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OPTIMAL RATE OF CONVERGENCE TO THE MOTION BY MEAN CURVATURE WITH A DRIVING FORCE

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Abstract. We consider a singularly perturbed parabolic problem with a small parameter $\varepsilon > 0$. This problem can be regarded as an approximation of the motion of a hypersurface by its mean curvature with a driving force. In this paper we derive a rate of convergence of an order ε^2 for the motion of a smooth and compact hypersurface by its mean curvature with a driving force. We also consider the special case of a circle evolving by its curvature and show that our rate is optimal.

1. Introduction

It is known that the initial value problem for a singularly perturbed parabolic equation with the double well potential $\Psi(r) = (1-r^2)^2/4$ and the forcing term g,

$$\begin{cases} u_t^\varepsilon - \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2}\psi(u^\varepsilon) = \dfrac{c_0}{\varepsilon} g & \text{in } (0,T)\times\mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

provides an approximation to the motion by mean curvature with a driving force (MMCDF). Here $\varepsilon > 0$ is a small parameter, $T > 0$, $\psi(r) = \Psi'(r)$, $c_0 = \int_{-1}^1 \sqrt{\Psi(r)}dr/\sqrt{2} = \sqrt{2}/3$ and u_0^ε is a bounded and continuous function on $\mathbb{R}^N$ which will be specified later. (See Figure 1.1 for the shapes of Ψ and ψ.)

In [1] Allen and Cahn proposed the above equation with $g \equiv 0$ to describe the motion of a curved antiphase boundary of a crystalline material. Also, they gave a formal observation that the set $\Gamma^\varepsilon(t) := \{x \in \mathbb{R}^N : u^\varepsilon(t,x) = 0\}$ called the interface propagates with the normal velocity equal to its mean curvature. This phenomenon was mathematically studied by many people. Bronsard–Kohn [4], Chen [5] and de Mottoni–Schatzman [6] proved that as $\varepsilon \searrow 0$, $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ converges to the motion of a smooth and compact

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hypersurface by its mean curvature. Evans–Soner–Souganidis [7], Barles–Soner–Souganidis [2], Ilmanen [10] and Soner [12] showed the convergence of $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ to the generalized or Brakke's motion by mean curvature.

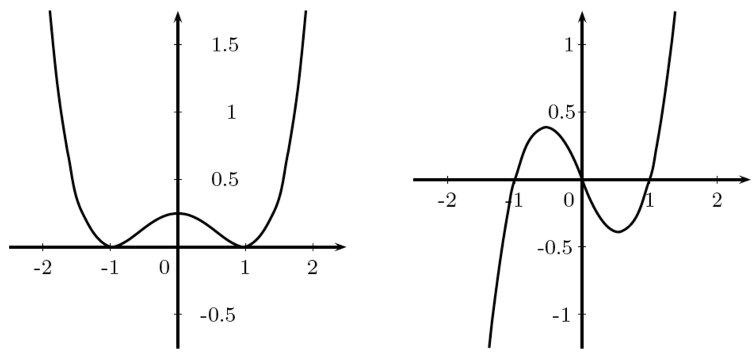

Figure 1.1 : Functions Ψ (left) and ψ (right)

From the viewpoint of an approximation of the MMCDF $\{\Gamma(t)\}_{0\le t\le T}$, it is natural to consider the rate of convergence of $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ to $\{\Gamma(t)\}_{0\le t\le T}$ as $\varepsilon \searrow 0$. The result of Chen [5] implies that the rate of convergence is an order of $\varepsilon|\log\varepsilon|$. Bellettini–Paolini [3] obtained the estimate of an order of $\varepsilon^2|\log\varepsilon|^2$. These results are valid before the onset of singularities, but not optimal.

The purpose of this paper is to derive the optimal rate of convergence of $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ to $\{\Gamma(t)\}_{0\le t\le T}$, valid before the onset of singularities, for the Hausdorff distance between $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ and $\{\Gamma(t)\}_{0\le t\le T}$. In fact, assuming that $\{\Gamma(t)\}_{0\le t<T_0}$ is the smooth and compact MMCDF, we prove that for any $T < T_0$,

$$\sup_{t\in[0,T]} d_H(\Gamma(t), \Gamma^\varepsilon(t)) \le L\varepsilon^2,$$

where L is a constant depending on T, but independent of $\varepsilon > 0$. This estimate improves the results by [5] and [3] and is optimal in the case of radial symmetry.

Both of the orders in ε and the optimality are a consequence of the maximum principle and the explicit constructions of sub- and supersolutions of (1.1), which are inspired by, and indeed rely on, the formal asymptotics of solutions of (1.1) (see, e.g., Fife [8] and Paolini–Verdi [11]). The use of the asymptotic expansion, as in [6], would lead to a rate of convergence of the interfaces by the nondegeneracy properties both of the continuous and truncated solutions. We can expect from the rigorous asymptotic analysis due to [6] that an order of ε^2 is the rate of convergence of $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ to $\{\Gamma(t)\}_{0\le t\le T}$ under some regularity assumption on $\{\Gamma(t)\}_{0\le t\le T}$. Regardless of the convergence, the formal asymptotics provide valuable information on the shape of u^ε and it plays an important role in our analysis, together with modified distance functions combined with a vertical shift and the nondegeneracy property of sub- and supersolutions of (1.1).

This paper is organized as follows. In Section 2 we prepare some notation and make some assumptions. In addition, we briefly mention the existence and comparison lemma of solutions of (1.1). In Section 3 we discuss the formal asymptotics of the solution of (1.1) as $\varepsilon \searrow 0$. Subsection 3.1 deals with the nonradial case and Subsection 3.2 does with the radial case. In Section 4, based on [3, Section 3], we mention the existence and some estimates of solutions of ordinary differential equations derived from the formal asymptotics in Section 3. In Section 5 we construct sub- and supersolutions of (1.1), using some functions provided in Sections 3 and 4. Subsection 5.2 is devoted to the nonradial case and Subsection 5.3 is to the radial case. The arguments are quite similar to those in [3, Section 6]. A different point from [3] is to introduce a correction term $\varepsilon^3 S^\varepsilon$. This point is briefly explained in Subsection 5.2. In Section 6 we state our results. In Subsection 6.1 we give a rate of convergence of $\{\Gamma^\varepsilon(t)\}_{0\le t\le T}$ to the smooth and compact MMCDF $\{\Gamma(t)\}_{0\le t\le T}$. In Subsection 6.2 we consider the special case of a circle evolving by its curvature (without a driving force) and show the optimality of our estimate.

2. Preliminaries

In this section we prepare some notation, make some assumptions and give a brief review of the classical solutions of (1.1).

Let $\{\Gamma(t)\}_{0\le t<T_0}$ be a family of smooth and compact hypersurfaces moving by its ($(N-1)$-times) mean curvature with a driving force $-g$; that is, $\Gamma(t)$ satisfies

$$V = \kappa - g \quad \text{on } \Gamma(t),\ t>0. \tag{2.1}$$

Here, $V = V(t,x)$ is the inner normal velocity of $\Gamma(t)$ at $x \in \Gamma(t)$ and $\kappa = \kappa(t,x)$ is the $(N-1)$-times mean curvature of $\Gamma(t)$ at $x \in \Gamma(t)$ with respect to the inner normal direction.

For each $t \in [0,T_0)$, the signed distance function $d = d(t,x)$ to $\Gamma(t)$ is defined by

$$d(t,x) = \begin{cases} \operatorname{dist}(x,\Gamma(t)), & x \in I(t),\\ 0, & x\in\Gamma(t),\\ -\operatorname{dist}(x,\Gamma(t)), & x \in O(t), \end{cases} \tag{2.2}$$

where $I(t)$ = inside of $\Gamma(t)$ and $O(t)$ = outside of $\Gamma(t)$. Recalling that $V = -d_t$ and $\kappa = -\Delta d$ on $\Gamma(t)$, we easily see that (2.1) is equivalent to

$$d_t = \Delta d + g \quad \text{on } \Gamma(t),\ t > 0.$$

For any $T \in (0, T_0)$ and $\delta > 0$, let $\mathcal{N}$ be the tubular neighborhood of $\{(t,x) \in [0,T] \times \mathbb{R}^N : x \in \Gamma(t)\}$:

$$\mathcal{N} := \{(t,x) \in [0,T] \times \mathbb{R}^N : |d(t,x)| \le \delta\}.$$

We make the following assumptions.

(A.1) $\{\Gamma(t)\}_{0 \le t < T_0}$ is so smooth that, for any $T < T_0$, there exists a $\delta > 0$ such that

$$d \in C^{2,4}(\mathcal{N}) \cap W^{2,4,\infty}(\mathcal{N}).$$

(A.2) g is a smooth function satisfying

$$g \in C^{2,4}((0,T_0) \times \mathbb{R}^N) \cap W^{2,4,\infty}((0,T_0) \times \mathbb{R}^N),$$
$$g_t \in W^{0,2,\infty}((0,T_0) \times \mathbb{R}^N).$$

It follows from (A.1) that for any $(t,x) \in \mathcal{N}$, there is a unique $y(t,x) \in \Gamma(t)$ such that

$$|d(t,x)| = |x - y(t,x)|.$$

Let $\lambda = \lambda(t,x)$ be the sum of the squares of all principal curvatures of $\Gamma(t)$ at $x \in \Gamma(t)$ and $\mu = \mu(t,x)$ the sum of the cubes of them. Define

$$\overline{\omega} = \overline{\omega}(t,x) := \omega(t, y(t,x)) \quad \text{for } (t,x) \in \mathcal{N} \quad (\omega = \kappa,\ \lambda,\ \mu,\ g).$$

It follows from Gilbarg–Trudinger [9, Chapter 14] that

$$-\Delta d(t,x) = \overline{\kappa}(t,x) + \overline{\lambda}(t,x) d(t,x) + \overline{\mu}(t,x)(d(t,x))^2 + \cdots \quad \text{on } \mathcal{N}. \tag{2.3}$$

Taylor expansion and the fact $Dd(t, y(t,x)) = Dd(t,x)$ on $\mathcal{N}$ yield that

$$\begin{aligned}\Big| g(t,x) - \overline{g}(t,x) - \langle D\overline{g}(t,x), Dd(t,x)\rangle d(t,x) \\ - \tfrac{1}{2}\langle D^2\overline{g}(t,x) Dd(t,x), Dd(t,x)\rangle (d(t,x))^2 \Big| \le K_g (d(t,x))^3 \quad \text{on } \mathcal{N},\end{aligned} \tag{2.4}$$

where $K_g := \sum_{i,j,k=1}^N \|g_{x_i x_j x_k}\|_{L^\infty(\mathcal{N})}$. Since $-d_t(t,x) = V(t, y(t,x)) = \overline{\kappa}(t,x) - \overline{g}(t,x)$ on $\mathcal{N}$, combining with the above expansions, we get

$$|d_t - \Delta d - c_1 d - c_2 d^2 - g| \le K_0 d^3 \quad \text{on } \mathcal{N}, \tag{2.5}$$

where

$$\begin{cases} c_1(t,x) := \overline{\lambda}(t,x) - \langle D\overline{g}(t,x), Dd(t,x)\rangle, \\ c_2(t,x) := \overline{\mu}(t,x) - \frac{1}{2}\langle D^2\overline{g}(t,x) Dd(t,x), Dd(t,x)\rangle \end{cases} \tag{2.6}$$

and K_0 is a constant depending on (A.1) and K_g. The inequality (2.5) will be used in Section 5.

We briefly review the problem (1.1). Assume that u_0^ε is bounded and continuous on $\mathbb{R}^N$. The standard theory for parabolic equations implies the existence and uniqueness of a bounded classical solution $u^\varepsilon = u^\varepsilon(t,x)$ of (1.1). Besides, the following comparison lemma holds.

Lemma 2.1. *Let u, $v \in C^{1,2}((0,T)\times\mathbb{R}^N)\cap C([0,T)\times\mathbb{R}^N)\cap L^\infty([0,T)\times\mathbb{R}^N)$ be, respectively, a subsolution and a supersolution of* (1.1) *in the classical sense. If $u(0,x)\le v(0,x)$ for all $x\in\mathbb{R}^N$, then $u\le v$ in $[0,T)\times\mathbb{R}^N$.*

Proof. Let $M=\max\{\|u\|_{L^\infty([0,T)\times\mathbb{R}^N)},\|v\|_{L^\infty([0,T)\times\mathbb{R}^N)}\}$ and choose $\lambda = 2\|\psi'\|_{L^\infty(-M,M)}/\varepsilon^2$. Set $u^\lambda(t,x)=e^{-\lambda t}u(t,x)$ and $v^\lambda(t,x)=e^{-\lambda t}v(t,x)$. Then $u^\lambda(t,x)$ and $v^\lambda(t,x)$ satisfy, respectively,

$$u_t^\lambda-\Delta u^\lambda+\lambda u^\lambda+\frac{e^{-\lambda t}}{\varepsilon^2}\psi(e^{\lambda t}u^\lambda)\le e^{-\lambda t}g \quad \text{in } (0,T)\times\mathbb{R}^N,$$
$$v_t^\lambda-\Delta v^\lambda+\lambda v^\lambda+\frac{e^{-\lambda t}}{\varepsilon^2}\psi(e^{\lambda t}v^\lambda)\ge e^{-\lambda t}g \quad \text{in } (0,T)\times\mathbb{R}^N,$$

in the classical sense and $u^\lambda(0,x)\le v^\lambda(0,x)$ for all $x\in\mathbb{R}^N$. By the choice of λ, it is easily verified that the function

$$r\mapsto \lambda r+\frac{e^{-\lambda t}}{\varepsilon^2}\psi(e^{\lambda t}r)$$

is strictly increasing. Thus we obtain the result by the standard comparison argument. □

3. Formal asymptotics

In this section we discuss the formal asymptotic expansion to the solution of (1.1) as $\varepsilon\searrow 0$, based on Paolini–Verdi [11]. Even if the presentation is only formal, it shows us several crucial aspects for the constructions of sub- and supersolutions of (1.1).

3.1. **Nonradial case.** Let $\Gamma^0(0)$ be a smooth and compact hypersurface in $\mathbb{R}^N$ and $I^0(0)$ (respectively, $O^0(0)$) the inside (respectively, the outside) of $\Gamma^0(0)$. We choose the initial data $u_0^\varepsilon(x)$ as a smooth function satisfying $\|u_0^\varepsilon\|_{L^\infty(\mathbb{R}^N)}\le 1$ and

$$u_0^\varepsilon(x)>0 \quad \text{in } I^0(0),\ u_0^\varepsilon(x)<0 \quad \text{in } O^0(0),$$
$$\Gamma^0(0)=\Gamma^\varepsilon(0)=\{x\in\mathbb{R}^N : u_0^\varepsilon(x)=0\}.$$

Let $u^\varepsilon=u^\varepsilon(t,x)$ be a solution of (1.1) and set $\Gamma^\varepsilon:=\{(t,x)\in[0,T]\times\mathbb{R}^N : x\in\Gamma^\varepsilon(t)\}$, $\Gamma^\varepsilon(t):=\{x\in\mathbb{R}^N : u^\varepsilon(t,x)=0\}$, $I^\varepsilon(t)=$ inside of $\Gamma^\varepsilon(t)$ and $O^\varepsilon(t)=$ outside of $\Gamma^\varepsilon(t)$. In addition, assume that for each $t>0$, $u^\varepsilon(t,x)>0$

in $I^\varepsilon(t)$ and $u^\varepsilon(t,x) < 0$ in $O^\varepsilon(t)$. We define the signed distance function $d^\varepsilon(t,x)$ to $\Gamma^\varepsilon(t)$ by

$$d^\varepsilon(t,x) = \begin{cases} \mathrm{dist}(x,\Gamma^\varepsilon(t)), & x \in I^\varepsilon(t), \\ 0, & x \in \Gamma^\varepsilon(t), \\ -\mathrm{dist}(x,\Gamma^\varepsilon(t)), & x \in O^\varepsilon(t). \end{cases} \tag{3.1}$$

For ε, $\delta > 0$, let $\mathcal{T}$ be the tubular neighborhood of Γ^ε:

$$\mathcal{T} := \{(t,x) \in [0,T] \times \mathbb{R}^N : x \in \mathcal{T}(t)\},\ \mathcal{T}(t) := \{x \in \mathbb{R}^N : |d^\varepsilon(t,x)| \le \delta\}.$$

We introduce the stretched variable $\rho := d^\varepsilon(t,x)/\varepsilon$. Assume that u^ε is expressed in terms of ε as follows (the inner expansion):

$$u^\varepsilon(t,x) = \sum_{i=0}^{3} \varepsilon^i U_i(t,x,\rho) + O(\varepsilon^4) \quad \text{for all } (t,x,\rho) \in \mathcal{T} \times \mathbb{R}. \tag{3.2}$$

Since $u^\varepsilon(t,x) = 0$ for all $(t,x) \in \Gamma^\varepsilon$, U_i's must satisfy

$$U_0(t,x,0) = U_1(t,x,0) = U_2(t,x,0) = U_3(t,x,0) = 0 \quad \text{for all } (t,x) \in \Gamma^\varepsilon.$$

For each $t \in [0,T]$, we suppose that $\Gamma^\varepsilon(t)$ converges to a smooth and compact hypersurface $\Gamma^0(t)$ as $\varepsilon \searrow 0$ and that d^ε is expanded as

$$d^\varepsilon(t,x) = d_0(t,x) + \varepsilon d_1(t,x) + \varepsilon^2 d_2(t,x) + O(\varepsilon^3) \quad \text{as } \varepsilon \searrow 0. \tag{3.3}$$

Here d_0 is the signed distance function to $\Gamma^0(t)$ given by (3.1) with $\varepsilon = 0$. Moreover, we assume that $\Gamma^0(t) \subset \mathcal{T}(t)$ for all $t \in [0,T]$ and that the d_k's $(k = 1,2,3)$ are smooth functions. The fact that $d^\varepsilon(0,x) = d_0(0,x)$ follows from $\Gamma^\varepsilon(0) = \Gamma^0(0)$, and hence the d_k's satisfy

$$d_k(0,\cdot) \equiv 0 \quad \text{in } \mathcal{T}(0). \tag{3.4}$$

Since

$$|Dd^\varepsilon|^2 = \sum_{k=0}^{2} \varepsilon^{2k}|Dd_k|^2 + 2\sum_{0\le k<l\le 2} \varepsilon^{k+l}\langle Dd_k, Dd_l\rangle + O(\varepsilon^3) \tag{3.5}$$

and $|Dd^\varepsilon|^2 = |Dd_0|^2 = 1$, we have

$$\langle Dd_0, Dd_1\rangle = |Dd_1|^2 + 2\langle Dd_0, Dd_2\rangle = 0 \quad \text{on } \mathcal{T}. \tag{3.6}$$

We substitute (3.2) into (1.1). Using (3.3), (3.5), (3.6) and the fact that $|Dd_0|^2 = 1$, we compute that in $\mathcal{T}$,

$$0 = u^\varepsilon_t - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}\psi(u^\varepsilon) - \frac{c_0}{\varepsilon} g = -\frac{1}{\varepsilon^2}\{U_{0,\rho\rho} - \psi(U_0)\} \tag{3.7}$$
$$- \frac{1}{\varepsilon}\{U_{1,\rho\rho} - \psi'(U_0)U_1 - (d_{0,t} - \Delta d_0)U_{0,\rho} + 2\langle DU_{0,\rho}, Dd_0\rangle + c_0 g\}$$

$$
\begin{aligned}
&- \Big\{ U_{2,\rho\rho} - \psi'(U_0)U_2 - (d_{0,t} - \Delta d_0)U_{1,\rho} + 2\langle DU_{1,\rho}, Dd_0\rangle \\
&\quad - \frac{1}{2}\psi''(U_0)U_1^2 - |Dd_1|^2 U_{0,\rho\rho} - (d_{1,t} - \Delta d_1)U_{0,\rho} \\
&\quad + 2\langle DU_{0,\rho}, Dd_1\rangle - (U_{0,t} - \Delta U_0)\Big\} \\
&- \varepsilon\Big\{ U_{3,\rho\rho} - \psi'(U_0)U_3 - (d_{0,t} - \Delta d_0)U_{2,\rho} + 2\langle DU_{2,\rho}, Dd_0\rangle \\
&\quad - \psi''(U_0)U_1U_2 - \frac{1}{6}\psi'''(U_0)U_1^3 - (d_{1,t} - \Delta d_1)U_{1,\rho} \\
&\quad + 2\langle DU_{1,\rho}, Dd_1\rangle + 2\langle Dd_1, Dd_2\rangle U_{0,\rho\rho} \\
&\quad - (d_{2,t} - \Delta d_2)U_{0,\rho} + 2\langle DU_{0,\rho}, Dd_2\rangle - (U_{1,t} - \Delta U_1)\Big\} + O(\varepsilon^2),
\end{aligned}
$$

where $U_{i,\rho} = \partial U_i/\partial\rho$, and $U_{i,\rho\rho} = \partial^2 U_i/\partial\rho^2$ $(i = 0, 1, 2, 3)$.

We compare the coefficients of ε^i $(i = -2, -1, 0, 1)$ and derive the equations for the U_i's. At first, we consider the ε^{-2}-term. We have

$$
\begin{cases}
U_{0,\rho\rho} - \psi(U_0) = 0 \quad \text{in } \mathcal{T} \times \mathbb{R}, \\
U_0(t,x,\rho) \longrightarrow \begin{cases} 1 & \text{as } \rho \to +\infty, \\ -1 & \text{as } \rho \to -\infty, \end{cases} \\
U_0(t,x,0) = 0 \quad \text{for any } (t,x) \in \Gamma^\varepsilon.
\end{cases}
\tag{3.8}
$$

(We do not explain how to determine the condition at the infinity, although it is a delicate issue that involves the matching with the outer expansion.) Then the solution of this problem is given by

$$
U_0(t,x,\rho) = P(\rho) = \tanh\left(\frac{\rho}{\sqrt{2}}\right).
$$

As for the ε^{-1}-term, we obtain

$$
\begin{cases}
U_{1,\rho\rho} - \psi'(P)U_1 = (d_{0,t} - \Delta d_0)P' - c_0 g \quad \text{in } \mathcal{T} \times \mathbb{R}, \\
U_1(t,x,0) = 0 \quad \text{for any } (t,x) \in \Gamma^\varepsilon.
\end{cases}
\tag{3.9}
$$

For this problem to be solvable, a compatibility condition between the right-hand side of the above equation and the span$\{P'\}$ must be enforced (the Fredholm alternative), where the span$\{P'\}$ is the kernel of the operator $U_{\rho\rho} - \psi'(P)U$ subject to the condition $U(t,x,\rho) \longrightarrow 0$ as $\rho \to \pm\infty$. Hence, we need

$$
(d_{0,t} - \Delta d_0)\int_{\mathbb{R}} |P'(\rho)|^2 d\rho - c_0 g \int_{\mathbb{R}} P'(\rho)d\rho = 0.
$$

Since we have chosen $c_0 = \int_{-1}^{1} \sqrt{\Psi(r)}dr/\sqrt{2} = \|P'\|^2_{L^2(\mathbb{R})}/\|P'\|_{L^1(\mathbb{R})}$, we see that

$$
d_{0,t} - \Delta d_0 = g \quad \text{on } \mathcal{T}(t),\ t > 0.
$$

This equation implies that at least formally, $\Gamma^0(t)$ moves by its mean curvature with a driving force $-g$ (cf. Section 2). Moreover, (3.9) is turned into

$$\begin{cases} U_{1,\rho\rho} - \psi'(P)U_1 = (P' - c_0)g \quad \text{in } \mathcal{T} \times \mathbb{R}, \\ U_1(t,x,0) = 0 \quad \text{for any } (t,x) \in \Gamma^\varepsilon. \end{cases} \tag{3.10}$$

By the form of the equation, we may consider $U_1(t,x,\rho) = g(t,x)Q(\rho)$. Then Q satisfies

$$\begin{cases} Q'' - \psi'(P)Q = P' - c_0 \quad \text{in } \mathbb{R}, \\ Q(0) = 0. \end{cases} \tag{3.11}$$

Once we have a solution Q of this problem, then $U_1(t,x,\rho) = g(t,x)Q(\rho)$ is the desired solution of (3.10).

We estimate $(d_{0,t} - \Delta d_0)U_{0,\rho}/\varepsilon$ and $(d_{0,t} - \Delta d_0)U_{1,\rho}$ of (3.7) in $\mathcal{T}$ in terms of d^ε, and the curvatures of $\Gamma^0(t)$, etc. In the sequel, we use the same notation as in Section 2, replacing $\Gamma(t)$ and d and $\Gamma^0(t)$ and d_0, respectively. By the same argument as in Section 2, we have

$$d_{0,t} - \Delta d_0 - g - c_1 d_0 - c_2(d_0)^2 = O((d_0)^3) \quad \text{on } \mathcal{T}.$$

Here c_1 and c_2 are given by (2.6). From (3.3), we get

$$\begin{aligned} d_{0,t} - \Delta d_0 = {} & g + c_1 d^\varepsilon + c_2(d^\varepsilon)^2 - \varepsilon(c_1 + 2c_2 d_0)d_1 \\ & - \varepsilon^2\{(c_1 + 2c_2 d_0)d_2 + d_1^2\} + O((d^\varepsilon)^3 + \varepsilon^3(d_1^3 + d_2^3)) \qquad \text{on } \mathcal{T}. \end{aligned}$$

Therefore, using $d^\varepsilon = \varepsilon\rho$, we obtain

$$\begin{aligned} \frac{1}{\varepsilon}(d_{0,t} - \Delta d_0)U_{0,\rho} = {} & \frac{g}{\varepsilon}P' + \{-(c_1 + 2c_2 d_0)d_1 + c_1\rho\}P' \\ & + \varepsilon\{-(c_1 + 2c_2 d_0)d_2 - d_1^2 + c_2\rho^2\}P' + O(\varepsilon^2) \end{aligned} \tag{3.12}$$

$$(d_{0,t} - \Delta d_0)U_{1,\rho} = g^2 Q' + \varepsilon\{-(c_1 + 2c_2 d_0)d_1 + c_1\rho\}gQ' + O(\varepsilon^2). \tag{3.13}$$

In the case of the ε^0-term, by using these estimates, we get

$$\begin{cases} U_{2,\rho\rho} - \psi'(P)U_2 = \{d_{1,t} - \Delta d_1 - (c_1 + 2c_2 d_0)d_1\}P' \\ \qquad\qquad + |Dd_1|^2 P'' + c_1\rho P' \\ \qquad\qquad + c_2 Q' + c_3\psi''(P)Q^2 \quad \text{in } \mathcal{T} \times \mathbb{R}, \\ U_2(t,x,0) = 0 \quad \text{for any } (t,x) \in \Gamma^\varepsilon, \end{cases} \tag{3.14}$$

where

$$c_3(t,x) := \tfrac{1}{2}(g(t,x))^2, \tag{3.15}$$

Noting that P'', $\rho P'$, Q' and $\psi''(P)Q^2$ are orthogonal to P' in the L^2 sense, to use the Fredholm alternative we have to solve the following equation for d_1:

$$d_{1,t} - \Delta d_1 - (c_1 + 2c_2 d_0)d_1 = 0 \quad \text{in } \mathcal{T}.$$

In view of (3.4), (3.6) and this equation, we may set $d_1 \equiv 0$ in $\mathcal{T}$. Therefore, (3.14) becomes

$$\begin{cases} U_{2,\rho\rho} - \psi'(P)U_2 = c_1\rho P' + c_2 Q' + c_3\psi''(P)Q^2 \quad \text{in } \mathcal{T}\times\mathbb{R}, \\ U_2(t,x,0) = 0 \quad \text{for any } (t,x)\in\Gamma^\varepsilon. \end{cases} \tag{3.16}$$

We consider the following problems:

$$\begin{cases} R_1'' - \psi'(P)R_1 = \rho P' \quad \text{in } \mathbb{R}, \\ R_1(\rho) \longrightarrow 0 \quad \text{as } \rho\to\pm\infty,\ R_1(0)=0, \end{cases} \tag{3.17}$$

$$\begin{cases} R_2'' - \psi'(P)R_2 = Q' \quad \text{in } \mathbb{R}, \\ R_2(\rho) \longrightarrow 0 \quad \text{as } \rho\to\pm\infty,\ R_2(0)=0, \end{cases} \tag{3.18}$$

$$\begin{cases} R_3'' - \psi'(P)R_3 = \psi''(P)Q^2 \quad \text{in } \mathbb{R}, \\ R_3(0)=0. \end{cases} \tag{3.19}$$

If these problems have solutions R_1, R_2 and R_3, respectively, then

$$U_2(t,x,\rho) = \sum_{i=1}^{3} c_i(t,x)R_i(\rho)$$

is the unique solution of (3.16). In addition, we can have

$$\varepsilon\{(d_{0,t}-\Delta d_0)U_{2,\rho} - 2\langle DU_{2,\rho}, Dd_0\rangle\} = \varepsilon\sum_{i=1}^{3}(gc_i - 2\langle Dc_i, Dd_0\rangle)R_i' + O(\varepsilon^2).$$

As for the ε-term, using (3.12), (3.13) and this estimate we see that

$$\begin{cases} U_{3,\rho\rho} - \psi'(P)U_3 = \{d_{2,t} - \Delta d_2 - (c_1 + 2c_2 d_0)d_2\}P' \\ \qquad\qquad + f_1\rho^2 P' + f_2\rho Q' + f_3 Q + \sum_{j=4}^{6} f_j R_{j-3}' \\ \qquad\qquad + \psi''(P)Q\sum_{j=7}^{9} f_j R_{j-6} + f_{10}Q^3 \quad \text{in } \mathcal{T}\times\mathbb{R}, \\ U_3(t,x,0) = 0 \quad \text{for any } (t,x)\in\Gamma^\varepsilon. \end{cases} \tag{3.20}$$

Here the f_j's $(j=1,2,\dots,10)$ are defined by

$$\begin{cases} f_1(t,x) := \overline{\mu}(t,x) - g_2(t,x), \\ f_2(t,x) := g(t,x)c_1(t,x), \\ f_3(t,x) := g_t(t,x) - \Delta g(t,x), \\ f_j(t,x) := g(t,x)c_{j-3}(t,x) - 2\langle Dc_{j-3}(t,x), Dd_0(t,x)\rangle \\ \qquad\qquad\qquad\qquad (j=4,5,6), \\ f_j(t,x) := g(t,x)c_{j-6}(t,x)\ (j=7,8,9), \\ f_{10}(t,x) := (g(t,x))^3, \end{cases} \tag{3.21}$$

and the c_i's $(i = 1, 2, 3)$ are given by (2.6), (3.15), respectively. To apply the Fredholm alternative, we need to solve the equation for d_2:

$$d_{2,t} - \Delta d_2 - (c_1 + 2c_2 d_0)d_2 = -\sum_{j=1}^{10} K_j f_j, \tag{3.22}$$

where the K_j's are given by

$$K_1 := \frac{\|\rho P'\|^2_{L^2(\mathbb{R})}}{\|P'\|^2_{L^2(\mathbb{R})}},\ K_2 := \frac{\int_{\mathbb{R}} \rho Q' P' d\rho}{\|P'\|^2_{L^2(\mathbb{R})}},\ K_3 := \frac{\int_{\mathbb{R}} Q P' d\rho}{\|P'\|^2_{L^2(\mathbb{R})}},$$
$$K_j := \frac{\int_{\mathbb{R}} R'_{j-3} P' d\rho}{\|P'\|^2_{L^2(\mathbb{R})}} \quad (j = 4, 5, 6),$$
$$K_j := \frac{\int_{\mathbb{R}} \psi''(P) Q R'_{j-6} P' d\rho}{\|P'\|^2_{L^2(\mathbb{R})}} \quad (j = 7, 8, 9)$$
$$K_{10} := \frac{\int_{\mathbb{R}} Q^3 P' d\rho}{\|P'\|^2_{L^2(\mathbb{R})}}.$$

If we have a solution of (3.22) satisfying (3.4) and (3.6), then (3.20) is turned into

$$\begin{cases} U_{3,\rho\rho} - \psi'(P)U_3 = f_1(\rho^2 - K_1)P' + f_2(\rho Q' - K_2 P') \\ \qquad\qquad + f_3(Q - K_3 P') + \sum_{j=4}^{6} f_j (R'_{j-3} - K_j P') \\ \qquad\qquad + \sum_{j=7}^{9} f_j \psi''(P) Q (R_{j-6} - K_j P') \\ \qquad\qquad + f_{10}(Q^3 - K_{10} P') \quad \text{in } \mathcal{T} \times \mathbb{R}, \\ U_3(t, x, 0) = 0 \quad \text{for any } (t, x) \in \Gamma^\varepsilon. \end{cases} \tag{3.23}$$

To construct the solution U_3 of this problem, we solve the following ODE's:

$$\begin{cases} S_1'' - \psi'(P)S_1 = (\rho^2 - K_1)P' \quad \text{in } \mathbb{R}, \\ S_1(\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty,\ S_1(0) = 0, \end{cases} \tag{3.24}$$

$$\begin{cases} S_2'' - \psi'(P)S_2 = \rho Q' - K_2 P' \quad \text{in } \mathbb{R}, \\ S_2(\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty,\ S_2(0) = 0, \end{cases} \tag{3.25}$$

$$\begin{cases} S_3'' - \psi'(P)S_3 = Q - K_3 P' \quad \text{in } \mathbb{R}, \\ S_3(0) = 0, \end{cases} \tag{3.26}$$

$$\begin{cases} S_j'' - \psi'(P)S_j = R'_{j-3} - K_j P' \quad \text{in } \mathbb{R}, \\ S_j(\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty,\ S_j(0) = 0\ (j = 4, 5, 6), \end{cases} \tag{3.27}$$

$$\begin{cases} S_j'' - \psi'(P)S_j = \psi''(P)QR_{j-6} - K_jP' \quad \text{in } \mathbb{R}, \\ S_j(\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty, \; S_j(0) = 0 \; (j = 7, 8), \end{cases} \tag{3.28}$$

$$\begin{cases} S_9'' - \psi'(P)S_9 = \psi''(P)QR_3 - K_9P' \quad \text{in } \mathbb{R}, \\ S_9(0) = 0, \end{cases} \tag{3.29}$$

$$\begin{cases} S_{10}'' - \psi'(P)S_{10} = Q^3 - K_{10}P' \quad \text{in } \mathbb{R}, \\ S_{10}(0) = 0. \end{cases} \tag{3.30}$$

Then, setting $U_3(t, x, \rho) := \sum_{j=1}^{10} f_j(t, x)S_j(\rho)$, we obtain the unique solution of (3.23).

3.2. **Radial case.** We consider the formal asymptotics of the radially symmetric solution of (1.1) under the assumptions $g \equiv 0$ and $N = 2$. The radial version of (1.1) is the following:

$$\begin{cases} u_t^\varepsilon - u_{rr}^\varepsilon - \dfrac{1}{r}u_r^\varepsilon + \dfrac{1}{\varepsilon^2}\psi(u^\varepsilon) = 0 & \text{in } (0, T) \times (0, +\infty), \\ u^\varepsilon(0, r) = u_0^\varepsilon(r) & \text{for } r \in (0, +\infty), \\ u_r^\varepsilon(t, 0) = 0 & \text{for } t \in (0, T). \end{cases} \tag{3.31}$$

We select the initial data $u_0^\varepsilon(r)$ as a smooth function satisfying $\|u_0^\varepsilon\|_{L^\infty(0,+\infty)} \leq 1$ and

$$u_0^\varepsilon(r) > 0 \; (0 \leq r < 1), \; u_0^\varepsilon(r) < 0 \; (r > 1),$$
$$\{x \in \mathbb{R}^2 : u_0^\varepsilon(|x|) = 0\} = \{x \in \mathbb{R}^2 : |x| = 1\}.$$

Let $u^\varepsilon = u^\varepsilon(t, r)$ be a solution of (3.31) and set $\Gamma^\varepsilon(t) := \{x \in \mathbb{R}^2 : u^\varepsilon(t, |x|) = 0\}$ for each $t \geq 0$. Then $\Gamma^\varepsilon(t)$ is a circle centered at the origin and we denote by $\phi^\varepsilon = \phi^\varepsilon(t)$ its radius. We define the signed distance function to $\Gamma^\varepsilon(t)$ by $d^\varepsilon(t, r) := \phi^\varepsilon(t) - r$. Put the stretched variable $\rho := d^\varepsilon(t, r)/\varepsilon$ and assume that u^ε and ϕ^ε are expressed in terms of ε near $\Gamma^\varepsilon(t)$ as follows.

$$u^\varepsilon(t, r) = \sum_{i=0}^{3} \varepsilon^i U_i(t, \rho) + O(\varepsilon^4), \; \phi^\varepsilon(t) = \sum_{j=0}^{2} \varepsilon^j \phi_j(t) + O(\varepsilon^3). \tag{3.32}$$

Since $u^\varepsilon(t, \phi^\varepsilon(t)) = 0$ for all $t > 0$, the U_i's must satisfy

$$U_0(t, 0) = U_1(t, 0) = U_2(t, 0) = U_3(t, 0) = 0 \quad \text{for all } t > 0.$$

It follows from the fact that $\Gamma^\varepsilon(0) = \{x \in \mathbb{R}^N : |x| = 1\}$ that $\phi_0(0) = 1$, $\phi_1(0) = \phi_2(0) = 0$. Since we easily see that

$$\begin{aligned} d_t^\varepsilon - \Delta d^\varepsilon &= d_t^\varepsilon - d_{rr}^\varepsilon - \frac{d_r^\varepsilon}{r} \\ &= \phi_0' + \varepsilon\phi_1' + \varepsilon^2\phi_2' + \frac{1}{\phi_0 + \varepsilon(\phi_1 - \rho) + \varepsilon^2\phi_2 + O(\varepsilon^3)} \end{aligned}$$

$$= \Big(\phi_0' + \frac{1}{\phi_0}\Big) + \varepsilon\Big(\phi_1' - \frac{\phi_1 - \rho}{\phi_0^2}\Big) + \varepsilon^2\Big(\phi_2' - \frac{\phi_2}{\phi_0^2} + \frac{\rho^2}{\phi_0^3}\Big) + O(\varepsilon^3),$$

substituting (3.32) into (3.31), we obtain

$$\begin{aligned}
0 &= u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}\psi(u^\varepsilon) \\
&= -\frac{1}{\varepsilon^2}\{U_{0,\rho\rho} - \psi(U_0)\} - \frac{1}{\varepsilon}\Big\{U_{1,\rho\rho} - \psi'(U_0)U_1 - \Big(\phi_0' + \frac{1}{\phi_0}\Big)U_{0,\rho}\Big\} \\
&- \Big\{U_{2,\rho\rho} - \psi'(U_0)U_2 - \Big(\phi_0' + \frac{1}{\phi_0}\Big)U_{1,\rho} - \frac{1}{2}\psi''(U_0)U_1^2 \\
&\qquad - \Big(\phi_1' - \frac{\phi_1 - \rho}{\phi_0^2}\Big)U_{0,\rho} - U_{0,t}\Big\} \\
&- \varepsilon\Big\{U_{3,\rho\rho} - \psi'(U_0)U_3 - \Big(\phi_0' + \frac{1}{\phi_0}\Big)U_{2,\rho} - \psi''(U_0)U_1U_2 \\
&\qquad - \frac{1}{6}\psi'''(U_0)U_1^3 - \Big(\phi_1' - \frac{\phi_1 - \rho}{\phi_0^2}\Big)U_{1,\rho} \\
&\qquad - \Big(\phi_2' - \frac{\phi_2}{\phi_0^2} + \frac{\rho^2}{\phi_0^3}\Big)U_{0,\rho} - U_{1,t}\Big\} + O(\varepsilon^2).
\end{aligned}$$

We see by the same arguments as in the previous subsection that

$$U_0(t,\rho) = P(\rho),\ U_1 \equiv 0,\ \phi_0(t) = \sqrt{1-2t}.$$

In the case of the ε^0-term, we have

$$\begin{cases} U_{2,\rho\rho} - \psi'(P)U_2 = \Big(\phi_1' - \dfrac{\phi_1 - \rho}{\phi_0^2}\Big)P' & \text{in } (0,T)\times\mathbb{R}, \\ U_2(t,\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty,\ U_2(t,0) = 0 & \text{for any } t > 0. \end{cases} \tag{3.33}$$

To apply the Fredholm alternative, we need the L^2 orthogonality between the right-hand side of the above equation and P'. From this condition we get the initial value problem for ϕ_1.

$$\phi_1' - \frac{\phi_1}{\phi_0^2} = 0,\ \phi_1(0) = 0.$$

Therefore, $\phi_1 \equiv 0$ and (3.33) is turned to

$$\begin{cases} U_{2,\rho\rho} - \psi'(P)U_2 = \dfrac{1}{\phi_0^2}\rho P' & \text{in } (0,T)\times\mathbb{R}, \\ U_2(t,\rho) \longrightarrow 0 \quad \text{as } \rho \to \pm\infty,\ U_2(t,0) = 0 & \text{for any } t > 0. \end{cases}$$

Then $U_2(t,\rho) = R_1(\rho)/(\phi_0(t))^2$ is a unique solution of this problem, where R_1 is given by (3.17).

As for the ε-term, we see that

$$\begin{cases} U_{3,\rho\rho} - \psi'(P)U_3 = \Big(\phi_2' - \dfrac{\phi_2}{\phi_0^2} + \dfrac{\rho^2}{\phi_0^3}\Big)P' \quad \text{in } (0,T)\times\mathbb{R}, \\ U_3(t,\rho) \longrightarrow 0 \quad \text{as } \rho\to\pm\infty,\ U_3(t,0)=0 \quad \text{for any } t>0. \end{cases} \tag{3.34}$$

By the same reason as above, we easily see that ϕ_2 satisfies

$$\phi_2' - \frac{\phi_2}{\phi_0^2} = -\frac{K_1}{\phi_0^3},\ \phi_2(0)=0,\ K_1 := \frac{\|\rho P'\|^2_{L^2(\mathbb{R})}}{\|P'\|^2_{L^2(\mathbb{R})}}. \tag{3.35}$$

Then we have

$$\phi_2(t) = \frac{K_1 \log\phi_0(t)}{\phi_0(t)}\ (<0). \tag{3.36}$$

Consequently, (3.34) becomes

$$\begin{cases} U_{3,\rho\rho} - \psi'(P)U_3 = \dfrac{1}{\phi_0^3}(\rho^2 - K_1)P' \quad \text{in } (0,T)\times\mathbb{R}, \\ U_3(t,\rho) \longrightarrow 0 \quad \text{as } \rho\to\pm\infty,\ U_3(t,0)=0 \quad \text{for any } t>0. \end{cases}$$

Then $U_3(t,\rho) := S_1(\rho)/(\phi_0(t))^3$ is a unique solution of this problem, where S_1 is a solution of (3.24).

4. Existence and estimates of solutions of ODE's

In the previous section we have formally derived some ordinary differential equations. However, it seems to be difficult to get the explicit solutions of these equations. In this section, based on Bellettini–Paolini [3, Section 3], we mention the existence and some estimates of their solutions. In this section and Sections 5 and 6, we denote by C various constants depending only on known ones.

Let $\mathcal{A} : H^1(\mathbb{R}) \longrightarrow H^{-1}(\mathbb{R})$ be defined by $\mathcal{A}V := V'' - \psi'(P)V$. Then we see that $\mathcal{A}$ is a Fredholm operator and that $\mathrm{Ker}(\mathcal{A}) = \mathrm{span}\{P'\}$. Hence we can show that for each $F \in L^2(\mathbb{R})$, $\mathcal{A}V = F$ has a solution $V \in H^2(\mathbb{R})$ if and only if $\int_{\mathbb{R}} FP'd\rho = 0$. For the detail, see [3, Section 3].

To obtain the solutions of (3.11), (3.19), (3.26), (3.29) and (3.30), we need the following lemmas due to [3].

Lemma 4.1. *Let $F \in L^2_{loc}(\mathbb{R})$ be a function such that there are $F^\pm \in \mathbb{R}$ with $F - F^+ \in L^2(1,+\infty)$ and $F - F^- \in L^2(-\infty,-1)$. Set $G^\pm := -F^\pm/\psi'(1)$. If $\int_{\mathbb{R}} FP'd\rho = 0$, then there exists a $G \in H^2_{loc}(\mathbb{R})$ satisfying*

$$\mathcal{A}G = F \text{ in } \mathbb{R},\ G - G^+ \in H^2(1,+\infty),\ G - G^- \in H^2(-\infty,-1). \tag{4.1}$$

Remark 4.1. (1) In [3, Lemma 3.1] we define the $G^{\pm}$ as the polynomial solutions of $(G^{\pm})'' - \psi'(1)G^{\pm} = F^{\pm}$. Since, in the case where $F^{\pm}$ are constants, the polynomial solutions of these equations must be constants, we can simplify [3, Lemma 3.1] in the above way.

(2) Once we have a solution G of (4.1), then by adding cP' with a suitable constant c, we see that $G(0) = 0$ and that the solution of (4.1) satisfying $G(0) = 0$ is unique.

Lemma 4.2. *Let $V \in H^2(\mathbb{R})$ satisfy $|\mathcal{A}V| \leq C_1(1 + |\rho|^m)P'$ in $\mathbb{R}$ for some $C_1 > 0$ and $m \in \mathbb{N} \cup \{0\}$. Then there exists a $C_2 > 0$ such that*

$$|V| \leq C_2(1 + |\rho|^{m+1})P' \quad \text{in } \mathbb{R}.$$

If, in addition, $V \in H^3(\mathbb{R})$ and it satisfies $|(\mathcal{A}V)'| \leq C_3(1 + |\rho|^m)P'$ in $\mathbb{R}$ for some $C_3 > 0$ and $m \in \mathbb{N} \cup \{0\}$, then there exists a $C_4 > 0$ such that

$$|V'| \leq C_4(1 + |\rho|^{m+1})P' \quad \text{in } \mathbb{R}.$$

It is obvious that

$$1 - P^2 = \sqrt{2}P', \ P'' = \sqrt{2}PP'', \ |1 - |P|| \leq 4P'. \tag{4.2}$$

These lemmas and estimates are used in the following part of this section.

We consider the problem (3.11). Applying Lemma 4.1 with $F = P' - c_0$ and $F^{\pm} = -c_0$, we have a unique solution $Q \in H^2_{loc}(\mathbb{R})$ of (3.11). In addition, it follows from (4.1) that

$$Q(\rho) \longrightarrow Q^{\infty} := \tfrac{1}{2}c_0 \quad \text{as } \rho \to \pm\infty.$$

Since we see by the fact that $\psi'(P) = 3P^2 - 1$ and (4.2) that

$$|\mathcal{A}(Q - Q^{\infty})| = |P' + 3\sqrt{2}Q^{\infty}P'| \leq CP' \quad \text{in } \mathbb{R},$$

we use Lemma 4.2 to obtain

$$|Q - Q^{\infty}| \leq C(1 + |\rho|)P' \quad \text{in } \mathbb{R}. \tag{4.3}$$

Furthermore, the elliptic regularity theory yields $Q \in H^3_{loc}(\mathbb{R})$ and it is observed by (4.2) that

$$|(\mathcal{A}(Q - Q^{\infty}))'| = |P'' + 3\sqrt{2}Q^{\infty}P''| \leq CP'.$$

Hence, applying Lemma 4.2 and (3.11), we get

$$|Q'|, \ |Q''| \leq C(1 + |\rho|)P' \quad \text{in } \mathbb{R}. \tag{4.4}$$

Q is even with respect to the origin because P' is even.

Next we treat the problem (3.17). Since it is clear that $\int_{\mathbb{R}} \rho|P'(\rho)|^2 d\rho = 0$, we use the Fredholm alternative to show the existence of a unique solution

$R_1 \in H^2(\mathbb{R})$ of (3.17). By elliptic regularity theory we observe that $R_1 \in H^3(\mathbb{R})$, and hence the following estimates hold by Lemma 4.2.

$$|R_1|,\ |R_1'|,\ |R_1''| \leq C(1+|\rho|^2)P' \quad \text{in } \mathbb{R}. \tag{4.5}$$

Since $\rho P'$ is odd with respect to the origin, so is R_1.

As for (3.18), by the same argument as in the case of (3.17), we get a unique solution $R_2 \in H^3(\mathbb{R})$, and it satisfies

$$|R_2|,\ |R_2'|,\ |R_2''| \leq C(1+|\rho|^2)P' \quad \text{in } \mathbb{R}. \tag{4.6}$$

Note that R_2 is odd because of the oddness of Q'.

To consider (3.19), we notice that $Q \in H^3_{loc}(\mathbb{R})$ and that (4.3) holds. Then, from Lemma 4.1 with $F = \psi''(P)Q^2$ and $F^+ = -F^- = 3c_0^2/4$, we have a unique solution $R_3 \in H^2_{loc}(\mathbb{R})$ of (3.19) satisfying

$$R_3(\rho) \longrightarrow \pm R_3^\infty \quad \text{as } \rho \to \pm\infty,\ R_3^\infty := -\frac{3c_0^2}{4} = -3(Q^\infty)^2.$$

In addition, $R_3 \in H^3_{loc}(\mathbb{R})$ and R_3 is odd because $\psi''(P)Q^2$ is odd. We derive some estimates for R_3. It is easily seen by (4.2) and (4.3) that

$$\begin{aligned}
|\mathcal{A}(R_3 - R_3^\infty)| &= |6PQ^2 - 3(3P^2-1)(Q^\infty)^2| \leq C(1+|\rho|)P' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } [1,+\infty),\\
|\mathcal{A}(R_3 + R_3^\infty)| &= |6PQ^2 + 3(3P^2-1)(Q^\infty)^2| \leq C(1+|\rho|)P' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (-\infty,-1].
\end{aligned}$$

Applying the proof of [3, Lemma 3.2], we have

$$\begin{aligned}
|R_3 - R_3^\infty| &\leq C(1+|\rho|^2)P' \quad \text{on } [1,+\infty),\\
|R_3 + R_3^\infty| &\leq C(1+|\rho|^2)P' \quad \text{on } (-\infty,-1].
\end{aligned}$$

Hence we obtain

$$||R_3| + R_3^\infty| \leq C(1+|\rho|^2)P' \quad \text{in } \mathbb{R}.$$

Furthermore, we easily observe from (4.3) and (4.4) that

$$\begin{aligned}
|(\mathcal{A}(R_3 - R_3^\infty))'| &= 6|P'Q^2 + 2PQQ' - PP'R_3^\infty| \leq C(1+|\rho|)P' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } [1,+\infty),\\
|(\mathcal{A}(R_3 + R_3^\infty))'| &= 6|P'Q^2 + 2PQQ' + PP'R_3^\infty| \leq C(1+|\rho|)P' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (-\infty,-1].
\end{aligned}$$

Thus using the proof of [3, Lemma 3.2], we get

$$|R_3'| \leq C(1+|\rho|^2)P' \quad \text{in } (-\infty,-1] \cup [1,+\infty).$$

Therefore by the equation of (3.19), (4.2) and this estimate, we obtain

$$|R_3'|,\ |R_3''| \le C(1+|\rho|^2)P' \quad \text{in } \mathbb{R}. \tag{4.7}$$

Finally, we consider the problems (3.24)–(3.30). We apply the Fredholm alternative to show the existence of unique solutions $S_j \in H^2(\mathbb{R})$ $(j=1,2,4,5,6,7,8)$ of (3.24), (3.25), (3.27), and (3.28), respectively. Furthermore, it follows from elliptic regularity theory that $S_j \in H^3(\mathbb{R})$. Thus, by Lemma 4.2 we obtain

$$|S_j|,\ |S_j'|,\ |S_j''| \le C(1+|\rho|^3)P' \quad \text{in } \mathbb{R} \quad (j=1,2,4,5,6,7,8). \tag{4.8}$$

As for (3.26) (respectively, (3.29), (3.30)), we apply Lemma 4.1 with $F = Q - K_3P'$ (respectively, $F = \psi''(P)QR_3 - K_9P'$, $F = Q^3 - K_{10}P'$) and $F^{\pm} = c_0/2$ (respectively, $F^{\pm} = -9c_0^3/4$, $F^{\pm} = c_0^3/8$) and Lemma 4.2 to obtain a unique solution $S_j \in H^2_{loc}(\mathbb{R})$ $(j=3,9,10)$ of (3.26) (respectively, (3.29), (3.30)) satisfying

$$|S_j - S_j^\infty|,\ |S_j'|,\ |S_j''| \le C(1+|\rho|^3)P' \quad \text{in } \mathbb{R}, \quad (j=3,9,10)$$
$$S_3^\infty := -\tfrac{1}{4}c_0,\ S_9^\infty := \tfrac{9}{8}c_0^3,\ S_{10}^\infty := -\tfrac{1}{16}c_0^3.$$

Note that S_j $(j=1,2,\ldots,10)$ is even. See Figure 4.1 for the rough shapes Q, R_1 and S_1.

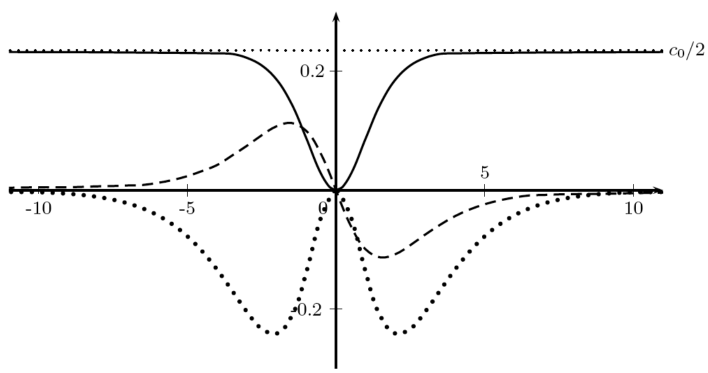

Figure 4.1 : Functions Q, R_1 (dashed curve), and S_1 (dotted curve)

5. Subsolutions and supersolutions

In this section we construct suitable sub- and supersolutions of (1.1) by utilizing the functions obtained in Sections 3 and 4. Throughout this section we fix $m \ge 4$ and set $\rho^\varepsilon = \sqrt{2}m|\log\varepsilon|$ for $\varepsilon > 0$.

5.1. **Modifications of functions.** Before the constructions of sub- and supersolutions of (1.1), we modify P, Q, R_i and S_j $(i=1,2,3,\ j=1,\dots,10)$.

Note that for any $\varepsilon\in(0,1/2)$,

$$\begin{cases} -2\varepsilon^{2m}\le P(\rho^\varepsilon)-1\le-\varepsilon^{2m},\ \dfrac{2\varepsilon^{2m}}{1+\varepsilon^{2m}}\le P'(\rho^\varepsilon)\le 2\varepsilon^{2m},\\ -2\varepsilon^{2m}\le P''(\rho^\varepsilon)\le -\frac{1}{4}\varepsilon^{2m}.\end{cases}\tag{5.1}$$

Using the estimates for Q, R_i, S_j $(i=1,2,3,\ j=1,\dots,10)$ obtained in Section 4 and the above ones, we get

$$\begin{cases} |Q(\rho^\varepsilon)-Q^\infty|,\ |Q'(\rho^\varepsilon)|,\ |Q''(\rho^\varepsilon)|\le C\varepsilon^{2m}|\log\varepsilon|,\\ |R_i(\rho^\varepsilon)-R_i^\infty|,\ |R_i'(\rho^\varepsilon)|,\ |R_i''(\rho^\varepsilon)|\le C\varepsilon^{2m}|\log\varepsilon|^2,\\ |S_j(\rho^\varepsilon)-S_j^\infty|,\ |S_j'(\rho^\varepsilon)|,\ |S_j''(\rho^\varepsilon)|\le C\varepsilon^{2m}|\log\varepsilon|^3,\end{cases}\tag{5.2}$$

where $R_1^\infty=R_2^\infty=S_j^\infty=0$ $(j=1,2,4,5,6,7,8)$.

Set $P^\infty=1$. We construct the functions $P^\varepsilon,\ Q^\varepsilon,\ R_i^\varepsilon,\ S_j^\varepsilon\in C^2(\mathbb{R})\cap W^{2,\infty}(\mathbb{R})$ satisfying

$$\begin{aligned} &P^\varepsilon=P,\ Q^\varepsilon=Q,\ R_i^\varepsilon=R_i,\ S_j^\varepsilon=S_j\quad\text{in }(-\rho^\varepsilon,\rho^\varepsilon),\\ &P^\varepsilon=P^\infty,\ Q^\varepsilon=Q^\infty,\ R_i^\varepsilon=R_i^\infty,\ S_j^\varepsilon=S_j^\infty\quad\text{in }(2\rho^\varepsilon,+\infty),\\ &P^\varepsilon=-P^\infty,\ Q^\varepsilon=Q^\infty,\ R_i^\varepsilon=-R_i^\infty,\ S_j^\varepsilon=S_j^\infty\quad\text{in }(-\infty,-2\rho^\varepsilon).\end{aligned}$$

For $F=P$, R_i $(i=1,2,3)$ and $G=Q$, S_j $(j=1,2,\dots,10)$, put

$$F^\varepsilon(\rho):=\begin{cases} F(\rho) & |\rho|<\rho^\varepsilon\\ \widetilde{F}(\rho) & \rho^\varepsilon\le\rho<2\rho^\varepsilon\\ -\widetilde{F}(-\rho) & -2\rho^\varepsilon<\rho\le-\rho^\varepsilon\\ F^\infty & \rho\ge 2\rho^\varepsilon\\ -F^\infty & \rho\le-2\rho^\varepsilon\end{cases}$$

$$G^\varepsilon(\rho):=\begin{cases} G(\rho) & |\rho|<\rho^\varepsilon\\ \widetilde{G}(\rho) & \rho^\varepsilon\le\rho<2\rho^\varepsilon\\ \widetilde{G}(-\rho) & -2\rho^\varepsilon<\rho\le-\rho^\varepsilon\\ G^\infty & |\rho|\ge 2\rho^\varepsilon.\end{cases}$$

Here $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{R}_i$ and $\widetilde{S}_j$ are defined as follows. For $V=P,Q,R_3,S_3,S_9,S_{10}$ and $W=R_1,R_2,S_j$ $(j=1,2,4,5,6,7,8)$, define

$$\begin{aligned}\widetilde{V}(\rho)=&\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^5\{6(V(\rho^\varepsilon)-V^\infty)+3\rho^\varepsilon V'(\rho^\varepsilon)+\tfrac{1}{2}(\rho^\varepsilon)^2V''(\rho^\varepsilon)\}\\ &-\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^4\{15(V(\rho^\varepsilon)-V^\infty)+7\rho^\varepsilon V'(\rho^\varepsilon)+(\rho^\varepsilon)^2V''(\rho^\varepsilon)\}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^3\{10(V(\rho^\varepsilon)-V^\infty)+4\rho^\varepsilon V'(\rho^\varepsilon)+\tfrac{1}{2}(\rho^\varepsilon)^2V''(\rho^\varepsilon)\}\\
&+V^\infty\\
\widetilde{W}(\rho)=&\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^5\left\{6W(\rho^\varepsilon)+3\rho^\varepsilon W'(\rho^\varepsilon)+\tfrac{1}{2}(\rho^\varepsilon)^2W''(\rho^\varepsilon)\right\}\\
&-\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^4\{15W(\rho^\varepsilon)+7\rho^\varepsilon W'(\rho^\varepsilon)+(\rho^\varepsilon)^2W''(\rho^\varepsilon)\},\\
&+\Big(\frac{2\rho^\varepsilon-\rho}{\rho^\varepsilon}\Big)^3\left\{10W(\rho^\varepsilon)+4\rho^\varepsilon W'(\rho^\varepsilon)+\tfrac{1}{2}(\rho^\varepsilon)^2W''(\rho^\varepsilon)\right\},
\end{aligned}
$$

for $\rho\in[\rho^\varepsilon,2\rho^\varepsilon]$. Then we observe that P^ε, Q^ε, R_i^ε, $S_j^\varepsilon\in C^2(\mathbb{R})\cap W^{2,\infty}(\mathbb{R})$ for each $\varepsilon\in(0,1/2)$, $i=1,2,3$ and $j=1,\dots,10$. Moreover, direct calculations yield that

$$
\begin{aligned}
&\|\widetilde{V}-V^\infty\|_{L^\infty(\rho^\varepsilon,2\rho^\varepsilon)}\le C\{|V^\infty-V(\rho^\varepsilon)|+\rho^\varepsilon|V'(\rho^\varepsilon)|+(\rho^\varepsilon)^2|V''(\rho^\varepsilon)|\},\\
&\|\widetilde{V}^{(k)}\|_{L^\infty(\rho^\varepsilon,2\rho^\varepsilon)}\le\frac{C}{(\rho^\varepsilon)^k}\{|V^\infty-V(\rho^\varepsilon)|+\rho^\varepsilon|V'(\rho^\varepsilon)|+(\rho^\varepsilon)^2|V''(\rho^\varepsilon)|\}\\
&\hspace{20em}(k=1,2),\\
&\|W^{(l)}\|_{L^\infty(\rho^\varepsilon,2\rho^\varepsilon)}\le\frac{C}{(\rho^\varepsilon)^l}\{|W(\rho^\varepsilon)|+\rho^\varepsilon|W'(\rho^\varepsilon)|+(\rho^\varepsilon)^2|W''(\rho^\varepsilon)|\}\\
&\hspace{20em}(l=0,1,2).
\end{aligned}
$$

Therefore, using (5.1), (5.2) and these estimates, we get

$$
\left\{
\begin{aligned}
&\||P^\varepsilon|-P^\infty\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^2,\\
&\|(P^\varepsilon)'\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|,\\
&\|(P^\varepsilon)''\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m},
\end{aligned}
\right.
\tag{5.3}
$$

$$
\left\{
\begin{aligned}
&\|Q^\varepsilon-Q^\infty\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^3,\\
&\|(Q^\varepsilon)'\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^2,\\
&\|(Q^\varepsilon)''\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|,
\end{aligned}
\right.
\tag{5.4}
$$

$$
\left\{
\begin{aligned}
&\||R_i^\varepsilon|-R_i^\infty\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^4,\\
&\|(R_i^\varepsilon)'\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^3,\\
&\|(R_i^\varepsilon)''\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^2.
\end{aligned}
\right.
\tag{5.5}
$$

$$
\left\{
\begin{aligned}
&\|S_j^\varepsilon-S_j^\infty\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^5,\\
&\|(S_j^\varepsilon)'\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^4,\\
&\|(S_j^\varepsilon)''\|_{L^\infty(\mathbb{R}\setminus(-\rho^\varepsilon,\rho^\varepsilon))}\le C\varepsilon^{2m}|\log\varepsilon|^3.
\end{aligned}
\right.
\tag{5.6}
$$

It follows from the estimates for P, Q, R_i, S_j $(i=1,2,3,\ j=1,\dots,10)$ obtained in Section 4 and the above ones that

$$\sup_{\varepsilon>0}\Big(\|P^\varepsilon\|_{W^{2,\infty}(\mathbb{R})}+\|Q^\varepsilon\|_{W^{2,\infty}(\mathbb{R})} + \sum_{i=1}^{3}\|R_i^\varepsilon\|_{W^{2,\infty}(\mathbb{R})}+\sum_{j=1}^{10}\|S_j^\varepsilon\|_{W^{2,\infty}(\mathbb{R})}\Big)<+\infty. \tag{5.7}$$

5.2. **Nonradial case.** In this subsection we construct a sub- and a supersolution of (1.1). We use the notation defined in Sections 2 and 3 throughout this subsection.

Assume (A.1) and (A.2). For $\varepsilon>0$, we slightly modify the signed distance function d defined by (2.2). Set

$$d_\pm^\varepsilon(t,x):=d(t,x)\pm\varepsilon^2\varphi(t)\quad\text{for } (t,x)\in\mathcal{N},$$

where $\varphi(t):=M_1e^{(1+K)t}$, $K:=\|c_1\|_{L^\infty(\mathcal{N})}+1$ and M_1 is chosen later, independently of $\varepsilon>0$. For any $T<T_0$, define

$$\mathcal{N}_\pm^\varepsilon:=\{(t,x)\in[0,T]\times\mathbb{R}^N:|d_\pm^\varepsilon(t,x)|<2\sqrt{2}m\varepsilon|\log\varepsilon|\},$$

Then it is seen that there exists an $\overline{\varepsilon}=\overline{\varepsilon}(m,T,\|\varphi\|_{L^\infty(0,T)})>0$ such that $\mathcal{N}_\pm^\varepsilon\subset\mathcal{N}$ for any $\varepsilon\in(0,\overline{\varepsilon})$.

Using (2.5) and the choice of K, we calculate

$$\begin{aligned} d_{-,t}^\varepsilon-\Delta d_-^\varepsilon &= d_t-\Delta d-\varepsilon^2\varphi'\le c_1d+c_2d^2+g+\kappa_0d^3-\varepsilon^2\varphi' \\ &\le c_1d_-^\varepsilon+c_2(d_-^\varepsilon)^2+g+\kappa_0(d_-^\varepsilon)^3+\varepsilon^2\overline{\varphi}\quad\text{in }\mathcal{N}, \end{aligned}\tag{5.8}$$

where c_1, c_2 are given by (2.6) and

$$\overline{\varphi}:=-M_1+\{-1+(2c_2+3K_0d_-^\varepsilon)d_-^\varepsilon-\varepsilon^2(1+3K_0d_-^\varepsilon)\varphi+\varepsilon^4K_0\varphi^2\}\varphi. \tag{5.9}$$

We specify the initial data u_0^ε of (1.1).

$$\begin{aligned} u_0^\varepsilon(x) \;:=\; & P\Big(\frac{d(0,x)}{\varepsilon}\Big)+\varepsilon g(0,x)Q\Big(\frac{d(0,x)}{\varepsilon}\Big) \\ &+\frac{\varepsilon^2(g(0,x))^2}{2}R_3\Big(\frac{d(0,x)}{\varepsilon}\Big) \end{aligned}\tag{5.10}$$

for $x\in\mathbb{R}^N$. Note that $u_0^\varepsilon\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $\Gamma(0)=\{x\in\mathbb{R}^N:u_0^\varepsilon(x)=0\}$ for all $\varepsilon>0$. Suggested by the formal asymptotics in Section 3, we define

$\underline{v}^\varepsilon = \underline{v}^\varepsilon(t,x)$ as follows. Set $\rho_-^\varepsilon = d_-^\varepsilon(t,x)/\varepsilon$.

$$\underline{v}^\varepsilon(t,x) = \begin{cases} P^\varepsilon(\rho_-^\varepsilon) + \varepsilon g(t,x) Q^\varepsilon(\rho_-^\varepsilon) + \varepsilon^2 R^\varepsilon(t,x,\rho_-^\varepsilon) \\ \quad +\varepsilon^3 \left\{ S^\varepsilon(t,x,\rho_-^\varepsilon) - M_2 \right\} \qquad \text{if } |d_-^\varepsilon(t,x)| \le 2\sqrt{2} m\varepsilon |\log\varepsilon|, \\ 1 + \varepsilon Q^\infty g(t,x) + \dfrac{\varepsilon^2}{2} R_3^\infty (g(t,x))^2 \\ \quad +\varepsilon^3 \Big\{ \displaystyle\sum_{j=3,9,10} f_j(t,x) S_j^\infty - M_2 \Big\} \\ \qquad\qquad\qquad \text{if } d_-^\varepsilon(t,x) \ge 2\sqrt{2} m\varepsilon |\log\varepsilon|, \\ -1 + \varepsilon Q^\infty g(t,x) - \dfrac{\varepsilon^2}{2} R_3^\infty (g(t,x))^2 \\ \quad +\varepsilon^3 \Big\{ \displaystyle\sum_{j=3,9,10} f_j(t,x) S_j^\infty - M_2 \Big\} \\ \qquad\qquad\qquad \text{if } d_-^\varepsilon(t,x) \le -2\sqrt{2} m\varepsilon |\log\varepsilon|. \end{cases}$$

Here M_2 is a constant selected later, independently of $\varepsilon > 0$, and R^ε, S^ε are defined by

$$R^\varepsilon(t,x,\rho) = \sum_{i=1}^{3} c_i(t,x) R_i^\varepsilon(\rho), \;\; S^\varepsilon(t,x,\rho) := \sum_{j=1}^{10} f_j(t,x) S_j^\varepsilon(\rho),$$

where the c_i's and f_j's are given by (2.6), (3.15) and (3.21), respectively. Then for any small $\varepsilon > 0$, $\underline{v}^\varepsilon \in C^{1,2}((0,T) \times \mathbb{R}^N) \cap C([0,T) \times \mathbb{R}^N) \cap L^\infty((0,T) \times \mathbb{R}^N)$.

We prove the following proposition.

Proposition 5.1. *Assume* (A.1) *and* (A.2). *Let* $T \in (0, T_0)$ *and let* u_0^ε *and* $\underline{v}^\varepsilon$ *be defined above. Then for large* M_1, M_2 *and small* $\varepsilon_2 \in (0, \overline{\varepsilon})$ *depending on* T, *we have*

$$\underline{v}_t^\varepsilon - \Delta \underline{v}^\varepsilon + \frac{1}{\varepsilon^2} \psi(\underline{v}^\varepsilon) \le \frac{c_0}{\varepsilon} g \quad \text{in } (0,T) \times \mathbb{R}^N, \tag{5.11}$$

$$\underline{v}^\varepsilon(0,x) \le u_0^\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{5.12}$$

for each $\varepsilon \in (0, \varepsilon_2)$.

We show (5.11) and (5.12) separately.

Proof of (5.12). For each $\varepsilon \in (0, 1/2)$, define $\rho := d_-^\varepsilon(0,x)/\varepsilon$ and

$$w^\varepsilon(x) \;\; := \;\; P(\rho) + \varepsilon g(0,x) Q(\rho) + \varepsilon^2 \sum_{i=1}^{3} c_i(0,x) R_i(\rho)$$

$$+\varepsilon^3\Big\{\sum_{j=1}^{10} f_j(0,x)S_j(\rho) - \frac{1}{2}M_2\Big\}.$$

To obtain (5.12), it is enough to show that there is an $\varepsilon_{2,1} \in (0,1/2)$ such that

$$w^\varepsilon(x) \le u_0^\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } \varepsilon \in (0,\varepsilon_{2,1}). \tag{5.13}$$

Indeed, it is observed by (5.3)–(5.6) that

$$\Big|\underline{v}^\varepsilon(0,x) - w^\varepsilon(x) + \frac{\varepsilon^3}{2}M_2\Big| \le C\varepsilon^{2m}|\log\varepsilon|^5.$$

Thus, if (5.13) holds, then taking $M_2 \ge 2C$, we obtain

$$\underline{v}^\varepsilon(0,x) \le w^\varepsilon(x) - \frac{\varepsilon^3}{2}M_2 + C\varepsilon^{2m}|\log\varepsilon|^5 \le u_0^\varepsilon(x)$$

for all $x \in \mathbb{R}^N$ and $\varepsilon \in (0,\varepsilon_{2,1})$.

We prove (5.13). Set $\eta = d(0,x)/\varepsilon$ $(= \rho + M_1\varepsilon)$. It follows from the definitions of w^ε and u_0^ε that

$$\begin{aligned} w^\varepsilon(x) - u_0^\varepsilon(x) &= (P(\rho) - P(\eta)) + \varepsilon g(0,x)(Q(\rho) - Q(\eta)) \\ &\quad + \varepsilon^2\Big(\sum_{i=1}^{3} c_i(0,x)R_i(\rho) - \frac{g^2(0,x)}{2}R_3(\eta)\Big) \\ &\quad + \varepsilon^3\Big(\sum_{j=1}^{10} f_j(0,x)S_j(\rho) - \frac{1}{2}M_2\Big). \end{aligned}$$

At first, choosing $M_2 \ge 4\sum_{j=1}^{10}\|f_j\|_{L^\infty(\mathcal{N})}\|S_j\|_{L^\infty(\mathbb{R})}$, we get

$$\varepsilon^3\Big(\sum_{j=1}^{10} f_j(0,x)S_j(\rho) - \frac{1}{2}M_2\Big) \le -\frac{1}{4}M_2\varepsilon^3 \tag{5.14}$$

for all $\rho \in \mathbb{R}$ and $\varepsilon \in (0,1/2)$.

We estimate $P(\rho) - P(\eta)$. By Bellettini–Paolini [3, Lemma 6.1], we can find an $\varepsilon_{2,2} \in (0,1/2)$ such that

$$\frac{1}{2}P'(\rho) \le P'(\zeta) \le 2P'(\rho) \quad \text{for all } \zeta \in [\rho,\eta] \text{ and } \varepsilon \in (0,\varepsilon_{2,2}). \tag{5.15}$$

It follows from this estimate and the fact that $P' > 0$ in $\mathbb{R}$ that

$$P(\rho) - P(\eta) \le -\frac{1}{2}P'(\rho)M_1\varepsilon < 0 \quad \text{for all } \rho \in \mathbb{R} \text{ and } \varepsilon \in (0,\varepsilon_{2,2}). \tag{5.16}$$

We use (4.4), (4.7) and (5.15) to obtain

$$|Q(\rho) - Q(\eta)| = |Q'(\rho + \theta M_1\varepsilon)|M_1\varepsilon \quad (0 < \theta < 1)$$

$$\begin{aligned} &\le C(1+|\rho|+M_1\varepsilon)P'(\rho)M_1\varepsilon \\ |R_3(\rho)-R_3(\eta)| &= |R_3'(\rho+\widetilde{\theta}M_1\varepsilon)|M_1\varepsilon \quad (0<\widetilde{\theta}<1) \\ &\le C(1+\rho^2+M_1^2\varepsilon^2)P'(\rho)M_1\varepsilon. \end{aligned}$$

Combining these estimates with (A.2), we get

$$\begin{aligned} &|g(0,x)||Q(\rho)-Q(\eta)|+\frac{(g(0,x))^2}{2}|R_3(\rho)-R_3(\eta)| \\ &\le CM_1\varepsilon(1+\rho^2+M_1^2\varepsilon^2)P'(\rho). \end{aligned} \tag{5.17}$$

Here we have used the inequality $|a|\le 1+a^2$ for $a\in\mathbb{R}$. Therefore, from (4.5), (4.6), (5.14), (5.16), (5.17) and the boundedness of c_i ($i=1,2$), we conclude that

$$\begin{aligned} w^\varepsilon(x)-u_0^\varepsilon(x) &\le -\frac{1}{2}P'(\rho)M_1\varepsilon \\ &+C\varepsilon^2\{1+|\log\varepsilon|^2+M_1(1+\rho^2+M_1^2\varepsilon^2)\}P'(\rho)-\frac{1}{4}M_2\varepsilon^3. \end{aligned} \tag{5.18}$$

Case 1. $|\rho|\le\rho^\varepsilon$.

It is easily seen from (5.18) that

$$\begin{aligned} w^\varepsilon(x)-u_0^\varepsilon(x) \le\ &\varepsilon P'(\rho)\Big[M_1\Big\{-\frac{1}{4}+C\varepsilon(1+|\log\varepsilon|^2+M_1^2\varepsilon^2)\Big\} \\ &+\Big\{-\frac{1}{4}M_1+C\varepsilon(1+|\log\varepsilon|^2)\Big\}\Big]. \end{aligned}$$

Choosing $M_1\ge 4C$ and then $\varepsilon_{2,3}\in(0,\varepsilon_{2,2})$ small, we have (5.13) for all $|\rho|\le\rho^\varepsilon$ and $\varepsilon\in(0,\varepsilon_{2,3})$.

Case 2. $\rho>|\rho^\varepsilon|$.

It follows from (5.18) and (5.3) that

$$w^\varepsilon(x)-u_0^\varepsilon(x)\le C\varepsilon^{2m+2}(M_1+1)(1+|\log\varepsilon|^2+M_1^2\varepsilon^2)-\frac{1}{4}M_2\varepsilon^3.$$

Taking $\varepsilon_{2,4}\in(0,\varepsilon_{2,2})$ sufficiently small, we have (5.13) for all $|\rho|>\rho^\varepsilon$ and $\varepsilon\in(0,\varepsilon_{2,4})$.

Setting $\varepsilon_{2,1}:=\min\{\varepsilon_{2,3},\varepsilon_{2,4}\}$, we complete the proof of (5.12). □

Before proving (5.11), we roughly explain the reason why we introduce a correction term $\varepsilon^3S^\varepsilon$. This also gives the outline of the proof of (5.11). We consider the case of $g\equiv 0$ to simplify the description. Set $\rho=d_-^\varepsilon/\varepsilon$ and assume $|d_-^\varepsilon|\le\sqrt{2}m\varepsilon|\log\varepsilon|$. Then $Q^\varepsilon=0$, $R^\varepsilon=\overline{\lambda}R_1^\varepsilon$ and $S^\varepsilon=\overline{\mu}S_1$. It follows from the same calculations below that

$$\underline{v}_t^\varepsilon-\Delta\underline{v}^\varepsilon-\frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon)$$

$$\leq \varepsilon[\overline{\mu}\{\rho^2 P' - (S_1'' - \psi'(P)S_1)\} + (-M_1 + 3\sqrt{2}M_2)P'] - \varepsilon M_2 + C\varepsilon^2|\log\varepsilon|^3.$$

We want to choose M_1 independently of $\varepsilon > 0$ to derive the optimal rate of convergence of $\Gamma^\varepsilon(t)$. For this purpose, the point is how to deal with $\overline{\mu}\rho^2 P'$ appearing from $(d^\varepsilon_{-,t} - \Delta d^\varepsilon_-)P'/\varepsilon$ and (5.8) (cf. (3.12)). Indeed, if we take $S_1 \equiv 0$, then we have to replace M_1 with $M_1|\log\varepsilon|^2$ because of the fact that $\rho = O(|\log\varepsilon|)$. Thus we cannot obtain our desired rate of convergence. However, selecting S_1 as the solution of (3.24), we get

$$\begin{aligned}&\underline{v}^\varepsilon_t - \Delta\underline{v}^\varepsilon - \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon)\\ &\leq \varepsilon(-M_1 + \|\overline{\mu}\|_{L^\infty(\mathcal{N})}K_1 + 3\sqrt{2}M_2)P' - \varepsilon M_2 + C\varepsilon^2|\log\varepsilon|^3.\end{aligned}$$

Hence, taking $M_1 > 0$ large and $\varepsilon > 0$ small, we obtain (5.11).

Proof of (5.11). Put $d^\varepsilon = d^\varepsilon_-$ and $\mathcal{N}^\varepsilon = \mathcal{N}^\varepsilon_-$ for the notational simplicity. Set $\rho := d^\varepsilon/\varepsilon$ and assume $\varepsilon \in (0, \overline{\varepsilon})$. First we prove (5.11) in $\mathcal{N}^\varepsilon$. It is observed by direct computations that in $\mathcal{N}^\varepsilon$

$$\begin{aligned}\underline{v}^\varepsilon_t - \Delta\underline{v}^\varepsilon = &-\frac{1}{\varepsilon^2}(P^\varepsilon)'' + \frac{1}{\varepsilon}(d^\varepsilon_t - \Delta d^\varepsilon)(P^\varepsilon)' - \frac{g}{\varepsilon}(Q^\varepsilon)''\\ &+(d^\varepsilon_t - \Delta d^\varepsilon)g(Q^\varepsilon)' - 2(Q^\varepsilon)'\langle Dg, Dd^\varepsilon\rangle - R^\varepsilon_{\rho\rho}\\ &+\varepsilon\{(g_t - \Delta g)Q^\varepsilon - 2\langle DR^\varepsilon_\rho, Dd^\varepsilon\rangle + (d^\varepsilon_t - \Delta d^\varepsilon)R^\varepsilon_\rho - S^\varepsilon_{\rho\rho}\}\\ &+\varepsilon^2\{R^\varepsilon_t - \Delta R^\varepsilon - 2\langle DS^\varepsilon_\rho, Dd^\varepsilon\rangle + (d^\varepsilon_t - \Delta d^\varepsilon)S^\varepsilon_\rho\}\\ &+\varepsilon^3(S^\varepsilon_t - \Delta S^\varepsilon)\\ \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) = &\frac{1}{\varepsilon^2}\psi(P^\varepsilon) + \frac{g}{\varepsilon}\psi'(P^\varepsilon)Q^\varepsilon + \psi'(P^\varepsilon)R^\varepsilon + \frac{g^2}{2}\psi''(P^\varepsilon)(Q^\varepsilon)^2\\ &+\varepsilon\{\psi'(P^\varepsilon)(S^\varepsilon - M_2) + g\psi''(P^\varepsilon)Q^\varepsilon R^\varepsilon + g^3(Q^\varepsilon)^3\} + \varepsilon^2\widetilde{\psi}.\end{aligned}$$

Here the $\widetilde{\psi}$ is the remainder term of $\psi(\underline{v}^\varepsilon)$.

Using (5.7) and (5.8), we have

$$\underline{v}^\varepsilon_t - \Delta\underline{v}^\varepsilon - \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) - \frac{c_0}{\varepsilon}g \leq J := J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$\begin{aligned}J_1 := &-\frac{1}{\varepsilon^2}((P^\varepsilon)'' - \psi(P^\varepsilon)) - \frac{g}{\varepsilon}\{(Q^\varepsilon)'' - \psi'(P^\varepsilon)Q^\varepsilon - (P^\varepsilon)' + c_0\}\\ &+\sum_{i=1}^{3} c_k\{-(R^\varepsilon_i)'' + \psi'(P^\varepsilon)R^\varepsilon_i\} + c_1\rho(P^\varepsilon)' + c_2(Q^\varepsilon)'\\ &+c_3\psi''(P^\varepsilon)(Q^\varepsilon)^2,\end{aligned}$$

$$J_2 := \varepsilon\Big\{\sum_{j=1}^{10} f_j\{-(S_j^\varepsilon)'' + \psi'(P^\varepsilon)S_j^\varepsilon\} + f_1\rho^2(P^\varepsilon)' + f_2\rho(Q^\varepsilon)' + f_3Q^\varepsilon + \sum_{j=4}^{6} f_j(R_{j-3}^\varepsilon)' + \sum_{j=7}^{9} f_j\psi''(P^\varepsilon)Q^\varepsilon R_{j-6}^\varepsilon + f_{10}(Q^\varepsilon)^3\Big\},$$

$$J_3 := \varepsilon\overline{\varphi}(P^\varepsilon)',$$

$$J_4 := -\varepsilon M_2\psi'(P^\varepsilon),$$

$$J_5 := \varepsilon^4\{g(Q^\varepsilon)' + \varepsilon R_\rho^\varepsilon + \varepsilon^2 S_\rho^\varepsilon\}\overline{\varphi} + C\varepsilon^2(1+|\rho|^3) + \varepsilon^2\widetilde{\psi}.$$

We divide our considerations into two cases.

Case 1. $|\rho| \le \rho^\varepsilon$.

In this case, $P^\varepsilon = P$, $Q^\varepsilon = Q$, $R_i^\varepsilon = R_i$ and $S_j^\varepsilon = S_j$ $(i = 1,2,3,\ j = 1,\dots,10)$. Besides, note that

$$(P^\varepsilon)' = P' > 0 \text{ in } (-\rho^\varepsilon, \rho^\varepsilon). \tag{5.19}$$

We easily see by (3.8), (3.11) and (3.17)–(3.19) that $J_1 = 0$ in $(-\rho^\varepsilon, \rho^\varepsilon)$. Moreover, we use (3.24)–(3.30) to obtain

$$J_2 \le \varepsilon P' \sum_{j=1}^{10} \|f_j\|_{L^\infty(\mathcal{N})}K_j. \tag{5.20}$$

We estimate J_3. By the definition (5.9) of $\overline{\varphi}$, the fact $|d^\varepsilon| \le \sqrt{2}m\varepsilon|\log\varepsilon|$ and (5.19), we have

$$J_3 \le \varepsilon[-M_1 + \{-1 + C(\varepsilon|\log\varepsilon| + \varepsilon^2(|\log\varepsilon|^2 + \varepsilon^2\varphi^2))\}\varphi]P' \quad \text{in } (-\rho^\varepsilon, \rho^\varepsilon). \tag{5.21}$$

Here we have used the inequality $\varepsilon|\log\varepsilon|\varphi \le |\log\varepsilon|^2 + \varepsilon^2\varphi^2$.

As for J_4, since we can find an $\varepsilon_{2,5} > 0$ such that

$$3\sqrt{2}(P^\varepsilon)' + \psi'(P^\varepsilon) \ge 1 \quad \text{in } \mathbb{R} \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,5}), \tag{5.22}$$

we obtain

$$J_4 \le \varepsilon M_2(3\sqrt{2}(P^\varepsilon)' - 1) \quad \text{in } \mathbb{R} \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,5}). \tag{5.23}$$

From (5.20), (5.21) and this estimate, we have

$$J \le \varepsilon[(-M_1 + C + 3\sqrt{2}M_2) + \{-1 + C(\varepsilon|\log\varepsilon| + \varepsilon^2(|\log\varepsilon|^2 + \varepsilon^2\varphi^2))\}\varphi]P' + J_5 - \varepsilon M_2 \qquad \text{in } (-\rho^\varepsilon, \rho^\varepsilon).$$

Choosing $M_2 \ge 1$, $M_1 \ge C + 3\sqrt{2}M_2$ and then $\varepsilon_{2,6} \in (0, \min\{\overline{\varepsilon}, \varepsilon_{2,5}\})$ small enough, we get

$$J \le J_5 - \varepsilon M_2 \quad \text{in } (-\rho^\varepsilon, \rho^\varepsilon) \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,6}).$$

Case 2. $\rho^\varepsilon < |\rho| \le 2\rho^\varepsilon$.

It follows from (5.3)–(5.6) that

$$J_1 + J_2 \le C\varepsilon^{2m-2}|\log\varepsilon|^5 \quad \text{in } \mathcal{I}^\varepsilon := (-2\rho^\varepsilon, 2\rho^\varepsilon)\backslash(-\rho^\varepsilon, \rho^\varepsilon).$$

As for J_3, we see by (5.3) that

$$J_3 \le C\varepsilon^{2m+1}|\log\varepsilon|^5[M_1 + 1 + C\{1+\varepsilon|\log\varepsilon| + \varepsilon^2(|\log\varepsilon|^2 + \varepsilon^2\varphi^2)\}]\varphi \quad \text{in } \mathcal{I}^\varepsilon$$

for small $\varepsilon > 0$. Hence combining the above estimates with (5.23), we have

$$\begin{aligned} J \quad \le \quad & C\varepsilon^{2m-2}|\log\varepsilon|^5[M_1 + 1 + 3\sqrt{2}M_2 + C\{1+\varepsilon|\log\varepsilon| \\ & +\varepsilon^2(|\log\varepsilon|^2 + \varepsilon^2\varphi^2)\}]\varphi + J_5 - \varepsilon M_2 \qquad \text{in } \mathcal{I}^\varepsilon. \end{aligned}$$

Therefore, we can find an $\varepsilon_{2,7} \in (0, \varepsilon_{2,6})$ such that

$$J \le J_5 - \frac{\varepsilon}{2}M_2 \quad \text{in } \mathcal{I}^\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,7}).$$

From Cases 1 and 2, we obtain this estimate in $(-2\rho^\varepsilon, 2\rho^\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_{2,7})$. The choices of M_1 and M_2, and (5.7) yield that $J_5 \le M\varepsilon^2|\log\varepsilon|^3$ for some $M > 0$. Hence, replacing $\varepsilon_{2,7}$ with a smaller one if necessary, we conclude that

$$J \le 0 \quad \text{in } (-2\rho^\varepsilon, 2\rho^\varepsilon) \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,7}).$$

Therefore, we have proved (5.11) in $\mathcal{N}^\varepsilon$ for any $\varepsilon \in (0, \varepsilon_{2,7})$.

We show that (5.11) holds in $((0,T)\times\mathbb{R}^N)\backslash\mathcal{N}^\varepsilon$. It is easily seen that

$$\underline{v}^\varepsilon_t - \Delta\underline{v}^\varepsilon \le C\varepsilon. \tag{5.24}$$

On the other hand, we observe by direct calculations and the choices $Q^\infty = c_0/2$ and $R_3^\infty = -3(Q^\infty)^2$ that

$$\frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) = \frac{c_0}{\varepsilon}g + \varepsilon\Big[2\sum_{j=3,9,10} f_j S_j^\infty + \{(Q^\infty)^3 - 3Q^\infty(R_3^\infty)^2\}g^3 - 2M_2\Big] + \varepsilon^2\overline{\psi}.$$

Here, $\overline{\psi}$ is the remainder term of $\psi(\underline{v}^\varepsilon)$. Hence, in view of (A.2) and (5.24), we can choose M_2 sufficiently large and then $\varepsilon_{2,8} \in (0, \overline{\varepsilon})$ sufficiently small. Thus we have

$$\underline{v}^\varepsilon_t - \Delta\underline{v}^\varepsilon + \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) \le \frac{c_0}{\varepsilon}g - \frac{\varepsilon}{2}M_2 + \varepsilon^2\overline{\psi} \le \frac{c_0}{\varepsilon}g \quad \text{for all } \varepsilon \in (0, \varepsilon_{2,8}).$$

Therefore, we obtain (5.11) in $((0,T)\times\mathbb{R}^N)\backslash\mathcal{N}^\varepsilon$ for all $\varepsilon \in (0, \varepsilon_{2,8})$. Hence we complete the proof of (5.11). □

Consequently, choosing M_1, M_2 large enough and $\varepsilon_2 = \min\{\varepsilon_{2,1}, \varepsilon_{2,7}, \varepsilon_{2,8}\}$, we complete the proof of Proposition 5.1.

We define $\rho^\varepsilon_+ := d^\varepsilon_+/\varepsilon$ and

$$\overline{v}^\varepsilon(t,x) = \begin{cases} P^\varepsilon(\rho^\varepsilon_+) + \varepsilon g(t,x) Q^\varepsilon(\rho^\varepsilon_+) + \varepsilon^2 R^\varepsilon(t,x,\rho^\varepsilon_+) \\ \quad +\varepsilon^3 \left\{ S^\varepsilon(t,x,\rho^\varepsilon_+) + M_2 \right\} \qquad \text{if } |d^\varepsilon_+(t,x)| \le 2\sqrt{2} m\varepsilon |\log\varepsilon|, \\ 1 + \varepsilon Q^\infty g(t,x) + \dfrac{\varepsilon^2}{2} R^\infty_3 (g(t,x))^2 \\ \quad +\varepsilon^3 \Big\{ \displaystyle\sum_{j=3,9,10} f_j(t,x) S^\infty_j + M_2 \Big\} \\ \qquad\qquad\qquad \text{if } d^\varepsilon_+(t,x) \ge 2\sqrt{2} m\varepsilon |\log\varepsilon|, \\ -1 + \varepsilon Q^\infty g(t,x) - \dfrac{\varepsilon^2}{2} R^\infty_3 (g(t,x))^2 \\ \quad +\varepsilon^3 \Big\{ \displaystyle\sum_{j=3,9,10} f_j(t,x) S^\infty_j + M_2 \Big\} \\ \qquad\qquad\qquad \text{if } d^\varepsilon_+(t,x) \le -2\sqrt{2} m\varepsilon |\log\varepsilon|. \end{cases}$$

Then, by the same arguments as above, we can show that $\overline{v}^\varepsilon \in C^{1,2}((0,T) \times \mathbb{R}^N) \cap C([0,T) \times \mathbb{R}^N) \cap L^\infty((0,T) \times \mathbb{R}^N)$ and that it is a supersolution of (1.1) satisfying $\overline{v}^\varepsilon(0,x) \ge u^\varepsilon_0(x)$ for all $x \in \mathbb{R}^N$ and small $\varepsilon > 0$.

5.3. **Radial case.** In this subsection we construct a sub- and a supersolution of (3.31).

Let $\Gamma(t) := \{x \in \mathbb{R}^2 : |x| = \phi_0(t)\}$ ($\phi_0(t) = \sqrt{1-2t}$). Then $\Gamma(t)$ moves by its curvature and shrinks to the origin at $t = 1/2$. Define the signed distance function d to $\Gamma(t)$ by $d(t,r) = \phi_0(t) - r$.

For $\varepsilon > 0$, we introduce the modified distance functions $d^\varepsilon_\pm(t,r)$ by

$$d^\varepsilon_\pm(t,r) := d(t,r) + \varepsilon^2 \phi_2(t) \pm \varepsilon^3 |\log\varepsilon|^3 \varphi(t),$$

where $\phi_2(t)$ is the same function as given by (3.36), $\varphi(t) := M_1 e^{(1+K)t}$, $K := 2\sqrt{2} m \|\phi_2/\phi_0^3\|_{L^\infty(0,T)} + \|\phi_0^{-2}\|_{L^\infty(0,T)} + 2^{3/2} m^3$ and M_1 is chosen later, independently of $\varepsilon > 0$. For any $T \in (0,1/2)$, set

$$\mathcal{N}^\varepsilon_\pm := \{(t,r) \in [0,T] \times \mathbb{R} : |d^\varepsilon_\pm(t,r)| \le 2\sqrt{2} m\varepsilon |\log\varepsilon|\}.$$

Then we easily see that there exists an $\widetilde{\varepsilon} = \widetilde{\varepsilon}(m,T,\|\varphi\|_{L^\infty(0,T)}) > 0$ such that $\mathcal{N}^\varepsilon_\pm \subset \mathcal{N}$ for any $\varepsilon \in (0,\widetilde{\varepsilon})$.

Put $\rho = d^\varepsilon_-/\varepsilon$. Taking $\widetilde{\varepsilon} > 0$ smaller if necessary, we observe by a calculation similar to that in subsection 3.2 that in $\mathcal{N}^\varepsilon_-$,

$$\begin{aligned} d^\varepsilon_{-,t} - d^\varepsilon_{-,rr} - \frac{d^\varepsilon_{-,r}}{r} \quad &\le \quad \phi_0' + \frac{1}{\phi_0} + \frac{\varepsilon\rho}{\phi_0^2} + \varepsilon^2 \Big(\frac{\rho^2}{\phi_0^3} - \frac{\phi_2}{\phi_0^2} + \phi_2' \Big) - \varepsilon^3 M_1 \\ &\quad + \varepsilon^{7/2} |\log\varepsilon|^3 \widetilde{\phi} \end{aligned}$$

for all $\varepsilon \in (0, \widetilde{\varepsilon})$. Here $\widetilde{\phi}$ is the remainder term. Since $\phi_0' + 1/\phi_0 = 0$ and ϕ_2 satisfies (3.35), we obtain

$$d^\varepsilon_{-,t} - d^\varepsilon_{-,rr} - \frac{d^\varepsilon_{-,r}}{r} \leq \frac{\varepsilon\rho}{\phi_0^2} + \frac{\varepsilon^2}{\phi_0^3}(\rho^2 - K_1) + \varepsilon^3(-M_1 + \sqrt{\varepsilon}|\log\varepsilon|^3\widetilde{\phi}). \quad (5.25)$$

We specify the initial data of (3.31) as follows:

$$u_0^\varepsilon(r) := P\Big(\frac{d(0,r)}{\varepsilon}\Big). \quad (5.26)$$

Then $u_0^\varepsilon \in C(\mathbb{R})$, $|u_0^\varepsilon(r)| < 1$ for all $r \in [0, +\infty)$ and $\varepsilon > 0$ and $\Gamma(0) = \{x \in \mathbb{R}^2 : u_0^\varepsilon(|x|) = 0\}$. Put $\rho^\varepsilon_- := d^\varepsilon/\varepsilon$. We define $\underline{v}^\varepsilon = \underline{v}^\varepsilon(t,r)$ by

$$\underline{v}^\varepsilon(t,r) = \begin{cases} P^\varepsilon(\rho^\varepsilon_-) + \dfrac{\varepsilon^2 R_1^\varepsilon(\rho^\varepsilon_-)}{(\phi_0(t))^2} + \dfrac{\varepsilon^3 S_1^\varepsilon(\rho^\varepsilon_-)}{(\phi_0(t))^3} - M_2\varepsilon^4|\log\varepsilon|^3 & \\ & \text{if } |d^\varepsilon_-(t,r)| \leq 2\sqrt{2}m\varepsilon|\log\varepsilon|, \\ 1 - M_2\varepsilon^4|\log\varepsilon|^3 & \text{if } d^\varepsilon_-(t,r) \geq 2\sqrt{2}m\varepsilon|\log\varepsilon|, \\ -1 - M_2\varepsilon^4|\log\varepsilon|^3 & \text{if } d^\varepsilon_-(t,r) \leq -2\sqrt{2}m\varepsilon|\log\varepsilon|, \end{cases}$$

where M_2 is a constant chosen later, independently of $\varepsilon > 0$. It is clear by (5.7) that for any small $\varepsilon > 0$, $\underline{v}^\varepsilon \in C^{1,2}((0,T) \times \mathbb{R}) \cap C([0,T) \times \mathbb{R}) \cap L^\infty((0,T) \times \mathbb{R})$.

We show the following proposition.

Proposition 5.2. *Let $T \in (0, 1/2)$ and let u_0^ε and $\underline{v}^\varepsilon$ be defined above. Then for large M_1, M_2 and small $\varepsilon_3 \in (0, \widetilde{\varepsilon})$ depending on T, we have*

$$\underline{v}^\varepsilon_t - \underline{v}^\varepsilon_{rr} - \frac{1}{r}\underline{v}^\varepsilon_r + \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) \leq 0 \quad \text{in } (0,T) \times \mathbb{R}, \quad (5.27)$$

$$\underline{v}^\varepsilon(0,r) \leq u_0^\varepsilon(r) \quad \text{for all } r \in \mathbb{R}, \quad (5.28)$$

$$\underline{v}^\varepsilon_r(t,0) = 0 \quad \text{for all } t \in (0,T). \quad (5.29)$$

for each $\varepsilon \in (0, \varepsilon_3)$.

Since we can easily check that $\underline{v}^\varepsilon$ satisfies (5.29) for any small $\varepsilon > 0$, we give only the proofs of (5.27) and (5.28).

Proof of (5.28). Since the proof is totally similar to that of (5.12), we omit the details. For each $\varepsilon \in (0, 1/2)$, put $\rho = d^\varepsilon_-(0,r)/\varepsilon$ and

$$w^\varepsilon(r) := P(\rho) + \varepsilon^2 R_1(\rho) + \varepsilon^3 S_1(\rho) - \frac{1}{2}M_2\varepsilon^4|\log\varepsilon|^3.$$

To prove (5.28), it is enough to show that there is an $\varepsilon_{3,1} \in (0, 1/2)$ such that

$$w^\varepsilon(r) \leq u_0^\varepsilon(r) \quad \text{for all } r \in [0, +\infty) \text{ and } \varepsilon \in (0, \varepsilon_{3,1}). \quad (5.30)$$

Set $\eta = d(0,r)/\varepsilon$ $(= \rho + M_1\varepsilon^2|\log\varepsilon|^3)$. It follows from the definitions of w^ε and u_0^ε that

$$w^\varepsilon(r) - u_0^\varepsilon(r) = (P(\rho) - P(\eta)) + \varepsilon^2 R_1(\rho) + \varepsilon^3 S_1(\rho) - \frac{1}{2}M_2\varepsilon^4|\log\varepsilon|^3.$$

Using (5.15) with $\eta = \rho + M_1\varepsilon^2|\log\varepsilon|^3$, we can find an $\varepsilon_{3,1} \in (0,1/2)$ such that

$$P(\rho) - P(\eta) < -\frac{1}{2}P'(\rho)M_1\varepsilon^2|\log\varepsilon|^3 \quad \text{for all } \varepsilon \in (0,\varepsilon_{3,1}).$$

By (4.5) and (4.8), we see that

$$\begin{aligned} w^\varepsilon(r) - u_0^\varepsilon(r) \quad &\leq \quad \frac{\varepsilon^2}{2}\{-M_1|\log\varepsilon|^3 P'(\rho) + C(1+|\rho|^3)P'(\rho) \\ &\qquad -M_2\varepsilon^2|\log\varepsilon|^3\} \qquad \text{for all } \rho \in \mathbb{R} \text{ and } \varepsilon \in (0,\varepsilon_{3,1}), \end{aligned}$$

where we have used $\rho^2 \leq 1 + |\rho|^3$ for all $\rho \in \mathbb{R}$. By arguments similar to those in the proof of (5.12), we obtain (5.30). Hence the proof of (5.28) is completed. □

Proof of (5.27). We easily see by the definition of $\underline{v}^\varepsilon$ that (5.27) holds in $((0,T)\times\mathbb{R})\backslash\mathcal{N}_-^\varepsilon$ for small $\varepsilon > 0$. Hence, in the sequel we prove (5.27) in $\mathcal{N}_-^\varepsilon$ for small $\varepsilon > 0$. Put $d^\varepsilon = d_-^\varepsilon$ and $\mathcal{N}^\varepsilon = \mathcal{N}_-^\varepsilon$ for the notational simplicity. Set $\rho := d^\varepsilon/\varepsilon$ and assume $\varepsilon \in (0,\widetilde{\varepsilon})$.

Direct computations yield that in $\mathcal{N}^\varepsilon$,

$$\begin{aligned} \underline{v}_t^\varepsilon - \underline{v}_{rr}^\varepsilon - \frac{1}{r}\underline{v}_r^\varepsilon = &-\frac{(P^\varepsilon)''}{\varepsilon^2} + \frac{(P^\varepsilon)'}{\varepsilon}\Big(d_t^\varepsilon - d_{rr}^\varepsilon - \frac{d_r^\varepsilon}{r}\Big) - \frac{(R_1^\varepsilon)''}{\phi_0^2} \\ &-\frac{\varepsilon(S_1^\varepsilon)''}{\phi_0^3} + \frac{\varepsilon(R_1^\varepsilon)'}{\phi_0^2}\Big(d_t^\varepsilon - d_{rr}^\varepsilon - \frac{d_r^\varepsilon}{r}\Big) \\ &+\frac{\varepsilon^2 R_1^\varepsilon}{\phi_0^4} + \frac{\varepsilon^2(S_1^\varepsilon)'}{\phi_0^3}\Big(d_t^\varepsilon - d_{rr}^\varepsilon - \frac{d_r^\varepsilon}{r}\Big) + \frac{\varepsilon^3 S_1^\varepsilon}{\phi_0^5}, \\ \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) = \frac{1}{\varepsilon^2}\psi(P^\varepsilon) &+ \frac{1}{\phi_0^2}\psi'(P^\varepsilon)R_1^\varepsilon + \frac{\varepsilon}{\phi_0^3}\psi'(P^\varepsilon)S_1^\varepsilon \\ &-M_2\varepsilon^2|\log\varepsilon|^3\psi'(P^\varepsilon) + \varepsilon^2\widetilde{\psi}. \end{aligned}$$

Here the $\widetilde{\psi}$ is the remainder term of $\psi(\underline{v}^\varepsilon)$. It follows from (5.7), (5.25) and these equalities that

$$\underline{v}_t^\varepsilon - \underline{v}_{rr}^\varepsilon - \frac{1}{r}\underline{v}_r^\varepsilon + \frac{1}{\varepsilon^2}\psi(\underline{v}^\varepsilon) \leq J := J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 := -\frac{1}{\varepsilon^2}((P^\varepsilon)'' - \psi(P^\varepsilon)) - \frac{1}{\phi_0^2}\{(R_1^\varepsilon)'' - \psi'(P^\varepsilon)R_1^\varepsilon - \rho(P^\varepsilon)'\},$$

$$-\frac{\varepsilon}{\phi_0^3}\{(S_1^\varepsilon)'' - \psi'(P^\varepsilon)S_1^\varepsilon - (\rho^2 - K_1)(P^\varepsilon)'\},$$

$$J_2 := \varepsilon^2(-M_1 + \sqrt{\varepsilon}|\log\varepsilon|^3\widetilde{\phi})(P^\varepsilon)',$$

$$J_3 := -M_2\varepsilon^2|\log\varepsilon|^3\psi'(P^\varepsilon),$$

$$J_4 := \frac{\varepsilon^4}{\phi_0^2}\left\{(R_1^\varepsilon)' + \frac{\varepsilon}{\phi_0}(S_1^\varepsilon)'\right\}(-M_1 + \sqrt{\varepsilon}|\log\varepsilon|^3\widetilde{\phi}) + C\varepsilon^2 + \varepsilon^2\widetilde{\psi}.$$

Case 1. $|\rho| \leq \rho^\varepsilon$.

In this case, $P^\varepsilon = P$, $R_1^\varepsilon = R_1$ and $S_1^\varepsilon = S_1$. Hence by (3.8), (3.17) and (3.24) we have $J_1 = 0$ in $(-\rho^\varepsilon, \rho^\varepsilon)$.

As for J_3, from (5.22) we can obtain

$$J_3 \leq M_2\varepsilon^2|\log\varepsilon|^3(3\sqrt{2}(P^\varepsilon)' - 1) \quad \text{in } \mathbb{R} \quad \text{for small } \varepsilon > 0. \tag{5.31}$$

Hence we get from this estimate

$$J \leq \varepsilon^2|\log\varepsilon|^3(-M_1 + 3\sqrt{2}M_2 + \sqrt{\varepsilon}\widetilde{\phi})P' + J_4 - M_2\varepsilon^2|\log\varepsilon|^3 \quad \text{in } (-\rho^\varepsilon, \rho^\varepsilon).$$

We choose $M_2 \geq 1$. Then since $\sup_{0<\varepsilon<1}|\widetilde{\phi}| < +\infty$, taking $M_1 \geq 3\sqrt{2}M_2 + \sup_{0<\varepsilon<1}|\widetilde{\phi}|$, we have from (5.19)

$$J \leq J_4 - M_2\varepsilon^2|\log\varepsilon|^3 \quad \text{in } (-\rho^\varepsilon, \rho^\varepsilon) \quad \text{for all } \varepsilon \in (0, \widetilde{\varepsilon}). \tag{5.32}$$

Case 2. $\rho^\varepsilon < |\rho| \leq 2\rho^\varepsilon$.

It follows from (5.3)–(5.5) that

$$J_1 + J_2 \leq C\varepsilon^{2m-2}|\log\varepsilon|^5(1 + M_1 + \sqrt{\varepsilon}\widetilde{\phi}) \quad \text{in } \mathcal{I}^\varepsilon := (-2\rho^\varepsilon, 2\rho^\varepsilon)\backslash(-\rho^\varepsilon, \rho^\varepsilon).$$

Thus, combining this estimate with (5.31) and (5.3), it is easily seen that

$$J \leq C\varepsilon^{2m-2}|\log\varepsilon|^5\{1 + 3\sqrt{2}M_2 + M_1 + \varphi + \sqrt{\varepsilon}\widetilde{\phi}\} + J_4 - M_2\varepsilon^2|\log\varepsilon|^3 \quad \text{in } \mathcal{I}^\varepsilon.$$

Taking $\varepsilon_{3,2} \in (0, \widetilde{\varepsilon})$ sufficiently small, we obtain

$$J \leq J_4 - \frac{1}{2}M_2\varepsilon^2|\log\varepsilon|^3 \quad \text{in } \mathcal{I}^\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_{3,2}).$$

From Cases 1 and 2 we conclude that this estimate holds in $(-2\rho^\varepsilon, 2\rho^\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_{3,2})$

Finally, since it follows from (5.7) that $\widetilde{\phi}$ and $\widetilde{\psi}$ are bounded in $(-2\rho^\varepsilon, 2\rho^\varepsilon)$ uniformly in $\varepsilon \in (0, 1/2)$, there exists an $M > 0$ such that $J_4 \leq M\varepsilon^2$ in $(-2\rho^\varepsilon, 2\rho^\varepsilon)$ for all $\varepsilon \in (0, 1/2)$.

Therefore we can find an $\varepsilon_{3,3} \in (0, \varepsilon_{3,2})$ such that

$$J \leq 0 \quad \text{in } \mathcal{N}^\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_{3,3}),$$

and the proof of (5.27) is completed. □

Consequently, choosing M_1, M_2 sufficiently large and $\varepsilon_3 = \min\{\varepsilon_{3,1}, \varepsilon_{3,3}\}$ we complete the proof of Proposition 5.2.

We define $\rho_+^\varepsilon = d_+(t,r)/\varepsilon$ and

$$\overline{v}^\varepsilon(t,r) = \begin{cases} P^\varepsilon(\rho_+^\varepsilon) + \dfrac{\varepsilon^2 R_1^\varepsilon(\rho_+^\varepsilon)}{(\phi_0(t))^2} + \dfrac{\varepsilon^3 S_1^\varepsilon(\rho_+^\varepsilon)}{(\phi_0(t))^3} + M_2\varepsilon^4|\log\varepsilon|^3 & \\ & \text{if } |d_-^\varepsilon(t,r)| \le 2\sqrt{2}m\varepsilon|\log\varepsilon|, \\ 1 + M_2\varepsilon^4|\log\varepsilon|^3 & \text{if } d_-^\varepsilon(t,r) \ge 2\sqrt{2}m\varepsilon|\log\varepsilon|, \\ -1 + M_2\varepsilon^4|\log\varepsilon|^3 & \text{if } d_-^\varepsilon(t,r) \le -2\sqrt{2}m\varepsilon|\log\varepsilon|. \end{cases}$$

Then, by the same arguments as above, we can show that $\overline{v}^\varepsilon \in C^{1,2}((0,T)\times\mathbb{R}) \cap C([0,T)\times\mathbb{R}) \cap L^\infty((0,T)\times\mathbb{R})$ and that it is a supersolution of (1.1) satisfying $\overline{v}^\varepsilon(0,r) \ge u_0^\varepsilon(r)$ for all $r \in \mathbb{R}$ and small $\varepsilon > 0$.

6. Results

In this section we state our results. Subsection 6.1 is devoted to the rate of convergence to the smooth and compact MMCDF and subsection 6.2 is to the optimality of our rate in the radial case.

6.1. **Rate of convergence.** Let $\{\Gamma(t)\}_{0\le t<T_0}$ be the smooth and compact MMCDF and let $d = d(t,x)$ be the signed distance function to $\Gamma(t)$ defined by (2.2). Let u_0^ε be given by (5.10) and u^ε be the unique solution of (1.1). Set $\Gamma^\varepsilon(t) = \{x \in \mathbb{R}^N : u^\varepsilon(t,x) = 0\}$.

Theorem 6.1. *Assume* (A.1) *and* (A.2)*. Then, for any* $T \in (0, T_0)$*, there exist an* $L = L(T) > 0$ *and an* $\varepsilon_0 = \varepsilon_0(T) > 0$ *such that*

$$\sup_{t\in[0,T]} d_H(\Gamma(t), \Gamma^\varepsilon(t)) \le L\varepsilon^2 \quad \textit{for all } \varepsilon \in (0,\varepsilon_0).$$

Here d_H *is the Hausdorff distance defined by*

$$d_H(A,B) := \max\left\{\sup_{x\in B} \operatorname{dist}(x,A), \sup_{x\in A} \operatorname{dist}(x,B)\right\}$$

for $A, B \subset \mathbb{R}^N$.

Proof. Let $\underline{v}^\varepsilon$ and $\overline{v}^\varepsilon$ be, respectively, a subsolution and a supersolution of (1.1) constructed in subsection 5.2. Since we have $\underline{v}^\varepsilon(0,x) \le u_0^\varepsilon(x) \le \overline{v}^\varepsilon(0,x)$ for all $x \in \mathbb{R}^N$, we apply Lemma 2.1 to obtain

$$\underline{v}^\varepsilon \le u^\varepsilon \le \overline{v}^\varepsilon \quad \text{in } [0,T]\times\mathbb{R}^N. \tag{6.1}$$

First, we prove that there exist an $\varepsilon_{0,1} > 0$ and an $L_1 > 0$ such that

$$\sup_{x\in\Gamma^\varepsilon(t)} \operatorname{dist}(x,\Gamma(t)) \le L_1\varepsilon^2 \quad \text{for all } t \in [0,T] \text{ and } \varepsilon \in (0,\varepsilon_{0,1}). \tag{6.2}$$

Fix $t \in [0,T]$ and $x \in \Gamma^\varepsilon(t)$. Then we have by (6.2)

$$\underline{v}^\varepsilon(t,x) \le u^\varepsilon(t,x) = 0 \le \overline{v}^\varepsilon(t,x).$$

In view of (A.2), (5.7) and this inequality, we can find an $\varepsilon_{0,1} \in (0,1/2)$ such that

$$P^\varepsilon\Big(\frac{d^\varepsilon_-(t,x)}{\varepsilon}\Big) \le C\varepsilon \le \frac{1}{2} \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,1}). \tag{6.3}$$

Using (5.1) and this inequality, we get

$$d^\varepsilon_-(t,x) < \sqrt{2}m\varepsilon|\log\varepsilon| \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,1}). \tag{6.4}$$

We show $d(t,x) \le C\varepsilon^2$ for small $\varepsilon > 0$. For this purpose, we may assume $d^\varepsilon_-(t,x) \ge 0$, because otherwise we would obtain $d(t,x) \le \|\varphi\|_{L^\infty(0,T)}\varepsilon^2$ for small $\varepsilon > 0$. It follows from the definition of P^ε, (6.3) and (6.4) that

$$0 \le P\Big(\frac{d^\varepsilon_-(t,x)}{\varepsilon}\Big) \le C\varepsilon\Big(\le \frac{1}{2}\Big) \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,1}).$$

Since P is smooth and strictly increasing and $P^{-1}(0) = 0$, we see that

$$\frac{d^\varepsilon_-(t,x)}{\varepsilon} \le P^{-1}(C\varepsilon) \le P^{-1}(0) + \sup_{|\rho|\le 1/2}(P^{-1})'(\rho)\cdot C\varepsilon \le C\varepsilon$$

for all $\varepsilon \in (0,\varepsilon_{0,1})$. Hence setting $L_1 \ge \|\varphi\|_{L^\infty(0,T)} + C$, we obtain

$$d(t,x) \le \varphi(t)\varepsilon^2 + C\varepsilon^2 \le L_1\varepsilon^2 \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,1}).$$

We can derive $d(t,x) \ge -L_1\varepsilon^2$ for all $\varepsilon \in (0,\varepsilon_{0,1})$ in a way similar to the above. Thus

$$\operatorname{dist}(x,\Gamma(t)) = |d(t,x)| \le L_1\varepsilon^2 \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,1}).$$

Consequently we get (6.2).

Next we show that there exist an $\varepsilon_{0,2} > 0$ and an $L_2 > 0$ such that

$$\sup_{x\in\Gamma(t)} \operatorname{dist}(x,\Gamma^\varepsilon(t)) \le L_2\varepsilon^2 \quad \text{for all } t \in [0,T] \text{ and } \varepsilon \in (0,\varepsilon_{0,2}). \tag{6.5}$$

Fix $t \in [0,T]$ and $x \in \Gamma(t)$. Take $\varepsilon_{0,3} > 0$ so small that $\|\varphi\|_{L^2(0,T)}\varepsilon \le 1/2$ for any $\varepsilon \in (0,\varepsilon_{0,3})$. Then $d^\varepsilon_-(t,x) = -\varphi(t)\varepsilon^2$ and we observe by (5.7) and the facts $P(0) = 0$, $P'(0) = 1/\sqrt{2}$ that $P(-\varphi(t)\varepsilon) \le -M_1\varepsilon/2$ for small $\varepsilon > 0$. In addition, it follows from the fact that $Q(0) = 0$ and (5.7) that $|Q(-\varphi(t)\varepsilon)| \le \|Q'\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(0,T)}\varepsilon$. Thus, replacing $\varepsilon_{0,3}$ with a smaller one if necessary, we get $|d^\varepsilon_-(t,x)| < m\varepsilon|\log\varepsilon|$ and

$$\underline{v}^\varepsilon(t,x) \le -\frac{1}{4}M_1\varepsilon \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,3}).$$

Let $n(t,x)$ be the inner unit normal of $\Gamma(t)$ at $x \in \Gamma(t)$ and set $L_{2,1} = 2\|\varphi\|_{L^\infty(0,T)}$, $y = x + L_{2,1}\varepsilon^2 n(t,x)$. Since $|d_-^\varepsilon(t,y)| < \sqrt{2}m\varepsilon|\log\varepsilon|$ for all small $\varepsilon > 0$, choosing $\varepsilon_{0,4} > 0$ small enough, we see that

$$\underline{v}^\varepsilon(t,y) \geq \frac{1}{8}L_{2,1}\varepsilon \quad \text{for all } \varepsilon \in (0,\varepsilon_{0,4}).$$

Note that $\langle D\underline{v}^\varepsilon(t, x + r\varepsilon^2 n(t,x)), n(t,x)\rangle \geq 1/4\varepsilon$ for small $\varepsilon > 0$ and any $r \in [-L_{2,1}, L_{2,1}]$. Hence we can find an $r_1 \in (0, L_{2,1})$ satisfying $\underline{v}^\varepsilon(t, x + r_1\varepsilon^2 n(t,x)) = 0$ for all $\varepsilon \in (0,\varepsilon_{0,4})$ by taking $\varepsilon_{0,4}$ smaller, if necessary.

We can also prove that there is an $r_2 \in (0, L_{2,1})$ satisfying $\overline{v}^\varepsilon(t, x - r_2\varepsilon^2 n(t,x)) = 0$ in the same way. Thus, combining these things with (6.1), we can check that $u^\varepsilon(t, x + r_3\varepsilon^2 n(t,x)) = 0$ for some $r_3 \in [-r_2, r_1]$. Therefore we have

$$\operatorname{dist}(x, \Gamma^\varepsilon(t)) \leq |(x + r_1\varepsilon^2 n(t,x)) - (x - r_2\varepsilon^2 n(t,x))| \leq 2L_{2,1}\varepsilon^2.$$

This shows (6.5) with $L_2 = 2L_{2,1}$ and $\varepsilon_{0,2} = \min\{\varepsilon_{0,3}, \varepsilon_{0,4}\}$. Setting $L = \max\{L_1, L_2\}$ and $\varepsilon_0 = \min\{\varepsilon_{0,1}, \varepsilon_{0,2}\}$, we complete the proof. □

6.2. **Optimality.** Let $\{\Gamma(t)\}_{0\leq t<1/2}$ and $d = d(t,r)$ be defined in subsection 5.3. Let u_0^ε be given by (5.26) and u^ε be the unique solution of (3.31).

Set $\Gamma^\varepsilon(t) = \{x \in \mathbb{R}^2 : u^\varepsilon(t,|x|) = 0\}$. It is easily seen that $\Gamma^\varepsilon(t)$ is a circle centered at the origin, and thus we can define its radius $\phi^\varepsilon(t)$ for each $t \geq 0$. Since $\Gamma(t)$ and $\Gamma^\varepsilon(t)$ have the same center, we easily see that

$$d_H(\Gamma(t), \Gamma^\varepsilon(t)) = |\phi^\varepsilon(t) - \phi_0(t)|.$$

Hence we have only to consider the behavior of ϕ^ε as $\varepsilon \searrow 0$.

Theorem 6.2. *For any $T \in (0,1/2)$, there exist an $L > 0$ and an $\varepsilon_1 > 0$ such that*

$$\sup_{t\in[0,T]} |\phi^\varepsilon(t) - (\phi_0(t) + \varepsilon^2\phi_2(t))| \leq L\varepsilon^3|\log\varepsilon|^3 \quad \textit{for all } \varepsilon \in (0,\varepsilon_1).$$

Here $\phi_2(t)$ is defined by (3.36).

This theorem implies that

$$d_H(\Gamma(t), \Gamma^\varepsilon(t)) = |\phi^\varepsilon(t) - \phi_0(t)| \geq \varepsilon^2|\phi_2(t)| - L\varepsilon^3|\log\varepsilon|^3 \geq \frac{\varepsilon^2}{2}\left|\phi_2\left(\tfrac{1}{4}\right)\right|$$

as $t \nearrow T$ for any $\varepsilon \in (0,\varepsilon_1)$. Thus the estimate in Theorem 6.1 is optimal to the order of ε.

Proof of Theorem 6.2. Fix $T \in (0,1/2)$. Let $\underline{v}^\varepsilon$ and $\overline{v}^\varepsilon$ be, respectively, a subsolution and a supersolution of (3.31) constructed in subsection 5.3.

Since $\underline{v}^\varepsilon(0,r) \le u_0^\varepsilon(r) \le \overline{v}^\varepsilon(0,r)$ for all $r \in \mathbb{R}$, we apply Lemma 2.1 to obtain

$$\underline{v}^\varepsilon \le u^\varepsilon \le \overline{v}^\varepsilon \quad \text{in } [0,T]\times\mathbb{R}. \tag{6.6}$$

Set

$$D_\pm^\varepsilon(t) := \{x \in \mathbb{R}^2 : |x| \le \phi_0(t) + \varepsilon^2\phi_2(t) \pm \varepsilon^3|\log\varepsilon|^3(\varphi(t)+1)\}.$$

Since $P(0)=0$ and $P'(0)=1/\sqrt{2}$, we observe that there is an $\varepsilon_{1,1}>0$ such that for any $\varepsilon \in (0,\varepsilon_{1,1})$, $t\in[0,T]$ and $x \in \partial D_+^\varepsilon(t)$,

$$\begin{aligned}\overline{v}^\varepsilon(t,|x|) &= P(-\varepsilon^2|\log\varepsilon|^3) + \frac{\varepsilon^2 R_1(-\varepsilon^2|\log\varepsilon|^3)}{(\phi_0(t))^2} + \frac{\varepsilon^3 S_1(-\varepsilon^2|\log\varepsilon|^3)}{(\phi_0(t))^3} \\ &\le -\frac{1}{2}\varepsilon^2|\log\varepsilon|^3 + C\varepsilon^2.\end{aligned}$$

Thus, replacing $\varepsilon_{1,1}$ smaller if necessary, we have $\overline{v}^\varepsilon(t,|x|)<0$ for all $t\in[0,T]$, $x\in\partial D_+^\varepsilon(t)$ and $\varepsilon\in(0,\varepsilon_{1,1})$. We can show in a similar way that $\underline{v}^\varepsilon(t,x)>0$ for all $t\in[0,T]$, $x\in\partial D_-^\varepsilon(t)$ and $\varepsilon\in(0,\varepsilon_{1,2})$, taking $\varepsilon_{1,2}$ sufficiently small. Moreover, we observe that $\overline{v}^\varepsilon(t,|x|)<0$ in $\mathbb{R}^2\backslash D_+^\varepsilon(t)$ and $\underline{v}^\varepsilon(t,|x|)>0$ in $D_-^\varepsilon(t)$ for all $t\in[0,T]$.

The inclusion $\Gamma^\varepsilon(t)\subset D_+^\varepsilon(t)\backslash D_-^\varepsilon(t)$ follows from (6.6) and the above observations. This implies that

$$|\phi^\varepsilon(t) - (\phi_0(t)+\varepsilon^2\phi_2(t))| \le \varepsilon^3|\log\varepsilon|^3(\varphi(t)+1)$$

for all $t\in[0,T]$ and $\varepsilon\in(0,\min\{\varepsilon_{1,1},\varepsilon_{1,2}\})$. Hence, taking $L=\|\varphi\|_{L^\infty(0,T)}+1$ and $\varepsilon_1=\min\{\varepsilon_{1,1},\varepsilon_{1,2}\}$, we obtain our desired result. □

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