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A COUPLED SYSTEM OF KORTEWEG–DE VRIES EQUATIONS AS SINGULAR LIMIT OF THE KURAMOTO–SIVASHINSKY EQUATIONS

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Abstract. We consider a coupled system of Kuramoto–Sivashinsky (KS) equations in a bounded interval depending on a suitable parameter $\nu > 0$. As ν tends to zero, we obtain a coupled system of Korteweg–de Vries (KdV) equations known to describe strong interactions of two long internal gravity waves in a stratified fluid. Existence and uniqueness of global solutions of the KS model is established as well.

1. Introduction

The Korteweg–de Vries (KdV) equation, Boussinesq equation and many other dispersive models can be formally derived as approximate equations for two-dimensional wave problems. In this work we consider a coupled system of the so-called generalized Kuramoto–Sivashinsky (KS) equations depending on a suitable parameter $\nu > 0$ and study its weak limit as $\nu \to 0$. We show that the limit system has the structure of a pair of KdV equations

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coupled through both dispersive and non-linear effects. Our results extend a previous work [13] where the same issue was addressed for the corresponding scalar models.

The Kuramoto–Sivashinsky system

$$U_t + UU_x + \alpha U_{xx} + \gamma U_{xxxx} = 0,$$

where α and γ are positive constants, describes angular phase turbulence for a system of reaction-diffusion equations, small thermal diffusive instabilities for laminar flame fronts, film layer flow on an inclined plane and several other interesting phenomena (see [2], [11], [12], [15] and [16]). Analysis shows that the negative diffusion term αU_{xx} is associated with instability, whereas the fourth-order diffusion term stabilizes the system. From a more general point of view, in many physical systems complexities may arise from nonlinear interaction between certain instability and a high-order dissipation mechanism (see [9] and the references therein). The KS equation is one of a few non-linear systems that have been extensively explored. Particularly, for better understanding of the non-linear interaction between negative diffusion and higher-order dissipation, efforts have been made on systems of KS type in more general form and multidimension (see [8], [17] and [18]).

Here, we are concerned with the generalized KS equations in a bounded domain depending on a parameter $\nu > 0$. Like previous model equations, this system also takes the basic mechanism of non-linear interaction between negative diffusion and high-order dissipation. More precisely, for $x \in [0, L]$, $t \geq 0$, $U = (u, v)$ and $\nu > 0$, we study the system

$$\begin{cases} u_t + u_x + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2(uv)_x \\ \qquad\qquad\qquad\qquad\qquad\qquad + \nu(u_{xx} + u_{xxxx}) = 0 \\ b_1 v_t + v_x + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x \\ \qquad\qquad\qquad\qquad\qquad\qquad + \nu(v_{xx} + v_{xxxx}) = 0, \end{cases} \tag{1.1}$$

where $0 < x < L$, $t > 0$, with boundary conditions

$$\begin{cases} u(0,t) = \nu u_{xx}(0,t) = u(L,t) = u_x(L,t) + \nu u_{xx}(L,t) = 0, \\ v(0,t) = \nu v_{xx}(0,t) = v(L,t) = v_x(L,t) + \nu v_{xx}(L,t) = 0, \end{cases} \tag{1.2}$$

$t > 0$, and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < L. \tag{1.3}$$

In (1.1), a_1, a_2, a_3, b_1 and b_2 are assumed to be real constants with $b_1 > 0$ and $b_2 > 0$. We also assume that

$$b_1 = b_2 = 1, \qquad 0 < a_3 < 1. \tag{1.4}$$

We introduced the parameter ν in the above system with the purpose of justifying in mathematical terms the "proximity" of the KS system to a non-linear system of KdV equations derived in [7]. The boundary condition $u_x(L,t) + \nu u_{xx}(L,t) = 0$ and $v_x(L,t) + \nu v_{xx}(L,t) = 0$, $t > 0$, could be seen as a regularization or singular perturbation of the boundary conditions $u_x(L,t) = v_x(L,t) = 0$, which is a natural boundary condition for the KdV-type equations. From the physical point of view, they mean an equilibrium between the inertial and viscous forces at the boundary.

In order to illustrate our motivation to study the behavior of $U = (u,v)$ when $\nu \to 0$, we observe that, formally, when $\nu = 0$, the above model can be written as

$$\begin{cases} u_t + u_x + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2(uv)_x = 0 & \text{in } Q \\ b_1 v_t + v_x + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x = 0 & \text{in } Q \end{cases} \tag{1.5}$$

satisfying the boundary conditions

$$u(0,t) = u(L,t) = 0,\ v(0,t) = v(L,t) = 0,\ u_x(L,t) = v_x(L,t) = 0,\ t > 0 \tag{1.6}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < L. \tag{1.7}$$

System (1.5)–(1.7) was derived by Gear and Grimshaw in [7] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and non-linear coupling terms. This somewhat complicated system has been the object of intensive research in recent years (see, for instance, [3], [4] and [14]).

The limiting process will be addressed in detail in Section 4. Let us now briefly explain the method we employ. Classical a priori estimates (energy estimates) of (1.1)–(1.3) provide uniform bounds with respect to ν on the solutions, which allow us to pass to the limit as $\nu \to 0$. The main difficulty when passing to the limit is the identification of the limit of the non-linear term. This is done using an ad hoc test function that depends on the boundary conditions on a sensitive way; i.e., this dependence is unstable and may lead rather to changes in the nature of the limit system. Our results extend those obtained in [13], from which we borrowed some ideas. However, since we are concerned with a coupled system of non-linear equations, in order to obtain the results, we need more delicate estimates. Such estimates are obtained using an approach that has some resemblance to the techniques applied by T. Kato in [10] for the study of well-posedness of the Cauchy problem associated to the KdV equation.

Clearly, in order to avoid ambiguities in the limiting process mentioned above, it is necessary to guarantee the existence and uniqueness of solutions for both models. Therefore, our first step in the analysis developed here was devoted to solving this problem. By means of the Faedo–Galerkin method with a special basis, we succeed in proving the well-posedness for system (1.1)–(1.3) in both the strong and the weak sense. We prefer the Faedo–Galerkin method because it allows us to treat both problems together: the singular limit and existence of solutions. Indeed, as will become clear during the proofs, our main result is based on a combination of local existence theory and global estimates. The global well-posedness for the KdV system was obtained in [3] and [6].

To our knowledge, the problem we are analyzing has not been addressed in the literature yet, and the existing development do not allow an immediate answer to it to be given. On the other hand, the results obtained here are only one example of a whole family of problems that arise in water waves propagation in a bounded channel. The connection between the various available models for a given "mechanical problem" may be often precisely described in mathematical terms by means of the analysis of the underlying singular perturbation problem. In this sense, the present work may be considered as a new contribution to the subject, this time in the context of dispersive systems.

The paper is organized as follows: In Section 2 we present our main results together with some preliminaries that will be used in the proofs. Section 3 is devoted to proving the global well-posedness for system (1.1)–(1.3), and in Section 4 we address the limiting process in details.

We close this section with an introduction of the notation that will be used throughout this paper.

Notation:

(1) $D_i = \frac{\partial^i}{\partial x^i}$: differentiation operator of order "i."
Due to the fact that the system in consideration has many different terms, we will use also the notation u_x, u_{xx} or $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ to denote D_1 or D_2, hoping this notation does not create extra confusion for the reader.

(2) $(u,v)(t) = \int_0^L u(x,t)v(x,t)dx$: inner product in $L^2(0,L)$;

(3) $\|u\|^2 = (u,u)(t)$: norm of u in $L^2(0,L)$;

(4) $H^m(0,L)$: Sobolev space $W^{m,2}(0,L)$;

(5) κ : imbedding constant $H^1(0,L) \hookrightarrow L^\infty(0,L)$;
(6) $\alpha = \max\{1, |a_1|, |a_2|\}$;
(7) C_p : Poincare's constant;
(8) $Q = (0,L) \times (0,T)$.

2. Statement of the main results and preliminaries

In the following we state the global well-posedness (existence and uniqueness of solutions) for the KS system. The proof will be given in Section 3.

Theorem 2.1. *Let a_3, b_1 and b_2 be as in (1.4), and suppose $u_0, v_0 \in H^4(0,L) \cap H_0^1(0,L)$ satisfy the following compatibility conditions:*

$$\begin{cases} u_0(0) = \nu u_{0,xx}(0) = u_0(L) = u_{0,x}(L) + \nu u_{0,xx}(L) = 0 \\ v_0(0) = \nu v_{0,xx}(0) = v_0(L) = v_{0,x}(L) + \nu v_{0,xx}(L) = 0. \end{cases} \tag{2.1}$$

Then, there exists a unique strong solution $\{u, v\}$ of (1.1)–(1.3) such that, for all $T > 0$,
$u, v \in L^2(0,T; H^4(0,L) \cap H_0^1(0,L)) \cap L^\infty(0,T; H^2(0,L))$;
$u_t, v_t \in L^\infty(0,T; L^2(0,L)) \cap L^2(0,T; H^2(0,L) \cap H_0^1(0,L))$;
$u_{tt}, v_{tt} \in L^2(0,T; H^{-2}(0,L))$.

When $\nu \to 0$, we obtain the main result of this paper:

Theorem 2.2. *Let $u_0, v_0 \in H^3(0,L) \cap H_0^1(0,L)$ satisfy the compatibility condition*

$$u_{0,x}(L) = v_{0,x}(L) = 0. \tag{2.2}$$

Then, there exists a unique strong solution $\{u, v\}$ of (1.5)–(1.7) such that, for all $T > 0$,
$u, v \in L^\infty(0,T; H^3(0,L) \cap H_0^1(0,L))$ *and* $u_t, v_t \in L^\infty(0,T; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$.

In order to prove Theorems 2.1 and 2.2, the following results will be needed:

Lemma 2.1. *For every $\nu > 0$ there exist eigenfunctions w_j of the problem*

$$\begin{cases} \nu D_4 w_j = \mu_j w_j \\ w_j(0) = \nu w_{j,xx}(0) = w_j(L) = w_{j,x}(L) + \nu w_{j,xx}(L) = 0 \end{cases} \tag{2.3}$$

and $\{w_j\}_{j\in\mathbb{N}}$ is a basis in $H^4(0,L)$ orthonormal in $L^2(0,L)$.

We note that, since the operator νD_4 is simultaneously positive and self-adjoint, the assertions of Lemma 2.1 follow from classical results (see for instance [5]).

Lemma 2.2. *The norms* $\|\cdot\|_{H^2(0,L)}$ *and* $\|D_2\cdot\|$ *are equivalent in* $H^2(0,L)\cap H_0^1(0,L)$*; i.e., there exists a positive constant* C_0 *such that*

$$\frac{1}{C_0}\|u\|_{H^2(0,L)} \leqslant \|D_2 u\| \leqslant \|u\|_{H^2(0,L)}, \quad \textit{for any} \quad u\in H^2(0,L)\cap H_0^1(0,L).$$

3. The KS system: global well-posedness

In order to prove the existence and uniqueness of global solutions for system (1.1)-(1.3) we employ the Faedo–Galerkin method. The result will be obtained following the steps below:

a) Approximate system; b) Estimate I; c) Estimate II; d) Estimate III; e) Convergence of the approximate solutions; f) Uniqueness and g) Verification of the initial data.

a) Approximate solution. Let $\{w_j\}_{j\in\mathbb{N}}$ be a basis of $H^4(0,L)\cap H_0^1(0,L)$ given by Lemma 2.1 and $[w_1,w_2,\ldots,w_N]$ be the space spanned by the first N eigenfunctions. We formulate the approximate problem as follows: Find $u^N(t), v^N(t)\in \text{span } V_m=[w_1,w_2,\ldots,w_m]$, i.e.,

$$u^N(x,t)=\sum_{i=1}^N g_i^N(t)w_i(x), \qquad v^N(x,t)=\sum_{i=1}^N h_i^N(t)w_i(x),$$

where $g_i^N(t)$ and $h_i^N(t)$ are solutions of the Cauchy problem of the system of ordinary differential equations

$$\begin{cases} \frac{d}{dt}g_j^N(t)+P_j(g_1^N,\ldots,g_N^N,h_1^N,\ldots,h_N^N)+Q_j(0,\ldots,0,h_1^N,\ldots,h_N^N) \\ \qquad\qquad\qquad = a_2R_j(g_1^N,\ldots,g_N^N,h_1^N,\ldots,h_N^N) \\ \frac{d}{dt}h_j^N(t)+P_j(h_1^N,\ldots,h_N^N,g_1^N,\ldots,g_N^N)+Q_j(g_1^N,\ldots,g_N^N,0,\ldots,0) \\ \qquad\qquad\qquad = a_1R_j(g_1^N,\ldots,g_N^N,h_1^N,\ldots,h_N^N) \\ g_j^N(0)=(u_0,w_j), \quad j=1,\ldots,N \\ h_j^N(0)=(v_0,w_j), \quad j=1,\ldots,N \end{cases} \tag{3.1}$$

for each $j=1,\ldots,N$, where P_j, Q_j and R_j are real functions defined from $\mathbb{R}^{2N}$ into $\mathbb{R}$ given by

$$P_j(x_1,\ldots,x_N,y_1,\ldots,y_N)=\sum_{i=1}^N x_i(D_1w_i+D_3w_i+\nu D_2w_i+\nu D_4w_i,w_j)$$
$$+a_3\sum_{i=1}^N y_i(D_3w_i,w_j)+\sum_{i=1}^N\sum_{k=1}^N x_ix_k(w_iD_1w_k,w_j)$$

$$Q_j(x_1,\dots,x_N,y_1,\dots,y_N)=\sum_{i=1}^{N}\sum_{k=1}^{N}(a_2x_ix_k+a_1y_iy_k)(w_iD_1w_k,w_j)$$

$$R_j(x_1,\dots,x_N,y_1,\dots,y_N)=\sum_{i=1}^{N}\sum_{k=1}^{N}x_iy_k(w_iw_k,D_1w_j).$$

The problem (3.1) is equivalent to

$$\begin{cases}(u_t^N(t),w_j)+(u_x^N(t),w_j)+(D_3u^N(t),w_j)+a_3(D_3v^N(t),w_j)\\ +(u^N(t)u_x^N(t),w_j)+a_1(v^N(t)v_x^N(t),w_j)+a_2\left(\frac{\partial}{\partial x}[u^N(t)v^N(t)],w_j\right)\\ +\nu(u_{xx}^N(t),w_j)+\nu(D_4u^N(t),w_j)=0 \quad \forall\, j=1,2,\dots,N,\end{cases} \tag{3.2}$$

$$\begin{cases}(v_t^N(t),w_j)+(v_x^N(t),w_j)+(D_3v^N(t),w_j)+a_3(D_3u^N(t),w_j)\\ +(v^N(t)v_x^N(t),w_j)+a_2(u^N(t)u_x^N(t),w_j)+a_1\left(\frac{\partial}{\partial x}[u^N(t)v^N(t)],w_j\right)\\ +\nu(v_{xx}^N(t),w_j)+\ \nu(D_4v^N(t),w_j)=0 \quad \forall\, j=1,2,\dots,N\end{cases} \tag{3.3}$$

$$\begin{cases} u^N(0,t)=\nu u_{xx}^N(0,t)=u^N(L,t)=u_x^N(L,t)+\nu u_{xx}^N(L,t)=0,\ 0<t<T\\ v^N(0,t)=\nu v_{xx}^N(0,t)=v^N(L,t)=v_x^N(L,t)+\nu v_{xx}^N(L,t)=0,\ 0<t<T\end{cases} \tag{3.4}$$

$$\begin{cases} u_0^N(0)=u_0^N\to u_0 \text{ strongly in } H^4(0,L)\cap H_0^1(0,L)\\ v_0^N(0)=v_0^N\to v_0 \text{ strongly in } H^4(0,L)\cap H_0^1(0,L)\end{cases} \tag{3.5}$$

as $N\to\infty$. Clearly, (3.2)–(3.5) is a first-order system of ODEs that can be written as

$$\frac{dX}{dt}=F(X), \qquad X(0)=X_0,$$

where $X=(g_1^N,g_2^N,\dots,g_N^N,h_1^N,h_2^N,\dots,h_N^N)$, $X(0)=X_0=(g_1^N(0),\dots,g_N^N(0),h_1^N(0),\dots,h_N^N(0))$ and F is given by

$$\begin{aligned} F:\mathbb{R}^{2N} &\mapsto \mathbb{R}^{2N}\\ X &\mapsto (F_1(X),F_2(X),\dots,F_{2N}(X))\end{aligned}$$

with each $F_l:\mathbb{R}^{2N}\mapsto\mathbb{R}$ being a polynomial in X independent of the variable t. Then, we can apply Caratheodory's theorem to obtain a solution

$$X=(g_1^N(t),\dots,g_N^N(t),h_1^N(t),\dots,h_N^N(t))$$

defined in some interval $(0,T_N)$, with $0<t<T_N$. This solution can be extended to the closed interval $[0,T]$, for all $T>0$, by using the estimate given below:

b) Estimate I. Replacing w_j by $2u^N$ and w_j by $2v^N$ in (3.2) and (3.3), respectively, we can add both equations to deduce that

$$
\begin{aligned}
&(u_t^N, 2u^N) + (u_x^N, 2u^N) + (D_3 u^N, 2u^N) + a_3(D_3 v^N, 2u^N) + (u^N u_x^N, 2u^N)\\
&+ a_1(v^N v_x^N, 2u^N) + a_2\Big(\frac{\partial}{\partial x}[u^N v^N], 2u^N\Big) + \nu(D_2 u^N, 2u^N) + \nu(D_4 u^N, 2u^N)\\
&+ (v_t^N, 2v^N) + (v_x^N, 2v^N) + (D_3 v^N, 2v^N) + a_3(D_3 u^N, 2v^N) + (v^N v_x^N, 2v^N)\\
&+ a_2(u^N u_x^N, 2v^N) + a_1\Big(\frac{\partial}{\partial x}[u^N v^N], 2v^N\Big) + \nu(D_2 v^N, 2v^N) + \nu(D_4 v^N, 2v^N)\\
&= 0.
\end{aligned}
$$

Thus, performing integration by parts and using the boundary conditions (3.4), we get

$$
\begin{aligned}
&\frac{d}{dt}\{\|u^N\|^2 + \|v^N\|^2\} + \big\{[u_x^N(0,t)]^2 + [v_x^N(0,t)]^2 - [u_x^N(L,t)]^2 - [v_x^N(L,t)]^2\big\}\\
&+ 2a_3\left\{[u_x^N(0,t)v_x^N(0,t)] - [u_x^N(L,t)v_x^N(L,t)]\right\}\\
&+ 2\nu\left\{(D_2 u^N, u^N) + (D_2 v^N, v^N)\right\} + 2\nu\left\{\|D_2 u^N\|^2 + \|D_2 v^N\|^2\right\}\\
&- 2\nu\left\{u_x^N(L,t) D_2 u^N(L,t) + v_x^N(L,t) D_2 v^N(L,t)\right\} = 0.
\end{aligned}
$$

Consequently, assumption (1.4) on the coefficient a_3, the Cauchy–Schwarz inequality and (3.4) enable us to get the following inequality:

$$
\begin{aligned}
\frac{d}{dt}\{\|u^N\|^2 + \|v^N\|^2\} &+ (1 - a_3)\left\{[u_x^N(0,t)]^2 + [v_x^N(0,t)]^2\right\}\\
&+\nu\left\{\|u_{xx}^N\|^2 + \|v_{xx}^N\|^2\right\} \leqslant \nu\left\{\|u^N\|^2 + \|v^N\|^2\right\}. \qquad (3.6)
\end{aligned}
$$

Now, integrating (3.6) over the interval $(0,t)$, with $0 < t < T_N$, and using the boundary conditions (2.3) on w_j we have

$$
\begin{aligned}
&\|u^N(t)\|^2 + \|v^N(t)\|^2 + (1 - a_3)\int_0^t \left\{[u_x^N(0,s)]^2 + [v_x^N(0,s)]^2\right\} ds\\
&\qquad + \nu\int_0^t \left\{\|u_{xx}^N(s)\|^2 + \|v_{xx}^N(s)\|^2\right\} ds\\
&\leqslant \|u_0^N\|^2 + \|v_0^N\|^2 + \int_0^t \left\{\|u^N(s)\|^2 + \|v^N(s)\|^2\right\} ds.
\end{aligned}
$$

Finally, applying Gronwall's inequality, we conclude that

$$
\|u^N(t)\|^2 + \|v^N(t)\|^2 + (1 - a_3)\int_0^t \left\{[u_x^N(0,s)]^2 + [v_x^N(0,s)]^2\right\} ds
$$

$$+\nu \int_0^t \left\{ \|u_{xx}^N(s)\|^2 + \|v_{xx}^N(s)\|^2 \right\} ds \leqslant C_1 \left\{ \|u_0\|^2 + \|v_0\|^2 \right\}, \quad (3.7)$$

where C_1 does not depend on N, $\quad t \in [0,T]$ and $0 < \nu < 1$.

c) Estimate II. Letting $w_j = \nu D_4 \mu_j^{-1} w_j$ in (3.2), multiplying by $g_j^N(t)$ and summing from $j = 1$ to N, we obtain the identity

$$\underbrace{(u_t^N, D_4u^N)}_{(1)} + \underbrace{(u_x^N, D_4u^N)}_{(2)} + \underbrace{(D_3u^N, D_4u^N)}_{(3)} + \underbrace{a_3(D_3v^N, D_4u^N)}_{(4)} \quad (3.8)$$

$$+ \underbrace{(u^N u_x^N, D_4u^N)}_{(5)} + \underbrace{a_1(v^N v_x^N, D_4u^N)}_{(6)} + \underbrace{a_2\left(\frac{\partial}{\partial x}[u^N v^N], D_4u^N\right)}_{(7)}$$

$$+ \underbrace{\nu(D_2u^N, D_4u^N)}_{(8)} + \underbrace{\nu(D_4u^N, D_4u^N)}_{(9)} = 0.$$

Similarly, letting $w_j = \nu D_4 \mu_j^{-1} w_j$, multiplying by $h_j^N(t)$ and summing from $j = 1$ up to $j = N$, we get

$$(v_t^N, D_4v^N) + (v_x^N, D_4v^N) + (D_3v^N, D_4v^N) + a_3(D_3u^N, D_4v^N) \quad (3.9)$$

$$+ (v^N v_x^N, D_4v^N) + a_2(u^N u_x^N, D_4v^N) + a_1\left(\frac{\partial}{\partial x}[u^N v^N], D_4v^N\right)$$

$$+ \nu(D_2v^N, D_4v^N) + \nu(D_4v^N, D_4v^N) = 0.$$

The next steps are devoted to estimating the terms which appear on the left-hand side of identity (3.8):

(1) Performing integration by parts and using the boundary conditions (3.4), we have

$$(u_t^N, D_4u^N) = \frac{1}{2}\frac{d}{dt}\left\{\nu|D_2u^N(L,t)|^2 + \|D_2u^N\|^2\right\}.$$

(2) First,we observe that by Lemma 2.2

$$\|u_x^N\|^2 \leqslant L\|u_x^N\|_{L^\infty(0,L)}^2 \leqslant L\kappa^2\|u_x^N\|_{H^1(0,L)}^2 \leqslant L\kappa^2\|u^N\|_{H^2(0,L)}^2$$
$$\leqslant L\kappa^2 C_0^2 \|D_2u^N\|^2.$$

Thus, for any $\delta > 0$, using Cauchy's and Young's inequalities we conclude that

$$(u_x^N, D_4u^N) \geqslant -\|u_x^N\|\,\|D_4u^N\| \geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\delta}{2\nu}\|u_x^N\|^2$$
$$\geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\delta L\kappa^2 C_0^2}{2\nu}\|D_2u^N\|^2.$$

(3)–(4) For any $\delta > 0$, we get

$$(D_3u^N, D_4u^N) \geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\delta}{2\nu}\|D_3u^N\|^2$$

and

$$a_3(D_3v^N, D_4u^N) \geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - |a_3|^2\frac{\delta}{2\nu}\|D_3v^N\|^2.$$

(5)–(6) From Lemma 2.2 and Estimate I, the following estimate holds:

$$\begin{aligned}\|u^Nu_x^N\| &\leqslant \|u_x^N\|_{L^\infty(0,L)}\ \|u^N\| \leqslant \kappa\|u_x^N\|_{H^1(0,L)}\ \|u^N\| \leqslant \kappa\|u^N\|_{H^2(0,L)}\ \|u^N\| \\ &\leqslant \kappa C_0\|D_2u^N\|\ \|u^N\| \leqslant \kappa C_0\sqrt{C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}}\|D_2u^N\|.\end{aligned}$$

Similarly,

$$\|v^Nv_x^N\| \leqslant \kappa C_0\sqrt{C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}}\|D_2v^N\|.$$

Thus, using Cauchy's and Young's inequalities we have

$$\begin{aligned}(u^Nu_x^N, D_4u^N) &\geqslant -\|u^Nu_x^N\|\ \|D_4u^N\| \\ &\geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\kappa^2C_0^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\|D_2u^N\|^2\end{aligned}$$

and

$$\begin{aligned}a_1(v^Nv_x^N, D_4u^N) &\geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - |a_1|^2\frac{\delta}{2\nu}\|v^Nv_x^N\|^2 \\ &\geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\kappa^2C_0^2\alpha^2C_1\{\|u_0\|^2+\|v_0\|^2\}\delta}{2\nu}\|D_2v^N\|^2,\end{aligned}$$

for any $\delta > 0$.

(7) Performing as in (5)–(6), we have

$$\begin{aligned}\|u^Nv_x^N\| &\leqslant \|v_x^N\|_{L^\infty(0,L)}\ \|u^N\| \leqslant \kappa\|v_x^N\|_{H^1(0,L)}\ \|u^N\| \leqslant \kappa\|v^N\|_{H^2(0,L)}\ \|u^N\| \\ &\leqslant \kappa C_0\|D_2v^N\|\ \|u^N\| \leqslant \kappa C_0\sqrt{C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}}\|D_2v^N\|\end{aligned}$$

and

$$\|v^Nu_x^N\| \leqslant \kappa C_0\sqrt{C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}}\|D_2u^N\|.$$

Thus,

$$\begin{aligned}&a_2\left(\frac{\partial}{\partial x}[u^Nv^N], D_4u^N\right) \geqslant -|a_2|\ \|u^Nv_x^N + u_x^Nv^N\|\|D_4u^N\| \\ &\geqslant -|a_2|\ \|u^Nv_x^N\|\ \|D_4u^N\| - |a_2|\|u_x^Nv^N\|\|D_4u^N\| \\ &\geqslant -\frac{\nu}{\delta}\|D_4u^N\|^2 - |a_2|^2\frac{\kappa^2C_0^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\|D_2v^N\|^2 \\ &- |a_2|^2\frac{\kappa^2C_0^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\|D_2u^N\|^2\end{aligned}$$

$$\geqslant -\frac{\nu}{\delta}\|D_4u^N\|^2 - \frac{\kappa^2C_0^2\alpha^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}[\|D_2u^N\|^2+\|D_2v^N\|^2].$$

(8)–(9) In this case it is straightforward to see that

$$\nu(D_2u^N, D_4u^N) \geqslant -\frac{\nu}{2\delta}\|D_4u^N\|^2 - \frac{\nu\delta}{2}\|D_2u^N\|^2,$$

for all $\delta > 0$ and

$$\nu(D_4u^N, D_4u^N) = \nu\|D_4u^N\|^2.$$

In order to estimate the terms on the left-hand side of (3.9) we follow closely the above steps to obtain

$$(v_t^N, D_4v^N) = \frac{1}{2}\frac{d}{dt}\left\{\nu|D_2v^N(L,t)|^2 + \|D_2v^N\|^2\right\}$$

$$(v_x^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - \frac{\delta L\kappa^2C_0^2}{2\nu}\|D_2v^N\|^2$$

$$(D_3v^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - \frac{\delta}{2\nu}\|D_3v^N\|^2$$

$$a_3(D_3u^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - |a_3|^2\frac{\delta}{2\nu}\|D_3u^N\|^2$$

$$(v^Nv_x^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - \frac{\kappa^2C_0^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\|D_2v^N\|^2$$

$$a_2(u^Nu_x^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - \frac{\kappa^2C_0^2\alpha^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\|D_2u^N\|^2$$

$$a_1\Big(\frac{\partial}{\partial x}[u^Nv^N], D_4v^N\Big) \geqslant$$

$$-\frac{\nu}{\delta}\|D_4v^N\|^2 - \frac{\kappa^2C_0^2\alpha^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{2\nu}\left\{\|D_2u^N\|^2+\|D_2v^N\|^2\right\}$$

$$\nu(D_2v^N, D_4v^N) \geqslant -\frac{\nu}{2\delta}\|D_4v^N\|^2 - \frac{\nu\delta}{2}\|D_2v^N\|^2$$

$$\nu(D_4v^N, D_4v^N) = \nu\|D_4v^N\|^2.$$

Now, putting the expressions above into (3.8) and (3.9), respectively, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left\{\nu|D_2u^N(L,t)|^2 + \nu|D_2v^N(L,t)|^2 + \|D_2u^N\|^2 + \|D_2v^N\|^2\right\} \tag{3.10}$$

$$+\,\nu(1-\frac{4}{\delta})\left\{\|D_4u^N\|^2+\|D_4v^N\|^2\right\} - \frac{\delta}{2\nu}(1+|a_3|^2)\{\|D_3u^N\|^2+\|D_3v^N\|^2\}$$

$$\leqslant\Big(\frac{2\alpha^2\kappa^2C_0^2C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\delta}{\nu} + \frac{L\kappa^2C_0^2\delta}{2\nu} + \frac{\nu\delta}{2}\Big)\{\|D_2u^N\|^2+\|D_2v^N\|^2\}.$$

Moreover, from Gagliardo–Nirenberg's inequality and by Estimate I, we deduce that

$$\begin{aligned}\|D_3u^N\|^2+\|D_3v^N\|^2 &\leqslant \epsilon\left\{\|D_4u^N\|^2+\|D_4v^N\|^2\right\}+C_2(\epsilon)\left\{\|u^N\|^2+\|v^N\|^2\right\}\\ &\leqslant \epsilon\left\{\|D_4u^N\|^2+\|D_4v^N\|^2\right\}\,+\,C_2(\epsilon)C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\end{aligned}$$

for any $\epsilon>0$, where C_2 is a positive constant independent of ϵ. Then, putting the above inequality into (3.10), and choosing $\delta=16$ and $\epsilon=\frac{\nu^2}{32(1+|a_3|^2)}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\nu|D_2u^N(L,t)|^2+\nu|D_2v^N(L,t)|^2+\|D_2u^N\|^2+\|D_2v^N\|^2\right\}\\ &+\nu\left\{\|D_4u^N\|^2+\|D_4v^N\|^2\right\}\leqslant C_3(\nu)\left\{1+\|D_2u^N\|^2+\|D_2v^N\|^2\right\},\end{aligned}\tag{3.11}$$

where C_3 does depend only on $\nu>0$. Finally, integrating (3.11) over $(0,t), 0<t<T$, applying Gronwall's inequality and taking into account the conditions (3.5), we conclude that

$$\begin{aligned}&\nu\left\{|D_2u^N(L,t)|^2+|D_2v^N(L,t)|^2\right\}+\left\{\|D_2u^N\|^2+\|D_2v^N\|^2\right\}\\ &+\nu\int_0^t\left\{\|D_4u^N(s)\|^2+\|D_4v^N(s)\|^2\right\}ds\\ &\leqslant \widetilde{C_4}(\nu)\left\{\nu\left[|D_2u_0(L)|^2+|D_2v_0(L)|^2\right]+\|u_0\|^2_{H^2(0,L)}+\|v_0\|^2_{H^2(0,L)}\right\}=C_4,\end{aligned}\tag{3.12}$$

where C_4 does not depend on $N, t\in[0,T]$ but depends on $\nu>0$.

d) Estimate III. This estimate is devoted to obtaining a uniform bound with respect to t for the terms u_t^N and v_t^N. In order to do that, we differentiate (3.2) and (3.3) with respect to t and replace w_j by $2u_t^N$ and $2v_t^N$, respectively, to obtain

$$\begin{aligned}&(u_{tt}^N,2u_t^N)+(u_{tx}^N,2u_t^N)+(D_3u_t^N,2u_t^N)+a_3(D_3v_t^N,2u_t^N)\\ &+\Big(\frac{\partial}{\partial t}[u^Nu_x^N],2u_t^N\Big)+a_1\Big(\frac{\partial}{\partial t}[v^Nv_x^N],2u_t^N\Big)+a_2\Big(\frac{\partial^2}{\partial x\partial t}[u^Nv^N],2u_t^N\Big)\\ &+\nu(D_2u_t^N,2u_t^N)+\nu(D_4u_t^N,2u_t^N)=0\end{aligned}$$

and

$$\begin{aligned}&(v_{tt}^N,2v_t^N)+(v_{tx}^N,2v_t^N)+(D_3v_t^N,2v_t^N)+a_3(D_3u_t^N,2v_t^N)\\ &+\Big(\frac{\partial}{\partial t}[v^Nv_x^N],2v_t^N\Big)+a_2\Big(\frac{\partial}{\partial t}[u^Nu_x^N],2v_t^N\Big)+a_1\Big(\frac{\partial^2}{\partial x\partial t}[u^Nv^N],2v_t^N\Big)\\ &+\nu(D_2v_t^N,2v_t^N)+\nu(D_4v_t^N,2v_t^N)=0.\end{aligned}$$

Performing integration by parts and using the boundary conditions (3.4) we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left\{\|u_t^N\|^2+\|v_t^N\|^2\right\}+\left\{[u_{xt}^N(0,t)]^2+2a_3u_{xt}^N(0,t)v_{xt}^N(0,t)+[v_{xt}^N(0,t)]^2\right\}\\
&-\left\{[u_{xt}^N(L,t)]^2+2a_3u_{xt}^N(L,t)v_{xt}^N(L,t)+[v_{xt}^N(L,t)]^2\right\} \qquad (3.13)\\
&+2\nu\left\{\|u_{xxt}^N\|^2+\|v_{xxt}^N\|^2\right\}-2\nu\left\{u_{xt}^N(L,t)u_{xxt}^N(L,t)+v_{xt}^N(L,t)v_{xxt}^N(L,t)\right\}\\
&=-2a_1\left(v_t^Nv_x^N,u_t^N\right)\ -2a_2\left(u_t^Nu_x^N,v_t^N\right)\left(-u_x^N-a_2v_x^N,[u_t^N]^2\right)\\
&+\left(-v_x^N-a_1u_x^N,[v_t^N]^2\right)\ -2\nu(u_t^N,u_{xxt}^N)\ -2\nu(v_t^N,v_{xxt}^N).
\end{aligned}$$

Consequently, in order to prove our claim, we need to estimate the terms on the right-hand side of (3.13). The next steps are devoted to solving this problem:

Using assumption (1.4) on the coefficient a_3 we have

$$\begin{aligned}
&[u_{xt}^N(0,t)]^2+2a_3u_{xt}^N(0,t)v_{xt}^N(0,t)+[v_{xt}^N(0,t)]^2\\
&\geqslant(1-a_3)\{[u_{xt}^N(0,t)]^2+[v_{xt}^N(0,t)]^2\}\geqslant 0
\end{aligned}$$

and

$$\begin{aligned}
&[u_{xt}^N(L,t)]^2+2a_3u_{xt}^N(L,t)v_{xt}^N(L,t)+[v_{xt}^N(L,t)]^2\\
&\geqslant(1-a_3)\{[u_{xt}^N(L,t)]^2+[v_{xt}^N(L,t)]^2\}\geqslant -2\{[u_{xt}^N(L,t)]^2+[v_{xt}^N(L,t)]^2\}.
\end{aligned}$$

On the other hand, from Young's inequality, Estimate II, Lemma 2.2 and the Sobolev imbedding $H^1(0,L)\hookrightarrow L^\infty(0,L)$ it follows that

$$\begin{aligned}
&-2a_1\left(v_t^Nv_x^N,u_t^N\right)\leqslant 2|a_1|\int_0^L|v_t^N|\,|v_x^N|\,|u_t^N|dx\\
&\leqslant\alpha\|v_x^N\|_{L^\infty(0,L)}\int_0^L 2|v_t^N|\,|u_t^N|dx\leqslant\alpha\kappa\|v_x^N\|_{H^1(0,L)}\int_0^L\left\{|u_t^N|^2+|v_t^N|^2\right\}dx\\
&\leqslant\alpha\kappa\|v^N\|_{H^2(0,L)}\left\{\|u_t^N\|^2+\|v_t^N\|^2\right\}\\
&\leqslant\alpha\kappa C_0\|D_2v^N\|\left\{\|u_t^N\|^2+\|v_t^N\|^2\right\}\leqslant\alpha\kappa C_0\sqrt{C_4}\left\{\|u_t^N\|^2+\|v_t^N\|^2\right\}
\end{aligned}$$

$$-2a_2\left(u_t^Nu_x^N,v_t^N\right)\leqslant\alpha\kappa C_0\sqrt{C_4}\left\{\|u_t^N\|^2+\|v_t^N\|^2\right\}$$

$$\begin{aligned}
&\left(-u_x^N-a_2v_x^N,[u_t^N]^2\right)\leqslant\alpha\left\{\|u_x^N\|_{L^\infty(0,L)}+\|v_x^N\|_{L^\infty(0,L)}\right\}\int_0^L|u_t^N|^2dx\\
&\leqslant\alpha\kappa\left\{\|u_x^N\|_{H^1(0,L)}+\|v_x^N\|_{H^1(0,L)}\right\}\|u_t^N\|^2\\
&\leqslant\alpha\kappa\{\|u^N\|_{H^2(0,L)}+\|v^N\|_{H^2(0,L)}\}\|u_t^N\|^2
\end{aligned}$$

$$\leqslant \alpha\kappa C_0 \left\{\|D_2 u^N\| + \|D_2 v^N\|\right\} \|u_t^N\|^2 \leqslant 2\alpha\kappa C_0\sqrt{C_4}\|u_t^N\|^2$$

$$\left(-v_x^N - a_1 u_x^N, [v_t^N]^2\right) \leqslant 2\alpha\kappa C_0\sqrt{C_4}\|v_t^N\|^2.$$

Finally, it is straightforward to see that

$$-2\nu(u_t^N, u_{xxt}^N) \leqslant 2\nu\|u_t^N\|\,\|u_{xxt}^N\| \leqslant \nu\|u_t^N\|^2 + \nu\|u_{xxt}^N\|^2$$

and

$$-2\nu(v_t^N, v_{xxt}^N) \leqslant 2\nu\|v_t^N\|\,\|v_{xxt}^N\| \leqslant \nu\|v_t^N\|^2 + \nu\|v_{xxt}^N\|^2.$$

Then, using (3.13) and (3.4), the above estimates enable us to get

$$\begin{aligned}&\frac{d}{dt}\left\{\|u_t^N\|^2 + \|v_t^N\|^2\right\} + \nu\left\{\|u_{xxt}^N\|^2 + \|v_{xxt}^N\|^2\right\}\\ &\leqslant [\nu + 4\alpha\kappa C_0\sqrt{C_4}]\left\{\|u_t^N\|^2 + \|v_t^N\|^2\right\}.\end{aligned}$$

Now, we would like to use Gronwall's inequality. But, in order to do that, we need to bound the terms $\|u_t^N(x,0)\|$ and $\|v_t^N(x,0)\|$. Again, to get the bounds, we replace w_j by $u_t^N(x,t)$ in (3.2) and w_j by $v_t^N(x,t)$ in (3.3). Then, letting $t=0$ we have

$$\begin{aligned}&(u_t^N(0), u_t^N(0)) = -(u_x^N(0), u_t^N(0)) - (D_3 u^N(0), u_t^N(0))\\ &- a_3(D_3 v^N(0), u_t^N(0)) - (u^N(0)u_x^N(0), u_t^N(0)) - a_1(v^N(0)v_x^N(0), u_t^N(0))\\ &- a_2\Big(\frac{\partial}{\partial x}[u^N(0)v^N(0)], u_t^N(0)\Big) - \nu(D_2 u^N(0), u_t^N(0)) - \nu(D_4 u^N(0), u_t^N(0))\end{aligned}$$

and

$$\begin{aligned}&(v_t^N(0), v_t^N(0)) = -(v_x^N(0), v_t^N(0)) - (D_3 v^N(0), v_t^N(0))\\ &- a_3(D_3 u^N(0), v_t^N(0)) - (v^N(0)v_x^N(0), v_t^N(0)) - a_2(u^N(0)u_x^N(0), v_t^N(0))\\ &- a_1\Big(\frac{\partial}{\partial x}[u^N(0)v^N(0)], v_t^N(0)\Big) - \nu(D_2 v^N(0), v_t^N(0)) - \nu(D_4 v^N(0), v_t^N(0)).\end{aligned}$$

Using Young's and Cauchy-Schwarz's inequalities, the terms of the first identity above may be estimated as follows:

$$\begin{aligned}-(u_x^N(0), u_t^N(0)) &\leqslant \frac{1}{4\delta}\|u_x^N(0)\|^2 + \delta\|u_t^N(0)\|^2\\ &\leqslant \frac{1}{4\delta}\|u^N(0)\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2\end{aligned}$$

$$\begin{aligned}-(D_3 u^N(0), u_t^N(0)) &\leqslant \frac{1}{4\delta}\|D_3 u^N(0)\|^2 + \delta\|u_t^N(0)\|^2\\ &\leqslant \frac{1}{4\delta}\|u^N(0)\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2\end{aligned}$$

$$-a_3(D_3 v^N(0), u_t^N(0)) \leqslant \frac{|a_3|^2}{4\delta}\|D_3 v^N(0)\|^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{1}{4\delta}\|v^N(0)\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2$$

$$-(u^N(0)u_x^N(0), u_t^N(0)) \leqslant \frac{1}{4\delta}\|u^N(0)\|^2_{L^\infty(0,L)}\|u_x^N(0)\|^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\kappa^2}{4\delta}\|u^N(0)\|^2_{H^1(0,L)}\|u^N(0)\|^2_{H^1(0,L)} + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\kappa^2}{4\delta}\|u^N(0)\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2$$

$$-a_1(v^N(0)v_x^N(0), u_t^N(0)) \leqslant \frac{|a_1|^2}{4\delta}\|v^N(0)\|^2_{L^\infty(0,L)}\|v_x^N(0)\|^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha^2\kappa^2}{4\delta}\|v^N(0)\|^2_{H^1(0,L)}\|v^N(0)\|^2_{H^1(0,L)} + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha^2\kappa^2}{4\delta}\|v^N(0)\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2$$

$$- a_2\Big(\frac{\partial}{\partial x}[u^N(0)v^N(0)], u_t^N(0)\Big)$$
$$\leqslant \frac{|a_2|}{\sqrt{2\delta}}\|u^N(0)v_x^N(0) + u_x^N(0)v^N(0)\|\sqrt{2\delta}\|u_t^N(0)\|$$
$$\leqslant \frac{\alpha}{4\delta}\left[\|u^N(0)v_x^N(0)\| + \|u_x^N(0)v^N(0)\|\right]^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha}{2\delta}\left[\|u^N(0)v_x^N(0)\|^2 + \|u_x^N(0)v^N(0)\|^2\right] + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha}{2\delta}\Big[\|u^N(0)\|^2_{L^\infty(0,L)}\|v_x^N(0)\|^2 + \|v^N(0)\|^2_{L^\infty(0,L)}\|u_x^N(0)\|^2\Big] + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha\kappa^2}{\delta}\Big[\|u^N(0)\|^2_{H^1(0,L)}\|v^N(0)\|^2_{H^1(0,L)}\|\Big] + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\alpha\kappa^2}{2\delta}\Big[\|u^N(0)\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \|v^N(0)\|^4_{H^3(0,L)\cap H_0^1(0,L)}\Big] + \delta\|u_t^N(0)\|^2$$

$$-\nu(D_2 u^N(0), u_t^N(0)) \leqslant \frac{\nu^2}{4\delta}\|D_2 u^N(0)\|^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{(1-a_3)^4}{64\delta}\|u^N(0)\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{1}{64\delta}\|u^N(0)\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2$$

$$-\nu(D_4u^N(0), u_t^N(0)) \leqslant \frac{\nu^2}{4\delta}\|D_4u^N(0)\|^2 + \delta\|u_t^N(0)\|^2$$
$$\leqslant \frac{\nu^2}{4\delta}\|u^N(0)\|^2_{H^4(0,L)\cap H_0^1(0,L)} + \delta\|u_t^N(0)\|^2.$$

Similarly, we can bound all terms of the second identity. Choosing $\delta > 0$ sufficiently small we deduce that

$$\begin{aligned}&\|u_t^N(0)\|^2 + \|v_t^N(0)\|^2\\ &\leqslant C_5\Big\{\|u_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|v_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|u_0\|^4_{H^3(0,L)\cap H_0^1(0,L)}\\ &+ \|v_0\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \nu^2\|u_0\|^2_{H^4(0,L)\cap H_0^1(0,L)} + \nu^2\|v_0\|^2_{H^4(0,L)\cap H_0^1(0,L)}\Big\},\end{aligned}$$

where the constant C_5 does not depend on $\nu > 0$. Consequently, from the analysis developed above we get

$$\begin{aligned}&\|u_t^N\|^2 + \|v_t^N\|^2 + \nu\int_0^t \left\{\|u_{xxs}^N(s)\|^2 + \|v_{xxs}^N(s)\|^2\right\} ds \\ &\leqslant C_6(\nu)\Big\{\|u_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|v_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|u_0\|^4_{H^3(0,L)\cap H_0^1(0,L)}\\ &+\|v_0\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \|u_0\|^2_{H^4(0,L)\cap H_0^1(0,L)} + \|v_0\|^2_{H^4(0,L)\cap H_0^1(0,L)}\Big\},\end{aligned} \tag{3.14}$$

where the constant C_6 does not depend on $N, t \in [0,T]$, but depends on $\nu > 0$.

e) Convergence of the approximate solutions. Observe that the previous estimates allow us to pass to the limit in (3.2)–(3.3). Indeed, according to (3.7), (3.12) and (3.14) the sequence $\{u^N, v^N\}_{N=1}^{\infty}$ satisfies the following properties:

$$\begin{aligned}&\{u^N\} \text{ and } \{v^N\} \text{ are bounded in} \\ &L^\infty(0,T;H^2\cap H_0^1(0,L)) \cap L^2(0,T;H^4\cap H_0^1(0,L)),\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\{u_t^N\} \text{ and } \{v_t^N\} \text{ are bounded in} \\ &L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L)).\end{aligned} \tag{3.16}$$

As a consequence of (3.15) and (3.16) we can extract subsequences of $\{u^N\}$ and $\{v^N\}$ that allows us to pass to the weak limit in the linear terms of (3.2) and (3.3). Thus, the next step is to identify the weak limit of the non-linear terms.

The boundedness (3.15) guarantees the existence of a subsequence of $\{u^N\}$ and $\{v^N\}$, still denoted by $\{u^N\}$ and $\{v^N\}$, such that

$$u^N \rightharpoonup u \text{ and } v^N \rightharpoonup v \quad \text{weakly in} \quad L^2(0,T;L^2(0,L)) = L^2(Q) \text{ as } N \to \infty$$
$$u^N \rightharpoonup u \text{ and } v^N \rightharpoonup v \quad \text{weakly in} \quad H^1(0,T;H^1(0,L)) = H^1(Q) \text{ as } N \to \infty.$$

On the other hand, by compactness of $H^1(Q)$ in $L^2(Q)$ we deduce that

$$u^N \to u \quad \text{and} \quad v^N \to v \quad \text{strongly in} \quad L^2(0,T;L^2(0,L)) \text{ as } N \to \infty, \tag{3.17}$$

and, in particular we have that

$$[u^N]^2 \to u^2 \quad \text{and} \quad [v^N]^2 \to v^2 \quad \text{a.e. in} \quad Q \text{ as } N \to \infty.$$

Moreover, since $H^1_0(0,L)$ is imbedded into $L^4(0,L)$, it follows from (3.15) that $\{u^N\}$ is bounded in $L^\infty(0,T;L^4(0,L))$; i.e.,

$$[u^N]^2 \text{ is bounded in } L^2(Q). \tag{3.18}$$

Consequently, from (3.17) and (3.18) we conclude that

$$[u^N]^2 \rightharpoonup u^2 \text{ weakly in } L^2(Q) \text{ as } N \to \infty.$$

Then, for all $w \in H^4(0,L) \cap H^1_0(0,L)$ and $\theta \in \mathcal{D}(0,T)$ we get

$$\lim_{N\to\infty} \int_0^T \left(u^N u^N_x, w\right) \theta \, dt = \lim_{N\to\infty} \int_0^T -\frac{1}{2}\left(|u^N|^2, w_x\right) \theta \, dt$$
$$= -\frac{1}{2}\int_0^T \left(u^2, w_x\right) \theta \, dt = \int_0^T \left(u^N u^N_x, w\right) \theta \, dt.$$

The same analysis allows us to prove that

$$\int_0^T \left(v^N v^N_x, w\right) \theta dt \longrightarrow \int_0^T (v v_x, w)\, \theta dt \quad \text{as} \quad N \to \infty,$$

for any $w \in H^4(0,L) \cap H^1_0(0,L)$ and $\theta \in \mathcal{D}(0,T)$. On the other hand, due to (3.17) and (3.18) $u^N v^N \to uv$ almost everywhere in Q as $N \to \infty$, $\{u^N v^N\}$ is bounded in $L^2(Q)$. Thus, $u^N v^N \rightharpoonup uv$ weakly in $L^2(Q)$ as $N \to \infty$ and, for all $w \in H^4(0,L) \cap H^1_0(0,L)$ and $\theta \in \mathcal{D}(0,T)$, we get

$$\lim_{N\to\infty} \int_0^T \Big(\frac{\partial}{\partial x}[u^N v^N], w\Big)\theta \, dt = \lim_{N\to\infty} \int_0^T -\Big(u^N v^N, w_x\Big)\theta \, dt$$
$$= \int_0^T -\Big(u^N v^N, w_x\Big)\theta \, dt = \int_0^T \Big(\frac{\partial}{\partial x}[uv], w\Big)\theta \, dt.$$

f) Uniqueness. Uniqueness is proved in the usual way using Gronwall's inequality; in fact, let $\{u, v\}$ and $\{z, w\}$ be solutions of (1.1)–(1.3), corresponding to the same initial data $\{u_0, v_0\}$. Then, $\phi = u - z$ and $\psi = v - w$ satisfy

ϕ and ψ are bounded in $L^\infty(0,T;H^2(0,L)\cap H_0^1(0,L))\cap L^2(0,T;H^4(0,L)\cap H_0^1(0,L))$, ϕ_t and ψ_t are bounded in $L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))$ and

$$\begin{cases} \phi_t+\phi_x+D_3\phi+a_3D_3\psi+[uu_x-zz_x]+a_1[vv_x-ww_x]+ \\ \qquad +a_2\dfrac{\partial}{\partial x}[uv-zw]+\nu D_2\phi+\nu D_4\phi=0,\ x\in(0,L),\ t\in(0,T) \\ \psi_t+\psi_x+D_3\psi+a_3D_3\phi+[vv_x-ww_x]+a_2[uu_x-zz_x]+ \\ \qquad +a_1\frac{\partial}{\partial x}[uv-zw]+\nu D_2\psi+\nu D_4\psi=0,\ x\in(0,L),\ t\in(0,T) \\ u(0,t)=\nu\phi_{xx}(0,t)=\phi(L,t)=\phi_x(L,t)+\nu\phi_{xx}(L,t)=0,\quad t\in(0,T) \\ \psi(0,t)=\nu\psi_{xx}(0,t)=\psi(L,t)=\psi_x(L,t)+\nu\psi_{xx}(L,t)=0,\quad t\in(0,T) \\ \phi(x,0)=\psi(x,0)=0,\quad x\in(0,L). \end{cases} \tag{3.19}$$

Proceeding as in Estimate I we obtain a positive constant C_8 satisfying

$$\begin{aligned} &\frac{d}{dt}\{\|\phi(t)\|^2+\|\psi(t)\|^2\}+\nu\left\{\|\phi_{xx}\|^2+\|\psi_{xx}\|^2\right\} \\ &\leqslant C_8\Big\{\|u_x\|_{L^\infty(0,L)}+\|z_x\|_{L^\infty(0,L)}+\|v_x\|_{L^\infty(0,L)} \\ &+\|w_x\|_{L^\infty(0,L)}+\|z\|_{L^\infty(0,L)}+\|w\|_{L^\infty(0,L)}+1\}\{\|\phi\|^2+\|\psi\|^2\Big\}. \end{aligned} \tag{3.20}$$

Integrating (3.20) over $(0,t)$ and applying Gronwall's inequality, from the previous estimates we conclude that $\phi\equiv\psi\equiv 0$. Hence $u=v$ and $v=w$.

g) Verification of initial data. From the above discussion we obtain a subsequence of $\{u^N,v^N\}$, still denoted by $\{u^N,v^N\}$, with the following property:

$$\int_0^T(u^N(t),w(t))dt\to\int_0^T(u(t),w(t))dt,\ \text{for any } w\in L^2(0,T;L^2(0,L)) \tag{3.21}$$

$$\int_0^T(u_t^N(t),w(t))dt\to\int_0^T(u_t(t),w(t))dt,\ \text{for any } w\in L^2(0,T;L^2(0,L)) \tag{3.22}$$

as $N\to\infty$. Let $\theta\in\mathcal{C}^1[0,T]$, such that $\theta(0)=1$ and $\theta(T)=0$. Let $w(t)=\theta'(t)z,\ z\in L^2(0,L)$, in (3.21) and $w(t)=\theta(t)z$ in (3.22) to obtain

$$\int_0^T\frac{d}{dt}[(u^N(t),z)\theta(t)]\,dt\longrightarrow\int_0^T\frac{d}{dt}[(u(t),z)\theta(t)]\,dt$$

for any $z\in L^2(0,L)$, as $N\to\infty$; i.e.,

$$\big(u^N(0),z\big)\longrightarrow(u(0),z)),\qquad\text{for any }\ z\in L^2(0,L),\ \text{as } N\to\infty. \tag{3.23}$$

From (3.5) and (3.23) we conclude that $u(0) = u_0$.

4. Asymptotic limit as ν approaches zero

This section is devoted to proving Theorem 2.2, i.e., to pass to the limit as ν goes to zero in (1.1)–(1.3). To that end, we need some estimates independent of ν.

We introduce the notation $\{u, v\} = \{u_\nu, v_\nu\}$, which will be used throughout this section.

Due to Theorem 2.1, for all $\nu > 0$ we have

$$(u_{\nu t}, w) + (u_{\nu x}, w) + (D_3 u_\nu, w) + a_3(D_3 v_\nu, w) + (u_\nu u_{\nu x}, w) \\ + a_1(v_\nu v_{\nu x}, w) + a_2\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], w\Big) + \nu(D_2 u_\nu, w) + \nu(D_4 u_\nu, w) = 0, \tag{4.1}$$

$\forall\ w \in L^2(0, L)$ almost everywhere in $(0, T)$ and

$$(v_{\nu t}, w) + (v_{\nu x}, w) + (D_3 v_\nu, w) + a_3(D_3 u_\nu, w) + (v_\nu v_{\nu x}, w) \\ + a_2(u_\nu u_{\nu x}, w) + a_1\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], w\Big) + \nu(D_2 v_\nu, w) + \nu(D_4 v_\nu, w) = 0, \tag{4.2}$$

$\forall\ w \in L^2(0, L)$ almost everywhere in $(0, T)$.

Lemma 4.1. *For all $0 < \nu < \frac{(1-a_3)^2}{4}$, the solutions $\{u_\nu, v_\nu\}$ of (4.1)–(4.2), with boundary conditions (1.2) and initial data (1.3), satisfy the inequality*

$$\|u_\nu(t)\|^2 + \|v_\nu(t)\|^2 + \int_0^t \left\{\|u_{\nu x}(s)\|^2 + \|v_{\nu x}(s)\|^2\right\} ds \\ + \nu \int_0^t \left\{\|u_{\nu xx}(s)\|^2 + \|v_{\nu xx}(s)\|^2\right\} ds \leqslant C_9 \left\{\|u_0\|^2 + \|v_0\|^2\right\},$$

where the constant $C_9 > 0$ does not depend on ν.

Proof. For some real number $\lambda > 0$, we replace $w = 2e^{\lambda x} u_\nu$ in (4.1) and $w = 2e^{\lambda x} v_\nu$ in (4.2) to obtain

$$(u_{\nu t}, 2e^{\lambda x} u_\nu) + (u_{\nu x}, 2e^{\lambda x} u_\nu) + (D_3 u_\nu, 2e^{\lambda x} u_\nu) + a_3(D_3 v_\nu, 2e^{\lambda x} u_\nu) \\ + (u_\nu u_{\nu x}, 2e^{\lambda x} u_\nu) + a_1(v_\nu v_{\nu x}, 2e^{\lambda x} u_\nu) + a_2\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], 2e^{\lambda x} u_\nu\Big) \\ + \nu(D_2 u_\nu, 2e^{\lambda x} u_\nu) + \nu(D_4 u_\nu, 2e^{\lambda x} u_\nu = 0,$$

and

$$(v_{\nu t}, 2e^{\lambda x} v_\nu) + (v_{\nu x}, 2e^{\lambda x} v_\nu) + (D_3 v_\nu, 2e^{\lambda x} v_\nu) + a_3(D_3 u_\nu, 2e^{\lambda x} v_\nu)$$

$$+ (v_\nu v_{\nu x}, 2e^{\lambda x} v_\nu) + a_2(u_\nu u_{\nu x}, 2e^{\lambda x} v_\nu) + a_1\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], 2e^{\lambda x} v_\nu\Big)$$
$$+ \nu(D_2 v_\nu, 2e^{\lambda x} v_\nu) + \nu(D_4 v_\nu, 2e^{\lambda x} v_\nu) = 0.$$

Performing integration by parts and using the boundary conditions (1.2), all terms in the above inequalities may be written as follows:

$$(u_{\nu t}, 2e^{\lambda x} u_\nu) = \frac{d}{dt}(e^{\lambda x}, u_\nu^2),$$
$$(u_{\nu x}, 2e^{\lambda x} u_\nu) = -\lambda(e^{\lambda x}, u_\nu^2),$$
$$(D_3 u_\nu, 2e^{\lambda x} u_\nu) = -\lambda^3(e^{\lambda x}, u_\nu^2) + 3\lambda(e^{\lambda x}, u_{\nu x}^2) - e^{\lambda L} u_{\nu x}^2(L,t) + u_{\nu x}^2(0,t),$$
$$+ a_3(D_3 v_\nu, 2e^{\lambda x} u_\nu) + a_3(D_3 u_\nu, 2e^{\lambda x} v_\nu) = -2a_3\lambda^3(e^{\lambda x}, u_\nu v_\nu)$$
$$+ 6a_3\lambda(e^{\lambda x}, u_{\nu x} v_{\nu x}) - 2a_3 e^{\lambda L} u_{\nu x}(L,t) v_{\nu x}(L,t) + 2a_3 u_{\nu x}(0,t) v_{\nu x}(0,t),$$
$$(u_\nu u_{\nu x}, 2e^{\lambda x} u_\nu) = -\frac{2\lambda}{3}(e^{\lambda x}, u_\nu^3),$$
$$a_1(v_\nu v_{\nu x}, 2e^{\lambda x} u_\nu) = 2a_1(e^{\lambda x} v_\nu v_{\nu x}, u_\nu),$$
$$a_2\left(\frac{\partial}{\partial x}[u_\nu v_\nu], 2e^{\lambda x} u_\nu\right) = -2a_2\lambda(e^{\lambda x}, u_\nu^2 v_\nu) - 2a_2(e^{\lambda x} u_\nu u_{\nu x}, v_\nu),$$
$$\nu(D_2 u_\nu, 2e^{\lambda x} u_\nu) = \nu\lambda^2(e^{\lambda x}, u_\nu^2) - 2\nu(e^{\lambda x}, u_{\nu x}^2),$$
$$\nu(D_4 u_\nu, 2e^{\lambda x} u_\nu) = \nu\lambda^4(e^{\lambda x}, u_\nu^2) - 4\nu\lambda^2(e^{\lambda x}, u_{\nu x}^2) + 2\nu\lambda e^{\lambda L} u_{\nu x}^2(L,t)$$
$$- 2\nu\lambda u_{\nu x}^2(0,t) + 2e^{\lambda L} u_{\nu x}^2(L,t) + 2\nu(e^{\lambda x}, u_{\nu xx}^2).$$

Moreover,

$$(v_{\nu t}, 2e^{\lambda x} v_\nu) = \frac{d}{dt}(e^{\lambda x}, v_\nu^2),$$
$$(v_{\nu x}, 2e^{\lambda x} v_\nu) = -\lambda(e^{\lambda x}, v_\nu^2),$$
$$(D_3 v_\nu, 2e^{\lambda x} v_\nu) = -\lambda^3(e^{\lambda x}, v_\nu^2) + 3\lambda(e^{\lambda x}, v_{\nu x}^2) - e^{\lambda L} v_{\nu x}^2(L,t) + v_{\nu x}^2(0,t),$$
$$(v_\nu v_{\nu x}, 2e^{\lambda x} v_\nu) = -\frac{2\lambda}{3}(e^{\lambda x}, v_\nu^3),$$
$$a_2(u_\nu u_{\nu x}, 2e^{\lambda x} v_\nu) = 2a_2(e^{\lambda x} u_\nu u_{\nu x}, v_\nu),$$
$$a_1\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], 2e^{\lambda x} v_\nu\Big) = -2a_1\lambda(e^{\lambda x}, v_\nu^2 u_\nu) - 2a_1(e^{\lambda x} v_\nu v_{\nu x}, u_\nu),$$
$$\nu(D_2 v_\nu, 2e^{\lambda x} v_\nu) = \nu\lambda^2(e^{\lambda x}, v_\nu^2) - 2\nu(e^{\lambda x}, v_{\nu x}^2),$$
$$\nu(D_4 v_\nu, 2e^{\lambda x} v_\nu) = \nu\lambda^4(e^{\lambda x}, v_\nu^2) - 4\nu\lambda^2(e^{\lambda x}, v_{\nu x}^2) + 2\nu\lambda e^{\lambda L} v_{\nu x}^2(L,t)$$
$$- 2\nu\lambda v_{\nu x}^2(0,t) + 2e^{\lambda L} v_{\nu x}^2(L,t) + 2\nu(e^{\lambda x}, v_{\nu xx}^2).$$

Summing the above identities we get

$$
\begin{aligned}
&\frac{d}{dt}\Big\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\Big\}+\underbrace{(\nu\lambda^2+\nu\lambda^4)\Big\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\Big\}}_{\geqslant 0}\\
&+2\nu\big\{(e^{\lambda x},u_{\nu xx}^2)+(e^{\lambda x},v_{\nu xx}^2)\big\}+(3\lambda-2\nu-4\nu\lambda^2)\big\{(e^{\lambda x},u_{\nu x}^2)+(e^{\lambda x},v_{\nu x}^2)\big\}\\
&-\lambda\Big\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\Big\}+(1-2\nu\lambda)u_{\nu x}^2(0,t)+2a_3u_{\nu x}(0,t)v_{\nu x}(0,t)\\
&+(1-2\nu\lambda)v_{\nu x}^2(0,t)+6a_3\lambda(e^{\lambda x},u_{\nu x}v_{\nu x}) \qquad (4.3)\\
&+e^{\lambda L}\Big\{(1+2\nu\lambda)u_{\nu x}^2(L,t)-2a_3u_{\nu x}(L,t)v_{\nu x}(L,t)+(1+2\nu\lambda)v_{\nu x}^2(L,t)\Big\}\\
&=\lambda^3\Big(e^{\lambda x},u_\nu^2+2a_3u_\nu v_\nu+v_\nu^2\Big)+\frac{2\lambda}{3}(e^{\lambda x},u_\nu^3+3a_2u_\nu^2v_\nu+3a_1u_\nu v_\nu^2+v_\nu^3).
\end{aligned}
$$

The next steps are devoted to estimating some terms which appear in (4.3):
From Estimate I we get

$$
\begin{aligned}
&\lambda\Big\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\Big\}\leqslant\lambda e^{\lambda L}\left\{\|u_\nu\|^2+\|v_\nu\|^2)\right\}\\
&\leqslant\lambda e^{\lambda L}C_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\\
&(1-2\nu\lambda)u_{\nu x}^2(0,t)+2a_3u_{\nu x}(0,t)v_{\nu x}(0,t)+(1-2\nu\lambda)v_{\nu x}^2(0,t)\\
&\geqslant(1-a_3-2\nu\lambda)\Big\{u_{\nu x}^2(0,t)+v_{\nu x}^2(0,t)\Big\}\\
&\geqslant(1-a_3)\Big[1-\frac{\lambda(1-a_3)}{2}\Big]\Big\{u_{\nu x}^2(0,t)+v_{\nu x}^2(0,t)\Big\}, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
&6a_3\lambda(e^{\lambda x},u_{\nu x}v_{\nu x})=3a_3\lambda\int_0^L e^{\lambda x}[2u_{\nu x}v_{\nu x}]\,dx\\
&\geqslant 3a_3\lambda\int_0^L e^{\lambda x}[-u_{\nu x}^2-v_{\nu x}^2]\,dx=-3a_3\lambda\Big\{(e^{\lambda x},u_{\nu x}^2)+(e^{\lambda x},v_{\nu x}^2)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&e^{\lambda L}\Big\{(1+2\nu\lambda)u_{\nu x}^2(L,t)-2a_3u_{\nu x}(L,t)v_{\nu x}(L,t)+(1+2\nu\lambda)v_{\nu x}^2(L,t)\Big\}\\
&\geqslant e^{\lambda L}\Big\{(1+2\nu\lambda)\left[u_{\nu x}^2(L,t)+v_{\nu x}^2(L,t)\right]-a_3\left[u_{\nu x}^2(L,t)+v_{\nu x}^2(L,t)\right]\Big\}\\
&=\underbrace{e^{\lambda L}(1-a_3+2\nu\lambda)\left\{u_{\nu x}^2(L,t)+v_{\nu x}^2(L,t)\right\}}_{\geqslant 0},
\end{aligned}
$$

$$
\lambda^3(e^{\lambda x},u_\nu^2+2a_3u_\nu v_\nu+v_\nu^2)\leqslant\lambda^3e^{\lambda L}\int_0^L[|u_\nu|^2+2|a_3||u_\nu|\,|v_\nu|+|v_\nu|^2]dx
$$

$$\leqslant \lambda^3 e^{\lambda L}\int_0^L [\,|u_\nu|+|v_\nu|\,]^2\ dx \leqslant 2\lambda^3 e^{\lambda L}\int_0^L \left[\,|u_\nu|^2+|v_\nu|^2\,\right]\ dx$$
$$= 2\lambda^3 e^{\lambda L}\Big\{\|u_\nu\|^2+\|v_\nu\|^2\Big\} \leqslant 2\lambda^3 e^{\lambda L} C_1\Big\{\|u_0\|^2+\|v_0\|^2\Big\}.$$

Now, combining Estimate I and Young's inequality, we get, for some $\delta > 0$,

$$\begin{aligned}
&\frac{2}{3}\lambda\left(e^{\lambda x}, u_\nu^3+3a_2u_\nu^2v_\nu+3a_1u_\nu v_\nu^2+v_\nu^3\right)\\
&\quad\leqslant \frac{2}{3}\lambda\int_0^L \left|e^{\lambda x}\left[u_\nu^3+3a_2u_\nu^2v_\nu+3a_1u_\nu v_\nu^2+v_\nu^3\right]\right|\ dx\\
&\quad\leqslant \frac{2}{3}\alpha\lambda e^{\lambda L}\int_0^L [\,|u_\nu|+|v_\nu|\,]^3\ dx\\
&\quad\leqslant \frac{2}{3}\alpha\lambda e^{\lambda L}\Big\{\|u_\nu\|_{L^\infty}+\|v_\nu\|_{L^\infty}\Big\}\int_0^L [\,|u_\nu|+|v_\nu|\,]^2\ dx\\
&\quad\leqslant \frac{4}{3}\alpha\lambda e^{\lambda L}\kappa\Big\{\|u_\nu\|_{H^1}+\|v_\nu\|_{H^1}\Big\}\int_0^L \left[\,|u_\nu|^2+|v_\nu|^2\,\right]\ dx\\
&\quad\leqslant \frac{4}{3}\alpha\kappa\lambda e^{\lambda L}C_p\Big\{\|u_{\nu x}\|+\|v_{\nu x}\|\Big\}\Big\{\|u_\nu\|^2+\|v_\nu\|^2\Big\}\\
&\quad\leqslant 2\Big\{\sqrt{\frac{\delta}{2}}\,[\,\|u_{\nu x}\|+\|v_{\nu x}\|\,]\Big\}\Big\{\sqrt{\frac{2}{\delta}}\frac{2}{3}\alpha\kappa\lambda e^{\lambda L}C_pC_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\Big\}\\
&\quad\leqslant \delta\Big\{\|u_{\nu x}\|^2+\|v_{\nu x}\|^2\Big\}+\frac{8}{9\delta}\Big[\alpha\kappa\lambda e^{\lambda L}C_pC_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\Big]^2\\
&\quad\leqslant \delta\Big\{(e^{\lambda x},u_{\nu x}^2)+(e^{\lambda x},v_{\nu x}^2)\Big\}+\frac{1}{\delta}\Big[\,\alpha\kappa\lambda e^{\lambda L}C_pC_1\left\{\|u_0\|^2+\|v_0\|^2\right\}\Big]^2.
\end{aligned}$$

Returning to (4.3), the above estimates allow us to conclude that

$$\begin{aligned}
&\frac{d}{dt}\left\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\right\}+2\nu\left\{(e^{\lambda x},u_{\nu xx}^2)+(e^{\lambda x},v_{\nu xx}^2)\right\}\\
&+[3(1-a_3)\lambda-2\nu(1+2\nu\lambda^2)-\delta]\left\{(e^{\lambda x},u_{\nu x}^2)+(e^{\lambda x},v_{\nu x}^2)\right\}\\
&+(1-a_3)\Big[1-\frac{\lambda(1-a_3)}{2}\Big]\left\{u_{\nu x}^2(0,t)+v_{\nu x}^2(0,t)\right\}\leqslant C_{10}\left\{\|u_0\|^2+\|v_0\|^2\right\},
\end{aligned}\tag{4.5}$$

for some constant $C_{10} > 0$ that does not depend on ν. Moreover, since $0 < \nu < \frac{(1-a_3)^2}{4}$ and $0 < a_3 < 1$, we can choose $\lambda = \frac{1}{1-a_3}$ and $\delta = \frac{1}{2}$ in (4.5) to obtain

$$\frac{d}{dt}\left\{(e^{\lambda x},u_\nu^2)+(e^{\lambda x},v_\nu^2)\right\}+\nu\left\{(e^{\lambda x},u_{\nu xx}^2)+(e^{\lambda x},v_{\nu xx}^2)\right\}$$

$$+\Big\{(e^{\lambda x}, u_{\nu x}^2 + (e^{\lambda x}, v_{\nu x}^2)\Big\} \leqslant C_{11}\Big\{\|u_0\|^2 + \|v_0\|^2\Big\}, \tag{4.6}$$

where $C_{11} > 0$ does not depend on ν. Then, integrating (4.6) over $0 < t < T$, we finally conclude that

$$\|u_\nu(t)\|^2 + \|v_\nu(t)\|^2 + \int_0^t \left\{\|u_{\nu x}(s)\|^2 + \|v_{\nu x}(s)\|^2\right\}\, ds$$
$$+ \nu \int_0^t \left\{\|u_{\nu xx}(s)\|^2 + \|v_{\nu xx}(s)\|^2\right\}\, ds \leqslant C_9 \left\{\|u_0\|^2 + \|v_0\|^2\right\},$$

where the constant $C_9 > 0$ does not depend on $\nu \in (0, \frac{(1-a_3)^2}{4})$. □

Now we want to obtain a uniform bound (with respect to ν) for the solution $\{u_\nu, v_\nu\}$. To that end, we follow closely the steps of Estimate III and the proof of Lemma 4.1.

Lemma 4.2. *For all* $0 < \nu < \frac{(1-a_3)^2}{4}$*, the solutions* $\{u_\nu, v_\nu\}$ *of* (4.1)–(4.2)*, with boundary conditions* (1.2) *and initial data* (1.3)*, satisfy the inequality*

$$\|u_{\nu t}\|^2 + \|v_{\nu t}\|^2 + \int_0^t \left\{\|u_{\nu xs}^N(s)\|^2 + \|v_{\nu xs}^N(s)\|^2\right\} ds$$
$$+ \nu \int_0^t \left\{\|u_{\nu xxs}^N(s)\|^2 + \|v_{\nu xxs}^N(s)\|^2\right\} ds$$
$$\leqslant C_{12} \Big\{\|u_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|v_0\|^2_{H^3(0,L)\cap H_0^1(0,L)} + \|u_0\|^4_{H^3(0,L)\cap H_0^1(0,L)}$$
$$+ \|v_0\|^4_{H^3(0,L)\cap H_0^1(0,L)} + \nu^2\|u_0\|^2_{H^4(0,L)\cap H_0^1(0,L)} + \nu^2\|v_0\|^2_{H^4(0,L)\cap H_0^1(0,L)}\Big\},$$

where the constant $C_{12} > 0$ *does not depend on* $\nu > 0$*.*

Proof. Consider $\lambda > 0$. Differentiating both equations (4.1) and (4.2) with respect to t and letting $w = 2e^{\lambda x}u_{\nu t}$ and $w = 2e^{\lambda x}v_{\nu t}$, we get

$$(u_{\nu tt}, 2e^{\lambda x}u_{\nu t}) + (u_{\nu xt}, 2e^{\lambda x}u_{\nu t}) + (D_3 u_{\nu t}, 2e^{\lambda x}u_{\nu t}) \tag{4.7}$$
$$+ a_3(D_3 v_{\nu t}, 2e^{\lambda x}u_{\nu t}) + \Big(\frac{\partial}{\partial t}[u_\nu u_{\nu x}], 2e^{\lambda x}u_{\nu t}\Big) + a_1\Big(\frac{\partial}{\partial t}[v_\nu v_{\nu x}], 2e^{\lambda x}u_{\nu t}\Big)$$
$$+ a_2\Big(\frac{\partial}{\partial t\partial x}[u_\nu v_\nu], 2e^{\lambda x}u_{\nu t}\Big) + \nu(D_2 u_{\nu t}, 2e^{\lambda x}u_{\nu t}) + \nu(D_4 u_{\nu t}, 2e^{\lambda x}u_{\nu t}) = 0$$

and

$$(v_{\nu tt}, 2e^{\lambda x}v_{\nu t}) + (v_{\nu xt}, 2e^{\lambda x}v_{\nu t}) + (D_3 v_{\nu t}, 2e^{\lambda x}v_{\nu t}) \tag{4.8}$$
$$+ a_3(D_3 u_{\nu t}, 2e^{\lambda x}v_{\nu t}) + \Big(\frac{\partial}{\partial t}[v_\nu v_{\nu x}], 2e^{\lambda x}v_{\nu t}\Big) + a_2\Big(\frac{\partial}{\partial t}[u_\nu u_{\nu x}], 2e^{\lambda x}v_{\nu t}\Big)$$

$$+ a_1\Big(\frac{\partial}{\partial t\partial x}[u_\nu v_\nu], 2e^{\lambda x}v_{\nu t}\Big) + \nu(D_2 v_{\nu t}, 2e^{\lambda x}v_{\nu t}) + \nu(D_4 v_{\nu t}, 2e^{\lambda x}v_{\nu t}) = 0.$$

Integration by parts and boundary conditions (1.2) give us that

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}[u_\nu u_{\nu x}], 2e^{\lambda x}u_{\nu t}\Big) = (e^{\lambda x}u_{\nu x}, u_{\nu t}^2) - \lambda(e^{\lambda x}u_\nu, u_{\nu t}^2),\\
&a_1\Big(\frac{\partial}{\partial t}[v_\nu v_{\nu x}], 2e^{\lambda x}u_{\nu t}\Big) = 2a_1(e^{\lambda x}v_\nu v_{\nu xt}, u_{\nu t}) + 2a_1(e^{\lambda x}v_{\nu x}, u_{\nu t}v_{\nu t}),\\
&a_2\Big(\frac{\partial}{\partial t\partial x}[u_\nu v_\nu], 2e^{\lambda x}u_{\nu t}\Big) = -a_2\lambda(e^{\lambda x}v_\nu, u_{\nu t}^2) - 2a_2\lambda(e^{\lambda x}u_\nu, u_{\nu t}v_{\nu t})\\
&\qquad + a_2(e^{\lambda x}v_{\nu x}, u_{\nu t}^2) - 2a_2(e^{\lambda x}v_{\nu t}, u_\nu u_{\nu xt}),\\
&\Big(\frac{\partial}{\partial t}[v_\nu v_{\nu x}], 2e^{\lambda x}v_{\nu t}\Big) = (e^{\lambda x}v_{\nu x}, v_{\nu t}^2) - \lambda(e^{\lambda x}v_\nu, v_{\nu t}^2),\\
&a_2\Big(\frac{\partial}{\partial t}[u_\nu u_{\nu x}], 2e^{\lambda x}v_{\nu t}\Big) = 2a_2(e^{\lambda x}u_\nu u_{\nu xt}, v_{\nu t}) + 2a_2(e^{\lambda x}u_{\nu x}, u_{\nu t}v_{\nu t}),\\
&a_1\Big(\frac{\partial}{\partial t\partial x}[u_\nu v_\nu], 2e^{\lambda x}v_{\nu t}\Big) = -a_1\lambda(e^{\lambda x}u_\nu, v_{\nu t}^2) - 2a_1\lambda(e^{\lambda x}v_\nu, u_{\nu t}v_{\nu t})\\
&\qquad + a_1(e^{\lambda x}u_{\nu x}, v_{\nu t}^2) - 2a_1(e^{\lambda x}u_{\nu t}, v_\nu v_{\nu xt}).
\end{aligned}$$

The remaining terms of (4.7)–(4.8) can be treated following the same steps of the proof of Lemma 4.1. Then, returning to (4.7)–(4.8) and taking all the above considerations into account we conclude that

$$\begin{aligned}
&\frac{d}{dt}\big\{(e^{\lambda x}, u_{\nu t}^2) + (e^{\lambda x}, v_{\nu t}^2)\big\} + \underbrace{(\nu\lambda^2 + \nu\lambda^4)\big\{(e^{\lambda x}, u_{\nu t}^2) + (e^{\lambda x}, v_{\nu t}^2)\big\}}_{\geqslant 0} \qquad (4.9)\\
&+ 2\nu\big\{(e^{\lambda x}, u_{\nu xxt}^2) + (e^{\lambda x}, v_{\nu xxt}^2)\big\}\\
&+ (3\lambda - 2\nu - 4\nu\lambda^2)\big\{(e^{\lambda x}, u_{\nu xt}^2) + (e^{\lambda x}, v_{\nu xt}^2)\big\}\\
&- \lambda\big\{(e^{\lambda x}, u_{\nu t}^2) + (e^{\lambda x}, v_{\nu t}^2)\big\} + (1-2\nu\lambda)u_{\nu xt}^2(0,t) + 2a_3u_{\nu xt}(0,t)v_{\nu xt}(0,t)\\
&+ (1-2\nu\lambda)v_{\nu xt}^2(0,t) + 6a_3\lambda(e^{\lambda x}, u_{\nu xt}v_{\nu xt})\\
&+ e^{\lambda L}\big\{(1+2\nu\lambda)u_{\nu xt}^2(L,t) - 2a_3u_{\nu xt}(L,t)v_{\nu xt}(L,t) + (1+2\nu\lambda)v_{\nu xt}^2(L,t)\big\}\\
&= \lambda^3\left(e^{\lambda x}, u_{\nu t}^2 + 2a_3u_{\nu t}v_{\nu t} + v_{\nu t}^2\right) + \left(e^{\lambda x}u_{\nu t}^2, \lambda u_\nu - u_{\nu x} + a_2\lambda v_\nu - a_2 v_{\nu x}\right)\\
&+ \left(e^{\lambda x}2u_{\nu t}v_{\nu t}, a_2\lambda u_\nu + a_1\lambda v_\nu - a_2u_{\nu x} - a_1v_{\nu x}\right)\\
&+ \left(e^{\lambda x}v_{\nu t}^2, \lambda v_\nu - v_{\nu x} + a_1\lambda u_\nu - a_1u_{\nu x}\right) + 2a_2(e^{\lambda x}v_{\nu t}, u_\nu v_{\nu xt})\\
&- 2a_2(e^{\lambda x}u_\nu u_{\nu xt}, v_{\nu t}).
\end{aligned}$$

Then, performing as in the proof of Lemma 4.1, we get

$$\lambda\Big\{(e^{\lambda x},u_{\nu t}^2)+(e^{\lambda x},v_{\nu t}^2)\Big\}\leqslant \lambda e^{\lambda L}\Big\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2)\Big\},$$

$$\begin{aligned}&(1-2\nu\lambda)u_{\nu xt}^2(0,t)+2a_3u_{\nu xt}(0,t)v_{\nu xt}(0,t)+(1-2\nu\lambda)v_{\nu xt}^2(0,t)\\&\qquad\geqslant(1-a_3)\Big[1-\frac{\lambda(1-a_3)}{2}\Big]\Big\{u_{\nu xt}^2(0,t)+v_{\nu xt}^2(0,t)\Big\},\end{aligned}$$

$$6a_3\lambda(e^{\lambda x},u_{\nu xt}v_{\nu xt})\geqslant -3a_3\lambda\Big\{(e^{\lambda x},u_{\nu xt}^2)+(e^{\lambda x},v_{\nu xt}^2)\Big\},$$

$$e^{\lambda L}\big\{(1+2\nu\lambda)u_{\nu xt}^2(L,t)-2a_3u_{\nu xt}(L,t)v_{\nu xt}(L,t)+(1+2\nu\lambda)v_{\nu xt}^2(L,t)\big\}\geqslant 0,$$

$$\lambda^3\left(e^{\lambda x},u_{\nu t}^2+2a_3u_{\nu t}v_{\nu t}+v_{\nu t}^2\right)\leqslant 2\lambda^3e^{\lambda L}\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\}.$$

Now, to estimate the last terms on the right-hand side of (4.9) we use Estimates I and II and Young's inequality to obtain

$$\begin{aligned}&\left(e^{\lambda x}u_{\nu t}^2,\lambda u_\nu-u_{\nu x}+a_2\lambda v_\nu-a_2v_{\nu x}\right)\\&+\left(e^{\lambda x}2u_{\nu t}v_{\nu t},a_2\lambda u_\nu+a_1\lambda v_\nu-a_2u_{\nu x}-a_1v_{\nu x}\right)\\&+\left(e^{\lambda x}v_{\nu t}^2,\lambda v_\nu-v_{\nu x}+a_1\lambda u_\nu-a_1u_{\nu x}\right)\\&\leqslant\max\{\lambda,1,\lambda|a_2|,|a_2|,\lambda|a_1|,|a_1|\}\int_0^L e^{\lambda x}\left\{u_{\nu t}^2\left[|u_\nu|+|u_{\nu x}|+|v_\nu|+|v_{\nu x}|\right]\right.\\&\left.2|u_{\nu t}||v_{\nu t}|\left[|u_\nu|+|v_\nu|+|u_{\nu x}|+v_{\nu x}\right]+v_{\nu t}^2\left[|v_\nu|+|v_{\nu x}|+|u_\nu|+u_{\nu x}|\right]\right\}\,dx\\&\leqslant\alpha(1+\lambda)\left\{\|u_\nu\|_{L^\infty}+\|v_\nu\|_{L^\infty}+\|u_{\nu x}\|_{L^\infty}+\|v_{\nu x}\|_{L^\infty}\right\}\times\\&\qquad\qquad\times\int_0^L e^{\lambda x}\left[|u_{\nu t}|^2+2|u_{\nu t}|\,|v_{\nu t}|+|v_{\nu t}|^2\right]\,dx\\&\leqslant\alpha(1+\lambda)\kappa\left\{\|u_\nu\|_{H^1}+\|v_\nu\|_{H^1}+\|u_{\nu x}\|_{H^1}+\|v_{\nu x}\|_{H^1}\right\}\times\\&\qquad\qquad\times e^{\lambda L}\int_0^L\left[|u_{\nu t}|+|v_{\nu t}|\right]^2\,dx\\&\leqslant 4\alpha(1+\lambda)\kappa e^{\lambda L}\left\{\|u_\nu\|_{H^2}+\|v_\nu\|_{H^2}\right\}\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\}\\&\leqslant 4\alpha(1+\lambda)\kappa e^{\lambda L}C_0\left\{\|D_2u_\nu\|+\|D_2v_\nu\|\right\}\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\}\\&\leqslant 8\alpha(1+\lambda)\kappa e^{\lambda L}C_0\sqrt{C_4}\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\}.\end{aligned}$$

Combining the above estimates and (4.9) it follows that

$$\frac{d}{dt}\left\{(e^{\lambda x},u_{\nu t}^2)+(e^{\lambda x},v_{\nu t}^2)\right\}\tag{4.10}$$

$$
\begin{aligned}
&+[3(1-a_3)\lambda-2\nu(1+2\nu\lambda^2)]\left\{(e^{\lambda x},u^2_{\nu xt})+(e^{\lambda x},v^2_{\nu xt})\right\}\\
&+\nu\left\{(e^{\lambda x},u^2_{\nu xxt})+(e^{\lambda x},v^2_{\nu xxt})\right\}\\
&+(1-a_3)\left[1-\frac{\lambda(1-a_3)}{2}\right]\left\{u^2_{\nu xt}(0,t)+v^2_{\nu xt}(0,t)\right\}\\
&\leqslant\left[2\lambda^3e^{\lambda L}+\lambda e^{\lambda L}+8\alpha(1+\lambda)\kappa e^{\lambda L}C_0\sqrt{C_4}\right]\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\}.
\end{aligned}
$$

Then, choosing $\lambda=\frac{1}{1-a_3}$ in (4.10) we obtain the inequality

$$
\begin{aligned}
&\frac{d}{dt}\left\{(e^{\lambda x},u^2_{\nu t})+(e^{\lambda x},v^2_{\nu t})\right\}+\nu\left\{(e^{\lambda x},u^2_{\nu xxt})+(e^{\lambda x},v^2_{\nu xxt})\right\}\\
&\qquad+\left\{(e^{\lambda x},u^2_{\nu xt})+(e^{\lambda x},v^2_{\nu xt})\right\}\leqslant C_{13}\left\{\|u_{\nu t}\|^2+\|v_{\nu t}\|^2\right\},
\end{aligned}
\tag{4.11}
$$

for some constant $C_{13}>0$ that does not depend on $\nu>0$.

Now, in order to apply Gronwall's inequality, we need to bound the terms $\|u_{\nu t}(0)\|$ and $\|v_{\nu t}(0)\|$. We proceed as in Estimate III to obtain

$$
\begin{aligned}
&\|u_{\nu t}(0)\|^2+\|v_{\nu t}(0)\|^2\leqslant\\
&C_{13}\Big\{\|u_0\|^2_{H^3(0,L)\cap H^1_0(0,L)}+\|v_0\|^2_{H^3(0,L)\cap H^1_0(0,L)}+\|u_0\|^4_{H^3(0,L)\cap H^1_0(0,L)}\\
&+\|v_0\|^4_{H^3(0,L)\cap H^1_0(0,L)}+\nu^2\|u_0\|^2_{H^4(0,L)\cap H^1_0(0,L)}+\nu^2\|v_0\|^2_{H^4(0,L)\cap H^1_0(0,L)}\Big\},
\end{aligned}
\tag{4.12}
$$

where the constant C_{13} does not depend on $\nu>0$.

Finally, integrating (4.11) over $(0,T)$ and using (4.12), we conclude that

$$
\begin{aligned}
&\|u_{\nu t}\|^2+\|v_{\nu t}\|^2+\int_0^t\left\{\|u^N_{\nu xs}(s)\|^2+\|v^N_{\nu xs}(s)\|^2\right\}ds\\
&\quad+\nu\int_0^t\left\{\|u^N_{\nu xxs}(s)\|^2+\|v^N_{\nu xxs}(s)\|^2\right\}ds\\
&\quad\leqslant C_{12}\Big\{\|u_0\|^2_{H^3(0,L)\cap H^1_0(0,L)}+\|v_0\|^2_{H^3(0,L)\cap H^1_0(0,L)}+\|u_0\|^4_{H^3(0,L)\cap H^1_0(0,L)}\\
&\quad+\|v_0\|^4_{H^3(0,L)\cap H^1_0(0,L)}+\nu^2\|u_0\|^2_{H^4(0,L)\cap H^1_0(0,L)}+\nu^2\|v_0\|^2_{H^4(0,L)\cap H^1_0(0,L)}\Big\}
\end{aligned}
$$

where the constant C_{12} does not depend on $\nu>0$. □

Now we can prove Theorem 2.2:

Proof of Theorem 2.2. We recall that our objective is to pass to the limit in (4.1)–(4.2) as $\nu\to 0$.

Lemmas 4.1 and 4.2 imply that the sequence $\{u_\nu, v_\nu\}$ satisfies the following properties:

$$\begin{cases} \{u_\nu\} \text{ and } \{v_\nu\} \text{ are bounded in } L^\infty(0,T;L^2(0,L)) \hookrightarrow L^2(0,T;L^2(0,L)) \\ \{u_{\nu x}\} \text{ and } \{v_{\nu x}\} \text{ are bounded in } L^2(0,T;L^2(0,L)) \\ \{\nu^{\frac{1}{2}} u_{\nu xx}\} \text{ and } \{\nu^{\frac{1}{2}} v_{\nu xx}\} \text{ are bounded in } L^2(0,T;L^2(0,L)) \\ \{u_{\nu t}\} \text{ and } \{v_{\nu t}\} \text{ are bounded in } L^\infty(0,T;L^2(0,L)) \hookrightarrow L^2(0,T;L^2(0,L)) \\ \{u_{\nu xt}\} \text{ and } \{v_{\nu xt}\} \text{ are bounded in } L^2(0,T;L^2(0,L)) \\ \{\nu^{\frac{1}{2}} u_{\nu xxt}\} \text{ and } \{\nu^{\frac{1}{2}} v_{\nu xxt}\} \text{ are bounded in } L^2(0,T;L^2(0,L)). \end{cases} \tag{4.13}$$

From (4.13) we deduce that

$$\begin{cases} \{u_\nu\} \text{ and } \{v_\nu\} \text{ are bounded in } L^2(0,T;H_0^1(0,L)) \\ \{\nu^{\frac{1}{2}} u_\nu\} \text{ and } \{\nu^{\frac{1}{2}} v_\nu\} \text{ are bounded in } L^2(0,T;H^2(0,L)) \\ \{u_{\nu t}\} \text{ and } \{v_{\nu t}\} \text{ are bounded in } L^2(0,T;H_0^1(0,L)) \\ \{\nu^{\frac{1}{2}} u_{\nu t}\} \text{ and } \{\nu^{\frac{1}{2}} v_{\nu t}\} \text{ are bounded in } L^2(0,T;H^2(0,L)). \end{cases}$$

Hence,

$$\begin{cases} \{u_\nu\} \text{ and } \{v_\nu\} \text{ are bounded in } \mathcal{C}(0,T;H_0^1(0,L)) \\ \{\nu^{\frac{1}{2}} u_\nu\} \text{ and } \{\nu^{\frac{1}{2}} v_\nu\} \text{ are bounded in } \mathcal{C}(0,T;H^2(0,L)). \end{cases}$$

Moreover, from (4.1), (4.2) and the above discussion we deduce that

$$\{u_{\nu tt}\} \text{ and } \{v_{\nu tt}\} \text{ are bounded in } L^2(0,T;H^{-2}(0,L)). \tag{4.14}$$

The boundedness of the sequences (4.13)–(4.14) guarantees the existence of subsequences of $\{u_\nu\}$ and $\{v_\nu\}$ (still denoted by u_ν and v_ν) and some functions U and V (which depend on x and t) such that, as $\nu \to 0$,

$$\begin{cases} u_\nu \to U \text{ strongly in } \mathcal{C}([0,L]\times[0,T]) \\ v_\nu \to V \text{ strongly in } \mathcal{C}([0,L]\times[0,T]) \\ u_\nu \rightharpoonup U \text{ weakly } {}^* \text{ in } L^\infty(0,T;L^2(0,L)) \\ v_\nu \rightharpoonup V \text{ weakly } {}^* \text{ in } L^\infty(0,T;L^2(0,L)) \\ u_\nu \rightharpoonup U \text{ weakly in } L^2(0,T;H_0^1(0,L)) \\ v_\nu \rightharpoonup V \text{ weakly in } L^2(0,T;H_0^1(0,L)) \\ u_{\nu t} \rightharpoonup U_t \text{ weakly } {}^* \text{ in } L^\infty(0,T;L^2(0,L)) \\ v_{\nu t} \rightharpoonup V_t \text{ weakly } {}^* \text{ in } L^\infty(0,T;L^2(0,L)) \\ u_{\nu t} \rightharpoonup U_t \text{ weakly in } L^2(0,T;H_0^1(0,L)) \\ v_{\nu t} \rightharpoonup V_t \text{ weakly in } L^2(0,T;H_0^1(0,L)) \\ \nu u_{\nu xx} \rightharpoonup 0 \text{ weakly in } L^2(0,T;L^2(0,L)) \\ \nu v_{\nu xx} \rightharpoonup 0 \text{ weakly in } L^2(0,T;L^2(0,L)). \end{cases} \tag{4.15}$$

Consider now the space of functions

$$\mathcal{W} = \Big\{\omega \in L^\infty(0,T; H^2(0,L) \cap H^1_0(0,L)) \ : \ \omega_x(0,t) = 0, \quad \forall \quad 0 < t < T\Big\}.$$

Then, for all $\nu \in (0, \frac{(1-a_3^2)}{4})$ and $\omega \in L^\infty(0,T; L^2(0,L))$ Theorem 2.1 gives us that

$$\begin{aligned}(u_{\nu t}, \omega) &+ (u_{\nu x}, \omega) + (u_{\nu x}, \omega_{xx}) + a_3(D_3 v_\nu, \omega) + (u_\nu u_{\nu x}, \omega)\\ &+ a_1(v_\nu v_{\nu x}, \omega) + a_2\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], \omega\Big) + \nu(D_2 u_\nu, \omega) + \nu(D_2 u_\nu, \omega_{xx}) = 0,\end{aligned}$$

and

$$\begin{aligned}(v_{\nu t}, \omega) &+ (v_{\nu x}, \omega) + (v_{\nu x}, \omega_{xx}) + a_3(D_3 u_\nu, \omega) + (v_\nu v_{\nu x}, \omega)\\ &+ a_2(u_\nu u_{\nu x}, \omega) + a_1\Big(\frac{\partial}{\partial x}[u_\nu v_\nu], \omega\Big) + \nu(D_2 v_\nu, \omega) + \nu(D_2 v_\nu, \omega_{xx}) = 0.\end{aligned}$$

Finally, letting $\nu \to 0$ and performing as in the previous section, the convergences (4.15) give us that the weak limit $\{U, V\}$ satisfies

$$\left\{\begin{array}{l}(U_t, \omega) + (U_x, \omega) + (U_x, \omega_{xx}) + a_3(D_3 V, \omega) + (UU_x, \omega)\\ \qquad +a_1(VV_x, \omega) + a_2\Big(\frac{\partial}{\partial x}[UV], \omega\Big) = 0,\\ (V_t, \omega) + (V_x, \omega) + (V_x, \omega_{xx}) + a_3(D_3 U, \omega) + (VV_x, \omega)\\ \qquad +a_2(UU_x, \omega) + a_1\Big(\frac{\partial}{\partial x}[UV], \omega\Big) = 0.\end{array}\right. \tag{4.16}$$

It remains to analyze the following steps:

a) Regularity of $\{U, V\}$. In order to prove that the weak solution obtained is regular, we use the properties of U and V and rewrite the identities (4.16) as

$$\left\{\begin{array}{l}(U_x + a_3 V_x, \omega_{xx})(t) = (F, \omega)(t), \text{ for any } \omega \in \mathcal{W}, \quad 0 < t < T\\ (V_x + a_3 U_x, \omega_{xx})(t) = (G, \omega)(t), \text{ for any } \omega \in \mathcal{W}, \quad 0 < t < T,\end{array}\right. \tag{4.17}$$

where

$$\begin{aligned}F &= -U_t - U_x - UU_x - a_1 VV_x - a_2\frac{\partial}{\partial x}[UV] \ \in L^2(0,L),\\ G &= -V_t - V_x - VV_x - a_2 UU_x - a_1\frac{\partial}{\partial x}[UV] \ \in L^2(0,L).\end{aligned}$$

Thus, the pair $\{U, V\}$ is a weak solution of the system

$$\left\{\begin{array}{l}\begin{bmatrix} 1 & a_3 \\ a_3 & 1\end{bmatrix}\begin{bmatrix} U_{xxx} \\ V_{xxx}\end{bmatrix} = \begin{bmatrix} F(x) \\ G(x)\end{bmatrix}, \quad x \in (0,L)\\ \qquad \begin{bmatrix} U(0,t) \\ V(0,t)\end{bmatrix} = \begin{bmatrix} U(L,t) \\ V(L,t)\end{bmatrix} = \begin{bmatrix} U_x(L,t) \\ V_x(L,t)\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}.\end{array}\right. \tag{4.18}$$

We claim that the weak solution of (4.18) is uniquely defined. Indeed, by the linearity of system (4.18) we will prove that the homogeneous system in (4.18) has only the trivial solution. Taking $F = G = 0$ and

$$W = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} (1-\frac{x}{L})\int_0^x\int_0^s U(t)\,dt\,ds \\ (1-\frac{x}{L})\int_0^x\int_0^s V(t)\,dt\,ds \end{bmatrix},$$

we obtain, after performing some computations,

$$\begin{aligned} \frac{dW}{dx} &= \begin{bmatrix} \omega_{1,x} \\ \omega_{2,x} \end{bmatrix} = \begin{bmatrix} (1-\frac{x}{L})\int_0^x U(t)\,dt - \frac{1}{L}\int_0^x\int_0^s U(t)\,dt\,ds \\ (1-\frac{x}{L})\int_0^x V(t)\,dt - \frac{1}{L}\int_0^x\int_0^s V(t)\,dt\,ds \end{bmatrix} \\ \frac{d^2W}{dx^2} &= \begin{bmatrix} \omega_{1,xx} \\ \omega_{2,xx} \end{bmatrix} = \begin{bmatrix} (1-\frac{x}{L})U(x) - \frac{2}{L}\int_0^x U(t)\,dt \\ (1-\frac{x}{L})V(x) - \frac{2}{L}\int_0^x V(t)\,dt \end{bmatrix}. \end{aligned}$$

Moreover, $W(0) = W(L) = W_x(0) = 0$, which allows us to conclude that $\omega_1, \omega_2 \in \mathcal{W}$. Now, putting w_1 into the first equation in (4.17) and w_2 into the second one, we obtain

$$\begin{aligned} 0 &= \Big(U_x, (1-\frac{x}{L})U(x) - \frac{2}{L}\int_0^x U(t)\,dt\Big) \\ &+ \Big(V_x, (1-\frac{x}{L})V(x) - \frac{2}{L}\int_0^x V(t)\,dt\Big) + a_3\Big(V_x, -\frac{2}{L}\int_0^x U(t)\,dt\Big) \\ &+ a_3\Big(U_x, -\frac{2}{L}\int_0^x V(t)\,dt\Big) + a_3\Big(V_x, (1-\frac{x}{L})U\Big) + a_3\Big(U_x, (1-\frac{x}{L})V\Big). \end{aligned} \tag{4.19}$$

Using integration by parts, the terms on the right-hand side of (4.19) may be written as follows:

$$\begin{aligned} &\Big(U_x, (1-\frac{x}{L})U(x) - \frac{2}{L}\int_0^x U(t)\,dt\Big) = \frac{5}{2L}\|U\|^2, \\ &\Big(V_x, (1-\frac{x}{L})V(x) - \frac{2}{L}\int_0^x V(t)\,dt\Big) = \frac{5}{2L}\|V\|^2, \\ &a_3\Big(V_x, -\frac{2}{L}\int_0^x U(t)\,dt\Big) = a_3\Big(U_x, -\frac{2}{L}\int_0^x V(t)\,dt\Big) = \frac{2a_3}{L}(U,V), \\ &a_3\Big(V_x, (1-\frac{x}{L})U\Big) + a_3\Big(U_x, (1-\frac{x}{L})V\Big) = \frac{a_3}{L}(U,V). \end{aligned}$$

Then, from (4.19) and assumption (1.4) we deduce that

$$\begin{aligned} 0 &= \frac{5}{2L}\|U\|^2 + \frac{5}{2L}\|V\| + \frac{5a_3}{L}(U,V) \\ &\geqslant \frac{5}{2L}\|U\|^2 + \frac{5}{2L}\|V\|^2 - \frac{5}{2L}a_3\|U\|^2 - \frac{5}{2L}a_3\|V\|^2 \end{aligned}$$

$$= \frac{5}{2L}(1-a_3)\{\|U\|^2 + \|V\|^2\};$$

hence, $U(x) \equiv V(x) \equiv 0$.

Now, we define the functions $\widetilde{U}$ and $\widetilde{V}$ as follows:

$$\widetilde{U}(x) = k_1 x + k_2 x^2 + \frac{1}{2(1-a_3^2)} \int_0^x s^2 [F(s) - a_3 G(s)]\, ds$$
$$- \frac{x}{1-a_3^2} \int_0^x s[F(s) - a_3 G(s)]\, ds + \frac{x^2}{2(1-a_3^2)} \int_0^x [F(s) - a_3 G(s)]\, ds$$
$$\widetilde{V}(x) = k_3 x + k_4 x^2 + \frac{1}{2(1-a_3^2)} \int_0^x s^2 [G(s) - a_3 F(s)]\, ds$$
$$- \frac{x}{1-a_3^2} \int_0^x s[G(s) - a_3 F(s)]\, ds + \frac{x^2}{2(1-a_3^2)} \int_0^x [G(s) - a_3 F(s)]\, ds.$$

It is straightforward to see that

- $\widetilde{U}(x)$ and $\widetilde{V}(x)$ belong to $H^3(0,L)$, when $F, G \in L^2(0,L)$
- $\widetilde{U}(0) = \widetilde{V}(0) = 0$, when $F, G \in L^2(0,L)$
- $\widetilde{U}$ and $\widetilde{V}$ satisfy the system

$$\begin{cases} D_3\widetilde{U} + a_3 D_3 \widetilde{V} = F \\ a_3 D_3 \widetilde{U} + D_3 \widetilde{V} = G. \end{cases} \tag{4.20}$$

- Given $F, G \in L^2(0,L)$, the constants k_1, k_2, k_3, and k_4 can be chosen to satisfy the boundary conditions

$$\begin{cases} \widetilde{U}(L) = \widetilde{V}(L) = 0 \\ \widetilde{U}_x(L) = \widetilde{V}_x(L) = 0. \end{cases} \tag{4.21}$$

Now, multiplying system (4.20) by arbitrary $\omega \in \mathcal{W}$, integrating by parts and using condition (4.21) we obtain

$$\begin{cases} (\widetilde{U}_x + a_3 \widetilde{V}_x, \omega_{xx})(t) = (F, \omega)(t) \\ (a_3 \widetilde{U}_x + \widetilde{V}_x, \omega_{xx})(t) = (G, \omega)(t), \end{cases} \tag{4.22}$$

for almost every $t \in (0,T)$. Comparing (4.22) and (4.17), we conclude that

$$\begin{cases} \left([U_x - \widetilde{U}_x] + a_3[V_x - \widetilde{V}_x], \omega_{xx}\right)(t) = 0 \\ \left(a_3[U_x - \widetilde{U}_x] + [V_x - \widetilde{V}_x], \omega_{xx}\right)(t) = 0. \end{cases} \tag{4.23}$$

Then, by our claim about the uniqueness of system (4.23) already shown, we conclude that $U \equiv \widetilde{U}$ and $V \equiv \widetilde{V}$. Consequently, $U, V \in H^3(0,L)$, since $\widetilde{U}, \widetilde{V} \in H^3(0,L)$.

b) The limit $\{U, V\}$ takes the initial data $\{u_0, v_0\}$. In view of the convergences obtained above, $\{u_\nu, v_\nu\} \to \{U, V\}$ in $\mathcal{C}([0,T]; (L^2(0,L))^2)$, as $\nu \to 0$. Then, $\{u_0, v_0\} = \{u_\nu(x,0), v_\nu(x,0)\} \to \{U(x,0), V(x,0)\}$ in $(L^2(0,L))^2$ and, consequently, $U(x,0) = u_0(x)$ and $V(x,0) = v_0(x)$.

c) Compatibility conditions of the initial data. The functions $u_0, v_0 \in H^4(0,L) \cap H_0^1(0,L)$ given in Theorem 2.1 satisfy (2.1). Then, passing to the limit as ν goes to zero, (2.1) becomes

$$u_0(0) = u_0(L) = u_{0,x}(L) = v_0(0) = v_0(L) = v_{0,x}(L).$$

d) Boundary conditions. Obviously,

$$U(0,t) = U(L,t) = V(0,t) = V(L,t) = 0 \text{ and } U_x(L,t) = V_x(L,t) = 0,$$

are satisfied in the weak sense.

e) Uniqueness. It follows immediately from item (f) in Section 3. Indeed, it is sufficient to take $\nu = 0$ in system (3.19).

Thus, from (4.16) we can conclude that the pair $\{U, V\}$ is a strong solution of

$$\begin{cases} U_t + U_x + D_3U + a_3D_3V + UU_x + a_1VV_x + a_2\frac{\partial}{\partial x}[UV] = 0, & \text{in } Q \\ V_t + V_x + D_3V + a_3D_3U + VV_x + a_2UU_x + a_1\frac{\partial}{\partial x}[UV] = 0, & \text{in } Q \\ U(0,t) = U(L,t) = V(0,t) = V(L,t) = U_x(L,t) = V_x(L,t) = 0, & t > 0 \\ U(x,0) = u_0(x), \quad V(x,0) = v_0(x), \quad x \in (0,L). \end{cases}$$

Finally, observe that we proved the existence of regular solution to (1.5)–(1.7) when the initial data $u_0, v_0 \in H^4(0,L) \cap H_0^1(0,L)$ and satisfy (2.2). But we are interested in obtaining the result when the initial data belongs to $H^3(0,L) \cap H_0^1(0,L)$ and satisfy (2.2). For this, note that in Lemma 4.2, the term

$$\nu^2 \left\{ \|u_0\|^2_{H^4(0,L)\cap H_0^1(0,L)} + \|v_0\|^2_{H^4(0,L)\cap H_0^1(0,L)} \right\}$$

vanishes when ν goes to zero. Hence, we can approximate the initial data u_0, v_0 by elements u_0^n, v_0^n of $\in H^3(0,L) \cap H_0^1(0,L)$ such that $u_0^n, v_0^n \in H^4(0,L)$ $\cap H_0^1(0,L)$ which satisfy (2.2) as well and such that

$$\lim_{n\to\infty} u_0^n = u_0 \quad \text{and} \quad \lim_{n\to\infty} v_0^n = v_0 \text{ in } H^3(0,L) \cap H_0^1(0,L).$$

This concludes the proof Theorem 2.2. □

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References

[1] R. Adams, "Sobolev Spaces," Acad. Press, 1975.
[2] D.J. Benney, *Long waves in liquid film,* J. Math. Phys., 45 (1966), 150–155.
[3] E. Bisognin, V. Bisognin, and G.P. Menzala, *Exponential stabilization of a coupled system of Korteweg–de Vries equations with localized damping,* Adv. Diff. Eq., 8 (2003), 443–469.
[4] J. Bona, G. Ponce, J.C. Saut, and M.M. Tom, *A model system for strong interaction between internal solitary waves,* Comm. Math. Phys., 143 (1992), 287–313.
[5] E.A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," Tata MacGraw-Hill, New Delhi, 1977.
[6] M. Davila, *On the unique continuation property for a coupled system of Korteweg–de Vries equations*, Ph.D. thesis, Institute of Mathematics, Federal University of Rio de Janeiro, Brazil, 1994.
[7] J.A. Gear and R. Grimshaw, *Weak and strong interaction between internal solitary waves,* Studies in Appl. Math., 70 (1984), 235–258.
[8] B.L. Guo, *The existence and nonexistence of a global smooth solution for the initial value problem of generalized Kuramoto–Sivashinsky type equations,* J. Math. Res. Expos., 11 (1991), 57–70.
[9] D.Y. Hsieh, *Elementary mechanisms of hydrodynamic instabilities,* Acta Mech. Sinica, 10 (1994), 193–202.
[10] T. Kato, *On the Cauchy problem for the (generalized) Korteweg–de Vries equation,* Stud. Appl. Math. Adv., in Math. Suppl. Stud., 8 (1983), 93–128.
[11] Y. Kuramoto and T. Tsuzuki, *On the formation of dissipative structures in reaction-diffusion system*, Prog. Theoret. Phys., 54 (1975), 687–699.
[12] Y. Kuramoto and T. Yamada, *Turbulent state in chemical reactions,* Prog. Theoret. Phys., 56 (1982), 670.
[13] N.A. Larkin, *Korteweg–de Vries and Kuramoto–Sivashinsky equations in bounded domains,* J. Math. Anal. Appl., 297 (2004), 169–185.
[14] F. Linares and M. Panthee, *On the Cauchy problem for a coupled system of KdV equations,* Commun. Pure Appl. Anal., 3 (2004), 417–431.
[15] G. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations,* Acta Astronaut., 4 (1977), 1177–1206.
[16] G. Sivashinsky and D.M. Michelson, *On irregular wavy flow of a liquid down a vertical plane,* Prog. Theoret. Phys., 63 (1980), 2112–2114.
[17] L.H. Zhang, *Decay of solutions of the multidimensional generalized Kuramoto–Sivashinsky system,* IMA J. Appl. Math., 50 (1993), 29–42.
[18] H. Zhao and S. Tang, *Nonlinear stability and optimal decay rate for a multidimensional generalized Kuramoto–Sivashinsky system,* J. Math. Anal. Appl., 246 (2000), 423–445.