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ELLIPTIC PROBLEMS WITH A PARAMETER IN UNBOUNDED DOMAINS

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Abstract. Linear elliptic boundary-value problems with a parameter are studied. The Agranovich-Vishik method and specially introduced function spaces allow us to consider mixed-order problems in unbounded domains. We obtain a priori estimates and unique solvability for large values of the parameter. These results are used to study analytic semigroups and Fredholm property of general elliptic problems in unbounded domains.

1. Introduction

This paper is devoted to general elliptic problems with a parameter for mixed-order systems in unbounded domains $\Omega \subset \mathbb{R}^n$:

$$A(x, \lambda, D)u = f, \quad x \in \Omega, \tag{1.1}$$

$$B(x, \lambda, D)u = g, \quad x \in \partial\Omega. \tag{1.2}$$

Precise definitions are given in Section 2. The aim of this work is to obtain a priori estimates of solutions and to prove the existence and the uniqueness of solutions for λ in a given sector S, and $|\lambda| \geq \lambda_0 > 0$.

Elliptic problems with a parameter form an important class of elliptic problems because it is possible to prove for them the existence and the uniqueness of solutions. As it is well known, they are also used to study evolution problems. Moreover, it turns out that parameter-elliptic problems may be applied to study the Fredholm property of general elliptic problems in unbounded domains. It is proved in [18] that a general elliptic partial differential operator is Fredholm if its limiting operators at infinity are invertible.

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This is why we introduce in this work a new class of operators—elliptic operators with a parameter at infinity. They will be invertible for $\lambda \in S$, $|\lambda| \geq \lambda_0$ and some $\lambda_0 > 0$. It follows that elliptic operators in a domain Ω, which are parameter-elliptic at infinity, are Fredholm for $\lambda \in S$, $|\lambda| \geq \lambda_0$. Obviously, any parameter-elliptic operator in the domain Ω is also parameter-elliptic at infinity, but the corresponding values of λ_0 are different. This is essential in the analysis of the location of the Fredholm spectrum. We note also that the parameter-ellipticity at infinity plays an essential role in the index theory for elliptic operators in unbounded domains. These questions will be discussed in the subsequent works.

After the classical papers by Agmon [1] and by Agranovich and Vishik [3] ellipticity with a parameter for problem (1.1), (1.2) or for some of its special cases in bounded domains Ω was studied by Geymonat and Grisvard [11], Roitberg [14], Agranovich [2], and Denk and Volevich [8].

The most general problem (1.1), (1.2) in unbounded domains Ω, which has been studied till now, is the following problem:

$$Au := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u - \lambda^{2m} u = f \quad \text{in} \quad \Omega, \tag{1.3}$$

$$Bu := \sum_{|\alpha| \leq m_j} b_{j\alpha}(x) D^\alpha u = g \quad \text{on} \quad \partial\Omega, \quad j = 1, \ldots, r, \tag{1.4}$$

where m and m_j are some integers, and the sector S is of the form

$$S = \{\lambda : |arg\ \lambda| \leq \theta\}. \tag{1.5}$$

These results embrace parameter-elliptic problems, which are obtained from parabolic systems in the sense of Petrovskii, but some important classes of parameter elliptic problems are excluded, for example, first-order systems ($m = 1/2$), in which case the sector S is

$$S = \{\lambda : |arg\ \lambda| \leq \theta, \ \ |\pi - arg\ \lambda| \leq \theta, \ \ \theta < \tfrac{\pi}{2}\}.$$

There are three known methods to prove the existence of solutions of problem (1.1), (1.2). The first method uses formally adjoint problems, the second one is based on the theory of sectorial operators, and the third method relies on the direct construction of the inverse operator.

The first method stems from the paper [1]. In the case of a scalar equation ($N = 1$) it is supposed that the boundary conditions are normal. This means that the boundary $\partial\Omega$ is non-characteristic with respect to the boundary operator B_j, $m_j < 2m$, and the orders of the boundary operators are distinct.

The operator A with the domain

$$D(A)=\{u\in W^{2m,p}(\Omega),\ B_j=0,\ j=1,\dots,r,\ 1<p<\infty\},$$

is considered and a priori estimates with a parameter are obtained. It follows from these estimates that the operator A has a zero kernel and a closed range. As it is proved by Schechter [15], then there exists a formally adjoint elliptic problem with normal boundary conditions. By the above arguments it is obtained that the formally adjoint operator has a zero kernel, and hence the range of A coincides with $L^p(\Omega)$. A priori estimates and existence theorems in the case of unbounded domains are proved for these problems by Higuchi [12] and by Freeman and Schechter [10].

For systems of equations ($N\ge 1$) a priori estimates and existence results for problem (1.3), (1.4) were obtained by Amann [4] under the assumption that $u\in W^{2m+s,p}(\Omega)$, where s is a real non-negative number. Instead of the normality mentioned above in the scalar case, in the proof of the existence he supposed that the operator A was uniformly strongly elliptic and the boundary operators had the form

$$B_ju=\frac{\partial^{k+j-1}u}{\partial\beta_j^{k+j-1}}\ ,\ \ j=1,\dots,m,\ k=0,\dots,m,$$

with the lower-order boundary operators, where

$$(\beta_j(x),\nu(x))\ge c>0,\ \ \forall x\in\Gamma,\ j=1,\dots,m,$$

and $\nu(x)$ was the normal vector. As is shown in the present paper, the existence results are true without these restrictions, though this class of operators may be interesting in other considerations.

There is a large number of works devoted to sectorial operators in relation with elliptic problems (1.3), (1.4) (see [13], [7], [6] and the references therein). In the paper by Denk, Hieber and Prüss [7], problem (1.3), (1.4) is considered for equations with operator coefficients. Using the methods based on the theory of sectorial operators, they have obtained existence results for the case where $\Omega=\mathbb{R}^n$, $\Omega=R^n_+$, and $\Omega\subset\mathbb{R}^n$ is a domain with a compact boundary. It is supposed that the sector S is of the type (1.5) and that there are some restrictions on the function spaces (for example, in the case of $W^{l,p}$ the domain of the operator belongs to $W^{2m,p}$; $l>2m$ is excluded).

The third of the methods mentioned above is the method introduced by Agranovich and Vishik [3] for bounded domains Ω. This method does not require restrictions, which are needed for other methods. In Sobolev spaces they introduced norms depending on the parameter, obtained a priori estimates with constants independent of the parameter and proved existence of solutions of problem (1.3), (1.4) (the parameter λ entered the operator in a more general way) by a direct construction of the inverse operator. A generalization of these results for mixed-order systems in bounded domains has been obtained by Agranovich [2]. To our knowledge, there are no generalizations of these results to unbounded domains Ω.

In the present paper, we use the Agranovich-Vishik method for unbounded domains. We obtain these results not directly in Sobolev spaces but in the scale of spaces $W_q^{l,p}(1 < p < \infty, 1 \le q \le \infty)$ (see [19], [18] and Section 3 of this paper). The space $W_q^{l,p}$ coincides with the Sobolev space $W^{l,p}$ for $q = p$ $(1 < p < \infty)$ and with the Sobolev-Stepanov space for $q = \infty$, $1 < p < \infty$. For arbitrary domains $\Omega \subset \mathbb{R}^n$ with $C^{k+\alpha}$ boundary we obtain a priori estimates in the spaces $W_q^{l,p}(1 \le q \le \infty)$ and construct the inverse operator first for the case $q = \infty$. Then we use this result to prove the existence theorem for the case $q < \infty$. For this we consider functions f and g in (1.1), (1.2) with compact supports. The set of such functions is dense in the case $q < \infty$. It is known (see [18]) that any solution $u \in W_\infty^{l,p}$ of (1.1), (1.2) with such f and g decays exponentially at infinity and hence belongs to $W_q^{l,p}(q < \infty)$. From this and a priori estimates we obtain the desired existence result.

We note that these results are obtained for general mixed-order parameter-elliptic problems in uniformly regular unbounded domains Ω. Even in the framework of the theory of sectorial operators (for the case (1.3), (1.4)) these results are new since: 1. the boundary $\partial\Omega$ is not supposed to be compact, 2. we prove that the operators are sectorial in the spaces L_q^p $(1 < p < \infty$, $1 \le q \le \infty)$, which are more flexible than the spaces $L^p = L_p^p$. We note that a priori estimates do not take place in the space $W^{2m,p}$ for $p = 1$ and $p = \infty$ (see e.g. [16] for counter-examples even for the Laplace operator). However, they are obtained for the spaces $W_q^{2m,p}$ $(1 < p < \infty$, $1 \le q \le \infty)$. The behavior of the functions $u \in W_q^{2m,p}$ at infinity is determined by the value of q. For example, the behavior at infinity of the functions from L_1^p $(1 < p < \infty)$ is similar to that in L^1.

The contents of the paper is as follows. In the next two sections we define the operators and spaces. Model problems with constant coefficients in $\mathbb{R}^n$ are studied in Section 4, and in the half-space $\mathbb{R}_+^n$ in Section 5. Sections 6 and 7 are devoted to problems with variable coefficients, respectively in the whole space and in the half-space. Problems in unbounded domains are studied in

Section 8. In Section 9 we discuss generation of analytic semigroups, and in Section 10 introduce operators with a parameter at infinity.

2. Parameter-elliptic boundary-value problems

Consider the matrix operator $A(x, \lambda, D)$ with the elements

$$A_{ij}(x, \lambda, D) = \sum_{|\alpha|+\beta \leq \alpha_{ij}} a_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i, j = 1, \ldots, N,$$

in an unbounded domain Ω, and the boundary operator $B(x, \lambda, D)$ with the elements

$$B_{kj}(x, \lambda, D) = \sum_{|\alpha|+\beta \leq \beta_{kj}} b_{kj}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad k = 1, \ldots, r, \ j = 1, \ldots, N.$$

Consider a sector S of the complex plane, $S = \{\lambda : \sigma_1 \leq arg \ \lambda \leq \sigma_2\}$ (we do not exclude the case $\sigma_1 = \sigma_2$). Let for each $\lambda \in S$ fixed the operator $L = (A, B)$ be an elliptic operator in the Douglis-Nirenberg sense. We recall that this implies the existence of some integers s_i, t_j, and σ_k such that $s_i + t_j = \alpha_{ij}$ and $\sigma_k + t_j = \beta_{kj}$. We suppose that the coefficients of the operator are defined for $x \in \mathbb{R}^n$ and $a_{ij}^{\alpha\beta}(x) \in C^{l-s_i+\theta}(\mathbb{R}^n)$, $b_{kj}^{\alpha\beta}(x) \in C^{l-\sigma_k+\theta}(\mathbb{R}^n)$, $0 < \theta < 1$.

We assume also that the domain Ω satisfies the following condition.

Condition D. For each $x_0 \in \partial\Omega$ there exists a neighborhood $U(x_0)$ such that:

1. $U(x_0)$ contains a sphere with the radius δ and the center x_0, where δ is independent of x_0,
2. There exists a homeomorphism $\psi(x; x_0)$ of the neighborhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in R^n such that the images of $\Omega \bigcap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0, \ |y| < 1\}$ and $B_0 = \{y : y_n = 0, \ |y| < 1\}$ respectively,
3. The function $\psi(x; x_0)$ and its inverse belong to the Hölder space $C^{m_0+\theta}$, $0 < \theta < 1$, $m_0 = \max_{i,j,k}(l + t_i, l - s_j, l - \sigma_k)$. Their $\| \cdot \|_{m_0+\theta}$-norms are bounded uniformly in x_0 .

It can be proved that this condition is satisfied if and only if the domain is uniformly regular in the sense of [5]. The operator L is supposed to be uniformly elliptic with a parameter (see Definition 2.2 below).

For any $x \in \bar{\Omega}$ consider the matrix $A(x, \lambda, \xi)$ with the elements

$$A_{ij}(x, \lambda, \xi) = \sum_{|\alpha|+\beta \le \alpha_{ij}} a_{ij}^{\alpha\beta}(x) \lambda^\beta \xi^\alpha, \quad i, j = 1, \dots, N,$$

where $\xi = (\xi_1, \dots, \xi_n)$. For any $x \in \partial\Omega$ consider the local coordinates (ξ', ν), where $\xi' = (\xi_1, \dots, \xi_{n-1})$ are the coordinates in the tangential hyperspace, ν is the normal coordinate. Let $A(\lambda, \xi', \nu)$ and $B(\lambda, \xi', \nu)$ be the matrices with the elements

$$A_{ij}(\lambda, \xi', \nu) = \sum_{|\alpha'|+\alpha_n+\beta = \alpha_{ij}} a_{ij}^{\alpha'\alpha_n\beta} \lambda^\beta \xi'^{\alpha'} \nu^{\alpha_n}, \quad i, j = 1, \dots, N,$$

$$B_{ij}(\lambda, \xi', \nu) = \sum_{|\alpha'|+\alpha_n+\beta = \beta_{ij}} b_{ij}^{\alpha'\alpha_n\beta} \lambda^\beta \xi'^{\alpha'} \nu^{\alpha_n}, \quad i = 1, \dots, r, \ , j = 1, \dots, N$$

(the dependence of the coefficients on x is not indicated).

We recall that the Lopatinskii matrix is given by the equality

$$\Lambda(\lambda, \xi') = \int_{\gamma_+} B(\lambda, \xi', \mu) A^{-1}(\lambda, \xi', \mu) \Phi(\mu) d\mu,$$

where $\Phi(\mu) = (E, \mu E, \dots, \mu^{s-1} E)$, E is the identity matrix of order N, $s = \max_{i,j} \alpha_{ij}$, γ_+ is a Jordan curve in the half-plane $Im\ \mu > 0$ enclosing all the roots of det $A(\lambda, \xi', \mu)$ with positive imaginary parts. By virtue of the condition of proper ellipticity there are r such roots, and there are no roots on the real axis. The Lopatinskii condition implies that the rank of the matrix $\Lambda(\lambda, \xi')$ equals r for all $|\xi'| \ne 0$.

We will use the following notation:

$$e_d^0 = \inf_{x \in \Omega, |\xi|=1} |\det A(x, 0, \xi)|, \quad e_d = \inf_{x \in \Omega, |\xi|+|\lambda|=1, \lambda \in S} |\det A(x, \lambda, \xi)|,$$

$$M_d = \max_{|\alpha|+\beta \le \alpha_{ij}, i,j=1,\dots,N} \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}(\Omega)},$$

$$M_\Gamma = \max_{|\alpha|+\beta \le \beta_{ij}, i=1,\dots,r, j=1,\dots,N} \|b_{ij}^{\alpha\beta}\|_{C^{l-\sigma_i}(\Gamma)},$$

$$e_\Gamma^0 = \inf_{x \in \Gamma, |\xi'|=1} \sum_\alpha |\mu_\alpha(x, 0, \xi')|, \quad e_\Gamma = \inf_{x \in \Gamma, |\xi'|+|\lambda|=1, \lambda \in S} \sum_\alpha |\mu_\alpha(x, \lambda, \xi')|,$$

where $\mu_\alpha(x, \xi')$ are all r-minors of the Lopatinskii matrix in the local coordinates (ξ', ν) at the point x.

Definition 2.1. The operator $L(x, 0, D)$ is called uniformly elliptic if $e_d^0 > 0$, $e_\Gamma^0 > 0$, $M_d < \infty$, and $M_\Gamma < \infty$.

Definition 2.2. The operator $L(x,\lambda,D)$ is called uniformly elliptic with a parameter if $e_d > 0$, $e_\Gamma > 0$, $M_d < \infty$, and $M_\Gamma < \infty$.

3. Spaces

We consider the following function spaces: $E(\Omega) = \Pi_{i=1}^N W^{l+t_j,p}(\Omega)$, $F^d(\Omega) = \Pi_{j=1}^N W^{l-s_i-1/p,p}(\Omega)$, $F^b(\partial\Omega) = \Pi_{k=1}^r W^{l-\sigma_k,p}(\partial\Omega)$, and $F = F^d \times F^b$. The operator L can be considered as acting from E to F. In the case of unbounded domains it is convenient to use the spaces E_q, $1 \le q \le \infty$ introduced in [18]. Their norms are given by the equalities

$$\|u\|_{E_\infty(\Omega)} = \sup_i \|\phi_i u\|_{E(\Omega)},$$

$$\|u\|_{E_q(\Omega)} = \Big(\sum_i \|\phi_i u\|^q_{E(\Omega)} \Big)^{1/q}, \ \ 1 \le q < \infty.$$

We define similarly the spaces F_q, $1 \le q \le \infty$.

Here ϕ_i is a system of functions satisfying Condition 2.1.5 in [18]:

Condition 2.1.5 [18]**.** A system of functions ϕ_i satisfies the following conditions:

1. $\phi_i(x) \ge 0$, $\phi_i \in D$,
2. For any i there exist no more than N functions ϕ_j such that *supp* $\phi_j \cap$ *supp* $\phi_i \neq \oslash$,
3. $\sup_i \|\phi_i\|_M < \infty$,
4. $\phi(x) = \sum_{i=1}^\infty \phi_i(x) \ge m > 0$ for some constant m,
5. the following estimate holds:

$$\sup_x |D^\alpha \phi(x)| \le M_\alpha,$$

where D^α denotes the operator of differentiation, and M_α are positive constants.

Here $\|\phi\|_M$ is the norm of a multiplier ϕ:

$$\|\phi u\|_E \le \|\phi\|_M \ \|u\|_E, \ \ \forall u \in E.$$

For $E = W^{s,p}$ it is known that $\|\phi\|_M \le \|\phi\|_{C^{[|s|]+1}}$, where K is a positive constant. For partitions of unity $\{\phi_i\}$ we always suppose that $\sup_i \|\phi_i\|_M < \infty$. To study operators with a parameter, following [3] and [14] we introduce the norm

$$|||u|||_{W^{l,p}(\mathbb{R}^n)} = \|F^{-1}(1+|\xi|^2+|\lambda|^2)^{l/2} Fu\|_{L^p(\mathbb{R}^n)},$$

where F denotes the Fourier transform, $\xi = (\xi_1, \dots, \xi_n)$. For $\lambda = 0$ this norm is the usual $W^{l,p}$ norm; for any λ fixed these norms are equivalent. On the other hand, the norm $|||\cdot|||_{W^{l,p}(\mathbb{R}^n)}$ is equivalent to the norm

$$[u]_{W^{l,p}(\mathbb{R}^n)} = \|u\|_{W^{l,p}(\mathbb{R}^n)} + |\lambda|^l \|u\|_{L^p(\mathbb{R}^n)}.$$

The proof uses Mikhlin's theorem. It is also used to prove the following assertion (cf. [3] and [14]).

Proposition 3.1. (Interpolation inequality) *The estimate*

$$|\lambda|^{l-k} |||u|||_{W^{k,p}(\mathbb{R}^n)} \le c_{kl} |||u|||_{W^{l,p}(\mathbb{R}^n)} \tag{3.1}$$

holds for any $u \in W^{l,p}(\mathbb{R}^n)$, $0 \le k \le l$ *with some constants* c_{kl} *that depend on* k, l, n, *and* p *only.*

To define the $|||\cdot|||$-norm in domains we put

$$|||u|||_{W^{l,p}(\Omega)} = \inf |||u^c|||_{W^{l,p}(\mathbb{R}^n)},$$

where the infimum is taken with respect to all $u^c \in W^{l,p}(\mathbb{R}^n)$ such that the restriction of $u^c(x)$ to Ω coincides with $u(x)$.

Proposition 3.2. (Interpolation inequality) *The estimate*

$$|\lambda|^{l-k} |||u|||_{W^{k,p}(\Omega)} \le c_{kl} |||u|||_{W^{l,p}(\Omega)} \tag{3.2}$$

holds for any $u \in W^{l,p}(\Omega)$, $0 \le k \le l$ *with the same constants* c_{kl} *as in Proposition* 3.1.

We will use also the estimate given by the following proposition.

Proposition 3.3. *The estimate*

$$|||D^\alpha u|||_{W^{l,p}(\Omega)} \le c |||u|||_{W^{l+|\alpha|,p}(\Omega)}, \tag{3.3}$$

holds for any $u \in W^{l+|\alpha|,p}(\Omega)$ *with a constant* c *independent of* u *and* λ.

We consider next the spaces $W^{l-1/p,p}(\partial\Omega)$ with an integer $l \ge 1$, $1 < p < \infty$. We introduce the following norm:

$$|||\phi|||_{W^{l-1/p,p}(\partial\Omega)} = \inf |||u|||_{W^{l,p}(\Omega)},$$

where the infimum is taken over all $u \in W^{l,p}(\Omega)$ such that the restriction of u to $\partial\Omega$ coincides with ϕ.

Proposition 3.4. (Interpolation inequality) *The estimate*

$$|\lambda|^{l-k} |||\phi|||_{W^{k-1/p,p}(\partial\Omega)} \le c_{kl} |||\phi|||_{W^{l-1/p,p}(\partial\Omega)} \tag{3.4}$$

holds for any $\phi \in W^{l-1/p,p}(\partial\Omega)$, $1 \le k \le l$ *with the same constants* c_{kl} *as in Proposition* 3.1.

We define next the $|||\cdot|||$-norms for the ∞-spaces. Let ϕ_i be a system of functions satisfying Condition 2.1.5 in [18]. We put

$$|||u|||_{W^{l,p}_\infty(\Omega)} = \sup_i |||\phi_i u|||_{W^{l,p}(\Omega)}, \quad \forall u \in W^{l,p}_\infty(\Omega).$$

We note that $|||u|||_{W^{l,p}_\infty(\Omega)} < \infty$. It follows from the estimate

$$|||\phi_i u|||_{W^{l,p}(\Omega)} \le c(\lambda)\|\phi_i u\|_{W^{l,p}(\Omega)},$$

that holds for any λ.

Proposition 3.5. *Let ϕ_i^1 and ϕ_i^2 be two equivalent systems of functions satisfying Condition* 2.1.5 *(see* [18]*). Then the norms $|||\cdot|||^1_{W^{l,p}_\infty(\Omega)}$ and $|||\cdot|||^2_{W^{l,p}_\infty(\Omega)}$ corresponding to these systems of functions are equivalent:*

$$c_1|||u|||^2_{W^{l,p}_\infty(\Omega)} \le |||u|||^1_{W^{l,p}_\infty(\Omega)} \le c_2|||u|||^2_{W^{l,p}_\infty(\Omega)}.$$

Here c_1 and c_2 are positive constants independent of λ.

The definition of the space $W^{l-1/p,p}_\infty(\partial\Omega)$ and the previous proposition are similar.

4. Model problem in $\mathbb{R}^n$

Consider the equation with constant coefficients $A(D,\lambda)u = f$ in $\mathbb{R}^n$. Applying the Fourier transform, we obtain $A(\xi,\lambda)\tilde{u}(\xi) = \tilde{f}(\xi)$ (we use the notation $\tilde{\cdot}$ for the Fourier transform as well as $F\cdot$). Hence,

$$\tilde{u}(\xi) = A^{-1}(\xi,\lambda)\tilde{f}(\xi). \tag{4.1}$$

In the notation of the previous section we have

$$|||u|||_{E(\mathbb{R}^n)} = \sum_{j=1}^N \|F^{-1}(1+|\xi|^2+|\lambda|^2)^{(l+t_j)/2}Fu_j\|_{L^p(\mathbb{R}^n)},$$

$$|||f|||_{F^d(\mathbb{R}^n)} = \sum_{j=i}^N \|F^{-1}(1+|\xi|^2+|\lambda|^2)^{(l-s_i)/2}Ff_i\|_{L^p(\mathbb{R}^n)}.$$

Denote $\mu = \sqrt{1+|\xi|^2+|\lambda|^2}$. We have (cf. [17])

$$A(\mu\xi,\mu\lambda) = S(\mu)A(\xi,\lambda)T(\mu),$$

$$|||u|||_{E(\mathbb{R}^n)} = \|F^{-1}\mu^l T(\mu)Fu\|_{L^p(\mathbb{R}^n)},$$

$$|||f|||_{F^d(\mathbb{R}^n)} = \|F^{-1}\mu^l S^{-1}(\mu)Fu\|_{L^p(\mathbb{R}^n)},$$

where $S(\rho) = (\delta_{ij}\rho^{s_i})$ and $T(\rho) = (\delta_{ij}\rho^{t_j})$ are diagonal matrices. From (4.1),

$$F^{-1}\mu^l T(\mu)Fu = F^{-1}T(\mu)A^{-1}(\lambda,\xi)S(\mu)FF^{-1}\mu^l S^{-1}(\mu)Ff. \tag{4.2}$$

With the notation

$$\Phi(\xi) = T(\mu)A^{-1}(\lambda,\xi)S(\mu), \quad v(x) = F^{-1}\mu^l S^{-1}(\mu)Ff,$$

we rewrite (4.2) as

$$F^{-1}\mu^l T(\mu)Fu = F^{-1}\Phi(\xi)Fv.$$

Hence,

$$|||u|||_{E(\mathbb{R}^n)} = \|F^{-1}\Phi(\xi)Fv\|_{L^p(\mathbb{R}^n)}. \tag{4.3}$$

We note that

$$\Phi(\xi) = A^{-1}\Big(\frac{\lambda}{\mu}, \frac{\xi}{\mu}\Big),$$

and

$$\Delta(\xi) \equiv \det A\Big(\frac{\lambda}{\mu}, \frac{\xi}{\mu}\Big) \geq \Big(\frac{|\lambda|}{\mu} + \frac{|\xi|}{\mu}\Big)^{\sigma} e_d,$$

where $\sigma = s_1 + \cdots + s_N + t_1 + \cdots + t_N$. Taking into account that for $\lambda \geq \lambda_0$

$$\frac{|\lambda| + |\xi|}{\mu} \geq \lambda_1 \equiv \frac{\lambda_0}{\sqrt{1 + |\lambda_0|^2}},$$

we obtain

$$\Delta(\xi) \geq \lambda_1^{\sigma} e_d.$$

It can be verified that

$$\xi^{\alpha} D^{\alpha} \Phi_{ik}(\xi) = P\Big(\frac{\lambda}{\mu}, \frac{\xi}{\mu}\Big)\Delta^{-m}(\xi),$$

where $\Phi_{ik}(\xi)$ are the elements of the matrix $\Phi(\xi)$, α is the multi-index from Mikhlin's theorem, P is a polynomial, and m is a positive integer. Therefore, $\Phi_{ik}(\xi)$ are Fourier multipliers and

$$|\xi^{\alpha} D^{\alpha} \Phi_{ik}(\xi)| \leq \frac{c_{ik}}{\lambda_1^{\sigma} e_d^m},$$

with some constants c_{ik}. We conclude from (4.3) that

$$|||u|||_{E(\mathbb{R}^n)} \leq \frac{c}{\lambda_1^{\sigma} e_d^m} |||f|||_{F^d(\mathbb{R}^n)}.$$

We have proved the following theorem.

Theorem 4.1. *The equation*

$$A(D,\lambda)u = f, \;\; u \in E, \;\; f \in F^d,$$

has a unique solution u for any $\lambda \in S$, $\lambda \geq \lambda_0 > 0$. The estimate

$$|||u|||_{E(\mathbb{R}^n)} \leq \frac{c}{e_d^\kappa} |||f|||_{F^d(\mathbb{R}^n)}$$

holds with some constants c and κ independent of λ and u. The constant c depends on λ_0 and on M_d (see Section 2).

5. Model problem in a half-space

Consider the matrix operators $A(\lambda, D)$ and $B(\lambda, D)$ with the elements

$$A_{ij}(\lambda, D) = \sum_{|\alpha|+\beta=\alpha_{ij}} a_{ij}^{\alpha\beta} \lambda^\beta D^\alpha, \quad i, j = 1, \dots, N,$$

$$B_{kj}(\lambda, D) = \sum_{|\alpha|+\beta=\beta_{kj}} b_{kj}^{\alpha\beta} \lambda^\beta D^\alpha, \quad k = 1, \dots, r, \ j = 1, \dots, N,$$

respectively, in the half-space $\mathbb{R}^n_+$ $(x_n \geq 0)$. The coefficients of the operators are complex numbers. It is supposed that

$$A(\rho\xi, \rho\lambda) = S(\rho)A(\xi, \lambda)T(\rho), \ \ B(\rho\xi, \rho\lambda) = M(\rho)B(\xi, \lambda)T(\rho), \tag{5.1}$$

where

$$S(\rho) = (\delta_{ij}\rho^{s_i}), \ \ T(\rho) = (\delta_{ij}\rho^{t_j}), \ \ M(\rho) = (\delta_{ij}\rho^{\sigma_k}),$$

are diagonal matrices (cf. [17]).

We consider the operator $L = (A, B)$ from the space E to the space F with the $|||\cdot|||$-norms (see Section 3).

Proposition 5.1. *The operator $L : E \to F$ is bounded. The estimate*

$$|||Lu|||_F \leq c|||u|||_E, \quad \forall u \in E, \tag{5.2}$$

holds with a constant c independent of u and λ.

Proof. We have

$$|||Lu|||_F = |||Au|||_{F^d} + |||Bu|||_{F^b}.$$

Consider first the operator $Au = (A_1 u, \dots, A_N u)$,

$$|||A_i u|||_{W^{l-s_i,p}(\Omega)} \leq M_d \sum_{j=1}^N \sum_{|\alpha|+\beta=\alpha_{ij}} |\lambda|^\beta |||D^\alpha u_j|||_{W^{l-s_i,p}(\Omega)},$$

where M_d is defined in Section 2. By virtue of Proposition 3.3,

$$|||A_i u|||_{W^{l-s_i,p}(\Omega)} \leq cM_d \sum_{j=1}^N \sum_{|\alpha|+\beta=\alpha_{ij}} |\lambda|^\beta |||u_j|||_{W^{l-s_i+|\alpha|,p}(\Omega)},$$

and by the interpolation inequality (Proposition 3.2),

$$|||A_i u|||_{W^{l-s_i,p}(\Omega)} \le K_1 \sum_{j=1}^{N} |||u_j|||_{W^{l+t_j,p}(\Omega)}.$$

Hence,

$$|||Au|||_{F^d} \le K_2 |||u|||_E.$$

Here, the constants c, K_1 and K_2 are independent of u and λ. Similarly,

$$|||Bu|||_{F^b} \le K_3 |||u|||_E.$$

The last two estimates give (5.2). The proposition is proved.

Proposition 5.2. *The operator $L : E_\infty \to F_\infty$ is bounded. The estimate*

$$|||Lu|||_{F_\infty} \le c|||u|||_{E_\infty}, \quad \forall u \in E, \tag{5.3}$$

holds with a constant c independent of u and λ.

Proof. We have

$$|||Lu|||_{F_\infty} = |||Au|||_{F^d_\infty} + |||Bu|||_{F^b_\infty}.$$

Let us consider the first norm on the right-hand side,

$$|||Au|||_{F^d_\infty} = \sum_{i=1}^{N} |||A_i u|||_{W^{l-s_i,p}_\infty(\Omega)}.$$

Let further ϕ_k be a partition of unity and $\psi_k(x) = 1$ for $x \in$ *supp* ϕ_k. Then

$$|||\phi_k A_i u|||_{W^{l-s_i,p}(\Omega)} = |||\phi_k A_i(\psi_k u)|||_{W^{l-s_i,p}(\Omega)} \le c_1 |||A_i(\psi_k u)|||_{W^{l-s_i,p}(\Omega)},$$

where $c_1 = \sup_k \|\phi_k\|_M$ is independent of λ. Here $\|\cdot\|_M$ denotes the norm of multipliers (see Lemma 5.3 below). From the boundedness of the operator A it follows that

$$|||\phi_k A_i \psi_k u|||_{W^{l-s_i,p}(\Omega)} \le c_2 |||\psi_k u|||_E \le c_3 \sum_{j=1}^{N} |||u_j|||_{W^{l+t_j,p}_\infty(\Omega)} = c_3 |||u|||_{E_\infty},$$

where c_3 does not depend on λ. Taking the supremum with respect to k in the last estimate and a sum with respect to i, we obtain

$$|||Au|||_{F^d_\infty} \le c_3 |||u|||_{E_\infty}.$$

The estimates of the boundary operators are similar. The proposition is proved.

Lemma 5.3. *For any $u \in W^{l,p}$*

$$|||\phi u|||_{W^{l,p}} \le c\|\phi\|_{C^l} |||u|||_{W^{l,p}},$$

where the constant c is independent of ϕ, u, and λ.

Proof. Consider first the norm $[\,\cdot\,]$. We have

$$[\phi u]_{W^{l,p}} = \|\phi u\|_{W^{l,p}} + |\lambda|^l \|\phi u\|_{L^p}$$
$$\le c_4 \|\phi\|_{C^l} \|u\|_{W^{l,p}} + |\lambda|^l \|\phi\|_C \|u\|_{L^p} \le (c_4+1)\|\phi\|_{C^l} [u]_{W^{l,p}},$$

where c_4 does not depend on ϕ, u, and λ. It remains to note that the norms $[\,\cdot\,]$ and $|||\cdot|||$ are equivalent. The lemma is proved. □

Consider the problem

$$Lu = f, \quad u \in E, \quad f \in F. \tag{5.4}$$

The remaining part of this section is devoted to the following theorem.

Theorem 5.4. *For any $\lambda \in S$, $\lambda \neq 0$ and for any $f \in F$ there exists a unique solution $u \in E$ of equation* (5.4).

If $\lambda \in S$ and $|\lambda| \ge \lambda_0$, where λ_0 is an arbitrary positive number, then the following estimate holds:

$$|||u|||_E \le c|||f|||_F, \tag{5.5}$$

with a constant c that depends on the coefficients of the operator L only through M_d, M_Γ, e_d, e_Γ, and λ_0, and does not depend on λ and f.

The proof of this theorem is based on the method developed in [17] in order to obtain a priori estimates of solutions. In [18], it is adapted to prove the invertibility of model problems for pseudo-differential operators introduced as a modification of elliptic differential operators. The invertibility of these operators was proved earlier in [14] in a more general case. The method suggested in [18] allowed a simplification of the proof. The proof of Theorem 5.4 follows the same ideas. Since it is long and technical, we do not present it here.

6. Problem in $\mathbb{R}^n$

We consider the operators

$$A_{ij}(x,\lambda,D) = \sum_{|\alpha|+\beta=\alpha_{ij}} a_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i,j = 1,\ldots,N,$$

where $x \in \mathbb{R}^n$, and the matrix $A(x,\lambda,D)$ with the elements $A_{ij}(x,\lambda,D)$. Here, $a_{ij}^{\alpha\beta}(x) \in C^{l-s_i+\theta}(\mathbb{R}^n)$, $0 < \theta < 1$.

Theorem 6.1. *Suppose that the operator $A(0,\lambda,D)$ is elliptic. Then there exist constants $\epsilon > 0$, $\lambda_0 > 0$ and $K > 0$, which depend only on e_d and M_d, such that if*

$$|a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0)| < \epsilon, \quad i,j = 1,\dots,N, \ \ |\alpha| + \beta = \alpha_{ij},$$

for all x, then for all $\lambda \in S$, $|\lambda| \geq \lambda_0$ the operator $A(x,\lambda,D)$ has a right inverse R, for which the estimate $|||Rf|||_E \ \leq \ K \ |||f|||_{F^d}$ for all $f \in F^d$ holds.

Proof. According to Theorem 4.1 the operator $A(0,\lambda,D)$ has an inverse R_0 and

$$|||R_0 f|||_E \leq K_0 |||f|||_{F^d}, \tag{6.1}$$

where the constant $K_0 = |||R_0|||$ does not depend on f and λ. Consider the operator $A(x,\lambda,D)R_0$. We have

$$A(x,\lambda,D)R_0 = A(0,\lambda,D)R_0 + T = I + T, \tag{6.2}$$

where I is the identity operator in F^d and

$$T = A(x,\lambda,D)R_0 - A(0,\lambda,D)R_0.$$

Denote $u = R_0 f$. The elements of the vector Tf have the form

$$T_i f = \sum_{j=1}^{N} \sum_{|\alpha|+\beta=\alpha_{ij}} (a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0))\lambda^\beta D^\alpha u_j. \tag{6.3}$$

We need to estimate the norm $|||T_i f|||_{W^{l-s_i,p}}$. Consider first $\|T_i f\|_{W^{l-s_i,p}}$. We have

$$D^\gamma (a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0))\lambda^\beta D^\alpha u_j = \lambda^\beta \sum_{\tau+\sigma=\gamma} c_{\tau\sigma} D^\tau (a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0)) D^{\sigma+\alpha} u_j$$

$$= \lambda^\beta (a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0)) D^{\gamma+\alpha} u_j + \lambda^\beta \sum_{\tau+\sigma=\gamma, |\tau|>0} c_{\tau\sigma} D^\tau a_{ij}^{\alpha\beta}(x) D^{\sigma+\alpha} u_j.$$

Hence,

$$\|D^\gamma (a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0))\lambda^\beta D^\alpha u_j\|_{L^p} \tag{6.4}$$

$$\leq |\lambda^\beta| \epsilon \|D^{\gamma+\alpha} u_j\|_{L^p} + |\lambda^\beta| \sum_{\tau+\sigma=\gamma, |\tau|>0} |c_{\tau\sigma}| \|D^\tau a_{ij}^{\alpha\beta}(x) D^{\sigma+\alpha} u_j\|_{L^p} = S_1 + S_2.$$

Here, $|\gamma| \leq l - s_i$. We estimate S_2:

$$S_2 \leq \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}} |\lambda^\beta| \sum_{|\sigma|<|\gamma|} \|D^{\sigma+\alpha} u_j\|_{L^p} \leq \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}} |\lambda^\beta| \sum_{|\sigma|<|\gamma|} \|u_j\|_{W^{|\sigma|+|\alpha|,p}}$$

$$\leq \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}} \frac{c_1}{|\lambda|} \sum_{|\sigma|<|\gamma|} |||u_j|||_{W^{\beta+|\sigma|+|\alpha|+1,p}},$$

by the interpolation inequality. Here and below c with subscripts denotes constants independent of u and λ, and of the coefficients of the operator. We have $\beta + |\sigma| + |\alpha| + 1 \leq \alpha_{ij} + |\gamma| \leq \alpha_{ij} + l - s_i = l + t_j$. Therefore,

$$S_2 \leq \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}} \frac{c_2}{|\lambda|} |||u|||_E.$$

Further,

$$S_1 \leq \epsilon|\lambda|^\beta \|u_j\|_{W^{|\gamma|+|\alpha|,p}} \leq \epsilon c_3 |||u_j|||_{W^{\beta+|\gamma|+|\alpha|,p}}$$
$$\leq \epsilon c_3 |||u_j|||_{W^{l+t_j,p}} \leq \epsilon c_3 |||u|||_E.$$

From (6.4) we get

$$\|D^\gamma(a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0))\lambda^\beta D^\alpha u_j\|_{L^p} \leq c_4(\epsilon + \frac{1}{|\lambda|}\|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}})|||u|||_E. \quad (6.5)$$

From this estimate and (6.3) we obtain

$$\|T_i f\|_{W^{l-s_i,p}} \leq c_5(\epsilon + \frac{1}{|\lambda|}M_d) |||u|||_E. \quad (6.6)$$

We estimate now the expression $|\lambda|^{l-s_i}\|T_i f\|_{L^p}$. We have

$$|\lambda|^{l-s_i}\|(a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0))\lambda^\beta D^\alpha u_j\|_{L^p} \leq \epsilon|\lambda^{\beta+l-s_i}|\|D^\alpha u_j\|_{L^p}$$
$$\leq \epsilon|\lambda^{\beta+l-s_i}|\|u_j\|_{W^{|\alpha|,p}} \leq \epsilon|||u_j|||_{W^{|\alpha|+\beta+l-s_i,p}} = \epsilon|||u_j|||_{W^{l+t_j,p}}.$$

Therefore,

$$|\lambda|^{l-s_i}\|T_i f\|_{L^p} \leq \epsilon c_6 |||u|||_E. \quad (6.7)$$

It follows from (6.6) and (6.7) that

$$|||Tf|||_{W^{l-s_i,p}} \leq c_7(\epsilon + \frac{1}{|\lambda|}M_d) |||u|||_E \quad (6.8)$$

and

$$|||Tf|||_{F^d} \leq c_8(\epsilon + \frac{1}{|\lambda|}M_d) |||u|||_E \leq c_8(\epsilon + \frac{1}{|\lambda|}M_d) K_0|||f|||_{F^d}. \quad (6.9)$$

Thus, if

$$(\epsilon + \frac{1}{|\lambda|}M_d) K_0 \leq \frac{1}{2c_8}, \quad (6.10)$$

then $|||T||| \leq \frac{1}{2}$, the operator $I+T$ is invertible, and $|||(I+T)^{-1}||| \leq 2$. From (6.2),

$$A(x, \lambda, D) = (I + T)R_0^{-1}.$$

Hence, the operator $A(x,\lambda,D)$ is invertible, $R = (A(x,\lambda,D))^{-1} = R_0(I + T)^{-1}$, and $|||R||| \le 2|||R_0|||$. Since $|||R_0|||$ depends only on e_d and M_d, then $|||R|||$ depends also only on them. The theorem is proved.

7. Problem in $\mathbb{R}^n_+$

We consider the operators $A(x,\lambda,D)$ and $B(x,\lambda,D)$ in the half-space $\mathbb{R}^n_+ (x_n > 0)$. Here, $A(x,\lambda,D)$ is the matrix with the elements

$$A_{ij}(x,\lambda,D) = \sum_{|\alpha|+\beta=\alpha_{ij}} a_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i,j = 1,\dots,N,$$

$B(x,\lambda,D)$ is a rectangular matrix with the elements

$$B_{ij}(x,\lambda,D) = \sum_{|\alpha|+\beta=\beta_{ij}} b_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i = 1,\dots,r,\ j = 1,\dots,N,$$

$$a_{ij}^{\alpha\beta}(x) \in C^{l-s_i+\theta}(\mathbb{R}^n_+),\ \ b_{ij}^{\alpha\beta}(x) \in C^{l-\sigma_i+\theta}(\mathbb{R}^n_+), \quad 0<\theta<1.$$

Theorem 7.1. *Suppose that the operator $L(0) = (A(0,\lambda,D), B(0,\lambda,D))$ is elliptic. Then there exist constants $\epsilon > 0$, $\lambda_0 > 0$ and $K > 0$, which depend only on e_d, e_Γ, M_d and M_Γ, such that if*

$$|a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(0)| < \epsilon,\ \ x \in \mathbb{R}^n_+, \quad i,j = 1,\dots,N,\ \ |\alpha|+\beta = \alpha_{ij},$$

$$|b_{ij}^{\alpha\beta}(x) - b_{ij}^{\alpha\beta}(0)| < \epsilon,\ \ x \in \mathbb{R}^n_+, \quad i = 1,\dots,r,\ j = 1,\dots,N,\ \ |\alpha|+\beta = \beta_{ij},$$

then for all $\lambda \in S$, $|\lambda| \ge \lambda_0$ the operator $L(x) = (A(x,\lambda,D), B(x,\lambda,D))$ has a right inverse R, for which the following holds:

$$|||Rf|||_E \ \le\ K\ |||f|||_F,\ \forall f \in F.$$

Proof. According to Theorem 5.4 the operator $L(0)$ has inverse R_0 and

$$|||R_0 f|||_{E(\mathbb{R}^n_+)} \le K_0 |||f|||_{F(\mathbb{R}^n_+)}. \tag{7.1}$$

Consider the operator

$$L(x)R_0 = L(0)R_0 + T = I + T, \tag{7.2}$$

where I is the identity operator in F and

$$T = (L(x) - L(0))R_0. \tag{7.3}$$

Denote $u = R_0 f$. Then

$$Tf = (L(x) - L(0))u = (A(x,\lambda,D) - A(0,\lambda,D), B(x,\lambda,D) - B(0,\lambda,D))u.$$

Similarly, as it is done in $\mathbb{R}^n$ we get the estimate

$$|||A(x,\lambda,D) - A(0,\lambda,D)u|||_{F^d} \le c_1(\epsilon + \frac{1}{|\lambda|}M_d)\ |||u|||_E, \tag{7.4}$$

where c_1 is a constant independent of the coefficients of the operator $L(x)$. In the same way we obtain a similar estimate for the boundary operator:

$$|||B(x,\lambda,D) - B(0,\lambda,D)u|||_{F^b} \le c_2(\epsilon + \frac{1}{|\lambda|}M_b)\ |||u|||_E. \tag{7.5}$$

From (7.4) and (7.5),

$$|||Tf|||_F \le c_3(\epsilon + \frac{M_d + M_b}{|\lambda|})\ |||f|||_F.$$

Here c_3 is a constant independent of the coefficients of the operator $L(x)$. We choose ϵ and λ such that

$$c_3(\epsilon + \frac{M_d + M_b}{|\lambda|})\ \le \frac{1}{2}.$$

Then $|||T||| \le \frac{1}{2}$ and the inverse operator $R = R_0(I+T)^{-1}$ admits the estimate $|||R||| \le 2|||R_0|||$. The theorem is proved.

8. Problem in Ω

Let Ω be an unbounded domain in $\mathbb{R}^n$. It is supposed that Condition D is satisfied. We consider the operators $A(x,\lambda,D)$ and $B(x,\lambda,D)$ in the domain Ω. Here $A(x,\lambda,D)$ is the matrix with the elements

$$A_{ij}(x,\lambda,D) = \sum_{|\alpha|+\beta\le\alpha_{ij}} a_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i,j = 1,\dots,N,$$

$B(x,\lambda,D)$ is a rectangular matrix with the elements

$$B_{ij}(x,\lambda,D) = \sum_{|\alpha|+\beta\le\beta_{ij}} b_{ij}^{\alpha\beta}(x)\lambda^\beta D^\alpha, \quad i = 1,\dots,r,\ j = 1,\dots,N,$$

$$a_{ij}^{\alpha\beta}(x) \in C^{l-s_i+\theta}(\mathbb{R}^n),\ \ b_{ij}^{\alpha\beta}(x) \in C^{l-\sigma_i+\theta}(\mathbb{R}^n), \quad 0<\theta<1.$$

Denote $L = (A,B)$. We will consider first this operator as acting in ∞-spaces introduced in Section 3.

Proposition 8.1. *The operator L is a bounded operator from E_∞ to F_∞. Moreover,*

$$|||Lu|||_{F_\infty} \le K|||u|||_{E_\infty},$$

where K is a constant independent of u and λ. This is also true for the spaces E and F.

The proof of this proposition is similar to the proofs of Propositions 5.1 and 5.2.

Let $\phi_i(x)$ be a partition of unity in $\mathbb{R}^n$, and $\psi_i(x)$ be a system of functions, $\psi_i(x) \in D$ such that $\psi_i(x) = 1$ in a neighborhood of *supp* $\phi_i(x)$. We suppose that $\phi_i(x)$ and $\psi_i(x)$ satisfy the conditions which are specified for systems of functions in the construction of spaces E_q. Moreover, we suppose that *supp* ϕ_i either do not intersect the boundary Γ or belong to a given covering of Γ. Similar assumptions are made for ψ_i (cf. [3], p. 86). It is easy to prove that for uniformly regular domains such systems of functions can be constructed. Moreover, we can suppose that the support of the function $\phi_i(x)$ is a ball $B_i(r)$ with center at x_i and radius r, and the support of the function $\psi_i(x)$ is the ball $B_i(2r)$ with the same center and radius $2r$.

Denote by X_i the space $E(\Omega \cap B_i(2r))$ and by L_i the restriction of the operator L to X_i. The operator L_i acts from X_i into $Y_i = F(\Omega \cap B_i(2r)) = F^d(\Omega \cap B_i(2r)) \times F^b(\Omega \cap B_i(2r))$. We have

$$Lu = \sum_{i=1}^{\infty} \phi_i L(\psi_i u). \tag{8.1}$$

By construction, $L(\psi_i u) = L_i(\psi_i u)$. If the support *supp* ψ_i is sufficiently small, then the coefficients of the operator L_i are close to constant (cf. [3], p. 86).

We denote next by L_{i0} the principal part of the operator L_i. According to Theorem 4.1 and Theorem 5.4 there exists λ_0 such that for $|\lambda| \geq \lambda_0$ the supports of the functions ψ_i can be chosen so small that the operators L_{i0} have locally right inverse operators $R_i : Y_i \to X_i$; that is,

$$L_{i0} R_i f = f, \ \forall f \in Y_i.$$

The norms of the inverse operators $\||R_i\||_{F \to E}$ are bounded independently of i. It is supposed that the boundary of the domain is sufficiently smooth (see Condition D in Section 2) in order to map the problems under consideration into the problems in half-spaces.

For any $f \in F_\infty$, $R_i(\phi_i f)$ is defined. Therefore we can introduce the operator

$$Rf = \sum_{i=1}^{\infty} \psi_i R_i(\phi_i f), \quad f \in F_\infty. \tag{8.2}$$

This sum contains only a finite number of terms at any $x \in \bar{\Omega}$.

Proposition 8.2. *The operator R is a bounded operator from F_∞ to E_∞ with domain F_∞. Moreover,*

$$|||Rf|||_{E_\infty} \leq K|||f|||_{F_\infty},$$

where K is a constant independent of u and λ. It depends on M_d, M_b, e_d, e_Γ and the smoothness of the boundary.

Proof. Denote

$$u_i = R_i(\phi_i f), \quad f \in F_\infty \tag{8.3}$$

and $u = Rf$. Then

$$u(x) = \sum_{i=1}^{\infty} \psi_i(x) u_i(x). \tag{8.4}$$

Here, $u_i \in X_i$, and $\psi_i(x)u_i(x)$ is defined in Ω and equals 0 outside the set $\Omega \cap B_i(2r)$. It follows from (8.4) that

$$\phi_k u = \sum_{i=1}^{\infty} \phi_k \psi_i u_i = \sum_{i'}^{\infty} \phi_k \psi_{i'} u_{i'},$$

where i' are all values of i such that $\phi_k\psi_i \neq 0$. By virtue of the condition on systems of functions, the number of such i' is limited by some number N_0. We have

$$|||\phi_k u|||_E \leq \sum_{i'}^{\infty} |||\phi_k \psi_{i'} u_{i'}|||_E \leq N_0 \sup_i |||\phi_k \psi_i u_i|||_E \leq N_0 c_1 \sup_i |||u_i|||_{X_i},$$

where the constant $c_1 = \sup_k |||\phi_k|||_M \sup_i |||\psi_i|||_M$ does not depend on λ. From (8.3)

$$|||\phi_k u|||_E \leq N_0 c_1 K_1 \sup_i |||\phi_i f|||_{Y_i} \leq N_0 c_1 K_1 |||f|||_{F_\infty},$$

where $K_1 = \sup_i |||R_i|||_{Y_i \to X_i}$. Hence,

$$|||u|||_{E_\infty} \leq N_0 c_1 K_1 |||f|||_{F_\infty}.$$

The proposition is proved. □

It follows from Propositions 8.1 and 8.2 that the operator LR is a bounded operator in F_∞ defined on all of F_∞.

Consider the operator T given by the following expression:

$$Tf = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_i((L_i \psi_j \psi_i u_i - \psi_j \psi_i L_i u_i) + \psi_j \psi_i (L_i u_i - L_{i0} u_i)),$$

where u_i is defined by (8.3). The expression on the right-hand side contains a finite number of terms for every $x \in \bar{\Omega}$.

Proposition 8.3.

$$LRf = f + Tf, \quad \forall f \in F_\infty. \tag{8.5}$$

Proof. Let $f \in F_\infty$ and $u = Rf$. We have from (8.1) and (8.4)

$$Lu = \sum_{j=1}^{\infty} \phi_j L(\psi_j u) = \sum_{j=1}^{\infty} \phi_j L_j(\psi_j u) = \sum_{j=1}^{\infty} \phi_j \sum_{i=1}^{\infty} L_j \psi_j \psi_i u_i.$$

The number of terms of this series is finite for each $x \in \bar{\Omega}$. We have

$$L_j \psi_j \psi_i u_i = L \psi_j \psi_i u_i = L_i \psi_j \psi_i u_i.$$

Hence,

$$Lu = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_j L_i \psi_j \psi_i u_i.$$

Further,

$$L\psi_j\psi_i u_i = L\psi_j\psi_i u_i - \psi_j\psi_i L_i u_i + \psi_j\psi_i(L_i u_i - L_{i0} u_i) + \psi_j\psi_i L_{i0} u_i.$$

Taking into account (8.3) we get

$$\psi_j\psi_i L_{i0} u_i = \psi_j\psi_i L_{i0} R_i \phi_i f = \psi_j\psi_i\phi_i f = \psi_j \phi_i f.$$

Therefore,

$$Lu = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_j (L\psi_j\psi_i u_i - \psi_j\psi_i L_i u_i + \psi_j\psi_i(L_i u_i - L_{i0} u_i)) + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_j \phi_i f.$$

The last term on the right-hand side equals f. Thus,

$$Lu = Tu + f,$$

almost everywhere. The proposition is proved.

Proposition 8.4. *For $\lambda \in S$, $|\lambda| > \lambda_0$, where λ_0 is sufficiently large, the following estimate holds:*

$$|||Tf|||_{F_\infty} \le K|\lambda|^{-1} |||f|||_{F_\infty}, \quad \forall f \in F_\infty,$$

with a constant K determined by M_d, M_b, e_d, e_Γ and the smoothness of the boundary (Condition D), and independent of f and λ.

Proof. Consider the operators

$$T_1 f = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_j (L_i \psi_j \psi_i u_i - \psi_j \psi_i L_i u_i),$$

$$T_2 f = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \phi_j \psi_j \psi_i (L_i u_i - L_{i0} u_i),$$

where u_i is defined by (8.3). We get

$$Tf = T_1 f + T_2 f, \quad \forall f \in F_\infty.$$

Consider first $T_1 f$. We have

$$\phi_k T_1 f = \sum_{j'} \sum_{i'} \phi_k \phi_{j'} (L_{i'} \psi_{j'} \psi_{i'} u_{i'} - \psi_{j'} \psi_{i'} L_{i'} u_{i'}), \tag{8.6}$$

where i' and j' are all those values of i and j for which $\phi_j \phi_k \neq 0$ and $\phi_i \phi_k \neq 0$.

Let A_i and B_i be the restrictions of the operators A and B to X_i. Then $L_i = (A_i, B_i)$. It follows from (8.6) that

$$|||\phi_k T_1 f|||_F \leq \sum_{j'} \sum_{i'} |||\phi_k \phi_{j'} (A_{i'} \psi_{j'} \psi_{i'} u_{i'} - \psi_{j'} \psi_{i'} A_{i'} u_{i'})|||_{F^d} +$$

$$\sum_{j'} \sum_{i'} |||\phi_k \phi_{j'} (B_{i'} \psi_{j'} \psi_{i'} u_{i'} - \psi_{j'} \psi_{i'} B_{i'} u_{i'})|||_{F^b}. \tag{8.7}$$

Denote $u_i = (v_1, \ldots, v_N)^T$ (the subscript i is omitted) and $\omega = \psi_j \psi_i$, and consider the elements of the vector $(A_i \omega u_i - \omega A_i u_i)$:

$$\sum_{\tau=1}^{N} \sum_{|\alpha|+\beta \leq \alpha_{\sigma\tau}} \left(a_{\sigma\tau}^{\alpha\beta}(x) \lambda^\beta D^\alpha (\omega v_\tau) - a_{\sigma\tau}^{\alpha\beta}(x) \lambda^\beta \omega D^\alpha v_\tau \right).$$

We should estimate them in the norm $|||\cdot|||_{W^{l-s_\sigma,p}(\Omega_i)}$, where $\Omega_i = \Omega \cap B_i(2r)$.

We first estimate the $\|\cdot\|_{W^{l-s_\sigma,p}(\Omega_i)}$ norm:

$$\begin{aligned}
&\|a_{\sigma\tau}^{\alpha\beta}(x) \lambda^\beta D^\alpha (\omega v_\tau) - a_{\sigma\tau}^{\alpha\beta}(x) \lambda^\beta \omega D^\alpha v_\tau\|_{W^{l-s_\sigma,p}(\Omega_i)} \\
&\leq \|a_{\sigma\tau}^{\alpha\beta}(x)\|_{C^{l-s_\sigma}} |\lambda|^\beta \|D^\alpha(\omega v_\tau) - \omega D^\alpha v_\tau\|_{W^{l-s_\sigma,p}(\Omega_i)} \\
&\leq M_1 \|a_{\sigma\tau}^{\alpha\beta}(x)\|_{C^{l-s_\sigma}} |\lambda|^\beta \|v_\tau\|_{W^{l-s_\sigma+|\alpha|-1,p}(\Omega_i)} \\
&\leq c_1 \, \frac{M_1}{|\lambda|} \|a_{\sigma\tau}^{\alpha\beta}(x)\|_{C^{l-s_\sigma}} \|v_\tau\|_{W^{l+t_\tau,p}(\Omega_i)} \leq \frac{M_2}{|\lambda|} \|u_i\|_{E(\Omega_i)}.
\end{aligned}$$

Here c_1 is a constant independent of the coefficients of the operator A and B, and M_1 is a constant which depends on derivatives of ω. Hence, it depends on ϵ in Theorems 6.1 and 7.1 and, consequently, on M_d, M_b, e_d, and e_Γ.

In a similar way we can estimate the norm $|\lambda|^{l-s_\sigma}\|\cdot\|_{L^p(\tilde\Omega)}$. We get the same estimate as before. Hence,

$$\sum_{j'}\sum_{i'}|||\phi_k\phi_{j'}(A_{i'}\psi_{j'}\psi_{i'}u_{i'}-\psi_{j'}\psi_{i'}A_{i'}u_{i'})|||_{F^d}\le\sum_{j'}\sum_{i'}c_2\frac{M_2}{|\lambda|}|||u_{i'}|||_{E(\Omega_{i'})},\tag{8.8}$$

where c_2 is a constant independent of the coefficients of the operator A and B. We have

$$|||u_i|||_{X_i}=|||R_i(\phi_i f)|||_{X_i}\le K_1|||\phi_i f|||_{Y_i}\le c_3K_1|||\phi_i f|||_F\le c_3K_1|||f|||_{F_\infty(\Omega)},$$

where $K_1=\sup_i|||R_i|||_{Y_i\to X_i}$. Since the number of i' and j' is bounded by N, the last estimate together with (8.8) give

$$\sum_{j'}\sum_{i'}|||\phi_k\phi_{j'}(A_{i'}\psi_{j'}\psi_{i'}u_{i'}-\psi_{j'}\psi_{i'}A_{i'}u_{i'})|||_{F^d}\le c_4K_1\frac{M_2}{|\lambda|}|||f|||_{F_\infty(\Omega)}.\tag{8.9}$$

We next estimate the operator B. We have as above

$$|||b_{\nu\tau}^{\alpha\beta}(x)\lambda^\beta D^\alpha(\omega v_\tau)-b_{\nu\tau}^{\alpha\beta}(x)\lambda^\beta\omega D^\alpha v_\tau|||_{W^{l-s_\nu,p}(\Omega_i)}\le\frac{M_3}{|\lambda|}|||u_i|||_{E(\Omega_i)}.$$

Denote $\Omega_i'=\partial\Omega\cap B_i(2r)$ assuming that this intersection is not empty. Since $|||\cdot|||_{W^{l-s_\nu-1/p,p}(\Omega_i')}\le|||\cdot|||_{W^{l-s_\nu,p}(\Omega_i)}$, then

$$|||b_{\nu\tau}^{\alpha\beta}(x)\lambda^\beta D^\alpha(\omega v_\tau)-b_{\nu\tau}^{\alpha\beta}(x)\lambda^\beta\omega D^\alpha v_\tau|||_{W^{l-s_\nu-1/p,p}(\Omega_i')}\le\frac{M_3}{|\lambda|}|||u_i|||_{E(\Omega_i)}.$$

Therefore,

$$\begin{aligned}&\sum_{j'}\sum_{i'}|||\phi_k\phi_{j'}(B_{i'}\psi_{j'}\psi_{i'}u_{i'}-\psi_{j'}\psi_{i'}B_{i'}u_{i'})|||_{F^b}\\&\le c_5\frac{M_3}{|\lambda|}|||u_i|||_{E(\Omega_i)}\le c_6K_1\frac{M_3}{|\lambda|}|||f|||_{F_\infty(\Omega)},\end{aligned}$$

where, as above, M_3 depends on M_d, M_b, e_d, e_Γ, c_6 is independent of the coefficients of the operator A and B (this convention about the constants is used also below). The last estimate together with (8.9) give

$$|||\phi_kT_1f|||_{F(\Omega)}\le c_7K_1\frac{M_2+M_3}{|\lambda|}|||f|||_{F_\infty(\Omega)}.$$

Hence,

$$|||T_1f|||_{F_\infty(\Omega)}\le c_7K_1\frac{M_2+M_3}{|\lambda|}|||f|||_{F_\infty(\Omega)}.$$

Similarly, we obtain the estimate

$$|||T_2 f|||_{F_\infty(\Omega)} \le c_8 K_1 \frac{M_2 + M_3}{|\lambda|} |||f|||_{E_\infty(\Omega)}.$$

Therefore,

$$|||Tf|||_{F_\infty(\Omega)} \le \frac{K}{|\lambda|} \ |||f|||_{F_\infty(\Omega)}.$$

The proposition is proved. □

Theorem 8.5. *There exists $\lambda_0 > 0$ such that for any $\lambda \in S$, $|\lambda| \ge \lambda_0$ the equation*

$$Lu = f, \quad u \in E_\infty, \quad f \in F_\infty, \tag{8.10}$$

is uniquely solvable for any $f \in F_\infty$. Moreover, the following estimate

$$|||u|||_{E_\infty} \le \kappa |||f|||_{F_\infty}, \tag{8.11}$$

holds for the solution $u \in E_\infty$ of this problem. Here $\lambda_0 = 2K$, where K is the constant in Proposition 8.4, *and hence depends on M_d, M_b, e_d, e_Γ and the smoothness of the boundary. The constant $\kappa = 2|||R|||$ depends on the same constants and is independent of λ and f.*

Proof. According to (8.5) we have

$$LR = I + T, \tag{8.12}$$

where I is the identity operator in F_∞. From Proposition 8.4 we get $|||T||| \le \frac{1}{2}$ for $\lambda_0 = 2k$. Hence the operator $I + T$ is invertible and

$$|||(I+T)^{-1}||| \le 2. \tag{8.13}$$

From (8.12), $LR(I+T)^{-1}f = f$, for all $f \in F_\infty$. Denote

$$u = R(I+T)^{-1} f. \tag{8.14}$$

Then, $Lu = f$. From a priori estimates proved below it follows that the solution of this equation is unique. Hence, its solution has the form (8.14). It follows that (8.11) holds with $\kappa = 2|||R|||$. The theorem is proved. □

Lemma 8.6. *For any point $x_0 \in \bar{\Omega}$ there exists a neighborhood U of this point at a number $\lambda_0 > 0$ such that for any function $u \in E(\Omega)$ with support in $U \cap \bar{\Omega}$ the following estimate holds:*

$$|||u|||_{E(\Omega)} \le K |||Lu|||_{F(\Omega)}, \tag{8.15}$$

where $|\lambda| \ge \lambda_0$, $\lambda \in S$ and the constant K does not depend on u and λ.

Proof. The proof of this lemma follows from Theorems 6.1 and 7.1. For a function ϕ with support in U we have, by virtue of these theorems, the estimate

$$|||\phi u|||_{E(\Omega)} \leq K|||L(\phi u)|||_{F(\Omega)}, \tag{8.16}$$

for any $u \in E_\infty(\Omega)$. The lemma is proved. □

Lemma 8.7. *For a function ϕ with support in U the following estimate holds:*

$$|||\phi u|||_{E(\Omega)} \leq K_1\Big(|||\phi Lu|||_{F(\Omega)} + \frac{1}{|\lambda|}|||\psi u|||_{E(\Omega)}\Big), \quad \forall u \in E(\Omega), \tag{8.17}$$

where K_1 is a constant independent of u and λ, $|\lambda| \geq \lambda_0$ and λ_0 is sufficiently large. Here $\psi \in D$ is such that $\psi(x) = 1$ for $x \in supp\ \phi$.

Proof. We should estimate the norm $|||L(\phi u) - \phi Lu|||_{F(\Omega)}$. We have

$$L(\phi u) - \phi Lu = L(\phi\psi u) - \phi L(\psi u).$$

The required estimate can be obtained in the same way as the estimates in the proof of Proposition 8.4 (see (8.6)–(8.8)). The lemma is proved. □

Let ϕ_i be a system of functions in the definition of the space E_q, $1 \leq q \leq \infty$, and $\psi_i(x) = 1$ for $x \in supp\ \phi_i$. Then from (8.17),

$$|||\phi_i u|||_{E(\Omega)} \leq K_1\Big(|||\phi_i Lu|||_{F(\Omega)} + \frac{1}{|\lambda|}|||\psi_i u|||_{E(\Omega)}\Big), \quad \forall u \in E(\Omega). \tag{8.18}$$

Let first $q = \infty$. Taking the supremum with respect to i, we get

$$|||u|||_{E_\infty(\Omega)} \leq K_1\Big(|||Lu|||_{F_\infty(\Omega)} + \frac{c}{|\lambda|}|||u|||_{E_\infty(\Omega)}\Big).$$

The constant c appears as a result of equivalence of the norms E_∞ with the systems of functions ϕ_i and ψ_i. Let $|\lambda| \geq 2cK_1$. Then

$$|||u|||_{E_\infty(\Omega)} \leq 2K_1|||Lu|||_{F_\infty(\Omega)}. \tag{8.19}$$

Consider now the case $1 \leq q < \infty$. From (8.18)

$$|||\phi_i u|||^q_{E(\Omega)} \leq 2^q K_1^q\Big(|||\phi_i Lu|||^q_{F(\Omega)} + \frac{1}{|\lambda|^q}|||\psi_i u|||^q_{E(\Omega)}\Big).$$

Taking a sum with respect to i, we obtain

$$|||u|||^q_{E_q(\Omega)} \leq 2^q K_1^q\Big(|||Lu|||^q_{F_q(\Omega)} + \frac{c^q}{|\lambda|^q}|||u|||^q_{E_q(\Omega)}\Big).$$

Let $|\lambda| \geq 4cK_1$. Then

$$(1 - \frac{1}{2q})|||u|||^q_{E_q(\Omega)} \leq 2^q K_1^q|||Lu|||^q_{F_q(\Omega)}.$$

Hence,

$$|||u|||_{E_q(\Omega)} \le 4K_1|||Lu|||_{F_q(\Omega)}. \tag{8.20}$$

We have proved the following theorem.

Theorem 8.8. *The following estimate holds:*

$$|||u|||_{E_q(\Omega)} \le 4K_1|||Lu|||_{F_q(\Omega)}, \quad u \in E_q(\Omega), \;\; 1 \le q \le \infty, \tag{8.21}$$

with the constant K_1 *from Lemma* 8.7, $|\lambda| \ge \lambda_0$ *and* λ_0 *sufficiently large.*

We prove next the solvability in spaces E_q. Consider the equation

$$Lu = f, \;\; u \in E_q(\Omega), \;\; f \in F_q(\Omega), \;\; 1 \le q \le \infty. \tag{8.22}$$

Theorem 8.9. *If* $\lambda \in S$, $|\lambda| \ge \lambda_0$, *then equation* (8.22) *is uniquely solvable for any* $f \in F_q$.

Proof. From estimate (8.21) it follows that it is sufficient to prove the solvability of (8.22) for smooth f with compact support. Let $f \in D(\Omega) \times D(\Gamma)$. Then from Theorem 8.5 it follows that there exists a solution $u \in E_\infty(\Omega)$ of equation (8.22). Then ([18], Theorem 3.4.2) $u(x)\ e^{\mu\sqrt{1+|x|^2}} \in E_\infty(\Omega)$; hence, $u \in E_q(\Omega)$. The theorem is proved. □

9. Generation of analytic semigroups

We suppose that the operator A has homogeneous principal terms; that is, $\alpha_{ij} = m$ for some m. Let

$$A(x, \lambda, D)u = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u - \lambda u, \tag{9.1}$$

where $a_\alpha(x)$ are square $N \times N$ matrices and u is a vector.

We suppose further that the boundary operator $B(x, D)$ does not contain the parameter λ. Let $E_q(\Omega) = W_q^{m,p}(\Omega)$, $1 < p < \infty$, $1 \le q \le \infty$. It is assumed that the domain of the operator A is

$$D(A) = \{u \in E_q(\Omega), \; B(x, D) = 0\}.$$

It is clear that $D(A)$ is dense in $L_q^p(\Omega)$. Indeed, the set of infinitely differentiable functions, which vanish at the boundary with their derivatives, is dense in $L_q^p(\Omega)$.

Assumption 9.1. *Suppose that* $\lambda = \mu^n$ *in* (9.1) *and the operator* $A(x, \lambda, D)$ *with* μ *as a parameter is elliptic with respect to sector* S *such that the set* $\{\mu : Re\ \mu^m \ge \omega\}$ *belongs to* S.

Theorem 9.2. *Consider the operator A acting in $L_q^p(\Omega)$ with the domain $D(A)$. If Assumption* 9.1 *is satisfied, then it is a generator of an analytic semigroup.*

Proof. It is known that an operator A is sectorial if the resolvent set $\rho(A)$ contains a half-plane

$$\{\lambda \in \mathbb{C} : Re\ \lambda \geq \omega\}, \tag{9.2}$$

and

$$\|\lambda R(\lambda, A)\| \leq M,\ Re\ \lambda \geq \omega, \tag{9.3}$$

where $R(\lambda, A)$ is the resolvent of the operator A, and M and ω are some constants (cf. [18], Proposition 2.1.11).

It follows from Theorem 8.9 that $\rho(A)$ contains the half-plane (9.2) for $|\mu|$ sufficiently large. Estimate (9.3) follows from Theorem 8.8. The theorem is proved. □

10. Elliptic problems with a parameter at infinity

We consider the operators $A(x, \lambda, D)$ and $B(x, \lambda, D)$ defined in Section 5. We suppose that the sector S is given.

Definition 10.1. The operator $L(x, \lambda, D) = (A(x, \lambda, D), B(x, \lambda, D))$ is elliptic with a parameter at infinity if it is elliptic for $x \in \bar{\Omega}$, $\lambda = 0$ and elliptic with a parameter for $x \in \bar{\Omega}$, $|x| \geq R$ and $\lambda \in S$, where R is a sufficiently large number.

We will use the following notation:

$$e_d^0 = \inf_{x \in G, |\xi|=1} |\det A(X, 0, \xi)|, \quad e_d = \inf_{x \in G, |\xi|+|\lambda|=1, \lambda \in S} |\det A(X, \lambda, \xi)|,$$

$$M_d = \max_{|\alpha|+\beta \leq \alpha_{ij}, i,j=1,\ldots,N} \|a_{ij}^{\alpha\beta}\|_{C^{l-s_i}(G)},$$

$$M_\Gamma = \max_{|\alpha|+\beta \leq \beta_{ij}, i=1,\ldots,r, j=1,\ldots,N} \|b_{ij}^{\alpha\beta}\|_{C^{l-\sigma_i}(G\cap\Gamma)},$$

$$e_\Gamma^0 = \inf_{x \in G\cap\Gamma, |\xi'|=1} \sum_\alpha |\mu_\alpha(x, 0, \xi')|, \quad e_\Gamma = \inf_{x \in G\cap\Gamma, |\xi'|+|\lambda|=1, \lambda \in S} \sum_\alpha |\mu_\alpha(x, \lambda, \xi')|,$$

where $G \subset \bar{\Omega}$, $\mu_\alpha(x, \xi')$ are all r-minors of the Lopatinskii matrix in the local coordinates (ξ', ξ^n) at the point x.

Definition 10.2. The operator $L(x, 0, D)$ is called uniformly elliptic on the set $G \subset \bar{\Omega}$ if $e_d^0 > 0$, $e_\Gamma^0 > 0$, $M_d < \infty$, and $M_\Gamma < \infty$.

Definition 10.3. The operator $L(x, \lambda, D)$ is called uniformly elliptic on the set $G \subset \bar{\Omega}$ with a parameter if $e_d > 0$, $e_\Gamma > 0$, $M_d < \infty$, and $M_\Gamma < \infty$.

Definition 10.4. The operator $L(x, \lambda, D)$ is called uniformly elliptic with a parameter at infinity if the operator $L(x, 0, D)$ is uniformly elliptic in $\bar{\Omega}$ and $L(x, \lambda, D)$ is uniformly elliptic with a parameter in the domain $\{|x| \geq R\}$ for some R.

Proposition 10.5. *If the operator $L(x, \lambda, D)$ is uniformly elliptic with a parameter in the domain $\{x \in \Omega, |x| \geq R\}$, then all limiting operators are elliptic with a parameter with the same S and with the same values of the constants e_d, e_Γ, M_d, and M_Γ, and with the same constants as in Condition D.*

The proof of the theorem follows from the definition of limiting operators.

Theorem 10.6. *If the operator $L(x, \lambda, D)$ is uniformly elliptic with a parameter at infinity, then there exist λ_0 such that for $|\lambda| \geq \lambda_0$, $\lambda \in S$ the operator $L(x, \lambda, D)$ is Fredholm as acting from the space E_∞ to F_∞.*

Proof. From Proposition 10.5 it follows that all limiting operators are elliptic operators with a parameter with the same S and constants e_d, e_Γ, M_d, and M_Γ, and the same constants as in Condition D. According to Theorem 8.5 there exists a constant λ_0, which depends on the constants above, such that for $|\lambda| \geq \lambda_0$ all limiting operators are invertible. Hence, according to the results in [18] the operator $L(x, \lambda, D)$ is Fredholm from E_∞ to F_∞. The theorem is proved. □

Remark 10.7. From results in [18] it also follows that the operator $L(x, \lambda, D)$ is Fredholm from E_q to F_q for $q \leq p$ and in the properly chosen Hölder spaces if the coefficients and the boundary are sufficiently smooth. We recall that $E_q = E$ and $F_q = F$ for $q = p$. Therefore the Fredholm property is proved also for the usual Sobolev spaces.

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