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ON THE SOLVABILITY OF SOME NONCLASSICAL BOUNDARY-VALUE PROBLEM FOR THE LAPLACE EQUATION IN THE PLANE CORNER

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Abstract. We study the nonstationary boundary-value problem for the Poisson equation in a plane corner with a dynamic boundary condition on the one part of the corner and the Dirichlet condition on the other part. We prove one-to-one solvability of the problem in weighted Hölder spaces and obtain the corresponding coercive estimates. These estimates will be useful to solve a free boundary problem.

1. Introduction

We are interested in the existence and uniqueness of the solution $u(y,t)$ of the problem

$$\Delta_y u = f_0(y,t), (y,t) \in G_T; \quad u = f_2(y,t), (y,t) \in \overline{\Gamma_T}; \quad u(y,0) = 0, y \in \overline{G}, \tag{1.1}$$

$$r^{-\gamma}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} - h\frac{\partial u}{\partial r} = f_1(y,t),\, (y,t) \in \overline{g_T}, \tag{1.2}$$

where Δ_y is the Laplace operator with respect to (y_1,y_2), $h = \cot\alpha_1$, $\alpha_1 \in (0,\pi)$, and γ is a given constant, $\gamma > 1$.

Here, ω is a given angle and

$$\begin{aligned} G &= \{(y_1,y_2) : -y_1\tan\omega < y_2 < 0, \quad y_1 > 0\}, \qquad G_T = G \times (0,T);\\ g &= \{(y_1,y_2) : \qquad y_2 = -y_1\tan\omega, \quad y_1 > 0\}, \qquad g_T = g \times (0,T);\\ \Gamma &= \{(y_1,y_2) : y_2 = 0, \quad y_1 \in R^1\}, \qquad \Gamma_T = \Gamma \times (0,T); \end{aligned}$$

n is a unit outward normal to the domain G and (r,φ) is a polar coordinate system on the plane (y_1,y_2), with $r = (y_1^2 + y_2^2)^{1/2}$ and $\varphi = \arctan(y_2/y_1)$.

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The mathematical interest of this problem is due to the presence of dynamic condition (1.2) and the singular domain (plane corner) where problem (1.1)–(1.2) is formulated. In the case of upper half space in R^n and $f_1 = u^q$, a problem analogous to (1.1)–(1.2) has been studied recently in [1], in which a critical exponent for global existence of positive solutions has been found. As for the case of a bounded domain with a smooth boundary and nonlinear right parts in (1.1)–(1.2), these problems have been investigated in [2–5]. Note that paper [4] is devoted to the study of boundedness and a priori bounds of global solutions. Problems of regularity of classical solutions have been studied with the semigroup approach in [2, 3] and with the potential technique in [5]. We observe that problems like (1.1)–(1.2) arise there as a linearized version of free boundary problems. In the case of plane corner and $\gamma = 0$ (i.e., without the singular term at the derivative with respect to time in the dynamic boundary condition), a problem similar to (1.1)–(1.2) has been studied by V.A. Solonnikov and E.V. Frolova in [6], in which the existence and uniqueness of the solution have been proved in weighted Sobolev classes. However, we use a basically different approach to obtain the coercive estimates of the solution in Hölder spaces.

The solvability of problem (1.1)–(1.2) in the weighted Hölder classes plays a significant role in studying the free boundary problem for the Laplace equation (the Hele-Shaw problem) in the case of free and fixed boundaries forming corner points at the initial moment.

The Hele-Shaw problem is a mathematical model of the plane motion of a viscous incompressible liquid with a free boundary. Furthermore, it can be regarded as a version of the well-known Stefan problem for the Laplace equation because the conditions at the free boundary have the same form for both problems [7, 8].

Let us consider for every $t \in [0, T]$ a bounded domain $\Omega_t \subset R^2$ and its boundary consisting of two components Γ_t and γ_t, where Γ_t is a given curve and γ_t is an unknown curve (a free boundary).

The fixed and free boundaries form corners Θ_1 and Θ_2 in the neighborhood of the points O and A at the initial time $t = 0$ as in Figure 1.

It is necessary to find a function $u(y, t)$ and a shape of the free boundary γ_t satisfying the conditions

$$\begin{aligned} &\Delta_y u = 0, \qquad (y_1, y_2) \in \Omega_t, \quad t \in (0, T); \\ &u|_{\Gamma_t} = f(y, t), \quad u|_{\gamma_t} = 0, \quad V_n|_{\gamma_t} = -\mu^{-1}\tfrac{\partial u}{\partial n}, \quad \Omega_t|_{t=0} = \Omega_0. \end{aligned} \tag{1.3}$$

Here Δ_y is a Laplace operator with respect to (y_1, y_2), $f(y, t)$ is a given function, μ is a positive constant, n is the unit outward normal to Ω_t, V_n is

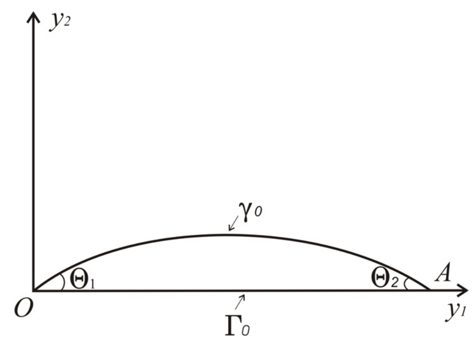

Figure 1

the velocity of the boundary γ_t in the direction of the normal, and Ω_0 is a given initial domain. We assume also that $f(y,t) > 0$, $(y,t) \in \Gamma_t/\{A,O\}$, and $f(y,t) = 0$ at the points A and O. The first condition on $f(y,t)$ ensures expansion of the domain Ω_t with respect to time [9] and guarantees well-posedness of the problem.

As noted by S.D. Howison [10], there is a vast literature on Hele-Shaw flows. In the past few years, this literature has included well-posedness proofs of the Hele-Shaw problem in various physical settings in the case of smooth initial data. The original investigation into the Hele-Shaw problem in the case of nonregular initial data like (1.3) starts with the paper [11], in which the problem has been considered when the initial free boundary is a plane corner. In particular, the comparison theory and self-similar solutions have allowed obtaining the "waiting time" in the case of acute angles. In [12], the solvability of the Hele-Shaw problem has been proved locally in time in the case of a corner point on the initial free boundary (the model problem corresponding to this case has been studied in [13]). Note that the case when the fixed boundary intersects with the free boundary has not yet been studied.

A method of proving solvability to problem (1.3) locally in time consists of reduction of this problem to a problem in a fixed domain for the two unknown functions corresponding to $u(y,t)$ and the free boundary shape. The Frechét derivative is constructed for the obtained nonlinear system of partial differential equations, and the one-valued solvability of the corresponding linear system is shown under certain conditions. After that the initial problem is reduced to the fixed-point problem for the obtained nonlinear operator. This method has been proposed by B.V. Bazaliy in [14] to study the Stefan problems in the case of regular initial data.

On this route the main analytical problems are connected with the proof of the one-to-one solvability of the corresponding linear problem. The essential part of this proof consists in the investigation of a model nonclassical boundary-value problem for a Poisson equation with a right part depending on time as a parameter and the dynamic boundary condition (1.1)–(1.2).

Our main purpose is to describe the weighted Hölder classes on which coercive estimates for the solution of model problem (1.1)–(1.2) hold and to prove these estimates, which will serve as the basis for the proof of the classical solvability of free boundary problem (1.3) with irregular boundary and for the investigation of the "waiting time" property.

The paper is organized as follows: in Section 2 we set our notation and assumptions on the data and state the main results. In Section 3, after the Fourier and Laplace transformations, we reduce problem (1.1)–(1.2) to a functional equation with a shift in the argument of the unknown function and, using some results from [6], construct the integral representation of the solution. Sections 4–6 are devoted to obtaining the coercive estimates of the solution and all its derivatives; and to prove the one-valued solvability of problem (1.1)–(1.2).

2. Statement of the problem and main results

Let us introduce a new coordinate system (x_1, x_2) in the plane corner G : $r = e^{-x_1}$, $\varphi = x_2$. Then the strip $B = \{(x_1, x_2) : x_2 \in (-\omega, 0),\, x_1 \in R^1\}$ is the image of G, $B_T = B \times (0, T)$, and $b = \{(x_1, x_2) : x_2 = -\omega,\, x_1 \in R^1\}$ is the image of g, $b_T = b \times (0, T)$, and Γ transforms to $\{(x_1, x_2) : x_2 = 0,$ $x_1 \in R^1\}$. Let $\gamma_1 = \gamma - 1$. In the new variables, problem (1.1)–(1.2) is rewritten as follows:

$$\Delta_x u = e^{-2x_1} f_0(x,t), (x,t) \in B_T; \quad u(x_1, 0, t) = f_2(x_1, t); u(x, 0) = 0, x \in \overline{B}, \tag{2.1}$$

$$e^{\gamma_1 x_1} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x_2} + h \frac{\partial u}{\partial x_1} = e^{-x_1} f_1(x_1, t),\ (x,t) \in \overline{b_T}, \tag{2.2}$$

where the original notation is preserved for the function $u(y(x), t)$ and the right-hand sides.

Let $\Omega \subset R^n$ and $\Omega_T = \Omega \times (0, T)$. Below, we use the Hölder spaces $C^{k+\alpha}(\overline{\Omega})$ [15] and the spaces $C^{k+\alpha,\beta,\delta}(\overline{\Omega_T})$, $k = \overline{0,2}$, $\alpha, \beta, \delta \in (0,1)$, with the following norm:

$$\|v\|_{C^{k+\alpha,\beta,\delta}(\overline{\Omega_T})} = \sum_{|l|=0}^{k} \Big[\max_{\overline{\Omega_T}} |D_x^l v| + \big\langle D_x^l v \big\rangle_{x,\overline{\Omega_T}}^{(\alpha)} + \big\langle D_x^l v \big\rangle_{t,\overline{\Omega_T}}^{(\beta)} + \big[D_x^l v \big]_{\overline{\Omega_T}}^{(\beta,\delta)} \Big]$$

where $l=(l_1,\dots,l_n)$, $D_x^l=\frac{\partial^{|l|}}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}}$, $|l|=l_1+\cdots+l_n$, $\langle v\rangle_{x,\overline{\Omega_T}}^{(\alpha)}$ and $\langle v\rangle_{t,\overline{\Omega_T}}^{(\beta)}$ are the Hölder constants with respect to the variables x and t, respectively, and

$$[v]_{\overline{\Omega_T}}^{(\beta,\delta)}=\sup_{\substack{x,z,t,\tau\\ x\neq z,t\neq\tau}}\frac{|v(x,t)-v(z,t)-v(x,\tau)+v(z,\tau)|}{|x-z|^{\delta}|t-\tau|^{\beta}} \tag{2.3}$$

is the seminorm introduced in [16]. We investigate problem (1.1)–(1.2) in the weighted spaces $E_s^{k+\alpha,\beta,\delta}$.

We say that $u(y,t)\in E_s^{k+\alpha,\beta,\delta}(\overline{G_T})$ (respectively, $u(y)\in E_s^{k+\alpha}(\overline{G})$) if $v(x,t)=e^{sx_1}u(y(x),t)\in C^{k+\alpha,\beta,\delta}(\overline{B_T})$ (respectively, $v(x)=e^{sx_1}u(y(x))\in C^{k+\alpha}(\overline{B})$).

We will use the space $E_{s,0}^{k+\alpha,\beta,\delta}(\overline{\Omega_T})$ or $C_0^{k+\alpha,\beta,\delta}(\overline{\Omega_T})$ for the function $u(y,t)\in E_s^{k+\alpha,\beta,\delta}(\overline{\Omega_T})$ or $u(y,t)\in C^{k+\alpha,\beta,\delta}(\overline{\Omega_T})$ if $u(y,0)=0$ $y\in\overline{\Omega}$. We assume that

$$f_0(y,t)\in E_{s,0}^{\alpha,\beta,\alpha}(\overline{G_T}),\quad f_1(y,t)\in E_{s+1,0}^{1+\alpha,\beta,\alpha}(\overline{g_T}),\quad f_2(y,t)\in E_{s+2,0}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T}) \tag{2.4}$$

for some real number $s>0$, and

$$f_0=0,\quad f_1=0,\quad f_2=0\quad\text{if either } t\leqslant 0 \text{ or } |y|>R_0. \tag{2.5}$$

Theorem 2.1. *Let $\alpha,\beta\in(0,1)$, $\omega\in(0,\pi)$, and f_0, f_1, and f_2 be as in (2.4) and (2.5) with $s\in(0,\frac{\alpha_1}{\omega}-2-\varepsilon)$, $0<\varepsilon<1$. Then there exists a unique solution $u(y,t)\in E_{s+2,0}^{2+\alpha,\beta,\alpha}(\overline{G_T})$ of problem (1.1)–(1.2) such that*

$$\begin{aligned}&\|u\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{G_T})}+\left\|r^{-\gamma_1}\frac{\partial u}{\partial t}\right\|_{E_{s+1}^{1+\alpha,\beta,\alpha}(\overline{g_T})}\\&\le c_1(\|f_0\|_{E_s^{\alpha,\beta,\alpha}(\overline{G_T})}+\|f_1\|_{E_{s+1}^{1+\alpha,\beta,\alpha}(\overline{g_T})}+\|f_2\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})}).\end{aligned} \tag{2.6}$$

If the functions $f_0(y,t)$ and $f_2(y,t)$ can be represented as

$$\begin{aligned}f_0(y,t)&=f_{01}(y,t)f_{02}(y,t)+f_{01}^*(y,t)f_0^*(y),\\ f_2(y,t)&=f_{21}(y,t)f_{22}(y,t)+f_{21}^*(y,t)f_2^*(y),\end{aligned} \tag{2.7}$$

where $f_{01}(y,t)\in C_0^{\alpha,\beta,\alpha}(\overline{G_T})$, $f_{02}(y,t)\in E_{s,0}^{\alpha,\beta,\alpha}(\overline{G_T})$, $f_{01}^*(y,t)\in C_0^{\alpha,\overline{\beta},\alpha}(\overline{G_T})$, $f_{02}^*(y)\in E^{\alpha}(\overline{G})$, $f_{21}(y,t)\in C_0^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})$, $f_{22}(y,t)\in E_{s+2,0}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})$, $f_{21}^*(y,t)\in C_0^{2+\alpha,\overline{\beta},\alpha}(\overline{\Gamma_T})$, and $f_{22}^*(y)\in E_{s+2}^{2+\alpha}(\overline{\Gamma})$, $0<\beta<\overline{\beta}<1$. Then the following inequality holds:

$$\|u\|_{E_{s+1}^{1+\alpha,\beta,\alpha}(\overline{G_T})}\le c_2T^{\delta_1}(C(f_0)+C(f_2)+\max\{1,R_0^{\beta\gamma_1}\}\,\|f_1\|_{E_{s+1}^{1+\alpha,\beta,\alpha}(\overline{g_T})}) \tag{2.8}$$

with $\delta_1 = \min\{\beta, \overline{\beta} - \beta\}$, *and*

$C(f_0) = \|f_{01}\|_{C^{\alpha,\beta,\alpha}(\overline{G_T})} \|f_{02}\|_{E_s^{\alpha,\beta,\alpha}(\overline{G_T})} + \|f_{01}^*\|_{C^{\alpha,\overline{\beta},\alpha}(\overline{G_T})} \|f_{02}^*\|_{E_s^{\alpha}(\overline{G})}$;

$C(f_2) = \|f_{21}\|_{C^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})} \|f_{22}\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})} + \|f_{21}^*\|_{C^{2+\alpha,\overline{\beta},\alpha}(\overline{\Gamma_T})} \|f_{22}^*\|_{E_{s+2}^{2+\alpha}(\overline{\Gamma})}$;

c_i, $i = 1, 2$, *are positive constants independent of* f_0, f_1, *and* f_2.

Note that inequality (2.8) is weaker then (2.6), since it does not give an estimate on $D_y^2 u$.

3. The integral representation of the solution to problem (2.1)–(2.2)

Let us consider the auxiliary problem

$$\Delta_y v = f_0(y,t), \quad (y,t) \in G_T; \quad v(y,t) = f_2(y,t), \quad (y,t) \in \overline{\Gamma_T};$$
$$v(y,0) = 0, \quad y \in G; \quad v|_{\overline{g_T}} = 0. \tag{3.1}$$

Grivard's results (see Chapter 6, Section 4, [17]) allow us to obtain that $v(y,t) \in E_{s+2}^{2+\alpha}(\overline{G})$ for every $t \in [0,T]$ if $f_0(y,t) \in E_s^{\alpha}(\overline{G})$ and $f_2(y,t) \in E_{s+2}^{2+\alpha}(\overline{\Gamma})$. The corresponding smoothness of the function $v(y,t)$ with respect to the parameter t is defined with the properties of the functions $f_0(y,t)$ and $f_2(y,t)$. The presence of a seminorm like (2.3) in the norms of $E_s^{\alpha,\beta,\alpha}(\overline{G_T})$ and $E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})$ makes it possible to prove the estimate

$$\|v\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{G_T})} \leq const.(\|f_0\|_{E_s^{\alpha,\beta,\alpha}(\overline{G_T})} + \|f_2\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{\Gamma_T})}). \tag{3.2}$$

This inequality means that $D_y v|_{\overline{g_T}} \in E_{s+1}^{1+\alpha,\beta,\alpha}(\overline{g_T})$. Moreover, if $f_i(y,t)$, $i = 0, 2$, satisfy (2.7), inequality (3.2) will lead to the estimate

$$\|v\|_{E_{s+2}^{2+\alpha,\beta,\alpha}(\overline{G_T})} \leq const. T^{\delta_1}(C(f_0) + C(f_2)). \tag{3.3}$$

Thus we can study problems (1.1)–(1.2) or (2.1)–(2.2) with $f_i(y,t) \equiv 0$, $i = 0, 2$.

We denote by $\widehat{u}(\lambda, x_2, \sigma)$ the Fourier and Laplace transformations of the function $u(x_1,\ x_2,\ t)$ with respect to the variables x_1 and t, respectively. Then

$$\frac{d^2\widehat{u}}{dx_2^2} - \lambda^2 \widehat{u} = 0, \quad x_2 \in (-\omega, 0); \quad \widehat{u}(\lambda, 0, \sigma) = 0,$$
$$\sigma \widehat{u}(\lambda + i\gamma_1, -\omega, \sigma) - \frac{d\widehat{u}}{dx_2}(\lambda, -\omega, \sigma) + ih\lambda \widehat{u}(\lambda, -\omega, \sigma) = \widehat{f}_1(\lambda - i, \sigma). \tag{3.4}$$

We conclude that

$$\widehat{u}(\lambda, x_2, \sigma) = m(\lambda, \sigma) \sinh(\lambda x_2),$$

and it follows from the last condition in (3.4) that the function $m(\lambda,\sigma)$ is the solution of the equation

$$\sigma m(\lambda+i\gamma_1,\sigma)\sinh(\omega(\lambda+i\gamma_1))+\lambda m(\lambda,\sigma)\sinh(\omega\lambda)[ih-\coth(\lambda\omega)] \\ =-\widehat{f}_1(\lambda-i,\sigma).$$

Let us denote $c(\lambda,\sigma):=m(\lambda,\sigma)\sinh(\lambda\omega)=-\widehat{u}(\lambda,-\omega,\sigma)$; then we have

$$\sigma c(\lambda+i\gamma_1,\sigma)+\lambda c(\lambda,\sigma)[ih-\coth(\lambda\omega)]=-\widehat{f}_1(\lambda-i,\sigma). \tag{3.5}$$

Thus, problem (2.1)–(2.2) is reduced to studying the properties of the solution of the functional equation (3.5).

We introduce a change of variables $\lambda=i\gamma_1 p$, $\theta=\gamma_1\omega$, $\sigma_1=\sigma/\gamma_1$, $g(p,\sigma)=\frac{-1}{\gamma_1}\widehat{f}_1(i\gamma_1(p-1)-i,\sigma)$, $v(p,\sigma)=c(i\gamma_1 p,\sigma)$. Then (3.5) reads as

$$\sigma_1 v(p+1,\sigma)-\Omega(p)v(p,\sigma)=g(p+1,\sigma), \qquad \Omega(p)=p[h-\cot(p\theta)]. \tag{3.6}$$

Equations like (3.6) have been studied thoroughly in [6], in which the integral representation of the solution for problem (3.6) was obtained. Here we use essentially this result. Let $\Gamma(p)$ be the gamma function and

$$L(p)=A(p)\sin(p-\frac{\alpha_1}{\theta})\pi;\quad \alpha_n=\frac{\pi n-\alpha_1}{\theta};\quad \beta_n=\frac{\pi n+\alpha_1}{\theta};\quad \eta_n=\frac{\pi n}{\theta},$$

$$n=1,2,\dots;\quad A(p)=\Gamma(p-\frac{\alpha_1}{\theta})\alpha_1^{-p+1/2}\prod_{n=1}^{\infty}\frac{\Gamma(p+\alpha_n)\Gamma(1-p+\eta_n)}{\Gamma(1-p+\beta_n)\Gamma(p+\eta_n)} \\ \times\Big(\frac{\eta_n^2}{\alpha_n\beta_n}\Big)^{p+1/2}\exp\{\beta_n(\ln\beta_n-1)+\alpha_n(1-\ln(\alpha_n))\}. \tag{3.7}$$

Using the property $\Gamma(p+1)=p\Gamma(p)$ of the gamma function, one can easily check that the function $L(p)$ is a solution of (3.6) with $g(p+1,\sigma)\equiv 0$. Hereinafter, we need the asymptotic representations of the functions $L(p)$ and $A(p)$, which follow from the well-known expansions of the gamma function as $|p|\to\infty$,

$$\Gamma(p)=c\exp\{(p-1/2)\ln p-p\}(1+O(1/|p|)), \tag{3.8}$$

and the obtained representation of the function $A(p)$ from [6],

$$A(p)=[\Omega(p)]^{p-1/2}\exp(\frac{\alpha_1}{\theta}\ln p)(1+O(1/|p|)) \qquad \text{for} \qquad |\operatorname{Im} p|\to\infty, \tag{3.9}$$

where $\Omega(p)=p\omega(p)$, $\omega(p)=h-\cot p\theta$, and in the case of $h<0$ we have $|\omega(p)|\to|h\pm i|$, $\arg\omega(p)\to\pm(\pi+\arctan\frac{1}{h})$, $\operatorname{Im} p\to\pm\infty$. Using (3.8) and (3.9) we have obtained the principal terms of the asymptotic expansions of the following functions for $|m|\to\infty$ and $\operatorname{Im} m=0$ if $h<0$:

$$\Gamma(im+d)\sin\pi(im+d)\approx b_1(\operatorname{sign} m,d)\exp\{im\ln|m|+\tfrac{\pi}{2}|m|-im-d\}|m|^{d-1/2},$$

$$L(im+d) \approx \tag{3.10}$$
$$b_2(\operatorname{sign} m, d)\exp\{im \ln\sqrt{1+h^2}|m| - (\tfrac{\pi}{2} + \arctan\tfrac{1}{h})|m|\}|m|^{d+\frac{\alpha_1}{\theta}-\frac{1}{2}},$$

where d is some positive number, and

$$b_1(\operatorname{sign} m, d) = \frac{-i \operatorname{sign} m}{2}\exp\{i(d-3\pi)\tfrac{\pi}{2}\operatorname{sign} m\},$$
$$b_2(\operatorname{sign} m, d) = \frac{-i \operatorname{sign} m}{2}\exp\{i(d-\tfrac{1}{2}+\tfrac{\alpha_1}{\theta})\arg(im+d) + \operatorname{sign} m[d\arctan\tfrac{1}{h}$$
$$+ \tfrac{(2\alpha_1-\theta)\pi - \theta\arctan\frac{1}{h}}{2\theta}] + (d-1/2)\ln\sqrt{1+h^2}\}. \tag{3.11}$$

Remark 3.1. Note that if one replaces $\arctan\frac{1}{h}$ with $(\arctan\frac{1}{h}-\pi)$ in (3.10) and (3.11) the asymptotic representations will be done in the case of $h>0$.

As follows from the results of [6], the solution of the nonhomogeneous equation (3.6) is represented with a contour integral

$$v(p,\sigma) = \int_{l_{-\delta}} \frac{L(p)}{L(p+1+\varsigma)}\frac{\sigma_1^{\varsigma} g(p+1+\varsigma,\sigma)}{e^{i\pi\varsigma}-e^{-i\pi\varsigma}}d\varsigma\,, \tag{3.12}$$

where $l_{-\delta}$ is a contour on the complex plane ς, and $l_{-\delta}$ coincides with the line $\operatorname{Re}\varsigma = -\delta$, $\delta\in(0,1)$. In the case of $\delta\to 1$ this contour goes around the pole $\varsigma=-1$ to the right along a circle of a small radius and the pole $\varsigma = 0$ to the left for $\delta\to 0$. The poles of the function $L(p)$ are real and located outside of the interval $(\frac{-\pi+\alpha_1}{\theta}; 1+\frac{\pi}{\theta})$. The zeros of the function $L(p)$ lie outside the interval $(-\frac{\pi}{\theta}; \frac{\alpha_1}{\theta}+1)$ of the real axis. Thus representation (3.12) will be true if $\operatorname{Re} p \in (\frac{-\pi+\alpha_1}{\theta}; \delta + \frac{\alpha_1}{\theta})$.

Note that representation (3.12) with $\operatorname{Re}\varsigma\to -1$ will be useful when the properties of $\frac{\partial^l u}{\partial x_1^l}$, $l=0,1$, are studied, and we will choose $\operatorname{Re}\varsigma\to 0$ to study the properties $\frac{\partial^2 u}{\partial x_1^2}$ in problem (2.1)–(2.2).

We have got from (3.12) the integral representation for the solution of problem (3.5):

$$c(\lambda,\sigma) = \frac{-1}{2\gamma_1}\int_{-\infty}^{+\infty}\frac{L(-i\lambda/\gamma_1)}{L(iy+1-\delta-i\lambda/\gamma_1)}\frac{(\sigma_1/\gamma_1)^{iy-\delta}\widehat{f}_1(\lambda-\gamma_1 y - i - i\delta\gamma_1,\sigma)}{\sin\pi(iy-\delta)}dy,$$

$$y = \operatorname{Im}\varsigma, \quad \lambda = \lambda_1 + i\lambda_2,$$

or, after inverse Fourier and Laplace transformations, we have got

$$c(x_1,t) = \frac{\gamma_1}{4\pi}\int_0^t d\tau[\gamma_1(t-\tau)]^{\delta-1}\int_{-\infty}^{\infty} d\xi f_1(x_1-\xi,\tau)e^{-(\delta\gamma_1+1)(x_1-\xi)}$$

$$\times \int_{-\infty}^{\infty} dy \frac{e^{-iy[\ln \gamma_1 (t-\tau) - \gamma_1 x_1]}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)} \int_{-\infty}^{\infty} \frac{L(-i\mu_1)}{L(iy+1-\delta-i\mu)} e^{i\gamma_1 \xi(\mu - y)} d\mu_1,$$

$$\mu = \frac{\lambda}{\gamma_1} = \mu_1 + i\mu_2, \qquad \operatorname{Im}\mu \in (\frac{-\pi+\alpha_1}{\theta}; \delta + \frac{\alpha_1}{\theta}). \tag{3.13}$$

4. The proof of estimate (2.8) for the solution of problem (2.1)–(2.2)

For the sake of clarity, we shall prove estimates (2.6) and (2.7) in the case of $h < 0$, since they can be obtained in the same way using Remark 3.1 and representation (3.13) if $h > 0$. Set

$$b(x,\xi,t) = \int_{-\infty}^{\infty} dy \frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)} \int_{-\infty}^{\infty} \frac{L(i(k-y))}{L(ik+1-\delta)} e^{-i\gamma_1 \xi k} dk_1,$$

$$q(t,x) = \ln \gamma_1 t - \gamma_1 x, \quad k = y - \mu \equiv k_1 + ik_2,$$

so that

$$-u(x_1,-\omega,t) = c(x_1,t) = \frac{\gamma_1}{4\pi}\int_0^t d\tau [\gamma_1(t-\tau)]^{\delta-1} \int_{-\infty}^{\infty} d\xi f_1(x_1-\xi,\tau)$$
$$\times e^{-(\delta\gamma_1+1)(x_1-\xi)} b(x,\xi,t). \tag{4.1}$$

The integral

$$E(y,\xi) = \int_{-\infty}^{\infty} \frac{L(i(k_1-y))}{L(ik_1+1-\delta)} e^{-i\gamma_1\xi k_1} dk_1 \tag{4.2}$$

does not converge absolutely since the integrand has the order $1/|k_1|^{1-\delta}$ at infinity and it should be expected that $E(y,\xi)$ has a singularity with respect to the parameters ξ of the order $1/|\xi|^{\delta}$.

The function $\frac{L(i(k-y))}{L(ik+1-\delta)}$ is analytical for $-\delta - \frac{\alpha_1}{\theta} < k_2 < \frac{\pi-\alpha_1}{\theta}$, as follows from the properties of the function $L(p)$. Hence the integrand in (4.2) is an analytical function for the above-mentioned k_2 and decreases for $|k_1| \to \infty$. We represent the function $b(x,\xi,t)$ in the form

$$b(x,\xi,t) = B_1(x,\xi,t) + B_2(x,\xi,t),$$

$$B_1(x,\xi,t) = \int_{-\infty}^{0} dy \frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)} E(y,\xi),$$

$$B_2(x,\xi,t) = \int_{0}^{\infty} dy \frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)} E(y,\xi); \tag{4.3}$$

then

$$c(x_1,t)=\frac{\gamma_1}{4\pi}\sum_{ij=1}^{2}c_{ij}(x_1,t),$$

where

$$c_{1j}(x_1,t)=\int_0^t d\tau[\gamma_1(t-\tau)]^{\delta-1}\int_{-\infty}^0 d\xi f_1(x_1-\xi,\tau)e^{-(\delta\gamma_1+1)(x_1-\xi)}B_j(x_1,\xi,t-\tau),$$
$$c_{2j}(x_1,t)=\int_0^t d\tau[\gamma_1(t-\tau)]^{\delta-1}\int_0^{+\infty} d\xi f_1(x_1-\xi,\tau)e^{-(\delta\gamma_1+1)(x_1-\xi)}B_j(x_1,\xi,t-\tau).$$

This representation is connected with the different behavior of the integrand in (4.2) for different subsets into domain of integration. For example, to estimate $c_{12}(x_1,t)$ we shall choose a value $|k_2|$, $k_2=\operatorname{Im}k<0$, arbitrarily small, and to evaluate the term $c_{22}(x_1,t)$ we shall take the value k_2 arbitrarily close to $(-\delta-\frac{\alpha_1}{\theta})$.

At the beginning we study the properties of the function $b(x,\xi,t)$ and $B_j(x,\xi,t)$, $j=1,2$. We decompose the domain of integration (y,k_1) into the planes $(y,k_1)^+$ and $(y,k_1)^-$, where $(y,k_1)^+=\{(y,k_1):y>0,\ k_1\in R^1\}$ and $(y,k_1)^-=\{(y,k_1):y\le 0,\ k_1\in R^1\}$.

Let $(y,k_1)^+=\bigcup_{i=1}^8 G_i^+$ and $(y,k_1)^-=\bigcup_{i=1}^8 G_i^-$ where

$$\begin{aligned}
G_1^+&=\{(y,k_1):y\in(0,2M),k_1\ge 3M\},\\
G_2^+&=\{(y,k_1):y\ge 2M,k_1\ge y+M\},\\
G_3^+&=\{(y,k_1):y\ge 2M,y-M\le k_1<y+M\},\\
G_4^+&=\{(y,k_1):y\ge 2M,M<k_1\le y-M\},\\
G_5^+&=\{(y,k_1):y\ge 2M,-M\le k_1<M\},\\
G_6^+&=\{(y,k_1):y\ge 2M,k_1<-M\},\\
G_7^+&=\{(y,k_1):y\in(0,2M),k_1\le -3M\},\\
G_8^+&=\{(y,k_1):y\in(0,2M),-3M\le k_1\le 3M\}.
\end{aligned}\tag{4.4}$$

Note that G_i^-, $i=\overline{1,8}$, can be obtained from corresponding G_i^+, $i=\overline{1,8}$, if we change y into $(-y)$ in G_i^+, $i=\overline{1,8}$ (see Figure 2).

First of all, we proceed to estimate the function $B_2(x,\xi,t-\tau)$:

$$B_2(x,\xi,t)=e^{\gamma_1\xi k_2}\int_0^\infty dy\frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)}\times\int_{-\infty}^\infty\frac{L(i(k_1-y)-k_2)}{L(ik_1+1-\delta-k_2)}e^{-i\gamma_1\xi k_1}dk_1\equiv e^{\gamma_1\xi k_2}D_2(x,\xi,t).\tag{4.5}$$

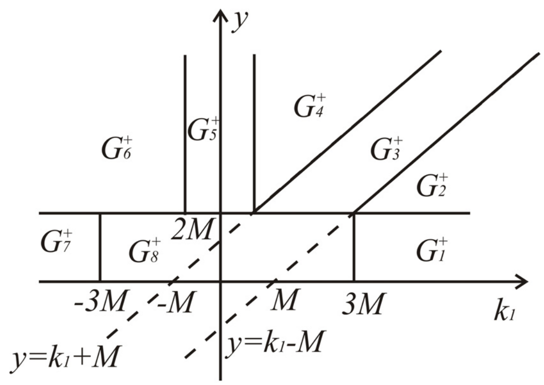

FIGURE 2

To evaluate the function $D_2(x,\xi,t)$ we take the sufficiently large number $M>0$ in (4.4), so that it could be possible to use the asymptotic expansions of the corresponding functions in the indicated domains. Write

$$D_2(x,\xi,t)=\sum_{i=1}^{8}D_{2i}(x,\xi,t),$$

and estimate every $D_{2i}(x,\xi,t)$, $i=\overline{1,8}$. Let us consider

$$D_{21}(x,\xi,t)=\int_0^{2M}dy\frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)} \times\int_{3M}^{\infty}\frac{L(i(k_1-y)-k_2)}{L(ik_1+1-\delta-k_2)}e^{-i\gamma_1\xi k_1}dk_1. \tag{4.6}$$

It is easy to show due to (3.10) and (3.11) that

$$\frac{L(i(k_1-y)-k_2)}{L(ik_1+1-\delta-k_2)} =const.(1-\frac{y}{k_1})^{-k_2-\frac{1}{2}+\frac{\alpha_1}{\theta}}k_1^{\delta-1}\exp\{ik_1\ln(1-\frac{y}{k_1})-iy\ln(k_1-y) +(\frac{\pi}{2}+\arctan\frac{1}{h})y\}(1+O(1/k_1)),\quad \text{if } (k_1,y)\in G_1, \tag{4.7}$$

where the constant depends on the functions like $b_2(\operatorname{sign} m,d)$ from (3.11).

The principal part of integral (4.6) corresponding to the higher term of asymptotic (4.7) is

$$I(x,\xi,t)=\int_0^{2M}dy\frac{e^{-iyq(t,x)}}{\Gamma(iy-\delta)\sin\pi(iy-\delta)}e^{(\frac{\pi}{2}+\arctan\frac{1}{h})y}\int_{3M}^{\infty}(1-\frac{y}{k_1})^{-k_2-\frac{1}{2}+\frac{\alpha_1}{\theta}}k_1^{\delta-1}$$

$$\times \exp\{ik_1 \ln(1 - \frac{y}{k_1}) - iy \ln(k_1 - y) - i\gamma_1 \xi k_1\} dk_1. \tag{4.8}$$

We consider the inner integral in (4.8) to estimate $I(x, \xi, t)$. Let

$$N(y, \xi) = \int_{3M}^{\infty} (1 - \frac{y}{k_1})^{-k_2 - \frac{1}{2} + \frac{\alpha_1}{\theta}} k_1^{\delta - 1} \times \exp\{ik_1 \ln(1 - \frac{y}{k_1}) - iy \ln(k_1 - y) - i\gamma_1 \xi k_1\} dk_1.$$

Note that $0 < y/k_1 < 2/3$ in G_1, and $(1-z)^{-k_2 - \frac{1}{2} + \frac{\alpha_1}{\theta}} = 1 + O(z)$, $\ln(1-z) = -z + O(z^2)$, $|z| \to 0$. Thus we have got the following representation for the main part of $N(y, \xi)$:

$$N_1(y, \xi) = \int_0^{\infty} k_1^{\delta - 1} \exp\{-iy \ln k_1 - i\gamma_1 \xi k_1\} dk_1.$$

Let us consider the function $N_1(y, \xi)$ in case of $\xi > 0$ (the case of $\xi < 0$ is studied in the same way). We perform a change of variables $\xi k_1 = u$ and obtain

$$N_1(y, \xi) = \frac{e^{iy \ln \xi}}{\xi^{\delta}} \int_0^{\infty} u^{\delta - 1} \exp\{-iy \ln u - i\gamma_1 u\} du = \frac{e^{iy \ln \xi}}{\xi^{\delta}} \sum_{j=1}^{4} I_j(y),$$

$$I_1(y) = \int_0^{\pi/\gamma_1} u^{\delta - 1} e^{-iy \ln u} \cos(\gamma_1 u) du,$$

$$I_2(y) = -i \int_0^{\pi/2\gamma_1} u^{\delta - 1} e^{-iy \ln u} \sin(\gamma_1 u) du,$$

$$I_3(y) = \int_{\pi/\gamma_1}^{\infty} u^{\delta - 1} e^{-iy \ln u} \cos(\gamma_1 u) du,$$

$$I_4(y) = -i \int_{\pi/2\gamma_1}^{\infty} u^{\delta - 1} e^{-iy \ln u} \sin(\gamma_1 u) du.$$

It is easy to see that the functions $I_1(y), I_2(y) \in C^{\infty}(0, 2M)$. Thus it is necessary to estimate $I_3(y)$ and $I_4(y)$. Integration by parts in $I_3(y)$ leads to

$$I_3(y) = \frac{1}{\gamma_1} \int_{\pi/\gamma_1}^{\infty} u^{\delta - 2} e^{-iy \ln u} [1 + iy - \delta] \sin(\gamma_1 u) du.$$

The functions $z^{\delta - 2} \ln^k z$, $k = \overline{0, 3}$, are bounded for all $z \geq \pi/\gamma_1$, and vanish to 0 as $z \to \infty$. Therefore

$$\Big| \frac{d^n I_3(y)}{dy^n} \Big| \leq const.(y + 1), \quad y \in (0, 2M), \quad n = 0, 1, 2.$$

The term $I_4(y)$ is evaluated using the same approach. The preceding estimates lead to

$$|N(y,\xi)| \le const.\Big(\frac{1}{|\xi|^\delta}+1\Big).$$

Moreover, the minor terms of asymptotic expansion (4.7) do not give the additional difficulties. As a result we obtain the following inequality:

$$|D_{21}(y,\xi)| \le const.\Big(\frac{1}{|\xi|^\delta}+1\Big).$$

The analogous estimates of the functions $D_{2i}(y,\xi)$, $i=\overline{2,8}$, have been obtained in a similarly way. Hence

$$|D_2(y,\xi)| \le const.\Big(\frac{1}{|\xi|^\delta}+1\Big).$$

Thus, altogether we obtain the following:

Lemma 4.1. *Let $\delta \in (0,1)$, $s \in (0, \frac{\alpha_1}{\omega}-2-|\varepsilon|)$, $0<|\varepsilon|<1$, and $-\frac{\alpha_1}{\omega}-\delta\gamma_1 < a < -2-s$ if $\xi>0$, and $-s-2<a<\frac{\pi-\alpha_1}{\omega}$ if $\xi<0$; then the following estimates hold:*

(1) $|B_2(x,\xi,t)| \le const.e^{a\xi}\Big(\frac{1}{|\xi|^\delta}+1\Big), \quad \forall (x,\xi)\in R^2,\ t\in[0,T]$;

(2) $|B_2(x_1,\xi,t)-B_2(x_2,\xi,t)| \le const.e^{a\xi}\Big(\frac{1}{|\xi|^\delta}+1\Big)|x_1-x_2|, \quad \forall x_1,x_2,\xi \in R^1,\ t\in[0,T]$;

(3)

$$\Big|\int_0^t d\tau\tau^{\delta-1}\int_{-\infty}^{+\infty} e^{-(\delta\gamma_1+s+2)\xi}B_2(x_1,\xi,t-\tau)d\xi\Big| \le const.\frac{t^\delta}{\delta}, \quad \forall t\in[0,T].$$

Let us return to the estimate of the functions $c_{i2}(x,t)$, $i=1,2$. The function $f_1(x_1,t)$ can be represented as

$$f_1(x_1,t)=e^{-(s+1)x_1}\varphi(x_1,t), \quad \varphi(x_1,t)\in C_0^{1+\alpha,\beta,\alpha}(\overline{b_T}), \tag{4.9}$$

because of $f_1(x_1,t)\in E^{1+\alpha,\beta,\alpha}_{s+1,0}(\overline{g_T})$ according to the condition of Theorem 2.1. The results of Lemma 4.1 and representation (4.9) lead to the following:

$$|c_{i2}(x_1,t)-c_{i2}(\overline{x}_1,t)| \le \frac{const.}{\delta}\Big(\max_{\overline{b_T}}|\varphi e^{-\delta\gamma_1 x}| + \Big\langle \varphi e^{-\delta\gamma_1 x}\Big\rangle^{(\alpha)}_{x,\overline{b_T}}\Big) \times T^\delta e^{-(s+2)x}|x_1-\overline{x}_1|^\alpha,$$

$$|c_{i2}(x_1,t)| \le \frac{const.}{\delta}T^\delta e^{-(s+2)x}\max_{\overline{b_T}}|\varphi e^{-\delta\gamma_1 x}|,\ i=1,2,\ x\in(x_1,\overline{x}_1). \tag{4.10}$$

To estimate the differences $S_i = |c_{i2}(x_1,t_1) - c_{i2}(x_1,t_2)|$, $i=1,2$, we set $\delta = \beta$, $i=2$, and use (4.9) and Lemma 4.1:

$$S_2 \le \int_0^{t_1} d\tau(\gamma_1\tau)^{\beta-1}\int_0^{+\infty}|f_1(x_1-\xi,t_1-\tau)-f_1(x_1-\xi,t_2-\tau)|e^{(\beta\gamma_1+1)(x_1-\xi)}$$
$$\times|B_2(x_1,\xi,\tau)|d\xi + \int_{t_1}^{t_2} d\tau(\gamma_1\tau)^{\beta-1}\int_0^{+\infty}|f_1(x_1-\xi,t_2-\tau)|e^{(\beta\gamma_1+1)(x_1-\xi)}$$
$$\times|B_2(x_1,\xi,\tau)|d\xi \le const.T^\beta e^{-(s+2)x}|t_1-t_2|^\beta\left\langle\varphi e^{-\beta\gamma_1 x_1}\right\rangle^{(\beta)}_{t,\overline{b_T}}.$$

The same inequality holds in the case of $i=1$. In conclusion, the results of Lemma 4.1 give

$$|c_{i2}(x_1,t_1)-c_{i2}(\overline{x}_1,t_1)-c_{i2}(x_1,t_2)+c_{i2}(\overline{x}_1,t_2)| \le \frac{const.}{\beta}T^\beta e^{-(s+2)x}|x_1-\overline{x}_1|^\alpha$$
$$\times|t_1-t_2|^\beta\left\|\varphi e^{-\beta\gamma_1 x_1}\right\|_{C^{\alpha,\beta,\alpha}(\overline{b_T})}, \quad i=1,2, \quad x\in(x_1,\overline{x}_1).$$

To estimate the functions $c_{i1}(x,t)$ we proceed as before, and find that

$$\left\|ce^{(s+2)x_1}\right\|_{C^{\alpha,\beta,\alpha}(\overline{b_T})} \le const.(\beta)T^\beta\max(1,d^{\beta\gamma_1})\,\|f_1\|_{E^{1+\alpha,\beta,\alpha}_{s+1}(\overline{g_T})}. \tag{4.11}$$

As it follows from representation (4.1), in order to estimate the function $c_{x_1}(x_1,t)$ it is possible to transfer the derivative with respect to x_1 to the function $f_1(x_1,t)$ and to repeat all arguments leading to estimate (4.11). As a result we have got

$$\left\|ue^{(s+2)x_1}\right\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})} \le const.(\beta)T^\beta\max(1,d^{\beta\gamma_1})\,\|f_1\|_{E^{1+\alpha,\beta,\alpha}_{s+1}(\overline{g_T})}. \tag{4.12}$$

This estimate can be extended into the domain $\overline{B_T}$. To that end, we apply the results of Theorem 6.2.3 from [17] to the function $U(x_1,t) = u(x_1,-\omega,t)e^{(s+2)x_1}$, and find

$$\|U\|_{C^{1+\alpha}(\overline{B_T})} \le const.(\beta)T^\beta\max(1,d^{\beta\gamma_1})\,\|f_1\|_{E^{1+\alpha,\beta,\alpha}_{s+1}(\overline{g_T})}. \tag{4.13}$$

Note that the corresponding estimates with respect to t for the function $u(x_1,x_2,t)$ will be obtained if one considers the Dirichlet problems for the Laplace equation in B_T for the functions $U_1 = u(x_1,x_2,t_1) - u(x_1,x_2,t_2)$ and $U_2 = u(x_1,x_2,t_1) - u(\overline{x_1},\overline{x_2},t_1) - u(x_1,x_2,t_2) + u(\overline{x_1},\overline{x_2},t_2)$ and applies inequalities (4.12) and (4.13). If we return to variables (y_1,y_2), we will obtain estimate (2.8) of Theorem 2.1, and $u(x_1,x_2,t)\in E^{1+\alpha,\beta,\alpha}_{s+2,0}(\overline{G_T})$.

5. Some estimates of the potential

This section is devoted to estimating the potential of the special form which is used to investigate the properties of the higher derivatives for the function $u(x_1, x_2, t)$.

Let M and a be positive constants, and $\alpha, \beta \in (0,1)$, $s(x,t) \in C_0^{\alpha,\beta,\alpha}(\overline{R_T^1})$, $R_T^1 = R^1 \times (0,T)$, $\Phi(y) \in C(-M, M)$,

$$B(y,x,\xi) = e^{-iy\gamma_1 x}\Phi(y)e^{-a|\xi|}\int_0^{+\infty} e^{-iy\ln k}e^{-i\gamma_1\xi k}dk,$$

$$F(y,t,x) = \int_{-\infty}^{+\infty} B(y,x,\xi)s(x-\xi,t)d\xi, \tag{5.1}$$

$$C(x,t) = \int_0^t d\tau \int_{-M}^{M} \frac{e^{-iy\ln(t-\tau)}}{t-\tau}[F(y,\tau,x) - F(y,t,x)]dy + \int_{-M}^{M} dy F(y,t,x)\int_0^t \frac{e^{-iy\ln(t-\tau)}}{t-\tau}d\tau. \tag{5.2}$$

We will show that $C(x,t) \in C^{\alpha,\beta,\alpha}(\overline{R_T^1})$ if the functions $s(x,t)$ and $\Phi(y)$ meet the conditions mentioned above. Now we formulate some properties of the function $F(y,t,x)$ which are used below. Their proof is unwieldy and given in the appendix (see Lemmas 7.1 and 7.2 and Proposition 7.2).

Lemma 5.1. *Let $\alpha, \beta \in (0,1)$, $s(x,t) \in C_0^{\alpha,\beta,\alpha}(\overline{R_T^1})$, and $\Phi(y) \in C[-M, M]$; then the following estimates hold:*

(1)

$$F(y,t,x) = F_1(y,t,x) + F_2(y,t,x) + 8\pi s(x,t)\delta(y) + s(x,t)\varphi_1(y), \tag{5.3}$$

$$F_1(y,t,x) = y\varphi_2(y)\Phi(y)e^{-iy(x+\ln\gamma_1)}\int_{-\infty}^{+\infty}\frac{e^{-|\xi|(a+iy)}}{\gamma_1|\xi|}[s(x-\xi,t) - s(x,t)]d\xi,$$

$$F_1(y,t,x) = [1+y]\varphi_3(y)\Phi(y)e^{-iy(x+\ln\gamma_1)}\int_{-\infty}^{+\infty}\frac{e^{-|\xi|(a+iy)}}{\gamma_1\xi}[s(x-\xi,t) - s(x,t)]d\xi,$$

where $\delta(y)$ is the Dirac delta function, $\varphi_1(y) \in C^1(-M,M)$, $\varphi_i(y) \in C^\alpha(-M,M)$, $i = 2,3$;

(2)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M} F_i(y,t,x)dy\Big| + \Big|\int_{-M}^{M} F(y,t,x)dy\Big| \le const.(\max_{\overline{R_T^1}}|s| + \langle s\rangle_{x,\overline{R_T^1}}^{(\alpha)}); \tag{5.4}$$

(3)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M}[F_i(y,t_1,x)-F_i(y,t_2,x)]dy\Big|+\Big|\int_{-M}^{M}[F(y,t_1,x)-F(y,t_2,x)]dy\Big|$$
$$\leq const.|t_1-t_2|^{\beta}(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}}+[s]^{(\alpha,\beta)}_{\overline{R^1_T}}),\ \forall t_1,t_2\in[0,T]; \tag{5.5}$$

(4)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M}[F_i(y,t,x_1)-F_i(y,t,x_2)]dy\Big|+\Big|\int_{-M}^{M}[F(y,t,x_1)-F(y,t,x_2)]dy\Big|$$
$$\leq const.|x_1-x_2|^{\alpha}(\max_{\overline{R^1_T}}|s|+\langle s\rangle^{(\alpha)}_{x,\overline{R^1_T}}),\ \forall x_1,x_2\in R^1, \tag{5.6}$$

Let $x_i\in R^1$, $t_i\in[0,T]$, $i=1,2$; $x_2>x_1$, $t_2>t_1$; and $\Delta x=x_2-x_1$, $\Delta t=t_2-t_1$.

Proposition 5.1. *Let the conditions of Lemma* 5.1 *be satisfied; then the following estimates hold:*

$$F(0,t,x)=\frac{\pi}{\gamma_1}\Phi(0)s(x,t)+\frac{i}{\gamma_1}\Phi(0)\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|}}{\xi}[s(x-\xi,t)-s(x,t)]d\xi;$$
$$|F(0,t_1,x)-F(0,t_2,x)|\leq const.(\Delta t)^{\beta}(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}}+[s]^{(\alpha,\beta)}_{\overline{R^1_T}});$$
$$|F(0,t,x_1)-F(0,t,x_2)|\leq const.(\Delta x)^{\alpha}(\max_{\overline{R^1_T}}|s|+\langle s\rangle^{(\alpha)}_{x,\overline{R^1_T}});$$

$$\Big|\int_{-M}^{M}\frac{e^{-iy\ln t}}{y}[F(y,t_2,x)-F(y,t_1,x)+F(0,t_1,x)-F(0,t_2,x)]dy\Big|$$
$$\leq const.(\Delta t)^{\beta}(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}}+[s]^{(\alpha,\beta)}_{\overline{R^1_T}}). \tag{5.7}$$

Let $D_T, D_{T\infty}\subset R^4$, and $A_T, d_T\subset R^3$:

$D_T=\{(x,y,t,\tau): x,y\in R^1,\ t,\tau\in(0,T)\}$, $D_{T\infty}=\{(x,y,t,\tau): x,y\in R^1,$ $t\in(0,T),\ \tau\in(-\infty,T)\}$, $A_T=\{(x,t,\tau): x\in R^1,\ t\in(0,T),\ \tau\in(-\infty,T)\}$, $d_T=\{(x,y,t): x,y\in R^1,\ t\in(0,T)\}$,

$i=1,2$, and introduce the following functions:

$$I_1(x,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty}B(y,x,\xi)\Psi_1(x-\xi,x,t)d\xi;$$
$$I_{2i}(x_1,x_2,t_1,t_2)=\int_{-M}^{M}dy\int_{t_1-2\Delta t}^{t_i}d\tau\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)\Psi_2(x_1,x_2,t_i,\tau)d\xi;$$

$$I_3(x_1,x_2,t_1,t_2)=\int_{-M}^{M}dy\int_{-\infty}^{t_1-2\Delta t}d\tau\left[\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau}-\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\right]\int_{-\infty}^{+\infty}B(y,x_1,\xi)\times\Psi_2(x_1,x_2,t_2,\tau)d\xi;$$

$$I_4(x_1,x_2,t_1,t_2)=\int_{-M}^{M}dy\int_{-\infty}^{t_1-2\Delta t}d\tau\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)\Psi_2(x_1,x_2,t_1,t_2)d\xi;$$

$$I_5(x_1,x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty}[B(y,x_1,\xi)-B(y,x_2,\xi)]\Psi_3(x_2,t,\tau)d\xi;$$

$$I_{6i}(x_1,x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\le 2\Delta x}B(y,x_1,x_1-\xi)\Psi_2(\xi,x_i,t,\tau)d\xi;$$

$$I_7(x_1,x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\ge 2\Delta x}\Psi_2(x_2,\xi,t,\tau)[B(y,x_2,x_1-\xi)-B(y,x_2,x_2-\xi)]d\xi;$$

$$I_8(x_1,x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\ge 2\Delta x}B(y,x_1,x_1-\xi)\Psi_2(x_1,x_2,t,\tau)d\xi;$$

$$I_9(x_1,x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\ge 2\Delta x}[B(y,x_1,x_1-\xi)-B(y,x_2,x_1-\xi)]\times\Psi_2(x_2,\xi,t,\tau)d\xi; \tag{5.8}$$

where $\Psi_1(\xi,x,t)$, $\Psi_2(z,x,t,\tau)$, and $\Psi_3(x,t,\tau)$ are some given functions which are continuous with respect to all the variables in the domains d_T, D_T or $D_{T\infty}$, and A_T, respectively.

Lemma 5.2. *Let the functions $\Psi_1(\xi,x,t)$, $\Psi_2(z,x,t,\tau)$, and $\Psi_3(x,t,\tau)$ meet the following conditions:*

$$\max_{\overline{d_T}}|\Psi_1|\le C_1(\Psi_1)|x-\xi|^\alpha;\qquad \max_{\overline{D_T}}|\Psi_2|\le C_2(\Psi_2)|x-\xi|^\alpha|t-\tau|^\beta\quad or$$

$$\max_{\overline{D_{T\infty}}}|\Psi_2|\le C_2(\Psi_2)|x-\xi|^\alpha|t-\tau|^\beta;$$

$$\Psi_3(x,t_2,\tau)-\Psi_3(x,t_1,\tau)=\Psi_3(x,t_2,t_1),\quad \max_{\overline{A_T}}|\Psi_3|\le C_3(\Psi_3)|t-\tau|^\beta;$$

where $C_i(\Psi_i)$, $i=\overline{1,3}$, are bounded positive constants depending on the functions Ψ_i. Then the following inequalities are true for the functions introduced with (5.8)*:*

$$I_1(x,t)=\frac{i\Phi(0)}{\gamma_1}\int_{-\infty}^{+\infty}\frac{e^{-a|x-\xi|}}{x-\xi}\Psi_1(x-\xi,x,t)d\xi+\frac{2\pi^2}{\gamma_1}\Psi_1(x,x,t);$$

$$\max_{\overline{D_T}}|I_{2i}|+\max_{\overline{D_T}}|I_j|\le const.C_2(\Psi_2)(\Delta x)^\alpha(\Delta t)^\beta,\qquad i=1,2,\quad j=3,4;$$

$$\langle I_5\rangle^{(\beta)}_{t,\overline{d_T}} \le const.C_3(\Psi_3)(\Delta x)^{\alpha}(\Delta t)^{\beta},$$

$$\langle I_{6i}\rangle^{(\beta)}_{t,\overline{d_T}} + \sum_{j=7}^{9}\langle I_j\rangle^{(\beta)}_{t,\overline{d_T}} \le const.C_2(\Psi_2)(\Delta x)^{\alpha}(\Delta t)^{\beta}, \qquad i=1,2. \tag{5.9}$$

Now we return to studying the properties of the function $C(x,t)$.

Lemma 5.3. *Let $s(x,t)\in C^{\alpha,\beta,\alpha}(\overline{R^1_T})$ and $\Phi(y)\in(-M,M)$; then $C(x,t)\in C^{\alpha,\beta,\alpha}(\overline{b_T})$, and*

$$\|C\|_{C^{\alpha,\beta,\alpha}(\overline{R_T 1})} \le const.\,\|s\|_{C^{\alpha,\beta,\alpha}(\overline{b_T})}\,. \tag{5.10}$$

Proof. First of all we estimate the regularity of the function $C(x,t)$ with respect to time. As follows from the results of Chapter II, Section 2 in [18],

$$\int_{-\infty}^{0} e^{-iy\tau}d\tau = \pi\delta(y) + iy^{-1}$$

in the distribution sense, and therefore,

$$\int_{-\infty}^{\ln t} e^{-iy\tau}d\tau = \pi\delta(y) + ie^{-iy\ln t}y^{-1}, \tag{5.11}$$

where $\delta(y)$ is the Dirac delta function. Then the function $C(x,t)$ can be written as

$$C(x,t) = \int_0^t d\tau\int_{-M}^{M}\frac{e^{-iy\ln(t-\tau)}}{t-\tau}[F(y,\tau,x)-F(y,t,x)]dy + i\int_{-M}^{M}F(y,t,x)\frac{e^{-iy\ln t}}{y}dy + \pi F(0,t,x).$$

Using this representation we get

$$C(x,t_2)-C(x,t_1) = \int_0^{t_2}d\tau\int_{-M}^{M}\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau}[F(y,\tau,x)-F(y,t_2,x)]dy - \int_0^{t_1}d\tau \times\int_{-M}^{M}\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}[F(y,\tau,x)-F(y,t_1,x)]dy + i\int_{-M}^{M}\frac{e^{-iy\ln t_2}}{y} \times[F(y,t_2,x)-F(y,t_1,x)]dy + i\int_{-M}^{M}\frac{e^{-iy\ln t_2 -}e^{-iy\ln t_1}}{y} \times F(y,t_1,x)dy + \pi[F(0,t_1,x)-F(0,t_2,x)] \equiv \sum_{i=1}^{5}J_i. \tag{5.12}$$

The results of Proposition 5.1 and estimates (5.4) together with the mean value theorem imply

$$|J_5| + |J_4| \le const.(\langle s\rangle^{(\beta)}_{t,R^1_T} + [s]^{(\alpha.\beta)}_{R^1_T})(\Delta t)^\beta. \tag{5.13}$$

In order to get

$$|J_3| \le const.(\langle s\rangle^{(\beta)}_{t,R^1_T} + [s]^{(\alpha,\beta)}_{R^1_T})(\Delta t)^\beta, \tag{5.14}$$

we represent J_3 as

$$\begin{aligned} J_3 = i\int_{-M}^{M} \frac{e^{-iy\ln t_2}}{y}[F(y,t_2,x) - F(0,t_2,x) + F(0,t_1,x) - F(y,t_1,x)]dy \\ + i\int_{-M}^{M} \frac{e^{-iy\ln t_2} - 1}{y}[F(0,t_2,x) - F(0,t_1,x)]dy + i\int_{-M}^{M} \frac{1}{y}[F(0,t_2,x) \\ - F(0,t_1,x)]dy, \end{aligned}$$

and apply Proposition 5.1. Now we show that J_1 and J_2 satisfy the inequality like (5.14).

$$\begin{aligned} J_1 + J_2 = \int_0^{t_1} d\tau \int_{-M}^{M} \Big[\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau} - \frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\Big][F(y,\tau,x) - F(y,t_2,x)]dy \\ + \int_0^{t_1} d\tau \int_{-M}^{M} \frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}[F(y,t_1,x) - F(y,t_2,x)]dy \\ + \int_{t_1}^{t_2} d\tau \int_{-M}^{M} \frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau}[F(y,\tau,x) - F(y,t_2,x)]dy \equiv \sum_{i=1}^{3} Q_i. \end{aligned} \tag{5.15}$$

Let $q = t_1 - \tau + l(t_2 - t_1)$, $l \in [0,1]$; then

$$\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau} - \frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau} = (t_1 - t_2)\int_0^1 \frac{\partial}{\partial q}\left(\frac{e^{-iy\ln q}}{q}\right) dl,$$

and

$$Q_1 = (t_2 - t_1)\int_0^1 dl \int_{-M}^{M} dy \int_0^{t_1} [1 + iy]\frac{e^{-iy\ln q}}{q^2}[F(y,\tau,x) - F(y,t_2,x)]d\tau.$$

If we apply inequality (5.5) from Lemma 5.1 to the function $F(y,\tau,x)$, the following estimate is obtained:

$$|Q_1| \le const.(\langle s\rangle^{(\beta)}_{t,R^1_T} + [s]^{(\alpha,\beta)}_{R^1_T})(\Delta t)^\beta. \tag{5.16}$$

Note that

$$\int_0^t \frac{e^{-iy\ln\tau}}{\tau} d\tau = (z = \ln\tau) = \int_{-\infty}^{\ln t} e^{-iyz} dz.$$

This equality together with (5.11) permit us to write Q_2 as

$$Q_2 = i\int_{-M}^{M} \frac{e^{-iy\ln t_1}}{y}[F(y,t_1,x) - F(y,t_2,x)]dy + \pi[F(0,t_1,x) - F(0,t_2,x)].$$

The first term in the sum is estimated like J_3. As for the second summand, its estimate follows from (5.7). Thus altogether we have got

$$|Q_2| \le const.(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}} + [s]^{(\alpha,\beta)}_{\overline{R^1_T}})(\Delta t)^{\beta}. \tag{5.17}$$

Note that the results of Lemma 5.1 give the analogous estimate for Q_3. Thus inequalities (5.13)–(5.17) and representation (5.12) lead to

$$\langle C\rangle^{(\beta)}_{t,\overline{R^1_T}} \le const.(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}} + [s]^{(\alpha,\beta)}_{\overline{R^1_T}}). \tag{5.18}$$

Note that $C(x,0) = 0$, $x \in R^1$, and therefore

$$\langle C\rangle^{(\alpha)}_{x,\overline{R^1_T}} \le const.T^{(\beta)}[C]^{(\alpha,\beta)}_{\overline{R^1_T}}. \tag{5.19}$$

Therefore, the proof of Lemma 5.3 will be completed if the estimate of $[C]^{(\alpha,\beta)}_{\overline{b_T}}$ is obtained. To this end we represent $C(x,t)$ as

$$\begin{aligned} C(x,t) &= \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau} \\ &\times \int_{-\infty}^{+\infty} B(y,x,\xi)[s(x-\xi,\tau) - s(x,\tau) - s(x-\xi,t) + s(x,t)]d\xi \\ &+ \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty} B(y,x,\xi)[s(x-\xi,t) - s(x,t)]d\xi \\ &+ \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty} B(y,x,\xi)[s(x,\tau) - s(x,t)]d\xi \\ &+ s(x,t)\int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty} B(y,x,\xi)d\xi \equiv \sum_{i=1}^{4} A_i(x,t); \end{aligned} \tag{5.20}$$

here we use that $s(x,t) \equiv 0$, $t \le 0$. Let us consider the difference

$$\begin{aligned} |A_1(x_1,t) - A_1(x_2,t)| &\le \Big| \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\le 2\Delta x} B(y,x_1,x_1-\xi)[s(\xi,\tau) \\ &-s(x_1,\tau) - s(\xi,t) + s(x_1,t)]d\xi \Big| + \Big| \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{|x_1-\xi|\le 2\Delta x} [s(\xi,\tau) \\ &-s(x_2,\tau) - s(\xi,t) + s(x_2,t)]B(y,x_2,x_2-\xi)d\xi \Big| + \Big| \int_{-M}^{M} dy\int_{-\infty}^{t} d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau} \end{aligned}$$

$$\times \int_{|x_1-\xi|\geq 2\Delta x} [B(y,x_2,x_1-\xi) - B(y,x_2,x_2-\xi)][s(\xi,\tau) - s(x_2,\tau) - s(\xi,t)$$
$$+s(x_2,t)]d\xi \Big| + \Big| \int_{-M}^{M} dy \int_{-\infty}^{t} d\tau \frac{e^{-iy\ln(t-\tau)}}{t-\tau} \int_{|x_1-\xi|\geq 2\Delta x} [s(x_1,\tau) - s(x_2,\tau) - s(x_1,t)$$
$$+s(x_2,t)]B(y,x_1,x_1-\xi)d\xi \Big| + \Big| \int_{-M}^{M} dy \int_{-\infty}^{t} d\tau \frac{e^{-iy\ln(t-\tau)}}{t-\tau} \int_{|x_1-\xi|\geq 2\Delta x} [s(\xi,\tau)$$
$$-s(x_2,\tau) - s(\xi,t) + s(x_2,t)][B(y,x_1,x_1-\xi) - B(y,x_2,x_1-\xi)]d\xi \Big|$$
$$\equiv \sum_{i=1}^{5} |a_i(x_1,x_2,t)|. \tag{5.21}$$

Note that the functions $a_i(x_1,x_2,t)$, $i = 1,2$, have the form of functions $I_{6i}(x_1,x_2,t)$ from (5.8) with

$$\Psi_2(\xi,x_i,t,\tau) = s(\xi,\tau) - s(x_i,\tau) - s(\xi,t) + s(x_i,t), \quad i = 1,2;$$

the functions $a_i(x_1,x_2,t)$, $i = \overline{3,5}$, are the functions $I_j(x_1,x_2,t)$, $j = \overline{7,9}$, where

$$\Psi_2(x_2,\xi,t,\tau) = s(\xi,\tau) - s(x_2,\tau) - s(\xi,t) + s(x_2,t)$$

in case of $a_3(x_1,x_2,t)$ and $a_5(x_1,x_2,t)$;

$$\Psi_2(x_1,x_2,t,\tau) = s(x_1,\tau) - s(x_2,\tau) - s(x_1,t) + s(x_2,t)$$

for $a_4(x_1,x_2,t)$. Note that the functions $\Psi_2(\xi,x_i,t,\tau)$, $\Psi_2(x_2,\xi,t,\tau)$ and $\Psi_2(x_1,x_2,t,\tau)$ fulfill the requirements of Lemma 5.2 with constant $C_2(\Psi_2) = [s]^{(\alpha,\beta)}_{R^1_T}$. Thus, estimates (5.9) lead to

$$|a_i(x_1,x_2,t_1) - a_i(x_1,x_2,t_2)| \leq const.[s]^{(\alpha,\beta)}_{R^1_T} (\Delta x)^{\alpha} (\Delta t)^{\beta}, \qquad i = \overline{1,5}.$$

These inequalities together with (5.21) give

$$[A_1]^{(\alpha,\beta)}_{R^1_T} \leq [s]^{(\alpha,\beta)}_{R^1_T}.$$

The function $A_2(x,t)$ looks like the function $I_1(x,t)$ from (5.8) with $\Psi_1(\xi,x, t) = s(\xi,t) - s(x,t)$, and the results of Lemma 5.2 give

$$A_2(x,t) = \frac{2\pi}{\gamma_1} \Phi(0) \int_{-\infty}^{\infty} \frac{e^{-a|x-\xi|}}{x-\xi} [s(\xi,t) - s(x,t)]d\xi.$$

Using this relation, we get

$$\frac{\gamma_1}{|\Phi(0)|2\pi} |A_2(x_1,t_1) - A_2(x_2,t_1) - A_2(x_1,t_2) + A_2(x_2,t_2)|$$
$$\leq \Big| \int_{|x_1-\xi|\leq 2\Delta x} \frac{e^{-a|x_1-\xi|}}{x_1-\xi} [s(\xi,t_1) - s(x_1,t_1) - s(\xi,t_2) + s(x_1,t_2)]d\xi \Big|$$

$$+\Big|\int_{|x_1-\xi|\le 2\Delta x}\frac{e^{-a|x_2-\xi|}}{x_2-\xi}[s(\xi,t_1)-s(x_2,t_1)-s(\xi,t_2)+s(x_2,t_2)]d\xi\Big|$$
$$+\Big|\int_{|x_1-\xi|\ge 2\Delta x}\Big[\frac{e^{-a|x_1-\xi|}}{x_1-\xi}-\frac{e^{-a|x_2-\xi|}}{x_2-\xi}\Big]$$
$$\times[s(\xi,t_1)-s(x_2,t_1)-s(\xi,t_2)+s(x_2,t_2)]d\xi\Big|$$
$$+\Big|\int_{|x_1-\xi|\ge 2\Delta x}\frac{e^{-a|x_1-\xi|}}{x_1-\xi}[s(x_1,t_1)-s(x_1,t_2)-s(x_2,t_1)+s(x_2,t_2)]d\xi\Big|$$
$$\le const.[s]^{(\alpha,\beta)}_{R^1_T}(\Delta x)^{\alpha}(\Delta t)^{\beta}. \tag{5.22}$$

Here we use the mean value theorem to estimate the third term in (5.22).

Now let us study the difference

$$A_3(x_1,t)-A_3(x_2,t)=\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)[s(x_1,\tau)-s(x_1,t)$$
$$-s(x_2,\tau)+s(x_2,t)]d\xi+\int_{-M}^{M}dy\int_{-\infty}^{t}d\tau\frac{e^{-iy\ln(t-\tau)}}{t-\tau}\int_{-\infty}^{+\infty}[B(y,x_1,\xi)-B(y,x_2,\xi)]$$
$$\times[s(x_2,\tau)-s(x_2,t)]d\xi\equiv i_1(x_1,x_2,t)+i_2(x_1,x_2,t). \tag{5.23}$$

The function $i_1(x_1,x_2,t)$ satisfies the inequality

$$|i_1(x_1,x_2,t_1)-i_1(x_1,x_2,t_2)|\le\Big|\int_{-M}^{M}dy\int_{t_1-2\Delta t}^{t_1}d\tau\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)[s(x_1,\tau)$$
$$-s(x_1,t_1)-s(x_2,\tau)+s(x_2,t_1)]d\xi\Big|+\Big|\int_{-M}^{M}dy\int_{t_1-2\Delta t}^{t_1}d\tau\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)$$
$$\times[s(x_1,\tau)-s(x_1,t_2)-s(x_2,\tau)+s(x_2,t_2)]d\xi\Big|+\Big|\int_{-M}^{M}dy\int_{-\infty}^{t_1-2\Delta t}d\tau\int_{-\infty}^{+\infty}B(y,x_1,\xi)$$
$$\times\left[\frac{e^{-iy\ln(t_2-\tau)}}{t_2-\tau}-\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\right][s(x_1,\tau)-s(x_1,t_2)-s(x_2,\tau)+s(x_2,t_2)]d\xi\Big|$$
$$+\Big|\int_{-M}^{M}dy\int_{-\infty}^{t_1-2\Delta t}d\tau\frac{e^{-iy\ln(t_1-\tau)}}{t_1-\tau}\int_{-\infty}^{+\infty}B(y,x_1,\xi)[s(x_1,t_1)-s(x_1,t_2)$$
$$-s(x_2,t_1)+s(x_2,t_2)]d\xi\Big|\equiv\sum_{i=1}^{4}|q_i(x_1,x_2,t_1,t_2)|. \tag{5.24}$$

Note that $q_i(x_1,x_2,t_1,t_2)$, $i=\overline{1,4}$, are the functions of the type $I_{2i}(x_1,x_2,t_1,t_2)$, $i=1,2$, $I_3(x_1,x_2,t_1,t_2)$ and $I_4(x_1,x_2,t_1,t_2)$ from (5.8), respectively, where $\Psi_2(x,y,t,\tau)=s(x,\tau)-s(y,\tau)-s(x,t)+s(y,t)$. Then, using Lemma 5.2 and inequality (5.24), we obtain

$$|i_1(x_1,x_2,t_1)-i_1(x_1,x_2,t_2)|\le const.[s]^{(\alpha,\beta)}_{R^1_T}(\Delta x)^{\alpha}(\Delta t)^{\beta}. \tag{5.25}$$

The same way of the proof gives the analogous estimate of $|i_2(x_1, x_2, t_1) - i_2(x_1, x_2, t_2)|$. Thus inequality (5.25) and representation (5.23) lead to

$$[A_3]^{(\alpha,\beta)}_{R^1_T} \leq const.[s]^{(\alpha,\beta)}_{R^1_T}. \tag{5.26}$$

We have, in the distribution sense,

$$\int_0^{+\infty} e^{-iy\ln t} t^{-1} dt = (\ln t = z) = \int_{-\infty}^{+\infty} e^{-iyz} dz = 2\pi\delta(y);$$

$$\int_{-\infty}^{+\infty} B(0, x, \xi) d\xi = \Phi(0) \int_{-\infty}^{+\infty} d\xi e^{-a|\xi|} \int_0^{+\infty} e^{-i\gamma_1 \xi k} dk$$

$$= \gamma_1^{-1} \Phi(0) \int_{-\infty}^{+\infty} e^{-a|\xi|} [\pi\delta(\xi) - i\xi^{-1}] d\xi.$$

That is why we can represent $A_4(x, t)$ as

$$A_4(x, t) = 2\pi^2 \gamma_1^{-1} \Phi(0) s(x, t),$$

and the estimates of $A_4(x, t)$ follow from the properties of the function $s(x, t)$.

To complete the proof of Lemma 5.3 we remark that representation (5.20) and estimates (5.22) and (5.26) lead to

$$[C]^{(\alpha,\beta)}_{R^1_T} \leq const.[s]^{(\alpha,\beta)}_{R^1_T}. \qquad \square$$

Remark 5.1. The results of Lemma 5.3 will hold for

$$C(x, t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{e^{-iy\ln(t-\tau)}}{t - \tau} [F(y, \tau, x) - F(y, t, x)] dy$$
$$+ \int_{-\infty}^{+\infty} dy F(y, t, x) \int_0^t \frac{e^{-iy\ln(t-\tau)}}{t - \tau} d\tau$$

if $\Phi(y) \in C(-\infty, \infty)$ and $\Phi(y) = o(|y|^{3+\varepsilon})$, $|y| \to \infty$, $\varepsilon \in (0, 1)$.

6. Completion of the proof of Theorem 2.1

In Section 4 we have shown that $u(y_1, y_2, t) \in E^{1+\alpha,\beta,\alpha}_{s+2,0}(\overline{G_T})$ and estimate (2.8) holds. Therefore, to complete the proof of Theorem 2.1 it is necessary to obtain the estimate of $\|D^2_y u\|_{E^{\alpha,\beta,\alpha}_{s,0}(\overline{G_T})}$.

According to inequality (4.12), we seek the solution of problem (2.1)–(2.2) (with $f_i(x, t) \equiv 0, i = 0, 2$) in the form $u(x, t) = e^{-(s+2)x_1} w(x, t)$, where the function $w(x, t)$ is a solution of the problem

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} - (2 + s)\frac{\partial w}{\partial x_1} + (2 + s)^2 w = 0, \qquad (x, t) \in B_T;$$

$$e^{\gamma_1 x_1}\frac{\partial w}{\partial t}-\frac{\partial w}{\partial x_2}+h\frac{\partial w}{\partial x_1}=e^{(s+1)x_1}f_1+h(s+2)w, \qquad (x,t)\in\overline{b_T};$$
$$w(x_1,0,t)=0, \qquad w(x,0)=0, \qquad x\in\overline{B}. \tag{6.1}$$

Let us set $w(x,t)=w_1(x,t)+w_2(x,t)$, where $w_1(x,t)$ is a solution of the problem

$$\frac{\partial^2 w_1}{\partial x_1^2}+\frac{\partial^2 w_1}{\partial x_2^2}-(2+s)\frac{\partial w_1}{\partial x_1}+(2+s)^2 w_1=(2+s)\frac{\partial w_2}{\partial x_1}-(2+s)^2 w_2$$
$$\equiv F_0, \qquad (x,t)\in B_T; \quad w|_{\partial B_T}=0, \qquad \partial B_T=\partial B\times[0,T], \tag{6.2}$$

where $\partial B=b\cup\{(x_1,x_2):x_2=0,\ x_1\in R^1\}$, and the function $w_2(x,t)$ satisfies the conditions

$$\frac{\partial^2 w_2}{\partial x_1^2}+\frac{\partial^2 w_2}{\partial x_2^2}=0, \quad (x,t)\in B_T; \quad w_2(x_1,0,t)=0, \quad w_2(x,0)=0, \quad x\in\overline{B};$$

$$e^{\gamma_1 x_1}\frac{\partial w_2}{\partial t}-\frac{\partial w_2}{\partial x_2}+h\frac{\partial w_2}{\partial x_1}=e^{(s+1)x_1}f_1+h(s+2)w_2+\frac{\partial w_1}{\partial x_2}\equiv F_1(x_1,t), \tag{6.3}$$

for $(x,t)\in\overline{b_T}$. First of all we analyze problem (6.3), which coincides with problem (2.1)–(2.2) for the given function $F_1(x_1,t)$. By repeating the reasoning of Section 3, we obtain the following form of a solution of problem (6.3) on the boundary $\overline{b_T}$:

$$w_2(x_1,-\omega,t)=-\frac{\gamma_1}{4\pi}\int_0^t d\tau\int_{-\infty}^{+\infty}dy$$
$$\times\int_{-\infty}^{+\infty}d\xi\frac{[\gamma_1(t-\tau)]^{-1-iy}}{\Gamma(-iy)\sin(i\pi y)}e^{i(x_1-\xi)\gamma_1 y}F_1(x_1-\xi,\tau)\int_{-\infty}^{+\infty}e^{i\lambda\xi}\frac{L(-i\lambda/\gamma_1)}{L(1+iy-i\lambda/\gamma_1)}d\lambda$$
$$-\frac{\gamma_1}{4\pi}\int_{-\infty}^{+\infty}e^{i\lambda x_1}\frac{\widehat{F_1}(\lambda,t)}{\Omega(-i\lambda/\gamma_1)}d\lambda=-\frac{\gamma_1}{4\pi}[a(x_1,t)+b(x_1,t)], \tag{6.4}$$

where $\widehat{F_1}(\lambda,t)$ is the Fourier transformation of the function $F_1(x_1,t)$ with respect to the variable x_1. To obtain (6.4) it is necessary to let δ tend to zero in (3.12), by passing the pole at the point $\xi=0$ from the left (this explains the appearance of the second term in (6.4)) and to apply the inverse Fourier and Laplace transformation to a convolution. As follows from representation (3.7) of the function $L(p)$ and its properties (see (3.10)), the function $\frac{e^{i\lambda\xi}L(-i\lambda/\gamma_1)}{L(1+iy-i\lambda/\gamma_1)}$ is analytic for $\frac{\alpha_1-\pi}{\theta}<\operatorname{Im}(\lambda/\gamma_1)<\frac{\alpha_1}{\theta}$ and decreases as $|\operatorname{Re}(\lambda/\gamma_1)|\to\infty$. Therefore one can easily obtain the relation for the first term in (6.4)

$$\frac{\partial^2 a(x_1,t)}{\partial x_1^2}=\int_0^t d\tau\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}d\xi\frac{\gamma_1^{-iy}(t-\tau)^{-1-iy}}{\Gamma(-iy)\sin(i\pi y)}$$

$$\times e^{i(x_1-\xi)\gamma_1 y}\frac{\partial F_1}{\partial\xi}(x_1-\xi,\tau)e^{-a_\pm\gamma_1\xi}\int_{-\infty}^{+\infty}e^{i\lambda_1\xi}\frac{(a_\pm-i\lambda_1/\gamma_1)L(a_\pm-i\lambda_1/\gamma_1)}{L(a_\pm+1+iy-i\lambda/\gamma_1)}d\lambda_1$$
$$+\int_0^t d\tau\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}d\xi\frac{iy\gamma_1^{1-iy}(t-\tau)^{-1-iy}}{\Gamma(-iy)\sin(i\pi y)}e^{i(x_1-\xi)\gamma_1 y}F_1(x_1-\xi,\tau)e^{-a_\pm\gamma_1\xi}$$
$$\times\int_{-\infty}^{+\infty}e^{i\lambda_1\xi}\frac{(a_\pm-i\lambda_1/\gamma_1)L(a_\pm-i\lambda_1/\gamma_1)}{L(a_\pm+1+iy-i\lambda/\gamma_1)}d\lambda_1\equiv a_1(x_1,t)+a_2(x_1,t), \qquad (6.5)$$

where $a_\pm=a/\gamma_1$ if $\xi\geq 0$ and $a_\pm=-a/\gamma_1$ if $\xi<0$, with a a positive constant, $0<a<1/2$. It is obvious that $a_2(x_1,t)$ can be estimated by means of the method of Section 4. Thus we study only the properties of $a_1(x_1,t)$ in this section.

Let $k=y-\lambda_1/\gamma_1$. Then the following representation of the function $a_1(x_1,t)$ is true:

$$a_1(x_1,t)=\int_0^t d\tau\int_{-\infty}^{+\infty}d\xi\frac{\partial F_1}{\partial\xi}(x_1-\xi,\tau)e^{-a|\xi|}\int_{-\infty}^{+\infty}dy\frac{\gamma_1^{1-iy}(t-\tau)^{-1-iy}}{\Gamma(-iy)\sin(i\pi y)}e^{ix_1\gamma_1 y}$$
$$\times\int_{-\infty}^{+\infty}e^{-i\gamma_1 k\xi}\frac{(a_\pm+i(k-y))L(a_\pm+i(k-y))}{L(a_\pm+1+ik)}dk.$$

Here it is convenient to use partition (4.4) of the plane (y,k_1). Let $\widetilde{G}_1^+\equiv G_1^+\cup G_1^-$, $\widetilde{G}_7^+\equiv G_7^+\cup G_7^-$, and $\widetilde{G}_8^+\equiv G_8^+\cup G_8^-$ (see (4.4)). According to this partition, we represent the function $a_1(x_1,t)$ in the form

$$a_1(x_1,t)=\sum_{i=1}^{8}A_i(x_1,t)+\sum_{i=2}^{6}A_i^-(x_1,t), \qquad (6.6)$$

where $A_i(x_1,t)$, $i=\overline{1,8}$, correspond to the domains $\widetilde{G}_1^+$, G_i, $i=\overline{2,6}$, $\widetilde{G}_7^+$, $\widetilde{G}_8^+$, and $A_i^-(x_1,t)$, $i=\overline{2,6}$, correspond to the domains G_i^-, $i=\overline{2,6}$.

Note that the estimates of $A_i^-(x_1,t)$, $i=\overline{2,6}$, are analogous to those of $A_i(x_1,t)$, $i=\overline{2,6}$. Now we restrict ourselves to the study of the term $A_7(x_1,t)$, because either the other terms in (6.4) can be considered the same way or their estimates are analogous to estimates of $\frac{\partial u}{\partial x_1}$ (see Section 4).

Using asymptotic behaviour (3.10) and (3.11), we have got the representation for the main part of $A_7(x_1,t)$:

$$bA(x_1,t)\equiv b\int_0^t d\tau\int_{-\infty}^{+\infty}d\xi\left[\frac{\partial F_1}{\partial\xi}(x_1-\xi,\tau)-\frac{\partial F_1}{\partial\xi}(x_1-\xi,t)\right]$$
$$\times e^{-a|\xi|}\int_{-2M}^{2M}dy(t-\tau)^{-1+iy}\times e^{ix_1\gamma_1 y}e^{-(\frac{\pi}{2}+\arctan\frac{1}{h})y}\int_0^{+\infty}\exp\{-iy\ln k+i\gamma_1\xi k\}dk$$
$$+b\int_0^t d\tau\int_{-\infty}^{+\infty}d\xi\frac{\partial F_1}{\partial\xi}(x_1-\xi,t)e^{-a|\xi|}\int_{-2M}^{2M}dy(t-\tau)^{-1+iy}e^{ix_1\gamma_1 y}e^{-(\frac{\pi}{2}+\arctan\frac{1}{h})y}$$

$$\times e^{ix_1\gamma_1 y}\int_0^{+\infty} \exp\{-iy\ln k + i\gamma_1\xi k\}dk,$$

where the constant b depends on functions like $b_1(\operatorname{sign} m, d)$ and $b_2(\operatorname{sign} m, d)$ from (3.11). Note that the function $A(x_1,t)$ looks like $C(x,t)$ from (5.2), where $s(x_1,t) = \frac{\partial F_1}{\partial x_1}(x_1,t) \in C^{\alpha,\beta,\alpha}(\overline{b_T})$. Thus, we can use the results of Lemma 5.3 and obtain

$$\langle A\rangle^{(\alpha)}_{x,\overline{b_T}} + \langle A\rangle^{(\beta)}_{t,\overline{b_T}} + [A]^{(\alpha,\beta)}_{\overline{b_T}} \le const. \left\|\frac{\partial F_1}{\partial x_1}\right\|_{C^{\alpha,\beta,\alpha}(\overline{b_T})}. \tag{6.7}$$

The same estimates are true for the lowest terms of $A_7(x_1,t)$ and the rest of $A_i(x_1,t)$ and $A_i^-(x_1,t)$. Inequality (6.7) together with representations (6.5) and (6.6) give

$$\left\langle\frac{\partial^2 a}{\partial x_1^2}\right\rangle^{(\alpha)}_{x,\overline{b_T}} + \left\langle\frac{\partial^2 a}{\partial x_1^2}\right\rangle^{(\beta)}_{t,\overline{b_T}} + \left[\frac{\partial^2 a}{\partial x_1^2}\right]^{(\alpha,\beta)}_{\overline{b_T}} \le const. \|F_1\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})}. \tag{6.8}$$

The second derivative of the function $b(x_1,t)$ in (6.4) can be evaluated with the same technique. Therefore the inequality

$$\left\langle\frac{\partial^2 w_2}{\partial x_1^2}\right\rangle^{(\alpha)}_{x,\overline{b_T}} + \left\langle\frac{\partial^2 w_2}{\partial x_1^2}\right\rangle^{(\beta)}_{t,\overline{b_T}} + \left[\frac{\partial^2 w_2}{\partial x_1^2}\right]^{(\alpha,\beta)}_{\overline{b_T}} \le const. \|F_1\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})} \tag{6.9}$$

is a direct consequence of estimate (6.8).

Note that, using the estimate of $\left\langle\frac{\partial^2 w_2}{\partial x_1^2}\right\rangle^{(\beta)}_{t,\overline{b_T}}$ from (6.9), one can evaluate $\max\limits_{\overline{b_T}}\left|\frac{\partial^2 w_2}{\partial x_1^2}\right|$; the lower seminorms of the function $w_2(x_1,-\omega,t)$ are estimated by analogy with (6.9). In that way we can establish the following result for $w_2(x_1,-\omega,t)$.

Lemma 6.1. *Let $F_1(x_1,t) \in C^{1+\alpha,\beta,\alpha}(\overline{b_T})$. Then the boundary value of the solution $w_2(x_1,-\omega,t)$ for problem* (6.3) *belongs to $C^{2+\alpha,\beta,\alpha}(\overline{b_T})$ and satisfies inequality*

$$\|w_2\|_{C^{2+\alpha,\beta,\alpha}(\overline{b_T})} \le const. \|F_1\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})}. \tag{6.10}$$

As follows from (6.10) and the result of Chapter 3, Section 2 in [15], the next estimate is true for every $t \in [0,T]$:

$$\|w_2\|_{C^{2+\alpha}(\overline{B})} \le const. \|F_1\|_{C^{1+\alpha}(\bar{b})}.$$

Returning to the definition of the function $F_1(x_1,t)$ in problem (6.3), we have

$$\|w_2\|_{C^{2+\alpha}(\overline{B})} \le const.\Big[\Big\|e^{(s+1)x_1}f_1\Big\|_{C^{1+\alpha}(\overline{b})} + \Big\|w_1\Big\|_{C^{1+\alpha}(\overline{b})} + \Big\|\frac{\partial w_1}{\partial x_2}\Big\|_{C^{1+\alpha}(\overline{b})}\Big], \tag{6.11}$$

where the solution $w_1(x_1,x_2,t)$ of problem (6.2) satisfies the inequality

$$\|w_1\|_{C^{2+\alpha}(\overline{B})} \le const.\, \|w_2\|_{C^{1+\alpha}(\overline{B})} \tag{6.12}$$

(estimate (6.12) follows from the results of Chapter 6, Section 4 in [17]).

Applying the interpolation inequality, we deduce from (6.11) and (6.12) that

$$\|w_2\|_{C^{2+\alpha}(\overline{B})} \le const.\Big\|e^{(s+1)x_1}f_1\Big\|_{C^{1+\alpha}(\overline{b})}.$$

Finally, differences of $[w(x,t_1)-w(x,t_2)]$ and $[w(x,t_1)-w(x,t_2)-w(\overline{x},t_1)+w(\overline{x},t_2)]$ are estimated with the equation from (6.1) and the properties of $f_1(x_1,t)$, and Lemma 6.1. Hence, we obtain the required estimates for the function $w(x,t)\in C^{2+\alpha,\beta,\alpha}(\overline{B_T})$:

$$\|w_2\|_{C^{2+\alpha,\beta,\alpha}(\overline{B_T})} \le const.\Big\|e^{(s+1)x_1}f_1\Big\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})}. \tag{6.13}$$

The boundary condition in (6.1) and inequality (6.13) lead to the embedding $e^{\gamma_1 x_1}\frac{\partial w}{\partial t}\in C^{1+\alpha,\beta,\alpha}(\overline{b_T})$, and

$$\Big\|e^{(s+1)x_1}\frac{\partial w}{\partial t}\Big\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})} \le const.\Big\|e^{(s+1)x_1}f_1\Big\|_{C^{1+\alpha,\beta,\alpha}(\overline{b_T})}. \tag{6.14}$$

Passing to the functions $u(r,\varphi,t)$ and $f_1(r,t)$ in (6.13) and (6.14), we obtain (2.6) and $u(r,\varphi,t)\in E^{2+\alpha,\beta,\alpha}_{s+2}(\overline{G_T})$, $r^{-\gamma_1}\frac{\partial u}{\partial t}\in E^{1+\alpha,\beta,\alpha}_{s+1}(\overline{g_T})$. The proof of Theorem 2.1 is complete.

7. Appendix

Let M and a be positive constants, and $\alpha,\beta\in(0,1)$, $s(x,t)\in C_0^{\alpha,\beta,\alpha}(\overline{R^1_T})$, $\Phi(y)\in C(-M,M)$,

$$B(y,x,\xi) = e^{-iy\gamma_1 x}\Phi(y)e^{-a|\xi|}\int_0^{+\infty} e^{-iy\ln k}e^{-i\gamma_1\xi k}dk,$$

$$F(y,t,x) = \int_{-\infty}^{+\infty} B(y,x,\xi)s(x-\xi,t)d\xi.$$

Proposition 7.1. *There hold*

$$\int_0^{+\infty} e^{-iy\ln k}\cos(\gamma_1\xi k)dk = \frac{e^{iy\ln|\gamma_1\xi|}}{|\gamma_1\xi|}y\varphi_1(y),$$

$$\int_0^{+\infty} e^{-iy\ln k}\sin(\gamma_1\xi k)dk = \frac{e^{iy\ln|\gamma_1\xi|}}{\gamma_1\xi}(1+y)\varphi_2(y), \tag{7.1}$$

where $\varphi_i(y)$, $i=1,2$, *are continuous and bounded functions for* $y\in(-M,M)$, *and* $\frac{d\varphi_i(y)}{dy} = a_1^i(y)+a_2^i(y)y+a_3^i(y)y^{-1}$, $i=1,2$; $a_j^i(y)$, $j=\overline{1,3}$, *are bounded and continuous functions.*

Proof. We prove the first relation in (7.1); the second one is dealt with in the same way. First of all, note that if

$$z = \begin{cases} \gamma_1\xi k, & \xi>0, \\ -\gamma_1\xi k, & \xi<0, \end{cases}$$

then

$$\int_0^{+\infty} e^{-iy\ln k}\cos(\gamma_1\xi k)dk = \begin{cases} \frac{e^{iy\ln\gamma_1\xi}}{\gamma_1\xi}\int_0^{+\infty}e^{-iy\ln z}\cos z\,dz, & \xi>0, \\ \frac{e^{iy\ln(-\gamma_1\xi)}}{(-\gamma_1\xi)}\int_0^{+\infty}e^{-iy\ln z}\cos z\,dz, & \xi<0, \end{cases}$$
$$= \frac{e^{iy\ln|\gamma_1\xi|}}{|\gamma_1\xi|}\int_0^{+\infty} e^{-iy\ln z}\cos z\,dz. \tag{7.2}$$

Decomposing the integral $\int_0^{+\infty}e^{-iy\ln z}\cos z\,dz$ into the sum of two integrals and integrating by parts twice in the second integral, we get

$$\begin{aligned}\int_0^{+\infty} e^{-iy\ln z}\cos z\,dz &= \int_0^{\pi} e^{-iy\ln z}\cos z\,dz + \int_\pi^{+\infty} e^{-iy\ln z}\cos z\,dz \\ &= I_1(y) - \frac{iye^{iy\ln\pi}}{\pi} + y(y-i)\int_\pi^{+\infty} e^{-iy\ln z}z^{-2}\cos z\,dz \\ &\equiv I_1(y) - \frac{iye^{iy\ln\pi}}{\pi} + y(y-i)I_2(y) \equiv y\varphi_1(y), \end{aligned} \tag{7.3}$$

where $I_j(y)\in C^1(-M,M)$, $j=1,2$, and $I_1(0)=0$. Here we use the following:

$$\int_0^{+\infty} e^{-iy\ln z}\cos z\,dz = \lim_{\delta\to 0}\int_0^{+\infty} e^{-iy\ln z}z^{-\delta}\cos z\,dz.$$

Thus, the proof of Proposition 7.1 follows from (7.2) and (7.3). □

Now, using results of Proposition 7.1, we represent the function $F(y,t,x)$ as

$$F(y,t,x) = F_1(y,t,x) + F_2(y,t,x), \tag{7.4}$$
$$F_1(y,t,x) = \Phi(y)y\varphi_1(y)\gamma_1^{-1}e^{-iyx\gamma_1}\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{|\xi|}s(x-\xi,t)d\xi,$$
$$F_2(y,t,x) = \Phi(y)[1+y]\varphi_2(y)\gamma_1^{-1}e^{-iyx\gamma_1}\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{\xi}s(x-\xi,t)d\xi.$$

The following estimates are true.

Lemma 7.1. *Let* $\alpha,\beta\in(0,1)$, $s(x,t)\in C_0^{\alpha,\beta,\alpha}(\overline{R_T^1})$, *and* $\Phi(y)\in C(-M,M)$. *Then*

(1)

$$F(y,t,x) = \Phi(y)\gamma_1^{-1}e^{-iyx\gamma_1}[y\varphi_1(y)\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{|\xi|}[s(x-\xi,t)-s(x,t)]d\xi$$
$$+[1+y]\varphi_2(y)\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{\xi}[s(x-\xi,t)-s(x,t)]d\xi$$
$$+8\pi\delta(y)s(x,t)+s(x,t)\varphi_3(y)], \tag{7.5}$$

where $\delta(y)$ is the Dirac delta function, $\varphi_3(y)\in C^1(-M,M)$;

(2)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M}F_i(y,t,x)dy\Big|+\Big|\int_{-M}^{M}F(y,t,x)dy\Big|\le const.(\langle s\rangle^{(\alpha)}_{x,\overline{R^1_T}}+\max_{\overline{R^1_T}}|s|); \tag{7.6}$$

(3)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M}[F_i(y,t_1,x)-F_i(y,t_2,x)]dy\Big|+\Big|\int_{-M}^{M}[F(y,t_1,x)-F(y,t_2,x)]dy\Big|$$
$$\le const.(\langle s\rangle^{(\beta)}_{t,\overline{R^1_T}}+[s]^{(\alpha,\beta)}_{\overline{R^1_T}})|t_1-t_2|^{\beta},\quad \forall t_1,t_2\in[0,T]; \tag{7.7}$$

(4)

$$\sum_{i=1}^{2}\Big|\int_{-M}^{M}[F_i(y,t,x_1)-F_i(y,t,x_2)]dy\Big|+\Big|\int_{-M}^{M}[F(y,t,x_1)-F(y,t,x_2)]dy\Big|$$
$$\le const.(\langle s\rangle^{(\alpha)}_{x,\overline{R^1_T}}+\max_{\overline{R^1_T}}|s|)|x_1-x_2|^{\alpha},\quad \forall x_1,x_2\in R^1. \tag{7.8}$$

Proof. The first statement of Lemma 7.1 will follow straight away from (7.4) if one notes

$$\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{|\xi|}d\xi = 2\Big[\int_0^{+\infty}\frac{e^{-a\xi}-1}{\xi}e^{iy\ln\xi}d\xi+\int_0^{+\infty}\frac{e^{iy\ln\xi}}{\xi}d\xi\Big]$$
$$=2[i_1(y)+i_2(y)], \tag{7.9}$$

where

$$\Big|\frac{d^k i_1(y)}{dy^k}\Big|\le const,\quad k=0,1, \tag{7.10}$$

$$i_2(y)=\int_0^{+\infty}\frac{e^{iy\ln\xi}}{\xi}d\xi=(z=\ln\xi)=\int_{-\infty}^{+\infty}e^{iyz}dz=2\pi\delta(y). \tag{7.11}$$

We show the validity of (7.6)–(7.8) for the functions $F_i(y,t,x), i=1,2$, then, as follows from (7.4), the functions $F(y,t,x)$ will satisfy (7.6)–(7.8).

Let us rewrite $F_1(y,t,x)$ as

$$F_1(y,t,x)=\Phi(y)\gamma_1^{-1}e^{-iyx\gamma_1}y\varphi_1(y)\ \Big[\int_{-\infty}^{+\infty}[s(x-\xi,t)-s(x,t)]\frac{e^{-a|\xi|}}{|\xi|}$$

$$\times e^{iy\ln|\xi|}d\xi + \int_{-\infty}^{+\infty} s(x,t)\frac{e^{-a|\xi|+iy\ln|\xi|}}{|\xi|}d\xi\ \Big] = \Phi(y)\gamma_1^{-1}e^{-iyx\gamma_1}y\varphi_1(y)$$
$$\times[F_{11}(y,t,x) + F_{12}(y,t,x)]. \tag{7.12}$$

It is easy to obtain for all $y \in [-M, M], t \in [0, T], x \in b$

$$|F_{11}(y,t,x)| \le const.\,\langle s\rangle^{(\alpha)}_{x,\overline{b_T}};$$

$$|F_{11}(y,t_1,x) - F_{11}(y,t_2,x)| \le const.[s]^{(\alpha,\beta)}_{\overline{b_T}}|t_1-t_2|^{\beta} \quad \forall t_1,t_2 \in [0,T];$$

$$|F_{11}(y,t,x_1) - F_{11}(y,t,x_2)| \le const.\,\langle s\rangle^{(\alpha)}_{x,\overline{b_T}}|x_1-x_2|^{\alpha} \quad \forall x_1,x_2 \in b. \tag{7.13}$$

Note that the first inequality follows right away from representation (7.12) and properties of the function $s(x,t)$. The procedure of the evaluation for the second inequality is based on the method presented in Chapter 4, Section 2 from [19]. As for the third inequality, this estimate is the analog of one for the double-layer potential for an elliptic boundary value problem (see, e.g., Chapter 3,Section 2 in [15]).

The relations (7.9)–(7.11) allow us to represent $F_{12}(y,t,x)$ as

$$F_{12}(y,t,x) = 2s(x,t)[i_1(y) + 2\pi\delta(y)]. \tag{7.14}$$

Thus, estimates (7.12)–(7.14) lead to inequalities (7.6)–(7.8) for the function $F_1(y,t,x)$. Note that estimates (7.6)–(7.8) for the function $F_2(y,t,x)$ can be obtained with a similar method if one uses the representation

$$F_2(y,t,x) = \Phi(y)\gamma_1^{-1}e^{-iyx\gamma_1}[1+y]\varphi_2(y)\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{\xi}[s(x-\xi,t)-s(x,t)]d\xi$$

and the properties of the functions $\Phi(y)$, $\varphi_2(y)$, and $s(x,t)$. This completes the proof of Lemma 7.1. □

Proposition 7.2. *Let the conditions of Lemma* 7.1 *be realized; then the following hold:*

(1)

$$F(0,t,x) = \pi\Phi(0)\gamma_1^{-1}s(x,t) + i\Phi(0)\gamma_1^{-1}\int_{-\infty}^{+\infty}\frac{e^{-a|\xi|+iy\ln|\xi|}}{\xi}[s(x-\xi,t)-s(x,t)]d\xi;$$

(2)

$$|F(0,t_1,x) - F(0,t_2,x)| \le const.([s]^{(\alpha,\beta)}_{\overline{b_T}} + \langle s\rangle^{(\beta)}_{t,\overline{b_T}})|t_1-t_2|^{\beta} \quad \forall t_1,t_2 \in [0,T];$$

(3)

$$|F(0,t,x_1) - F(0,t,x_2)| \le const.(\langle s\rangle^{(\alpha)}_{x,\overline{b_T}} + \max_{\overline{b_T}}|s|)|x_1-x_2|^{\alpha}, \quad \forall x_1,x_2 \in b;$$

(4)

$$\left|\int_{-M}^{M}\frac{e^{-iy\ln t}}{y}[F(y,t_1,x)-F(y,t_2,x)-F(0,t_1,x)+F(0,t_2,x)]dy\right|$$
$$\leq const.([s]_{\overline{b_T}}^{(\alpha,\beta)}+\langle s\rangle_{t,\overline{b_T}}^{(\beta)})|t_1-t_2|^{\beta}\quad\forall t_1,t_2\in[0,T]. \tag{7.15}$$

Proof. Note that, in the distribution sense,

$$\int_0^{+\infty}e^{-iy\ln k}e^{-i\gamma_1\xi k}dk\,\Big|_{y=0}=\int_0^{+\infty}e^{-i\gamma_1\xi k}dk=i\gamma_1^{-1}\xi^{-1}+\pi\gamma_1^{-1}\delta(\xi), \tag{7.16}$$

where $\delta(\xi)$ is the delta function. The first relation of Proposition 7.2 is obtained with the expression of $B(y,x,\xi)$ and (7.16). The second and third estimates follow from the representation of $F(0,t,x)$ and are based on the method from Chapter 4, Section 2 in [19] and Chapter 3, Section 2 in [15]. To prove inequality (7.15) we use representation (7.4). It is easy to see that $F_1(0,t,x)=0$; then estimate (7.7) leads to

$$\left|\int_{-M}^{M}\frac{e^{-iy\ln t}}{y}[F_1(y,t_1,x)-F_1(y,t_2,x)-F_1(0,t_1,x)+F_1(0,t_2,x)]dy\right|$$
$$\leq const.([s]_{\overline{b_T}}^{(\alpha,\beta)}+\langle s\rangle_{t,\overline{b_T}}^{(\beta)})|t_1-t_2|^{\beta}\quad\forall t_1,t_2\in[0,T]. \tag{7.17}$$

Now we consider the difference

$$F_2(y,t_1,x)-F_2(y,t_2,x)-F_2(0,t_1,x)+F_2(0,t_2,x)=-\int_{-\infty}^{+\infty}[s(x-\xi,t_2)$$
$$-s(x-\xi,t_1)-s(x,t_2)+s(x,t_1)]\frac{e^{-a|\xi|}}{\xi}[\Phi(y)\gamma_1^{-1}e^{-i\gamma_1xy}(1+y)$$
$$\times\varphi_2(y)e^{iy\ln|\xi|}-\Phi(0)\gamma_1^{-1}\varphi_2(0)]d\xi-\Phi(0)\gamma_1^{-1}\varphi_2(0)[s(x,t_1)-s(x,t_2)].$$

We apply the mean value theorem to the difference

$$[\Phi(y)\gamma_1^{-1}e^{-i\gamma_1xy+iy\ln|\xi|}(1+y)\varphi_2(y)-\Phi(0)\gamma_1^{-1}\varphi_2(0)]$$

and obtain

$$F_2(y,t_1,x)-F_2(y,t_2,x)-F_2(0,t_1,x)+F_2(0,t_2,x)$$
$$=-\Phi(0)\gamma_1^{-1}\varphi_2(0)[s(x,t_1)-s(x,t_2)]-\int_{-\infty}^{+\infty}[s(x-\xi,t_2)-s(x-\xi,t_1)$$
$$-s(x,t_2)+s(x,t_1)]\frac{e^{-a|\xi|}}{\xi}y[1+\ln|\xi|]P(\overline{y},x,\xi), \tag{7.18}$$

where $\overline{y}\in(0,y)$, and $P(\overline{y},x,\xi)$ is continuous and uniformly bounded with respect to each variable. Thus equalities (7.18) and (7.17) lead to (7.15). □

Let $D_T, D_{T\infty} \subset R^4$, and $A_T, d_T \subset R^3$, $i = 1, 2$:

$$D_T = \{(x, y, t, \tau) : x, y \in R^1, t, \tau \in (0, T)\},\ D_{T\infty} = \{(x, y, t, \tau) : x, y \in R^1,$$

$$t \in (0, T), \tau \in (-\infty, T)\},\ A_T = \{(x, t, \tau) : x \in R^1, t \in (0, T), \tau \in (-\infty, T)\},$$

$$d_T = \{(x, y, t) : x, y \in R^1, t \in (0, T)\}.$$

Lemma 7.2. *Let the functions $\Psi_1(\xi, x, t)$, $\Psi_2(z, x, t, \tau)$, and $\Psi_3(x, t, \tau)$ meet the conditions*

$$\max_{\overline{d_T}} |\Psi_1| \leq C_1(\Psi_1)|x - \xi|^\alpha; \quad \max_{\overline{D_T}} |\Psi_2| \leq C_2(\Psi_2)|x - \xi|^\alpha |t - \tau|^\beta \quad \textit{or}$$

$$\max_{\overline{D_{T\infty}}} |\Psi_2| \leq C_2(\Psi_2)|x - \xi|^\alpha |t - \tau|^\beta;$$

$$\Psi_3(x, t_2, \tau) - \Psi_3(x, t_1, \tau) = \Psi_3(x, t_2, t_1),\ \max_{\overline{A_T}} |\Psi_3| \leq C_3(\Psi_3)|t - \tau|^\beta;$$

where $C_i(\Psi_i)$, $i = \overline{1, 3}$, are bounded positive constants defined with the properties of the functions Ψ_i. Then the following inequalities are true for the functions introduced with (5.8)*:*

$$I_1(x, t) = \frac{i\Phi(0)}{\gamma_1} \int_{-\infty}^{+\infty} \frac{e^{-a|x-\xi|}}{x - \xi} \Psi_1(x - \xi, x, t) d\xi + \frac{2\pi^2}{\gamma_1} \Psi_1(x, x, t);$$

$$\max_{\overline{D_T}} |I_{2i}| + \max_{\overline{D_T}} |I_j| \leq const.C_2(\Psi_2)(\Delta x)^\alpha (\Delta t)^\beta, \qquad i = 1, 2, \quad j = 3, 4;$$

$$\langle I_5 \rangle^{(\beta)}_{t, \overline{d_T}} \leq const.C_3(\Psi_3)(\Delta x)^\alpha (\Delta t)^\beta,$$

$$\langle I_{6i} \rangle^{(\beta)}_{t, \overline{d_T}} + \sum_{j=7}^{9} \langle I_j \rangle^{(\beta)}_{t, \overline{d_T}} \leq const.C_2(\Psi_2)(\Delta x)^\alpha (\Delta t)^\beta, \qquad i = 1, 2. \tag{7.19}$$

Proof. As for the first equality in Lemma 7.2, it follows immediately from the relations

$$\int_{-\infty}^{t} \frac{e^{-iy \ln(t-\tau)}}{t - \tau} d\tau = 2\pi\delta(y),$$

$$\int_{-M}^{M} \int_{-\infty}^{t} d\tau \frac{e^{-iy \ln(t-\tau)}}{t - \tau} B(y, x, \xi) dy = 2\pi B(0, x, \xi) = \frac{2\pi\Phi(0)e^{-a|\xi|}}{\gamma_1} [i\xi^{-1} + \pi\delta(\xi)];$$

here we use Proposition 7.1 to estimate $B(0, x, \xi)$.

The results of Proposition 7.1 give

$$\int_{-\infty}^{+\infty} B(y, x, \xi) d\xi$$

$$= \frac{\Phi(y)e^{iy \ln \gamma_1} e^{-iyx\gamma_1}}{\gamma_1} \Big[2y\varphi_1(y) \int_0^{+\infty} \frac{e^{-a\xi} - 1}{\xi} e^{-iy \ln \xi} d\xi + [1 + y]\varphi_2(y)$$

$$\times \int_{-\infty}^{+\infty} \frac{e^{-a|\xi|}}{\xi} e^{-iy \ln |\xi|} d\xi + 2y\varphi_1(y) \int_0^{+\infty} \frac{e^{-iy \ln \xi}}{\xi} d\xi \Big]$$

$$= \frac{\Phi(y)e^{iy\ln\gamma_1}e^{-iyx\gamma_1}}{\gamma_1}[\widetilde{\varphi}(y) + 2\pi y\varphi_1(y)\delta(y)], \tag{7.20}$$

where $\widetilde{\varphi}(y) \in C[-M, M]$. Thus, using this representation, we have

$$|I_{21}(x_1, x_2, t_1, t_2)| = (t_1 - \tau = s) = \Big| \int_0^{2\Delta t} \frac{ds}{s} \Psi_2(x_1, x_2, t_1, t_1 - s) \int_{-M}^{M} \frac{e^{-iy\ln s}}{\gamma_1}$$

$$\times e^{-iy(x\gamma_1 - \ln\gamma_1)}\Phi(y)[\widetilde{\varphi}(y) + 2\pi y\varphi_1(y)\delta(y)]dy \Big| \leq C_2(\Psi_2)|x_1 - x_2|^{\alpha} \int_0^{2\Delta t} \frac{ds}{s^{1-\beta}}.$$

The function $I_{22}(x_1, x_2, t_1, t_2)$ can be estimated in the same way.

Now we evaluate the function $I_3(x_1, x_2, t_1, t_2)$. Let $z = t_1 - \tau$, $q = z + l\Delta t$, and $l \in [0, 1]$; then

$$e^{-iy\ln(t_2-\tau)}(t_2-\tau)^{-1} - e^{-iy\ln(t_1-\tau)}(t_1-\tau)^{-1} = -\Delta t \int_0^1 (1+iy)e^{-iy\ln q}q^{-2}dl. \tag{7.21}$$

This equality and (7.20) allow us to infer

$$|I_3(x_1, x_2, t_1, t_2)| = \Delta t \Big| \int_0^1 dlq^{-2} \int_{2\Delta t}^{+\infty} ds\Psi_2(x_1, x_2, t_2, t_1 - s) \int_{-M}^{M} dy[1 + iy]$$

$$\times e^{-iy\ln q} \int_{-\infty}^{+\infty} B(y, x_1, \xi)d\xi \Big| \leq const.\Delta t(\Delta x)^{\alpha} C_2(\Psi_2) \Big| \int_0^1 dl \int_{2\Delta t}^{+\infty} [s + \Delta t]^{\beta}$$

$$\times [s + l\Delta t]^{-2} ds \Big|,$$

and the estimate of $I_3(x_1, x_2, t_1, t_2)$ follows immediately from this inequality. Note that the estimate of $I_4(x_1, x_2, t_1, t_2)$ follows from (7.20) and the properties of the function $\Psi_2(x_1, x_2, t_1, t_2)$. Finally, we consider the function $I_5(x_1, x_2, t)$.

Let $\Delta_x B(y, x, \xi) = B(y, x_1, \xi) - B(y, x_2, \xi)$; then

$$|I_5(x_1, x_2, t_1) - I_5(x_1, x_2, t_2)| \leq \Big| \int_{-M}^{M} dy \int_{t_1-2\Delta t}^{t_1} d\tau\, e^{-iy\ln(t_1-\tau)}(t_1 - \tau)^{-1}$$

$$\times \int_{-\infty}^{+\infty} \Delta_x B(y, x, \xi)\Psi_3(x_2, t_1, \tau)d\xi \Big| + \Big| \int_{-M}^{M} dy \int_{t_1-2\Delta t}^{t_2} d\tau\, e^{-iy\ln(t_2-\tau)}(t_2 - \tau)^{-1}$$

$$\times \int_{-\infty}^{+\infty} \Delta_x B(y, x, \xi)\Psi_3(x_2, t_2, \tau)d\xi \Big| + \Delta t \Big| \int_{-M}^{M} dy \int_{-\infty}^{t_1-2\Delta t} d\tau \int_0^1 dl$$

$$\times (1 + iy)e^{-iy\ln q}q^{-2} \int_{-\infty}^{+\infty} \Delta_x B(y, x, \xi)\Psi_3(x_2, t_2, \tau)d\xi \Big| + \Big| \int_{-M}^{M} dy$$

$$\times \int_{-\infty}^{t_1-2\Delta t} d\tau\, e^{-iy\ln(t_1-\tau)}(t_1 - \tau)^{-1} \int_{-\infty}^{+\infty} \Delta_x B(y, x, \xi)\Psi_3(x_2, t_2, t_1)d\xi \Big|.$$

Here we have used (7.20) in the third term and the properties of the function $\Psi_3(x, t_2, t_1)$ in the last item. Further, applying the mean value theorem to $\Delta_x B(y, x, \xi)$, and (7.20), we have the estimate of $\langle I_5 \rangle_{t,d_T}^{(\beta)}$. Note that the corresponding estimates of I_{6i}, and I_j, $j = \overline{7,9}$, can be obtained with the same approach. □

References

[1] H. Amann and M. Fila, *A Fujita-type theorem for the Laplace equation with a dynamical boundary condition,* Acta Math. Univ. Comenianae, LXVI, 2 (1997), 321–328.

[2] J. Escher, *Nonlinear elliptic systems with dynamic boundary conditions,* Math. Z., 210 (1992), 413–439.

[3] J. Escher, *Smooth solutions of nonlinear elliptic systems with dynamic boundary conditions,* Lecture Notes Pure Appl. Math. 155, M. Dekker (1994), 173–183.

[4] M. Fila and P. Quittner, *Global solutions of the Laplace equation with a nonlinear dynamical boundary condition,* Math. Meth. Appl. Sciences, 20 (1997), 1325–1333.

[5] B.V. Bazaliy, *On the classical solvability of the Hele-Shaw problem with a free boundary,* Ukr. Mat. Zh., 50, 11 (1998), 1452–1462.

[6] V.A. Solonnikov and E.V. Frolova, *On the third boundary value problem for the Laplace equation in a plane sector and applications to parabolic problems,* Algebra Anal., 2 (1990), 213–241.

[7] C.M. Elliott and J.R. Ockendon, "Weak and Variational Methods for Moving Boundary Problem," London, Pitman, 1982.

[8] S. Richardson, *Hele-Shaw flow with a free boundary produced by the injection of fluid into a narrow channel,* J. Fluid Mech., 56 (1972), 609–618.

[9] E. Di Benedetto and A. Friedman, *The ill-posed Hele-Shaw model and Stefan problem for supercooled water,* Trans. Amer. Math. Soc., 282, 1 (1984), 183–203.

[10] S.D. Howison, www.maths.ox.ac.uk./howison/Hele-Shaw (2006).

[11] J.R. King, A.A. Lacey, and J.L. Vazquez, *Persistence of corners in free boundaries in Hele-Shaw flow,* European J. Appl. Math., 6 (1995), 455–490.

[12] N. Vasylyeva, *On the solvability of the Hele-Shaw problem in the case of nonsmooth initial data in weighted Hölder classes,* Ukr. Math. Bulletin, 2, 3 (2005), 323–349.

[13] B.V. Bazaliy and N.V. Vasil'eva, *On the solvability of the Hele-Shaw model problem in weighted Hölder spaces in a plane angle,* Ukrainian Math. J., 52 (2000), 1647–1660.

[14] B.V. Bazaliy, *On a Stefan Problem,* Dokl. AN USSR, Ser. A, 11 (1986), 3–7.

[15] O.A. Ladyzhenskaia and N.N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Nauka, Moscow, 1973 (in Russian).

[16] V.A. Solonnikov, *On the solvability of the second initial boundary value problem for a linear nonstationary system of Navier-Stokes equations,* Zap. Nauch. Sem. LOMI, 69 (1977), 200–218.

[17] P. Grisvard, "Elliptic Problems in Nonsmooth Domains," Pitman, Boston, 1985.

[18] V.S. Vladimirov, "General Function in Mathematical Physics," Nauka, Moscow, 1979 (in Russian).

[19] O.A. Ladyzhenskaia, V.A. Solonnikov, and N.N. Ural'tseva, "Linear and Quasilinear Parabolic Equations," Nauka, Moscow, 1967 (in Russian).