it follows that the solution $u(x,t)$ increases in t for every x fixed. Hence

$$u(x,t) \geq \rho(x), \quad x \in \Omega, \ t \geq 0. \tag{6.4}$$

On the other hand the solution converges to the travelling wave,

$$\|u(x,t) - w(x_1 + (c_1 - c)t, x')\|_{L_2(\Omega)} \to 0, \quad t \to \infty \tag{6.5}$$

(see Section 5). We show that (6.4) and (6.5) give a contradiction. Indeed, from (6.2) we have

$$\int_\Omega (\rho(x + he) - w(x + he)) f(x)\, dx \to 0, \quad h \to \pm\infty,$$

where $e = (1, 0, \ldots, 0)$, $f(x)$ is a nonnegative function with a finite support in Ω, and consequently

$$\lim_{h \to \pm\infty} \int_\Omega \rho(x + he) f(x)\, dx = \int_\Omega w_\pm f(x)\, dx. \tag{6.6}$$

From (6.4)

$$\int_\Omega u(x + he, t) f(x)\, dx \geq \int_\Omega \rho(x + he) f(x)\, dx, \tag{6.7}$$

and from (6.5)

$$\int_\Omega u(x + he, t) f(x)\, dx \to \int_\Omega w_+ f(x)\, dx, \quad t \to \infty. \tag{6.8}$$

We obtain from (6.7), (6.8)

$$\int_{\Omega} w_+ f(x)\, dx \geq \int_{\Omega} \rho(x+he) f(x)\, dx,$$

and from this, taking into account (6.6),

$$\int_{\Omega} w_+ f(x)\, dx \geq \int_{\Omega} w_- f(x)\, dx.$$

This inequality contradicts the assumption $w_- > w_+$. Thus the right inequality in (6.3) is proved.

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