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MOVEMENT OF HOT SPOTS ON THE EXTERIOR DOMAIN OF A BALL UNDER THE DIRICHLET BOUNDARY CONDITION

Kazuhiro Ishige
Mathematical Institute, Tohoku University
Aoba, Sendai 980-8578, Japan

(Submitted by: Daniele Andreucci)

Abstract. We consider the Cauchy-Dirichlet problem of the heat equation in the exterior domain of a ball, and study the movement of hot spots $H(t)$ as $t \to \infty$. In particular, we give a rate for the hot spots to run away from the boundary of the domain as $t \to \infty$. Furthermore we give a sufficient condition for the hot spots to consist of only one point after a finite time.

1. Introduction

We consider the Cauchy-Dirichlet problem of the heat equation in the exterior domain of a ball,

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $N \geq 2$, $L > 0$, and $\Omega = \{x \in \mathbf{R}^N : |x| > L\}$. Let u be the solution of (1.1). Then, under some suitable assumptions on the initial data ϕ, we may define

$$H(t) = \Big\{ x \in \overline{\Omega} \,:\, u(x,t) = \max_{y \in \overline{\Omega}} u(y,t) \Big\}, \qquad t > 0,$$

and call a point in $H(t)$ hot spot of the solution u at the time t. In this paper we study the movement of hot spots $H(t)$ of the solution u of (1.1) as $t \to \infty$.

Chavel and Karp [1] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbf{R}^N$, they proved that, for any nonzero, nonnegative initial

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data $\phi \in L^\infty_c(\mathbf{R}^N)$, the hot spots of the solution at each time $t > 0$ are contained in the closed convex hull of the support of ϕ, and the hot spots tend to the center of mass of ϕ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [7] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbf{R}^N_+$ and in the exterior domain of a ball, under the Dirichlet, the Neumann, and the Robin boundary conditions. In particular, for the Cauchy-Dirichlet problem (1.1) in the exterior domain of a ball in $\mathbf{R}^3$, they assumed that the initial data ϕ is a radially symmetric function belonging to $L^\infty_c(\Omega)$, and proved that there exist a positive constant T and a continuous function $r = r(t) \in C([T,\infty) : (L,\infty))$ such that $\lim_{t\to\infty} r(t)^3 t^{-1} = 2$ and

$$H(t) = \{x \in \mathbf{R}^N \,:\, |x| = r(t)\}, \quad t \ge T.$$

Recently, the author [6] studied the movement of the hot spots of the heat equation of the Cauchy-Neumann problem in the exterior domain $\Omega = \{x \in \mathbf{R}^N : |x| > L\}$ $(L > 0)$, and proved that, if $\int_\Omega \phi(x)dx > 0$ and

$$C_\phi \equiv \int_\Omega x\phi(x)\Big(1 + \frac{L^N}{(N-1)|x|^N}\Big)dx \Big/ \int_\Omega \phi(x)dx \neq 0,$$

then

$$\lim_{t\to\infty} \sup_{x\in H(t)} |x - x_\infty| = 0,$$

where $x_\infty = C_\phi$ if $C_\phi \in \Omega$ and $x_\infty = LC_\phi/|C_\phi|$ if $C_\phi \notin \Omega$.

In this paper we study the movement of hot spots for the Cauchy-Dirichlet problem (1.1), without the radial symmetry of the initial data ϕ. In particular, because of the Dirichlet boundary condition, the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$, and we study the rate and the direction for the hot spots to run away. Furthermore we give a sufficient condition for the hot spots to consist of only one point after a finite time.

Throughout this paper we assume

$$\Omega = \{x \in \mathbf{R}^N : |x| > L\}, \quad \phi \in L^2(\Omega, \rho\, dx), \tag{1.2}$$

where $\rho(x) = \exp(|x|^2/4)$. Put

$$m_\phi = \begin{cases} \displaystyle\int_\Omega \phi(x)\Big(1 - \frac{L^{N-2}}{|x|^{N-2}}\Big)dx & \text{if} \quad N \ge 3, \\ \displaystyle\int_\Omega \phi(x)\log\frac{|x|}{L}dx & \text{if} \quad N = 2. \end{cases}$$

We first give the following results on the asymptotic behavior of the solution u of (1.1), which imply that the hot spots run away from the boundary $\partial\Omega$ as $t \to \infty$ if $m_\phi > 0$.

Theorem 1.1. *Let u be a solution of the Cauchy-Dirichlet problem* (1.1) *under the condition* (1.2) *and $N \geq 3$. Then*

$$\lim_{t\to\infty} \int_\Omega u(x,t)dx = m_\phi \tag{1.3}$$

and

$$\lim_{t\to\infty} t^{\frac{N}{2}} u(x,t) = (4\pi)^{-\frac{N}{2}} m_\phi \Big(1 - \frac{L^{N-2}}{|x|^{N-2}}\Big) \tag{1.4}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Theorem 1.2. *Let u be a solution of the Cauchy-Dirichlet problem* (1.1) *under the conditions* (1.2) *and $N = 2$. Then there exists a constant C such that*

$$\|u(t)\|_{L^1(\Omega)} \leq C(\log t)^{-1} \|\phi\|_{L^2(\Omega, \rho\, dx)} \tag{1.5}$$

for all $t \geq 1$. Furthermore,

$$\lim_{t\to\infty} (\log t) \int_\Omega u(x,t)dx = 2m_\phi \tag{1.6}$$

and

$$\lim_{t\to\infty} t(\log t)^2\, u(x,t) = \frac{1}{\pi} m_\phi \log \frac{|x|}{L} \tag{1.7}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Remark 1.1. Collet, Martínes, and Martín [2] used the probability method to prove the asymptotic behavior of the Dirichlet heat kernel $G = G(x,y,t)$ on the exterior domain of a compact set as $t \to \infty$. In particular, for the exterior domain $\{x \in \mathbf{R}^N : |x| > L\}$ with $L > 0$, they obtained that

$$\lim_{t\to\infty} t^{\frac{N}{2}}\, G(x,y,t) = (4\pi)^{-\frac{N}{2}} \Big(1 - \frac{L^{N-2}}{|x|^{N-2}}\Big)\Big(1 - \frac{L^{N-2}}{|y|^{N-2}}\Big) \text{ if } N \geq 3, \tag{1.8}$$

$$\lim_{t\to\infty} t(\log t)^2\, G(x,y,t) = \frac{1}{\pi} \log \frac{|x|}{L} \log \frac{|y|}{L} \qquad \text{if } N = 2, \tag{1.9}$$

for all $x, y \in \Omega$ (see also [4]). By (1.4) and (1.7), we may obtain (1.8) and (1.9), and the proof of this paper is completely different from that of [2]. Furthermore, Herraiz [5] applied the comparison method to the Cauchy-Dirichlet problem (1.1) in the exterior domain of a compact set, and obtained some similar results to Theorems 1.1 and 1.2 for nonnegative initial data ϕ.

Next we give a result on the rate for the hot spots to run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 1.3. *Let u be a solution of the Cauchy-Dirichlet problem* (1.1) *under the condition* (1.2)*. Assume $m_\phi > 0$. Then, for any $t > 0$, $H(t) \neq \emptyset$ and*

$$\lim_{t\to\infty} \sup_{x\in H(t)} \left|\zeta(t)^{-1}|x|^N - 1\right| = 0, \tag{1.10}$$

where

$$\zeta(t) = 2(N-2)L^{N-2}t \quad \textit{if } N \geq 3, \quad \zeta(t) = 2t(\log t)^{-1} \quad \textit{if } N = 2.$$

Furthermore, there exists a positive constant T such that, if $x \in H(t)$ and $t \geq T$, then

$$H(t) \cap l_x = \{x\}, \tag{1.11}$$

where $l_x = \{kx/|x| \,:\, k \geq 0\}$.

Remark 1.2. Let u be a radial solution of the Cauchy-Dirichlet problem (1.1) under the condition (1.2) and $m_\phi > 0$. By the proof of Theorem 1.3 (see also (7.16) and (7.24)), we see that there exists a smooth curve $r = r(t) \in (L, \infty)$ such that

$$H(t) = \{x \in \mathbf{R}^N \,:\, |x| = r(t)\}$$

for all sufficiently large t.

Next we give a sufficient condition for the set $H(t)$ to consist of only one point $x(t)$ after a finite time. Furthermore, we give the limit of $x(t)/|x(t)|$ as $t \to \infty$.

Theorem 1.4. *Let u be a solution of the Cauchy-Dirichlet problem* (1.1) *under the condition* (1.2)*. Assume $m_\phi > 0$ and*

$$A_\phi \equiv \int_\Omega x\phi(x)\Big(1 - \frac{L^N}{|x|^N}\Big)dx \neq 0.$$

Then there exist a positive constant T and a smooth curve $x = x(t) \in C^\infty([T,\infty) : \Omega)$ such that $H(t) = \{x(t)\}$ for all $t \geq T$ and

$$\lim_{t\to\infty} \frac{x(t)}{|x(t)|} = \frac{A_\phi}{|A_\phi|}. \tag{1.12}$$

Therefore, under the conditions in Theorem 1.4, the set of hot spots $H(t)$ consists of one point $x(t)$ after a finite time, and

$$\lim_{t\to\infty} \zeta(t)^{-1/N}|x(t)| = 1, \qquad \lim_{t\to\infty} x(t)/|x(t)| = A_\phi/|A_\phi|.$$

Next we explain the idea of proving Theorems 1.1–1.4. As stated in [7], it is difficult to know the sign of the differential of the Dirichlet heat kernel even for the case that Ω is the exterior of a ball. So it seems difficult to obtain Theorems 1.3 and 1.4 by using the fundamental properties of the Dirichlet heat kernel. Following the strategy in [6], we divide the initial data ϕ into functions of the product by radial functions $\psi_k \in L^2(\Omega, \rho\, dx)$ and spherical harmonic functions $Q_{k,i}$ (see (7.1)), where $\rho(x) = \exp(|x|^2/4)$. Then we consider the radial solution v_k of the Cauchy-Dirichlet problem (L_k):

$$(L_k) \qquad \begin{cases} \partial_t v_k = \mathcal{L}_k v_k \equiv \Delta v_k - \dfrac{\omega_k}{|x|^2} v_k & \text{in} \quad \Omega \times (0,\infty), \\ v_k = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ v_k(x,0) = \psi_k(x) & \text{in} \quad \Omega, \end{cases}$$

where $k \in \mathbf{N} \cup \{0\}$ and $\omega_k = k(N+k-2)$. Here ω_k is the k-th eigenvalue of

$$-\Delta_{\mathbf{S}^{N-1}} Q = \omega Q \qquad \text{on} \quad \mathbf{S}^{N-1}, \tag{1.13}$$

where $\Delta_{\mathbf{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$. Furthermore, we define a rescaled function w_k of the solution v_k as follows:

$$w_k(y,s) = (1+t)^{\frac{N+k}{2}} v_k(x,t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t). \tag{1.14}$$

Then the function w_k satisfies

$$(P_k) \qquad \begin{cases} \partial_s w_k = P_k w_k + \dfrac{N+k}{2} w_k & \text{in} \quad W, \\ w_k = 0 & \text{on} \quad \partial W, \\ w_k(y,0) = \psi_k(y) & \text{in} \quad \Omega, \end{cases}$$

where

$$P_k w = \Delta_y w + \frac{y}{2} \cdot \nabla_y w - \frac{\omega_k}{|y|^2} w = \frac{1}{\rho} \mathrm{div}\,(\rho \nabla_y w) - \frac{\omega_k}{|y|^2} w,$$
$$\Omega(s) = e^{-s/2}\Omega, \ W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \ \partial W = \bigcup_{0<s<\infty} (\partial\Omega(s) \times \{s\}).$$

By studying the asymptotic behavior of $v_{k,i}$ and $w_{k,i}$, we obtain the asymptotic behavior of u and its derivatives as $t \to \infty$, and prove Theorems 1.1–1.4. The study of the asymptotic behaviors of $v_{k,i}$ and $w_{k,i}$, in particular for the case $N = 2$, is more complicated than that in [6]. Furthermore, in order to prove our theorems, we need more precise information on the asymptotic behavior of u than that in [6].

In order to prove Theorems 1.1–1.3, we study the asymptotic behavior of v_0 and its derivatives mainly. For the case $N \geq 3$, we study the asymptotic behavior of $w_0 = w_0(y,s)$ in the space L^2 with weight ρ, and obtain that of

$v_0 = v_0(x,t)$ for all $x \in \Omega$ with $|x| \sim t^{1/2}$ as $t \to \infty$. Furthermore, by using the radial symmetry of v_0 and (L_0), we obtain the asymptotic behavior of v_0 and its derivatives in the region $\{(x,t) : x \in \Omega,\ |x| = O(t^{1/2})\}$, and prove Theorem 1.1. For the case $N = 2$, the behavior of v_0 is different from that for the case $N \geq 3$. By using the similar arguments as in the case $N \geq 3$ several times, we obtain some estimates of $\partial_r v_0$ on $\partial\Omega$ and give the asymptotic behavior of v_0 and its derivatives in the region $\{(x,t) : x \in \Omega,\ |x| = O(t^{1/2})\}$. Then we obtain Theorem 1.2. Furthermore, we study the critical points of v_0 carefully, and obtain Theorem 1.3.

In order to prove Theorem 1.4, we study the asymptotic behavior of $x/|x|$ for all $x \in H(t)$, by using the asymptotic behavior of v_0 and v_1 (see Lemma 8.1). Furthermore, we compare $H(t)$ with the set of the hot spots for the radial solution of (1.1) with the initial data $\psi_0 \in L^2(\Omega, \rho\, dx)$ (see Lemma 8.2). Then we may prove that, if t is sufficiently large, the matrix $\{-\partial_{x_i}\partial_{x_j} u(x,t)\}_{i,j=1}^N$ is positive definite for all points near the hot spots $H(t)$, and complete the proof Theorem 1.4.

The rest of this paper is organized as follows: In Section 2 we recall preliminary lemmas on the decay rates of the solutions of (L_k) and on the eigenvalue problem for the operator P_0. In Section 3, by using the eigenfunctions of (P_0), we obtain the asymptotic behavior of w_k as $s \to \infty$, in the annulus $\{y \in \mathbf{R}^N : \epsilon \leq |y| \leq R\}$, where $0 < \epsilon < R$. In Sections 4 and 5 we study the behavior of the radial solution v_0 of (L_0) for the cases $N \geq 3$ and $N = 2$, respectively. In Section 6 we study the behavior of the radial solutions v_k of (L_k) for the case $k = 1, 2, \ldots$. In Section 7 we prove Theorems 1.1–1.3, by using the asymptotic behavior of v_0, v_1, and v_2 as $t \to \infty$. In Section 8 we study the asymptotic behavior of the matrix $\{-\partial_{x_i}\partial_{x_j} u(x,t)\}_{i,j=1}^N$ as $t \to \infty$, and prove Theorem 1.4.

2. Preliminaries

In this section we recall preliminary lemmas on the solutions of the Cauchy-Dirichlet problem (L_k) and on the eigenvalue problem for the operator P_0 in $\mathbf{R}^N$.

Let $k = 0, 1, 2, \ldots$ and $G_k = G_k(x,y,t)$ be the Green's function of the Cauchy-Dirichlet problem (L_k). Then G_k has the following properties:

$$G_k(x,y,t) = G_k(y,x,t), \quad t > 0; \tag{2.1}$$

$$G_k(x,y,t) = \int_\Omega G_k(x,z,t-s) G_k(z,y,s)\,dz, \quad t > s > 0; \tag{2.2}$$

$$0 < G_l(x,y,t) \leq G_k(x,y,t) \leq \Gamma(x,y,t), \quad t > 0, \tag{2.3}$$

for all $x, y \in \Omega$ and $l \geq k$, where $\Gamma = \Gamma(x, y, t)$ is the fundamental solution of the heat equation on $\mathbf{R}^N$. By (2.1)–(2.3), we have the following lemma (see also the proof of Lemma 2.2 in [6]).

Lemma 2.1. *Let v_k be a solution of (L_k) with $k = 0, 1, 2, \ldots$. Then there exists a positive constant C such that*

$$\|v_k(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{N}{4}}\|\psi_k\|_{L^1(\Omega)}, \tag{2.4}$$

$$\|v_k(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{4}}\|\psi_k\|_{L^2(\Omega)}, \tag{2.5}$$

$$\|\nabla_x v_k(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{2}}\|\psi_k\|_{L^2(\Omega)}, \tag{2.6}$$

$$\|\partial_t v_k(t)\|_{L^2(\Omega)} \leq Ct^{-1}\|\psi_k\|_{L^2(\Omega)}, \tag{2.7}$$

for all $t > 0$.

Next we consider the eigenvalue problem

$$(E) \qquad P_0\varphi \equiv \frac{1}{\rho}\mathrm{div}\,(\rho\nabla\varphi) = -\lambda\varphi \quad \text{in} \quad \mathbf{R}^N, \quad \varphi \in H^1(\mathbf{R}^N, \rho\, dy).$$

Let l_k be the dimension of the eigenspace of (1.13) corresponding to $\omega = \omega_k$ and $\{Q_{k,i}\}_{i=1}^{l_k}$ the eigenfunctions of (1.13) corresponding to $\omega = \omega_k$ such that $(Q_{k,i}, Q_{k,j})_{L^2(\mathbf{S}^{N-1})} = \delta_{ij}$, $i, j = 1, \ldots, l_k$. In particular, we may take

$$Q_{1,i}\left(\frac{x}{|x|}\right) = c_q \frac{x_i}{|x|}, \qquad i = 1, \ldots, N, \tag{2.8}$$

for some positive constant $c_q = c_q(N) > 0$. Furthermore, we have the following lemma on the eigenfunctions of (E) (see [3] and [9]).

Lemma 2.2. *Let $k = 0, 1, 2, \ldots$. Let $\{\lambda_{k,i}\}_{i=0}^\infty$ be the eigenvalues of*

$$(E_k) \qquad \begin{cases} P_k\varphi \equiv P_0\varphi - \dfrac{\omega_k}{|y|^2}\varphi = -\lambda\varphi & \text{in} \quad \mathbf{R}^N, \\ \varphi \text{ is a radial function in } \mathbf{R}^N, \\ \varphi \in L^2(\mathbf{R}^N, \rho\, dy), \end{cases}$$

such that $\lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \cdots$ and $\varphi_{k,i}$ the eigenfunction corresponding to $\lambda_{k,i}$ such that $\|\varphi_{k,i}\|_{L^2(\Omega, \rho\, dx)} = 1$. Then

$$\lambda_{k,i} = \frac{N+k}{2} + i, \qquad \varphi_{k,0}(y) = c_k|y|^k \exp\left(-\frac{|y|^2}{4}\right) \tag{2.9}$$

for some constants c_k. Furthermore, $\{\lambda_{k,i}\}_{k,i=0}^\infty$ give all eigenvalues of (E), and the eigenspace of (E) corresponding to the eigenvalue λ is spanned by the eigenfunctions $\{\varphi_{k,i}(y)Q_{k,j}(y/|y|)\}_{j=1}^{l_k}$ with $\lambda = \lambda_{k,i}$.

3. Asymptotic behavior of the solutions of (L_k)

In this section we study the asymptotic behaviors of w_k and v_k. In what follows, for any $s > 0$, we put $\|f\|_s = \|f\|_{L^2(\Omega(s),\rho\, dy)}$ for simplicity. Then we have the following lemma.

Lemma 3.1. *Let v_k be a radial solution of (L_k). Let w_k be a function defined by* (1.14)*. Then there exists a constant C such that*

$$\sup_{s>0} \|w_k(s)\|_s \le C\|\psi_k\|_{L^2(\Omega,\rho\, dx)}.$$

Proof. Put

$$W(s) = \int_{\Omega(s)} w_k(y,s)^2 \rho\, dy, \quad \tilde{w}_k(y,s) = \begin{cases} w_k(y,s) & (|y| \ge Le^{-s/2}), \\ 0 & (|y| < Le^{-s/2}). \end{cases}$$

By (P_k) and (2.9), for any $s > 0$, we have

$$\begin{aligned} W'(s) &= 2\int_{\Omega(s)} w_k(P_k w_k)\rho\, dy + (N+k)W(s) \\ &= -2\int_{\Omega(s)} \left\{ |\nabla w_k|^2 + \frac{N+k}{|y|^2} w_k^2 \right\} \rho\, dy + (N+k)W(s) \\ &= -2\int_{\Omega(s)} \left\{ |\nabla \tilde{w}_k|^2 + \frac{N+k}{|y|^2} \tilde{w}_k^2 \right\} \rho\, dy + (N+k)W(s) \le 0. \end{aligned}$$

This inequality gives Lemma 3.1. □

Then, by Theorem 10.1 of Chapter 4 in [8], Lemma 2.1, and Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let v_k be a radial solution of (L_k). Then there exists a constant C such that*

$$\|v_k(t)\|_{L^2(\Omega)} \le Ct^{-\frac{N}{4}-\frac{k}{2}} \|\psi_k\|_{L^2(\Omega,\rho\, dx)}, \tag{3.1}$$

$$\|v_k(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N+k}{2}} \|\psi_k\|_{L^2(\Omega,\rho\, dx)}, \tag{3.2}$$

$$\|\nabla v_k(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N+k}{2}} \|\psi_k\|_{L^2(\Omega,\rho\, dx)}, \tag{3.3}$$

for all sufficiently large t.

Next we give two lemmas on the asymptotic behavior of $w_k(s)$ as $s \to \infty$.

Lemma 3.3. *Assume the same conditions as in Lemma* 3.1*. Put*

$$a_k(s) = (w_k(s), \varphi_{k,0})_s, \quad s \ge 0.$$

Let $N \geq 3$ or $N = 2$ and $k \geq 1$. Then there exists a limit of $a_k(s)$ as $s \to \infty$ and

$$\lim_{s\to\infty} \left\| w_k(s) - \lim_{s\to\infty} a_k(s)\,\varphi_{k,0} \right\|_s = 0. \tag{3.4}$$

Proof. By (3.2) and (3.3), there exists a constant C such that

$$\|w_k(s)\|_{L^\infty(\Omega(s))} + e^{-s/2}\|\nabla w_k(s)\|_{L^\infty(\Omega(s))} \leq C \tag{3.5}$$

for all $s \geq 1$. By (P_k), (2.9) and (3.5), we have

$$a_k'(s) = \int_{\partial\Omega(s)} \partial_\nu w_k \varphi_{k,0} \rho\, d\sigma = O(e^{-(N+k-2)s/2}) \quad \text{as} \quad s \to \infty, \tag{3.6}$$

and we see the existence of the limit of $a_k(s)$ as $s \to \infty$.

Let $\tilde{w}_k$ be a function given in the proof of Lemma 3.1. Put

$$\hat{w}_k(s) = \tilde{w}_k - a_k(s)\varphi_{k,0}, \quad W(s) = \int_{\Omega(s)} \hat{w}_k(y,s)^2 \rho\, dy.$$

Then $\hat{w}_k$ satisfies

$$\partial_s \hat{w}_k = P_k \hat{w}_k + \frac{N+k}{2}\hat{w}_k - a_k'(s)\varphi_{k,0} \quad \text{in} \quad W.$$

By (3.5), we have

$$\begin{aligned} W'(s) &= 2(P_k\hat{w}_k, \hat{w}_k)_s + (N+k)W(s) \\ &\quad -2a_k'(s)W(s) - e^{-s/2}\int_{\partial\Omega(s)} \hat{w}_k^2 \rho \frac{y\cdot\nu}{|y|}\, d\sigma \\ &= 2(P_k\hat{w}_k, \hat{w}_k)_s + (N+k)W(s) + O(e^{-(N+k)s/2}) \\ &\leq -2W(s) + 2\int_{\partial\Omega(s)} \partial_\nu \hat{w}_k \hat{w}_k \rho\, d\sigma + O(e^{-(N+k)s/2}) \\ &= -2W(s) + O(e^{-(N+k-2)s/2}) + O(e^{-(N+k)s/2}) \end{aligned} \tag{3.7}$$

for all sufficiently large s. This implies (3.4), and the proof of Lemma 3.3 is complete. □

Lemma 3.4. *Assume the same conditions as in Lemma 3.3. Then, for any ϵ and R with $0 < \epsilon < R$,*

$$\lim_{s\to\infty} \|\partial_s^\alpha \partial_y^l (w_k(s) - a_k\varphi_{k,0})\|_{C(D(\epsilon,R))} = 0, \quad \alpha, l = 0,1,2,\ldots, \tag{3.8}$$

where $a_k = \lim_{s\to\infty} a_k(s)$ and $D(\epsilon,R) = \{y \in \mathbf{R}^N : \epsilon < |y| < R\}$.

Proof. By Lemma 3.1 and (P_k), for any ϵ and R with $0 < \epsilon < R$ and any α, $l \in \mathbf{N} \cup \{0\}$, there exist constants C and s_0 such that

$$\|(\partial_s^\alpha \nabla_y^l w)(s)\|_{C(D(\epsilon,R)\times[s_0,\infty))} \le C.$$

Let $\{s_j\}$ be any sequence with $\lim_{j\to\infty} s_j = \infty$. Put $w_j(y,s) = w(y,s+s_j)$. Then, by the Ascoli-Arzela theorem and a diagonal argument, taking a subsequence if necessary, we see that there exists a function $w_\infty \in C^\infty((\mathbf{R}^N \setminus \{0\}) \times [0,1])$ such that

$$\lim_{j\to\infty} \|\partial_s^\alpha \nabla_y^l w_j - \partial_s^\alpha \nabla_y^l w_\infty\|_{C(D(\epsilon,R)\times[0,1])} = 0$$

for all α, $l \in \mathbf{N} \cup \{0\}$ and all ϵ and R with $0 < \epsilon < R$. On the other hand, Lemma 3.3 implies that $w_\infty(y,s) = [\lim_{s\to\infty} a_k(s)]\varphi_{k,0}(y)$, and the limit function w_∞ is independent of the choice of the sequence $\{s_j\}$. So we have (3.8), and the proof of Lemma 3.4 is complete. $\square$

4. Radial solutions of (L_0) for the case $N \ge 3$

In this section we consider the asymptotic behavior of the radial solution v_0 of (L_0) for the case $N \ge 3$, as $t \to \infty$. For any radial function f in Ω, we put $f^*(r) = f(x)$ for all $r \ge L$ with $r = |x|$.

Lemma 4.1. *Let v_0 be a radial solution of (L_0) and $N \ge 3$. Then*

$$a_0 \equiv \lim_{s\to\infty} a_0(s) = \int_\Omega \psi_0(x) U_L^0(|x|)dx, \tag{4.1}$$

where

$$U_L^0(r) = c_0\Big(1 - \frac{L^{N-2}}{r^{N-2}}\Big).$$

Proof. By (L_0), we have

$$\int_\Omega v_0(x,t)U_L^0(|x|)dx = \int_\Omega \psi_0(x)U_L^0(|x|)dx, \quad t \ge 0. \tag{4.2}$$

By (1.14), we have

$$c_0 \int_{\Omega(s)} w_0(y,s)\Big(1 - \frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}\Big)dy = \int_\Omega v_0(x,t)U_L^0(|x|)dx \tag{4.3}$$

for all $s \ge 0$ with $s = \log(1+t)$. On the other hand, for any $\epsilon > 0$, we have

$$\int_{\Omega(s)\cap\{|y|\ge\epsilon\}} |w_0(y,s)| \frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}} dy \le \frac{(Le^{-s/2})^{N-2}}{|\epsilon|^{N-2}} \|w_0(s)\|_s.$$

By Lemma 3.3, we have

$$c_0 \lim_{s\to\infty} \int_{\Omega(s)\cap\{|y|\geq\epsilon\}} w_0(y,s)\Big(1-\frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}\Big)dy \tag{4.4}$$
$$= c_0 \lim_{s\to\infty} \int_{\Omega(s)\cap\{|y|\geq\epsilon\}} w_0(y,s)dy = c_0 a_0 \int_{\{|y|\geq\epsilon\}} \varphi_{0,0}(y)dy$$
$$= a_0 c_0^2 \int_{\mathbf{R}^N} e^{-|y|^2/4} dy + O(\epsilon^N) = a_0 + O(\epsilon^N).$$

Furthermore, we have

$$\Big| \int_{\Omega(s)\cap\{|y|\leq\epsilon\}} w_0(y,s)\Big(1-\frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}\Big)dy\Big| \tag{4.5}$$
$$\leq \int_{\Omega(s)\cap\{|y|\leq\epsilon\}} |w_0(y,s)|dy \leq \epsilon^{N/2}\|w(s)\|_s.$$

By Lemma 3.1, (4.4), (4.5), and the arbitrariness of ϵ, we have

$$c_0 \lim_{s\to\infty} \int_{\Omega(s)} w_0(y,s)\Big(1-\frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}\Big)dy = \lim_{s\to\infty} a_0(s). \tag{4.6}$$

Therefore, by (4.2), (4.3) and (4.6), we have (4.1), and the proof of Lemma 4.1 is complete. □

Lemma 4.2. *Let v_0 be a radial solution of (L_0) and $N \geq 3$. Then there exist functions $\zeta_1 = \zeta_1(t)$ and $\zeta_2 = \zeta_2(t)$ with*

$$\lim_{t\to\infty} t^{\frac{N+2}{2}}\zeta_1(t) = \frac{Na_0}{2}, \qquad \zeta_2(t) = O(t^{-\frac{N+4}{2}}) \text{ as } t\to\infty, \tag{4.7}$$

such that

$$(\partial_t v_0^*)(r,t) = -\zeta_1(t)U_L^0(r) + \zeta_2(t)O(r^2) \tag{4.8}$$

for all $r \geq L$ and $t \geq 1$.

Proof. Put $\tilde{v}_0^*(|x|,t) = \partial_t v_0^*(|x|,t) = \partial_t v_0(x,t)$. By (1.14) with $k = 0$, we have

$$\begin{aligned}(1+t)^{\frac{N+2}{2}}\tilde{v}_0^*(|x|,t) &= (1+t)^{\frac{N+2}{2}}(\partial_t v_0)(x,t)\\ &= -\frac{N}{2}w_0(y,s) - \frac{y}{2}\cdot\nabla_y w_0 + \partial_s w_0.\end{aligned}$$

Let ϵ be a sufficiently small positive constant. By Lemma 3.4, there exists a constant C, independent of ϵ, such that

$$\limsup_{t\to\infty} \Big|(1+t)^{\frac{N+2}{2}}\tilde{v}_0^*(\epsilon(1+t)^{\frac{1}{2}},t) + \frac{N}{2}c_0 a_0 e^{-\frac{\epsilon^2}{4}}\Big|$$

$$\leq \frac{\epsilon}{2} \lim_{s\to\infty} |\nabla_y w_0(y,s)| = \frac{\epsilon}{2} |\nabla_y \varphi_{0,0}(y)| \leq C\epsilon,$$

where $y \in \mathbf{R}^N$ with $|y| = \epsilon$. So we have

$$\lim_{t\to\infty} (1+t)^{\frac{N+2}{2}} \tilde{v}_0^*(\epsilon(1+t)^{\frac{1}{2}}, t) = -\frac{N}{2} c_0 a_0 + O(\epsilon). \tag{4.9}$$

By Lemmas 2.1 and 3.2, we have

$$\begin{cases} r^{N-1}\partial_t \tilde{v}_0^* = \partial_r (r^{N-1}\partial_r \tilde{v}_0^*), & \text{in} \quad (L,\infty)\times(0,\infty), \\ \tilde{v}_0^*(r,t) = 0 & \text{on} \quad \{L\}\times(0,\infty), \\ \|\partial_t \tilde{v}_0^*(t)\|_{L^\infty(L,\infty)} = O(t^{-\frac{N+4}{2}}) & \text{as} \quad t\to\infty, \end{cases}$$

where $r = |x|$. So we have

$$r^{N-1}(\partial_r \tilde{v}_0^*)(r,t) - L^{N-1}(\partial_r \tilde{v}_0^*)(L,t) \leq \frac{1}{N}(r^N - L^N)\|\partial_t \tilde{v}^*(t)\|_{L^\infty(L,\infty)},$$

and obtain

$$(\partial_r \tilde{v}_0^*)(r,t) = \Big(\frac{L}{r}\Big)^{N-1} (\partial_r \tilde{v}_0^*)(L,t) + \frac{r}{N}\Big(1 - \frac{L^N}{r^N}\Big) O(t^{-\frac{N+4}{2}}) \tag{4.10}$$

for all $r \in (L,\infty)$ and all $t \geq 1$. Put

$$\zeta_1(t) = \frac{L}{c_0(N-2)} (\partial_r \tilde{v}_0^*)(L,t). \tag{4.11}$$

By (4.10) and (4.11), we have

$$\tilde{v}_0^*(r,t) = \int_L^r (\partial_r \tilde{v}_0^*)(s,t)ds = -\zeta_1(t)U_L^0(r) + O(r^2)O(t^{-\frac{N+4}{2}}) \tag{4.12}$$

for all $r \geq L$ and $t \geq 1$. Furthermore, for any $\epsilon > 0$, by (4.9) and (4.12), we have

$$-t^{\frac{N+2}{2}} \zeta_1(t)(c_0 + o(1)) + O(\epsilon^2) = t^{\frac{N+2}{2}} \tilde{v}_0^*(\epsilon(1+t)^{1/2}, t) = -\frac{N}{2} c_0 a_0 + o(1) + O(\epsilon).$$

By the arbitrariness of ϵ, we obtain (4.7). Furthermore, by (4.7) and (4.12), we have (4.8), and the proof of Lemma 4.2 is complete. □

Lemma 4.3. *Let v_0 be a radial solution of (L_0) and $N \geq 3$. Then there exists a function $\zeta_3 = \zeta_3(t)$ with*

$$\lim_{t\to\infty} t^{\frac{N}{2}} \zeta_3(t) = a_0, \tag{4.13}$$

such that

$$v_0^*(r,t) = \zeta_3(t)U_L^0(r) + \zeta_1(t)O(r^2) + \zeta_2(t)O(r^4), \tag{4.14}$$

$$(\partial_r v_0^*)(r,t) = \zeta_3(t)\partial_r U_L^0(r) - \frac{c_0}{2} r\zeta_1(t)(1 + O(r^{-1})) + \zeta_2(t)O(r^3), \tag{4.15}$$

$$(\partial_r^2 v_0^*)(r,t) = \zeta_3(t)\partial_r^2 U_L^0(r) - U_L^0(r)\zeta_1(t) + O(r^2)\zeta_2(t), \tag{4.16}$$

for all $r \geq L$ and $t \geq 1$. Here $\zeta_1(t)$ and $\zeta_2(t)$ are given in Lemma 4.2.

Proof. Since v_0^* satisfies $r^{N-1}\partial_t v_0^* = \partial_r(r^{N-1}(\partial_r v_0^*))$, we have

$$(\partial_r v_0^*)(r,t) = \left(\frac{L}{r}\right)^{N-1}(\partial_r v_0^*)(L,t) + r^{1-N}\int_L^r \tau^{N-1}(\partial_t v_0^*)(\tau,t)d\tau. \tag{4.17}$$

Furthermore, by Lemma 4.2, we have

$$r^{1-N}\int_L^r \tau^{N-1}(\partial_t v_0^*)(\tau,t)d\tau = -\frac{c_0}{2}r\zeta_1(t)(1+O(r^{-1})) + \zeta_2(t)O(r^3) \tag{4.18}$$

for all $r \geq L$ and $t \geq 1$. Put

$$\zeta_3(t) = \frac{L}{c_0(N-2)}(\partial_r v_0^*)(L,t).$$

By (4.17) and (4.18), we have (4.15). By (L_0), Lemma 4.2, and (4.15), we have (4.16). Furthermore, by integrating (4.17) on $[0,r]$, we obtain (4.14).

On the other hand, for any $\epsilon > 0$, by Lemma 3.4, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}} v_0^*(\epsilon(1+t)^{1/2},t) = a_0 c_0 e^{-\epsilon^2/4}. \tag{4.19}$$

By (4.7) and (4.14), we have

$$t^{\frac{N}{2}} v_0^*(\epsilon(1+t)^{1/2},t) = t^{\frac{N}{2}}\zeta_3(t)(c_0+o(1)) + \epsilon^2 O(1) + \epsilon^4 O(1) \tag{4.20}$$

as $t \to \infty$. By the arbitrariness of ϵ, (4.19), and (4.20), we have (4.13), and the proof of Lemma 4.3 is complete. □

5. RADIAL SOLUTIONS OF (L_0) FOR THE CASE $N = 2$

In this section we consider the asymptotic behavior of the radial solution v_0 of (L_0) for the case $N = 2$, as $t \to \infty$. We first obtain an upper bound of $\partial_r v_0$ on $\partial\Omega$ for all sufficiently large t.

Lemma 5.1. *Let $N = 2$ and v_0 be a radial solution of (L_0). Then there exists a constant C such that*

$$|(\partial_r v_0^*)(L,t)| \leq Ct^{-1}(\log t)^{-1} \tag{5.1}$$

for all sufficiently large t.

Proof. By Lemmas 2.1 and 3.2, there exist positive constants C_1, C_2, and C_3 such that

$$\|(\partial_t v_0)(t)\|_{L^\infty(\Omega)} \leq C_1 t^{-\frac{1}{2}}\|(\partial_t v_0)(t/2)\|_{L^2(\Omega)} \tag{5.2}$$

$$\leq \quad C_2 t^{-\frac{3}{2}} \|v_0(t/4)\|_{L^2(\Omega)} \leq C_3 t^{-2}$$

for all sufficiently large t. This inequality together with (L_0) implies

$$\frac{1}{r}\partial_r(r(\partial_r v_0^*)) \leq \|(\partial_t v_0)(t)\|_{L^\infty(\Omega)} = O(t^{-2})$$

for all sufficiently large t. So we obtain

$$(\partial_r v_0^*)(r,t) = \frac{L}{r}(\partial_r v_0^*)(L,t) + \frac{1}{2r}(r^2 - L^2)O(t^{-2})$$

for all $r \geq L$ and all sufficiently large t. Putting $\zeta(t) = L(\partial_r v_0^*)(L,t)$, we have

$$v_0^*(r,t) = \zeta(t)\log\frac{r}{L} + O(r^2)O(t^{-2}) \tag{5.3}$$

for all $r \geq L$ and all sufficiently large t. By Lemma 3.2 and (5.3), we have

$$v_0^*(t^{1/2},t) = O(t^{-1}) = \frac{1}{2}\zeta(t)(\log t)(1+o(1)) + O(t^{-1}),$$

and obtain $\zeta(t) = O(t^{-1}(\log t)^{-1})$ as $t \to \infty$. This implies (5.1), and the proof of Lemma 5.1 is complete. □

Lemma 5.2. *Let $N = 2$ and v_0 be a radial solution of (L_0). Put*

$$\tilde{a}_0 = 2c_0 \int_\Omega \psi_0(x)\log\frac{|x|}{L}\,dx.$$

Then there exists a constant C such that

$$\|w_0(s)\|_s \leq C(1+s)^{-1}, \qquad s \geq 0, \tag{5.4}$$

$$\lim_{s\to\infty} s a_0(s) = \tilde{a}_0. \tag{5.5}$$

Proof. By (L_0) and (1.14), we have

$$\begin{aligned} \tilde{a}_0 &= 2c_0\int_\Omega v_0(x,t)\log\frac{|x|}{L}\,dx = 2c_0\int_{\Omega(s)} w_0(y,s)\log\frac{e^{s/2}|y|}{L}\,dy \\ &= s a_0(s) + 2c_0\int_{\Omega(s)} w_0(y,s)\log\frac{|y|}{L}\,ds \end{aligned} \tag{5.6}$$

for all $t \geq 0$ and $s \geq 0$ with $s = \log(1+t)$. By Lemma 3.1, there exists a constant C such that

$$\int_{\Omega(s)} |w_0(y,s)|\log\frac{|y|}{L}\,ds \leq C, \qquad s \geq 0.$$

This inequality together with (5.6) implies

$$|a_0(s)| = O(s^{-1}) \tag{5.7}$$

as $s \to \infty$. On the other hand, by Lemma 5.1, we have

$$e^{-s/2} \max_{y \in \partial\Omega(s)} |(\partial_\nu w_0)(y,s)| = O(s^{-1}) \tag{5.8}$$

as $s \to \infty$. Put $\hat{w}_0 = \tilde{w}_0 - a_0(s)\varphi_{0,0}$. By (5.7) and (5.8), we have

$$\begin{aligned} &\Big| \int_{\partial\Omega(s)} \partial_\nu \hat{w}_k \hat{w}_k \rho \, dy \Big| \qquad (5.9) \\ &= \Big| a_0(s)^2 \int_{\partial\Omega(s)} (\partial_\nu \varphi_{0,0}) \varphi_{0,0} \rho \, dy \Big| + \Big| a_0(s) \int_{\partial\Omega(s)} (\partial_\nu w_0) \varphi_{0,0} \rho \, dy \Big| = O(s^{-2}) \end{aligned}$$

as $s \to \infty$. Then, in the same way as in the proof of Lemma 3.3, we obtain

$$W'(s) \le -W(s) + O(s^{-2})$$

for all sufficiently large s, instead of (3.7). This inequality together with Lemma 3.1 implies

$$\|w_0(s) - a_0(s)\varphi_{0,0}\|_s = O(s^{-1})$$

for all sufficiently large s. Therefore, by (5.7), we obtain (5.4) and

$$\lim_{s \to \infty} \int_{\Omega(s)} w_0(y,s) \log \frac{|y|}{L} dy = 0.$$

Then, by (5.6), we have (5.5), and the proof of Lemma 5.2 is complete. □

By using Lemma 5.2, we improve the upper estimate (5.1) of $\partial_r v_0$.

Lemma 5.3. *Let $N = 2$ and v_0 be a radial solution of (L_0). Then there exists a constant C such that*

$$|(\partial_r v_0^*)(L,t)| \le C t^{-1} (\log t)^{-2} \tag{5.10}$$

for all sufficiently large t.

Proof. By (5.4), there exists a constant C_1 such that

$$\|v_0(t)\|_{L^1(\Omega)} \le C_1 (\log t)^{-1} \tag{5.11}$$

for all sufficiently large t. By Lemma 2.1, (5.2), and (5.11), there exist positive constants C_2, C_3, and C_4 such that

$$\begin{aligned} \|v_0(t)\|_{L^\infty(\Omega)} &\le C_2 t^{-\frac{1}{2}} \|v_0(t/4)\|_{L^2(\Omega)} \\ &\le C_3 t^{-1} \|v_0(t/8)\|_{L^1(\Omega)} \le C_4 t^{-1} (\log t)^{-1}, \\ \|(\partial_t v_0)(t)\|_{L^\infty(\Omega)} &\le C_2 t^{-\frac{3}{2}} \|v_0(t/4)\|_{L^2(\Omega)} \le C_4 t^{-2} (\log t)^{-1} \end{aligned}$$

for all sufficiently large t. Then, by the same argument as in the proof of Lemma 5.1, we put $\zeta(t) = L(\partial_r v_0^*)(L,t)$ and have

$$v_0^*(r,t) = \zeta(t)\log\frac{L}{r} + O(r^2)O(t^{-2}(\log t)^{-1}) = O(t^{-1}(\log t)^{-1})$$

for all $r \geq L$ and all sufficiently large t. So we have

$$v_0^*(t^{\frac{1}{2}},t) = \frac{1}{2}\zeta(t)(\log t)(1+o(1)) + O(t^{-1}(\log t)^{-1}) = O(t^{-1}(\log t)^{-1})$$

as $t \to \infty$. This implies that $L(\partial_r v_0^*)(L,t) = \zeta(t) = O(t^{-1}(\log t)^{-2})$ as $t \to \infty$. So we have (5.10), and the proof of Lemma 5.3 is complete. □

Next we give a result on the asymptotic behavior of w_0 as $s \to \infty$ for the case $N = 2$.

Lemma 5.4. *Let $N = 2$ and v_0 be a radial solution of (L_0). Then, for any ϵ and R with $0 < \epsilon < R$,*

$$\lim_{s\to\infty} \left\| \partial_s^\alpha \partial_y^l (sw_0(s) - \tilde{a}_0\varphi_{0,0}) \right\|_{C(D(\epsilon,R))} = 0, \quad \alpha, l = 0,1,2,\dots. \tag{5.12}$$

Proof. By Lemma 5.3, we have

$$e^{-s/2} \max_{y\in\partial\Omega(s)} |(\partial_\nu w_0)(y,s)| = O(s^{-2}) \quad \text{as} \quad s \to \infty,$$

instead of (5.8). Then we have

$$\left| \int_{\partial\Omega(s)} \partial_\nu \hat{w}_0 \hat{w}_0 \rho \, dy \right| = O(s^{-3})$$

as $s \to \infty$, instead of (5.9). Therefore, by the same argument as in the proof of Lemma 3.3, we have

$$\|w_0(s) - a_0(s)\varphi_{0,0}\|_s = O(s^{-3/2}) \quad \text{as} \quad s \to \infty.$$

This together with (5.5) implies that

$$\lim_{s\to\infty} \|sw_0(s) - \tilde{a}_0\varphi_{0,0}\|_s = 0. \tag{5.13}$$

On the other hand, by (5.4) and the parabolic regularity theorem, for any ϵ and R with $0 < \epsilon < R$ and $\alpha, l \in \mathbf{N} \cup \{0\}$, there exist constants C_1 and s_0 such that

$$\|\partial_s^\alpha \nabla_y^l w_0(y,s)\|_{C(D(\epsilon,R)\times[s,\infty))} \leq Cs^{-1}$$

for all $s \geq s_0$. So, for any ϵ and R with $0 < \epsilon < R$, we have

$$\|\partial_s^\alpha \nabla_y^l (sw_0)\|_{C(D(\epsilon,R)\times[s_0,\infty))} < \infty.$$

Therefore, by the Ascoli-Arzela theorem, a diagonal argument, and (5.13), we have (5.12), and the proof of Lemma 5.4 is complete. □

Next we study the asymptotic behavior of v_0 and $\partial_t v_0$ as $t \to \infty$ by using a similar argument to that of the case $N \geq 3$.

Lemma 5.5. *Let v_0 be a radial solution of (L_0) and $N = 2$. Then there exist functions $\zeta_1 = \zeta_1(t)$ and $\zeta_2 = \zeta_2(t)$ with*

$$\lim_{t\to\infty} t^2(\log t)^2\, \zeta_1(t) = 2\tilde{a}_0 c_0 \tag{5.14}$$

such that

$$(\partial_t v_0^*)(r,t) = -\Big(\log \frac{r}{L}\Big)\, \zeta_1(t) + O(r^2)O(t^{-3}(\log t)^{-1}) \tag{5.15}$$

for all $r \geq L$ and $t \geq 2$.

Proof. Put $\tilde{v}_0^* = \partial_t v_0^*$ for simplicity. By (L_0) and (1.14), we have

$$(1+t)^2 \log(1+t)\, \tilde{v}_0^*(|x|,t) = s\Big(-w_0(y,s) - \frac{y}{2}\cdot \nabla w_0 + \partial_s w_0\Big)$$

for all $t > 0$ and $s > 0$ with $s = \log(1+t)$. Let $\epsilon > 0$. Then, by Lemma 5.4, there exists a constant C, independent of ϵ, such that

$$\limsup_{t\to\infty} \Big|(1+t)^2 \log(1+t)\, \tilde{v}_0^*(\epsilon(1+t)^{\frac{1}{2}},t) + c_0\tilde{a}_0 e^{-\frac{\epsilon^2}{4}}\Big| \leq C\epsilon.$$

So we have

$$\lim_{t\to\infty} (1+t)^2 \log(1+t)\, \tilde{v}_0^*(\epsilon(1+t)^{\frac{1}{2}},t) = -c_0\tilde{a}_0 + O(\epsilon). \tag{5.16}$$

On the other hand, by Lemma 3.2 and (5.11), we have

$$\|\partial_t \tilde{v}_0^*(t)\|_{L^\infty(\Omega)} = O(t^{-3}(\log t)^{-1}) \quad \text{as} \quad t\to\infty.$$

By the same argument as in the proof of Lemma 5.1, we put $\zeta_1(t) = -L(\partial_r\tilde{v}_0^*)(L,t)$, and have (5.15). The equation (5.15) together with (5.16) implies that

$$\begin{aligned}&-\zeta_1(t)\cdot\frac{1}{2}(\log t)(1+o(1)) + \epsilon^2 O(t^{-2}(\log t)^{-1}) = \tilde{v}_0^*(\epsilon(1+t)^{\frac{1}{2}},t)\\ &= -c_0\tilde{a}_0 t^{-2}(\log t)^{-1}(1+o(1)) + t^{-2}(\log t)^{-1}O(\epsilon)\end{aligned}$$

as $t\to\infty$. Therefore, by the arbitrariness of ϵ, we have (5.14), and the proof of Lemma 5.5 is complete. □

Lemma 5.6. *Let v_0 be a radial solution of (L_0) and $N = 2$. Then there exists a function $\zeta_3 = \zeta_3(t)$ with*

$$\lim_{t\to\infty} t(\log t)^2 \zeta_3(t) = 2\tilde{a}_0 c_0, \tag{5.17}$$

such that

$$v_0^*(r,t) = \zeta_3(t)\log\frac{r}{L} + O(r^2\log r)\zeta_1(t) + O(r^4)O(t^{-3}(\log t)^{-1}), \tag{5.18}$$

$$(\partial_r v_0^*)(r,t) = \frac{\zeta_3(t)}{r} - \zeta_1(t) r\log r(1+o(1)) + O(r^3)O(t^{-3}(\log t)^{-1}), \tag{5.19}$$

$$(\partial_r^2 v_0^*)(r,t) = -\frac{\zeta_3(t)}{r^2} - U_L^0(r)\zeta_1(t) + O(r^2)O(t^{-3}(\log t)^{-1}), \tag{5.20}$$

for all $r \ge L$ and $t \ge 2$. Here $\zeta_1(t)$ is given in Lemma 5.5.

Proof. By (L_0) and Lemma 5.5, we have

$$\begin{aligned}(\partial_r v_0^*)(r,t) &= \frac{L}{r}(\partial_r v_0^*)(L,t) + r^{-1}\int_L^r \tau(\partial_t v_0^*)(\tau,t)d\tau \\ &= \frac{L}{r}(\partial_r v_0^*)(L,t) - \zeta_1(t) r\log r(1+o(1)) + O(r^3)\zeta_2(t)\end{aligned} \tag{5.21}$$

for all $r \ge L$ and $t \ge 2$. By an similar argument to that of the proof of Lemma 4.3, we put $\zeta_3(t) = L(\partial_r v_0^*)(L,t)$, and obtain (5.18)–(5.20). Furthermore, by (5.12) and (5.18), for any $\epsilon > 0$, we have

$$\begin{aligned}&\tilde{a}_0 c_0 e^{-\epsilon^2/4} = t(\log t)v_0^*(\epsilon t^{\frac{1}{2}},t) \\ &= t(\log t)\zeta_3(t)\tfrac{1}{2}(\log t)(1+o(1)) + \epsilon^2 t(\log t)\zeta_1(t)O(t\log t) + \epsilon^4 O(t^3\log t)\zeta_2(t) \\ &= t(\log t)\zeta_3(t)\tfrac{1}{2}(\log t)(1+o(1)) + \epsilon^2 O(1)\end{aligned}$$

as $t \to \infty$. By the arbitrariness of ϵ, we have (5.17), and the proof of Lemma 5.6 is complete. □

6. Radial solutions of (L_k) with $k \ge 1$

In this section we consider the asymptotic behavior of the radial solutions v_k of (L_k) as $t \to \infty$. For $k = 1, 2, \dots$, we put

$$U_L^k(r) = c_k r^k\Big(1 - \frac{L^{N+2k-2}}{r^{N+2k-2}}\Big).$$

We first give the following lemma on the asymptotic behavior of $a_1(s)$.

Lemma 6.1. *Let v_1 be a radial solution of (L_1) and $N \ge 2$. Then*

$$a_1 \equiv \lim_{s\to\infty} a_1(s) = \int_\Omega \psi_1(x)U_L^1(|x|)dx. \tag{6.1}$$

Proof. By (L_1) and (1.14), we have

$$a_1 = \int_\Omega \psi_1(x)U_L^1(|x|)dx = \int_\Omega v_1(x,t)U_L^1(|x|)dx \tag{6.2}$$

$$= \int_{\Omega(s)} w_1(y,s)c_1|y|\Big(1 - \frac{(Le^{-s/2})^{N-1}}{|y|^{N-1}}\Big)dy$$
$$= a_1(s) - c_1 Le^{-s/2}\int_{\Omega(s)} w(y,s)\frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}e^{-|y|^2/4}\rho\, dy$$

for all $t > 0$ and $s > 0$ with $s = \log(1+t)$. Furthermore, by Lemma 3.1, there exist constants C_1 and C_2 such that

$$\Big|\int_{\Omega(s)} w_1(y,s)\frac{(Le^{-s/2})^{N-2}}{|y|^{N-2}}e^{-|y|^2/4}\rho\, dy\Big| \tag{6.3}$$
$$\le \int_{\Omega(s)} |w_1(y,s)|e^{-|y|^2/4}\rho\, dy \le C_1\|w_1(s)\|_s \le C_2$$

for all $s > 0$. Therefore, by (6.2) and (6.3), we have (6.1), and the proof of Lemma 6.1 is complete. □

By an argument similar to that in Section 4, we give the following lemma on the asymptotic behavior of the solution v_1 of (L_1).

Lemma 6.2. *Let v_1 be a radial solution of (L_1) and $N \ge 2$. Then there exists a function $\zeta_4 = \zeta_4(t)$ with*

$$\lim_{t\to\infty} t^{\frac{N+2}{2}}\zeta_4(t) = a_1, \tag{6.4}$$

such that

$$\partial_r^j v_1^*(r,t) = \zeta_4(t)\partial_r^j U_L^1(r) + O(r^{2-j})O(t^{-\frac{N+3}{2}}) \tag{6.5}$$

for all $r \ge L$, $t > 1$, and $j = 0, 1, 2$. Furthermore, there holds that

$$\frac{\partial_r v_1^*(r,t)}{r} - \frac{v_1^*(r,t)}{r^2} = -\frac{N+1}{2(N-1)}a_1 t^{-\frac{N+4}{2}}(1+o(1))U_L^1(r) \tag{6.6}$$
$$+O(r^2)O(t^{-\frac{N+5}{2}}) + O(t^{-\frac{N+2}{2}}r^{-N-1})$$

for all $L \le r \le L(1+t)^{1/2}$ and $t > 1$.

Proof. Since $\partial_t v_1$ is also a solution of (L_1), by (2.5), (2.7), and (3.1), we have $\|\partial_t v_1(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N+3}{2}})$ as $t \to \infty$. Then, by (L_1), we have $\partial_t v_1^* = \partial_r^2 v_1^* + (N-1)\partial_r\Big(\dfrac{1}{r}v_1^*\Big)$, and obtain

$$(\partial_r v_1^*)(r,t) - (\partial_r v_1^*)(L,t) + \frac{N-1}{r}v_1^*(r,t) \le \|\partial_t v_1(t)\|_{L^\infty(\Omega)}(r-L)$$

for all $r \ge L$ and $t > 0$. So we have

$$\partial_r(r^{N-1}v_1^*) = r^{N-1}(\partial_r v^*)(L,t) + r^{N-1}(r-L)O(t^{-\frac{N+3}{2}})$$

for all $r \geq L$ and $t \geq 1$. Therefore, we put $\zeta_4(t) = (c_1 N)^{-1}(\partial_r v_1^*)(L,t)$, and obtain (6.5) by an argument similar to that in the proof of Lemma 4.3.

On the other hand, for any $\epsilon > 0$, Lemma 3.4 implies that

$$\epsilon^{-1} \lim_{t\to\infty} (1+t)^{\frac{N+1}{2}} v_1^*\Big(\epsilon(1+t)^{\frac{1}{2}}, t\Big) \tag{6.7}$$
$$= \epsilon^{-1} \lim_{s\to\infty} w_1(y,s) = \frac{a_1}{\epsilon}\varphi_{1,0}(y) = a_1 c_1 e^{-\epsilon^2/4}$$

where $y \in \mathbf{R}^N$ with $|y| = \epsilon$. Furthermore, by (6.5), we have

$$\epsilon^{-1} t^{\frac{N+1}{2}} v_1^*\Big(\epsilon(1+t)^{\frac{1}{2}}, t\Big) = t^{\frac{N+2}{2}}\zeta_4(t)(c_1 + o(1)) + \epsilon^2 O(1) \tag{6.8}$$

for all sufficiently large t. Then, by the arbitrariness of ϵ, (6.7), and (6.8), we have (6.4).

On the other hand, since $\partial_t v_1$ is also a solution of (L_1), in a way similar to that in the proof of Lemma 4.2, there exists a function $\tilde{\zeta}_4(t)$ such that

$$\partial_t v_1^*(r,t) = \tilde{\zeta}_4(t) U_L^1(r) + O(r^2)O(t^{-\frac{N+5}{2}}) \tag{6.9}$$

for all $r \geq L$ and $t > 1$ and

$$\lim_{t\to\infty} (1+t)^{\frac{N+4}{2}} \tilde{\zeta}_4(t) = -\frac{N+1}{2} a_1.$$

Then, by (L_1), we have

$$\frac{\partial_r v_1^*(r,t)}{r} - \frac{v_1^*(r,t)}{r^2} = (N-1)^{-1}[\partial_t v_1^*(r,t) - \partial_r^2 v_1^*(r,t)],$$

and, by (6.4), (6.5), (6.9), and the definition of U_L^1, we have (6.6). Therefore, the proof of Lemma 6.2 is complete. □

Next we assume $a_1 = 0$ and study the asymptotic behavior of v_1 again.

Lemma 6.3. *Let v_1 be a radial solution of (L_1) and $N \geq 2$. Assume that $a_1 = 0$. Then there exists a function ζ_4 with*

$$\zeta_4(t) = O(t^{-\frac{N+4}{2}}) \quad as \quad t \to \infty \tag{6.10}$$

such that

$$\partial_r^j v_1^*(r,t) = \zeta_4(t)\partial_r^j U_L^1(r) + O(r^{2-j})O(t^{-\frac{N+5}{2}}) \tag{6.11}$$

for all $r \geq L$, $t > 1$, and $j = 0, 1, 2$.

Proof. Put $\hat{w}_1 = \tilde{w}_1 - a_1(s)\varphi_{1,0}$. By (3.6) and $a_1 = 0$, we have $a_1(s) = O(e^{-s/2})$ as $s \to \infty$. Then, by Lemma 6.2, we have

$$\begin{aligned} \hat{w}_1(y,s) &= -a_1(s)\varphi_{1,0}(y) = O(e^{-s}), \\ \partial_\nu \hat{w}_1(y,s) &= \partial_\nu w_1(y,s) - a_1(s)\partial_\nu \varphi_{1,0}(y,s) = o(1) \end{aligned}$$

for all $y \in \partial\Omega(s)$ and all sufficiently large s. Then we apply the same argument as in the proof of Lemma 3.3 to $\hat{w}_1$, and have

$$W'(s) \le -2W(s) + O(e^{-3s/2}) + O(e^{-3s})$$

for all sufficiently large s. So we have $W(s) = O(e^{-3s/2})$ as $s \to \infty$, and obtain $\|v_1(t)\|_{L^2(\Omega)} = O(t^{-\frac{N}{4}-\frac{1}{2}-\frac{3}{4}})$ as $t \to \infty$. Then, by (2.5), we have $\|v_1(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N}{2}-\frac{1}{2}-\frac{3}{4}})$ as $t \to \infty$. Therefore, by (6.8), we have

$$\zeta_4(t) = O(t^{-\frac{N+2}{2}-\frac{3}{4}})$$

as $t \to \infty$, and obtain $\max_{x\in\partial\Omega} |\partial_r v_1(x,t)| = O(t^{-\frac{N+2}{2}-\frac{3}{4}})$ as $t \to \infty$. This implies that

$$\max_{y\in\partial\Omega(s)} |(\partial_\nu w_1)(y,s)| = O(e^{-3s/2}) \quad \text{as} \quad s \to \infty.$$

Then, by (3.6), we have $a_1(s) = O(e^{-5s/2})$ as $s \to \infty$, and obtain

$$\max_{y\in\partial\Omega(s)} |\hat{w}_1(y,s)| = e^{-3s}, \quad \max_{y\in\partial\Omega(s)} |(\partial_\nu \hat{w}_1)(y,s)| = O(e^{-3s/2})$$

as $s \to \infty$. Therefore, we apply the same argument as in the proof of Lemma 3.3 to $\hat{w}_1$ again, and obtain

$$W'(s) \le -2W(s) + O(e^{-5s}) + O(e^{-6s}e^{-Ns/2}).$$

So we have $W(s) = O(e^{-2s})$ as $s \to \infty$. This together with Lemma 2.1 implies that

$$\|v_1(t)\|_{L^2(\Omega)} = O(t^{-\frac{N}{4}-\frac{1}{2}-1}), \quad \|v_1(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N+1}{2}-1}),$$

as $t \to \infty$. Then, by (6.8), we have (6.10). Furthermore, applying the same argument with in the proof of Lemma 6.2 to v_1, we may obtain (6.11), and the proof of Lemma 6.3 is complete. □

Next we give a result on the asymptotic behavior on the solution v_k of (L_k) with $k \ge 2$.

Lemma 6.4. *Let $k \ge 2$, v_k be a radial solution of (L_k), and $N \ge 2$. Then there exists a function $\zeta_5 = \zeta_5(t)$ with*

$$\zeta_5(t) = O(t^{-\frac{N+k+2}{2}} \log t) \ \textit{as}\ t \to \infty, \tag{6.12}$$

such that

$$\partial_r^j v_k^*(r,t) = \zeta_5(t)\partial_r^j U_L^k(r) + O(t^{-\frac{N+k+2}{2}})r^{k-j} \log \frac{r}{L} \tag{6.13}$$

for all $r \ge L$, $t > 1$, and $j = 0, 1, 2$.

Proof. Let $k \geq 2$. Since $\partial_t v_k$ is also a solution of (L_k), by (2.5), (2.7), and (3.1), we have $\|\partial_t v_k(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N+k+2}{2}})$ as $t \to \infty$. So, by (L_k), we have

$$\partial_r\Big(\frac{1}{r^{k-1}}\partial_r v_k^*\Big) + (N+k-2)\partial_r\Big(\frac{v_k^*}{r^k}\Big) = \frac{1}{r^{k-1}}\partial_t v_2^* = \frac{1}{r^{k-1}}O(t^{-\frac{N+k+2}{2}})$$

and obtain

$$\frac{1}{r^{k-1}}(\partial_r v_k^*)(r,t) - \frac{1}{L^{k-1}}(\partial_r v_k^*)(L,t) + \frac{N+k-2}{r^k}v_2^*(r,t) = O(t^{-\frac{N+k+2}{2}})\log\frac{r}{L}$$

for all $r \geq L$ and $t > 0$. Furthermore, we have

$$\partial_r(r^{N+k-2}v_k^*(r,t)) = \frac{r^{N+2k-3}}{L^{k-1}}(\partial_r v_2^*)(L,t) + O(t^{-\frac{N+k+2}{2}})r^{N+2k-3}\log\frac{r}{L}.$$

Therefore, we put $\zeta_5(t) = (\partial_r v_k^*)(L,t)/c_2 L^{k-1}(N+2k-2)$ and obtain (6.13) by an argument similar to that in the proof of Lemma 4.3. Furthermore, since $\|v_2(t)\|_{L^\infty(\Omega)} = O(t^{-\frac{N+k}{2}})$ as $t \to \infty$, by (6.13) with $j = 1$ and $r = t^{1/2}$, we have (6.12), and the proof of Lemma 6.4 is complete. □

7. PROOF OF THEOREMS 1.1–1.3

Let u be a solution of (1.1) under the condition (1.2). By Lemma 2.2, there exist radial functions $\{\psi_{k,j}\}_{k\in\mathbf{N}\cup\{0\},j=1,\dots,l_k}$, such that $\psi_{k,i} \in L^2(\Omega,\rho\,dx)$ and

$$\lim_{n\to\infty}\Big\|\phi - \sum_{k=0}^{n}\sum_{j=1}^{l_k}\psi_{k,j}(|x|)Q_{k,j}\Big(\frac{x}{|x|}\Big)\Big\|_{L^2(\Omega,\rho\,dx)} = 0. \tag{7.1}$$

Put

$$\phi_0 = \psi_{0,1}, \quad \phi_k = \sum_{j=1}^{l_k}\psi_{k,j}Q_{k,j} \quad (k=1,2,3), \quad \phi_4 = \phi - \sum_{i=0}^{3}\phi_i.$$

Let u_i and $u_{k,j}$ be a solution of (1.1) and (L_k) with the initial data ϕ and $\phi_{k,j}$, respectively. Then we have $u = \sum_{k=0}^4 u_k$ and see that $u_{k,j} = u_{k,j}(x,t)$ is a radial function for all $t > 0$ and

$$u_k(x,t) = \sum_{j=1}^{l_k}Q_{k,j}\Big(\frac{x}{|x|}\Big)u_{k,j}(x,t), \quad k = 0,1,2,3. \tag{7.2}$$

Furthermore, we have

$$(u_4(t),\varphi\,Q_{k,j})_{L^2(\Omega)} = 0, \quad k = 0,1,2,3,\ j = 1,\dots,l_k \tag{7.3}$$

for all radial functions $\varphi \in L^2(\Omega, \rho\, dx)$. By (7.3), we apply the same argument as in the proof of Lemma 3.1 (see also Lemma 6.1 of [6]) to the solution u_4, and obtain the following lemma.

Lemma 7.1. *Assume* (1.2). *Let* ϕ_4 *be a function defined by* (7.2). *Let* u_4 *be a function of* (1.1) *with the initial data* ϕ_4. *Then there exists a constant* C *such that*

$$\|\nabla_x^k u_4(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N+4}{2}}, \quad k = 0, 1, 2, \tag{7.4}$$

for all sufficiently large t.

We are now ready to prove Theorems 1.1–1.3.

Proof of Theorem 1.1. By Lemmas 4.3, 6.1–6.4, and 7.1, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}} u_0(x,t) = aU_L^0(|x|), \quad \lim_{t\to\infty} t^{\frac{N}{2}} u_k(x,t) = 0, \ k = 1, 2, 3, 4,$$

where

$$a = \int_\Omega \psi_{0,1}(x) U_L^0(|x|) dx = c_0 \int_\Omega \phi(x) \Big(1 - \frac{L^{N-2}}{|x|^{N-2}}\Big) dx. \tag{7.5}$$

Therefore, we obtain (1.3), and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By (5.11) and Lemma 5.6, we have

$$\|u_0(t)\|_{L^1(\Omega)} = O((\log t)^{-1}), \quad \lim_{t\to\infty} t(\log t)^2 u_0(x,t) = 2\tilde{a} c_0 \log \frac{|x|}{L},$$

where

$$\tilde{a} = 2c_0 \int_\Omega \psi_{0,1}(x) \log \frac{|x|}{L} dx = 2c_0 \int_\Omega \phi(x) \log \frac{|x|}{L} dx. \tag{7.6}$$

Then, by an argument similar to that in the proof of Theorem 1.1, we obtain (1.5) and (1.7), and the proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3 for the case $N \ge 3$. Since

$$\frac{d}{dt} \int_\Omega u(x,t) \Big(1 - \frac{L^{N-2}}{|x|^{N-2}}\Big) dx = 0,$$

for any $t > 0$, we have $m_{u(t)} = m_\phi > 0$ and see that $u(x_0, t) > 0$ for some point $x_0 \in \mathbf{R}^N$. Furthermore, by (1.2) and (2.3), we have $\lim_{|x|\to\infty} u(x,t) = 0$. Therefore, we see that $H(t) \ne \emptyset$ for any $t > 0$.

Next we prove (1.10). By Theorem 1.1 and (2.3), we see easily that

$$\lim_{t\to\infty} \inf\{|x| : x \in H(t)\} = \infty, \tag{7.7}$$

$$\sup\{|x| \,:\, x \in H(t)\} = O(t^{1/2}) \tag{7.8}$$

for all sufficiently large t. Furthermore, by Lemmas 3.4 and 7.1, for any R_1 and R_2 with $L < R_1 < R_2$, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}} u_0\Big(rt^{1/2}\frac{x}{|x|}, t\Big) = ac_0 e^{-r^2/4}, \tag{7.9}$$

$$\lim_{t\to\infty} t^{\frac{N}{2}} u_k\Big(rt^{1/2}\frac{x}{|x|}, t\Big) = 0, \ k = 1,2,3,4, \tag{7.10}$$

uniformly for all $r \in [R_1, R_2]$, where $a = c_0 m_\phi > 0$. By (7.8)–(7.10), for any $\epsilon > 0$, there exists a constant T such that

$$\sup\{|x| : x \in H(t)\} \le \epsilon t^{1/2}, \quad t \ge T. \tag{7.11}$$

Let $b = b(t) \in H(t)$ and $t \ge T$. By Lemmas 6.2, 6.4, 7.1, and (7.11), we have

$$t^{\frac{N}{2}} \sum_{k=1}^{4} \partial_r u_k(b,t) = O(t^{-1}) \tag{7.12}$$

for all sufficiently large t. By Lemma 4.3 and (7.11), we have

$$t^{\frac{N}{2}} \partial_r u_0(b,t) = a(1+o(1))(\partial_r U_L^0)(|b|) - \frac{ac_0}{2}|b|(1+o(1)) + O(t^{-2})O(|b|^3) \tag{7.13}$$

for all sufficiently large t. By (7.12) and (7.13), we have

$$\begin{aligned} 0 = t^{\frac{N}{2}}(\partial_r u)(b,t) &= ac_0(1+o(1))(N-2)L^{N-2}|b|^{1-N} \\ &\quad -\frac{ac_0}{2t}(1+o(1))|b| + O(t^{-2})O(|b|^3) + O(t^{-1}). \end{aligned} \tag{7.14}$$

Therefore, by (7.11), (7.7) and (7.14), we have

$$2(N-2)L^{N-2}|b(t)|^{-N}(1+o(1)) = t^{-1}(1+o(1)) \tag{7.15}$$

as $t \to \infty$, and obtain (1.10). Furthermore, by Lemmas 6.1–6.3, 7.1, and (4.16), for any positive constants R_1' and R_2' with $R_1' < R_2'$, there exist positive constants C and T such that

$$\begin{aligned} (\partial_r^2 u)(x,t) &= a_0 t^{-\frac{N}{2}}(1+o(1))(\partial_r^2 U_L^0)(|x|) \\ &\quad -\frac{Na_0}{2} t^{-\frac{N+2}{2}}(1+o(1))U_L^0(r) + O(t^{\frac{2}{N}})O(t^{-\frac{N+4}{2}}) + o(t^{-\frac{N+2}{2}}) \\ &= -Ct^{-\frac{N+2}{2}}(1+o(1)) < 0 \end{aligned} \tag{7.16}$$

for all $(x,t) \in \Omega \times [T,\infty)$ with $R_1' \le t^{-\frac{1}{N}}|x| \le R_2'$. Then, by (1.10), we have (1.11), and the proof of Theorem 1.3 for the case $N \ge 3$ is complete. □

Proof of Theorem 1.3 for the case $N = 2$. Similarly to in the case $N \geq 3$, we see that $H(t) \neq \emptyset$ for all $t > 0$. On the other hand, by Lemma 5.4, for any R_1 and R_2 with $0 < R_1 < R_2$, we have

$$\lim_{t\to\infty} t(\log t)u_0^*(rt^{1/2}, t) = \tilde{a}c_0 e^{-r^2/4}$$

uniformly for all $r \in [R_1, R_2]$, where $\tilde{a} = 2c_0 m_\phi > 0$. By the similar way to in the case $N \geq 3$, we have (7.11), (7.7) and (7.12), and by Lemma 5.6, we obtain

$$\begin{aligned} 0 &= t(\log t)(\partial_r u)(b,t) = t(\log t)(\partial_r u_0)(b,t) + O(t^{-1}(\log t)) \\ &= 2\tilde{a}c_0(\log t)^{-1}(1+o(1))|b|^{-1} \\ &\quad -2\tilde{a}c_0 t^{-1}(\log t)^{-1}(1+o(1))|b|(\log|b|)(1+o(1)) \\ &\quad +O(|b|^3)O(t^{-2}) + O(t^{-1}(\log t)) \end{aligned}$$

for all $b = b(t) \in H(t)$. Therefore there exists a constant C_1 such that

$$\begin{aligned} &\left| |b|^{-2} - t^{-1}(1+o(1))(\log|b|) \right| \\ &\qquad \leq C_1|b|^2 t^{-2}(\log t) + C_1|b|^{-1}t^{-1}(\log t)^2 \end{aligned} \tag{7.17}$$

for all sufficiently large t.

Let T be a positive constant to be chosen later. Assume that there exists $t \geq T$ and $b \in H(t)$ such that $|b| \geq 4t^{\frac{1}{2}}(\log t)^{-\frac{1}{2}}$. Taking sufficiently large T if necessary, we have

$$\begin{aligned} |b|^{-2} - t^{-1}(1+o(1))(\log|b|) &\leq \frac{1}{16}t^{-1}\log t - \frac{3}{4}t^{-1}\log(4t^{\frac{1}{2}}(\log t)^{-\frac{1}{2}}) \\ &\leq -4^{-1}t^{-1}\log t, \end{aligned}$$

and obtain

$$\left| |b|^{-2} - t^{-1}(\log|b|) \right| \geq 4^{-1}t^{-1}\log t. \tag{7.18}$$

On the other hand, by (7.8), (7.11), and (7.18), taking sufficiently large T if necessary, we have

$$\begin{aligned} C_1|b|^2t^{-2}\log t + C_1 t^{-1}|b|^{-1}(\log t) &\leq 8^{-1}t^{-1}\log t \\ &< ||b|^{-2} - t^{-1}(1+o(1))(\log|b|)|, \end{aligned} \tag{7.19}$$

which contradicts (7.17). Therefore we see that

$$|b| < 4t^{\frac{1}{2}}(\log t)^{-\frac{1}{2}} \tag{7.20}$$

for all $b \in H(t)$ and sufficiently large t.

On the other hand, by (7.20), we have

$$t^{-1}(1+o(1))|b|(\log|b|) = O\Big(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}\Big), \tag{7.21}$$

$$|b|^3 t^{-2}\log t + t^{-1}(\log t) = O\Big(t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}\Big) \tag{7.22}$$

for all $b \in H(t)$ and all sufficiently large t. By (7.17), (7.21), and (7.22), we have

$$|b|^{-1} = O\Big(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}\Big) \tag{7.23}$$

for all $b \in H(t)$ and all sufficiently large t. Therefore, by (7.17), (7.20), and (7.23), we have

$$|b|^{-2} - \frac{1}{2}t^{-1}(\log t)(1+o(1)) = O(t^{-1})$$

for all $b \in H(t)$ and all sufficiently large t. This implies (1.10) for the case $N = 2$. Furthermore, by Lemmas 5.6, 6.2, 6.4, and 7.1, for any positive constants R_1 and R_2 with $R_1 < R_2$, there exist positive constants C_2 and T' such that

$$(\partial_r^2 u)(x,t) \le -C_2 t^{-2}(\log t)^{-1} < 0 \tag{7.24}$$

for all $(x,t) \in \Omega \times [T',\infty)$ with $R_1 \le t^{-1}(\log t)|x|^2 \le R_2$. This together with (1.10) implies (1.11), and the proof of Theorem 1.3 for the case $N = 2$ is complete. □

8. PROOF OF THEOREM 1.4.

Let $e_1 = (1,0,\dots,0)$. In order to prove Theorem 1.4, we may assume, without loss of generality, that

$$A_\phi = |A_\phi| e_1. \tag{8.1}$$

By Theorem 1.3, there exist functions $\eta_1(t)$ and $\eta_2(t)$ with $\eta_1(t) < \eta_2(t)$ and $\lim_{t\to\infty}\eta_i(t) = 1$ $(i = 1,2)$ such that

$$H(t) \subset B(t) \equiv \{x \in \mathbf{R}^N : \eta_1(t)\zeta(t) < |x|^N < \eta_2(t)\zeta(t)\} \tag{8.2}$$

for all $t \ge 2$, where $\zeta = \zeta(t)$ is the function given in Theorem 1.3. By Lemmas 4.3 and 5.6, we have

$$\begin{cases} u_0(x,t) = d_0 t^{-\frac{N}{2}}(1+o(1)), \\ \partial_r u_0(x,t) = O(t^{-\frac{N+2}{2}+\frac{1}{N}}), \\ \partial_r^2 u_0(x,t) = -d_1 t^{-\frac{N+2}{2}}(1+o(1)) \end{cases} \tag{8.3}$$

for all $x \in B(t)$ if $N \geq 3$ and

$$\begin{cases} u_0(x,t) = \tilde{d}_0 t^{-1}(\log t)^{-1}(1+o(1)), \\ \partial_r u_0(x,t) = O(t^{-\frac{3}{2}}(\log t)^{-\frac{3}{2}}), \\ \partial_r^2 u_0(x,t) = -\tilde{d}_1 t^{-2}(\log t)^{-1}(1+o(1)) \end{cases} \tag{8.4}$$

for all $x \in B(t)$ if $N = 2$, where d_0, d_1, $\tilde{d}_0$, and $\tilde{d}_1$ are positive constants. Furthermore, we have

$$(\partial_{x_i} u_0)(x,t) = (\partial_r u_0)(x,t)\frac{x_i}{|x|}, \tag{8.5}$$

$$(\partial_{x_i}\partial_{x_j} u_0)(x,t) = (\partial_r^2 u_0)(x,t)\frac{x_i x_j}{|x|^2} + (\partial_r u_0)(x,t)\frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3}, \tag{8.6}$$

for all $(x,t) \in \Omega \times (0,\infty)$. On the other hand, by Lemma 6.2 and (8.1), we have

$$\begin{cases} u_{1,1}(x,t) = d_{2,1} t^{-\frac{N+2}{2}}|x|(1+o(1)), \\ \partial_r u_{1,1}(x,t) = d_{2,1} t^{-\frac{N+2}{2}}(1+o(1)), \\ \partial_r^2 u_{1,1}(x,t) = O(t^{-\frac{N+3}{2}}), \\ \dfrac{\partial_r u_{1,1}(x,t)}{|x|} - \dfrac{u_{1,1}(x,t)}{|x|^2} = -d_{2,2} t^{-\frac{N+4}{2}}|x|(1+o(1)) \end{cases} \tag{8.7}$$

for all $x \in B(t)$. where $d_{2,1}$ and $d_{2,2}$ are positive constants. On the other hand, by Lemma 6.3 and (8.1), we have

$$\begin{cases} u_{1,i}(x,t) = O(t^{-\frac{N+4}{2}}|x|), \\ \partial_r u_{1,i}(x,t) = O(t^{-\frac{N+4}{2}}), \\ \partial_r^2 u_{1,i}(x,t) = O(t^{-\frac{N+5}{2}}) \end{cases} \tag{8.8}$$

for all $x \in B(t)$ and $i = 2, \ldots, N$. Furthermore, we have

$$\frac{(\partial_{x_i} u_1)(x,t)}{c_q} = \sum_{k=1}^{N}(\partial_r u_{1,k})(x,t)\frac{x_k x_i}{|x|^2} + \sum_{k=1}^{N} u_{1,k}(x,t)\frac{\delta_{ki}|x|^2 - x_k x_i}{|x|^3}, \tag{8.9}$$

$$\begin{aligned} \frac{(\partial_{x_i}\partial_{x_j} u_1)(x,t)}{c_q} &= \sum_{k=1}^{N}(\partial_r^2 u_{1,k})(x,t)\frac{x_k x_i x_j}{|x|^3} \\ &\quad + \sum_{k=1}^{N}\Big(\frac{(\partial_r u_{1,k})(x,t)}{|x|} - \frac{u_{1,k}(x,t)}{|x|^2}\Big) \\ &\qquad \times \frac{|x|^2(\delta_{ki}x_j + \delta_{kj}x_i + \delta_{ij}x_k) - 3x_i x_j x_k}{|x|^3}, \end{aligned} \tag{8.10}$$

for all $(x,t)\in\Omega\times(0,\infty)$. Then we prove the following lemma.

Lemma 8.1. *Assume the same conditions as in Theorem* 1.4 *and* (8.1). *Put*

$$\eta(t)=\sup_{b\in H(t)}\sqrt{1-\frac{b_1^2}{|b|^2}}.$$

Then

$$\lim_{t\to\infty}\sup_{b\in H(t)}\left(1-\frac{b_1}{|b|}\right)=0 \tag{8.11}$$

and

$$\eta(t)=o(t^{-\frac{1}{2}})\ \textit{if}\ N\ge 3,\quad \eta(t)=O(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}})\ \textit{if}\ N=2, \tag{8.12}$$

as $t\to\infty$.

Proof. We first consider the case $N\ge 3$. By Theorem 1.3, Lemma 6.4, (7.4), (8.7), and (8.8), we have

$$\begin{aligned}
&0\le u(b,t)-u(|b|e_1,t) \qquad (8.13)\\
&=-u_{1,1}(b,t)\Big(1-\frac{b_1}{|b|}\Big)+\sum_{i=2}^{N}u_{1,i}(b,t)\frac{b_i}{|b|}+\sum_{k=2}^{4}[u_k(b,t)-u_k(|b|e_1,t)]\\
&=-d_{2,1}t^{-\frac{N+2}{2}}|b|(1+o(1))\Big(1-\frac{b_1}{|b|}\Big)+o(t^{-\frac{N+3}{2}+\frac{1}{N}})
\end{aligned}$$

for all $b\in H(t)$. So we have (8.11). Furthermore, since

$$\max_{i=2,\dots,N}\frac{b_i^2}{|b|^2}\le 1-\frac{b_1^2}{|b|^2}=(1+o(1))\Big(1-\frac{b_1}{|b|}\Big)$$

for all $b\in H(t)$, by Theorem 1.3 and Lemma 6.4, there exist positive constants C_1 and C_2 such that

$$\begin{aligned}
&|u_2(b,t)-u_2(|b|e_1,t)|\le\sum_{j=1}^{l_2}|u_{2,j}(b,t)|\left|Q_{2,j}\Big(\frac{b}{|b|}\Big)-Q_{2,j}(e_1)\right| \qquad (8.14)\\
&\le C_1\max_{i=2,\dots,N}\frac{|b_i|}{|b|}\sum_{j=1}^{l_2}|u_{2,j}(b,t)|\le C_2t^{-\frac{N}{2}-2+\frac{2}{N}}\log t\sqrt{1-\frac{b_1}{|b|}}
\end{aligned}$$

for all $b\in H(t)$ and all sufficiently large t. By (3.1) with $k=3$ and (7.2), $\|u_{3,j}(t)\|_{L^\infty(\Omega)}=O(t^{-\frac{N+3}{2}})$ as $t\to\infty$, where $j=1,\dots,l_3$. So, in a way similar to (8.14), we have

$$|u_3(b,t) - u_3(|b|e_1,t)| = O(t^{-\frac{N+3}{2}})\sqrt{1-\frac{b_1}{|b|}} \tag{8.15}$$

for all $b \in H(t)$ and all sufficiently large t. Furthermore, by (8.8), we have

$$\Big|\sum_{i=2}^{N} u_{1,i}(b,t)\frac{b_i}{|b|}\Big| = O(t^{-\frac{N+3}{2}})\sqrt{1-\frac{b_1}{|b|}} \tag{8.16}$$

for all $b \in H(t)$ and all sufficiently large t. Therefore, by Lemma 7.1, (8.7), and (8.13)–(8.16), we have

$$\begin{aligned} 0 &\leq t^{\frac{N+2}{2}-\frac{1}{N}}\Big(u(b,t) - u(|b|e_1,t)\Big) \\ &\leq -C_3\Big(1-\frac{b_1}{|b|}\Big) + C_4(t^{-1+\frac{1}{N}}\log t + t^{-\frac{1}{2}-\frac{1}{N}})\sqrt{1-\frac{b_1}{|b|}} + C_5 t^{-1-\frac{1}{N}} \end{aligned}$$

for all $b \in H(t)$ and all sufficiently large t, where C_3, C_4, and C_5 are positive constants. So we have

$$\sqrt{1-\frac{b_1}{|b|}} = o(t^{-\frac{1}{2}})$$

for all $b \in H(t)$ and all sufficiently large t. This implies (8.12) for the case $N \geq 3$. Similarly, for the case $N = 2$, we obtain

$$0 \leq u(b,t) - u(|b|e_1,t) = -\tilde{d}_{2,1}t^{-2}|b|(1+o(1))\Big(1-\frac{b_1}{|b|}\Big) + O(t^{-2})$$

for all $b \in H(t)$. This implies (8.11). Furthermore, by Theorem 1.3, Lemma 6.4, Lemma 7.1, (8.7) and (8.8), we apply the same argument as in the proof for the case $N \geq 3$ to the solution u, and see that there exist positive constants C_6, C_7, and C_8 such that

$$\begin{aligned} 0 &\leq t^{\frac{3}{2}}(\log t)^{\frac{1}{2}}\Big(u(b,t) - u(|b|e_1,t)\Big) \\ &\leq -C_6\Big(1-\frac{b_1}{|b|}\Big) + C_7 t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}\sqrt{1-\frac{b_1}{|b|}} + C_8 t^{-\frac{3}{2}}(\log t)^{\frac{1}{2}} \end{aligned}$$

for all $b \in H(t)$ and all sufficiently large t. So we have

$$\sqrt{1-\frac{b_1}{|b|}} = O(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}})$$

for all $b \in H(t)$ and all sufficiently large t. This implies (8.12) for the case $N = 2$, and the proof of Lemma 8.1 is complete. □

Let

$$H_0(t) = \big\{x \in \overline{\Omega} : u_0(x,t) = \max_{y\in\overline{\Omega}} u_0(y,t)\big\}.$$

By Theorem 1.3 and the radial symmetry of the solution u_0, there exists a function $r = r(t)$ such that $\lim_{t\to\infty} r(t)^N/\zeta(t) = 1$ and

$$H_0(t) = \{x \in \mathbf{R}^N : |x| = r(t)\}$$

for all sufficiently large t. Then we have the following lemma.

Lemma 8.2. *Assume the same conditions as in Theorem* 1.4 *and* (8.1). *Put*

$$C(t) = \left\{x \in \Omega : x_1 > 0,\ |x| > r(t),\ \max_{i=2,\dots,N} \frac{|x_i|}{|x|} < 2\eta(t)\right\}.$$

Then there holds

$$H(t) \subset C(t) \tag{8.17}$$

for all sufficiently large t.

Proof. We first treat the case $N \geq 3$. Let T be a sufficiently large constant. Assume that there exist a point $b \in H(t)$ and $t \geq T$ such that $|b| \leq r(t)$. Since u_0 satisfies (7.16), by Theorem 1.3 and the fact that $\partial_r u_0(r(t),t) = 0$, we have $(\partial_r u_0)(b,t) \geq 0$. This together with (8.5) and (8.12) implies that

$$(\partial_{x_1} u_0)(b,t) \geq 0. \tag{8.18}$$

On the other hand, by Theorem 1.3, Lemma 8.1, (8.7)–(8.9), we have

$$(\partial_{x_1} u_1)(b,t) = Ct^{-\frac{N+2}{2}+\frac{1}{N}}(1+o(1)) > 0 \tag{8.19}$$

for all sufficiently large t, where C is a positive constant. Furthermore, by Lemmas 3.2 and 7.1, we have

$$(\partial_{x_1} u_k)(b,t) = O(t^{-\frac{N+2}{2}}), \quad k = 2,3,4, \tag{8.20}$$

for all sufficiently large t. Therefore, by (8.18)–(8.20) and Lemma 7.1, taking sufficiently large T if necessary, we have

$$(\partial_{x_1} u)(b,t) \geq (C/2)t^{-\frac{N+2}{2}+\frac{1}{N}} > 0.$$

This contradicts $b \in H(t)$. So we see that $|b| > r(t)$ for all sufficiently large t. Then, by Theorem 1.3 and Lemma 8.1, we have (8.17) for the case $N \geq 3$. Furthermore, by the same argument as in the case $N \geq 3$, we may prove (8.17) for the case $N = 2$, and the proof of Lemma 8.2 is complete. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4 for the case $N \geq 3$. By (7.16) and Lemma 8.2, we see that

$$(\partial_r u_0)(x,t) < 0 \tag{8.21}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t. So, by (8.3), (8.6), (8.12), and (8.21), there exist positive constants a_{11} and C_1 such that

$$\begin{cases} (\partial_{x_1}\partial_{x_1}u_0)(x,t) = -a_{11}t^{-\frac{N+2}{2}}(1+o(1)), \\ (\partial_{x_1}\partial_{x_j}u_0)(x,t) = o(t^{-\frac{N+2}{2}-\frac{1}{2}}) \quad (j=2,\dots,N), \\ (\partial_{x_i}\partial_{x_j}u_0)(x,t) = o(t^{-\frac{N+2}{2}-1}) \quad (i \neq j,\ i,j \neq 1), \\ -C_1 t^{-\frac{N+2}{2}} \leq (\partial_{x_i}\partial_{x_i}u_0)(x,t) \leq 0 \quad (i=2,\dots,N), \end{cases} \tag{8.22}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t. On the other hand, by (1.10), (8.7), (8.10), and (8.12) there exist positive constants b_i ($i = 2,\dots,N$) such that

$$\begin{cases} (\partial_{x_1}\partial_{x_1}u_1)(x,t) = o(t^{-\frac{N+2}{2}}), \\ (\partial_{x_i}\partial_{x_j}u_1)(x,t) = o(t^{-\frac{N+2}{2}-1+\frac{1}{N}}) \quad (i,j=1,\dots,N,\ i \neq j), \\ (\partial_{x_i}\partial_{x_i}u_1)(x,t) = -b_i t^{-\frac{N+2}{2}-1+\frac{1}{N}}(1+o(1)), \quad (i=2,\dots,N), \end{cases} \tag{8.23}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t. Since $|\nabla_x^l Q_{k,i}(x/|x|)| \leq C|x|^{-l}$ for some constant C, by Lemmas 6.4 and 7.1, we have

$$(\nabla_x^2 u_k)(x,t) = o(t^{-\frac{N+2}{2}-1+\frac{1}{N}}), \quad k=2,3,4 \tag{8.24}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t. Therefore, by (8.22)–(8.24), we have

$$u_{x_i x_j}(x,t) \begin{cases} = -a_{11}t^{-\frac{N+2}{2}}(1+o(1)) & (i=j=1), \\ \leq -b_i t^{-\frac{N+2}{2}-1+\frac{1}{N}}(1+o(1)) & (i=j\neq 1), \\ = o\Big(|u_{x_i x_i}(x,t)|^{1/2}|u_{x_j x_j}(x,t)|^{1/2}\Big) & (i \neq j), \end{cases}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t, where $u_{x_i x_j} = \partial_{x_i}\partial_{x_j}u$. This implies

$$(u_{x_i x_j}(x,t))_{i,j=1,\dots} < 0 \tag{8.25}$$

for all $x \in B(t) \cap C(t)$ and all sufficiently large t. This together with (8.17) implies that there exists a smooth curve $x = x(t) \in \Omega$ such that $H(t) = \{x(t)\}$ for all sufficiently large t. Furthermore, by Theorem 1.3 and Lemma 8.1, we have (1.12), and the proof of Theorem 1.4 is complete. □

Proof of Theorem 1.4 for the case $N = 2$. Similarly to the case $N \geq 3$, by (7.24) and Lemma 8.2, we have (8.21). Furthermore, by (8.6), (8.7), and

(8.21), there exist positive constants $\tilde{a}_{11}$ and $\tilde{C}_1$ such that

$$\begin{cases} (\partial_{x_1}\partial_{x_1}u_0)(x,t) = -\tilde{a}_{11}t^{-2}(\log t)^{-1}(1+o(1)), \\ (\partial_{x_1}\partial_{x_j}u_0)(x,t) = O(t^{-\frac{5}{2}}(\log t)^{-\frac{1}{2}}) \quad (j=2,\dots,N,\ i\neq j), \\ (\partial_{x_i}\partial_{x_j}u_0)(x,t) = o(t^{-\frac{5}{2}}(\log t)^{-\frac{1}{2}}) \quad (i\neq j,\ i,j\neq 1), \\ -\tilde{C}_1 t^{-2}(\log t)^{-1} \le (\partial_{x_i}\partial_{x_i}u_0)(x,t) \le 0 \quad (i=2,\dots,N), \end{cases} \tag{8.26}$$

for all $x \in B(t)\cap C(t)$ and all sufficiently large t. Furthermore, by (1.10), (8.7) and (8.10), there exist positive constants $\tilde{b}_i$, $(i=2,\dots,N)$ such that

$$\begin{cases} (\partial_{x_1}\partial_{x_1}u_1)(x,t) = o(t^{-2}(\log t)^{-1}), \\ (\partial_{x_i}\partial_{x_j}u_1)(x,t) = o(t^{-\frac{5}{2}}(\log t)^{-\frac{1}{2}}) \quad (i,j=1,\dots,N,\ i\neq j), \\ (\partial_{x_i}\partial_{x_i}u_1)(x,t) = -\tilde{b}_i t^{-\frac{5}{2}}(\log t)^{-\frac{1}{2}}(1+o(1)), \quad (i=2,\dots,N), \end{cases} \tag{8.27}$$

for all $x \in B(t)\cap C(t)$ and all sufficiently large t. By Lemmas 6.4 and 7.1, we have

$$(\nabla_x^2 u_k)(x,t) = o(t^{-\frac{5}{2}}(\log t)^{-\frac{1}{2}}), \quad k=2,3,4 \tag{8.28}$$

for all $x \in B(t)\cap C(t)$ and all sufficiently large t. Therefore, by (8.26)–(8.28), we have (8.25) for all $x \in B(t)\cap C(t)$ and all sufficiently large t. Therefore, by the same argument as for the case $N \ge 3$, we may complete the proof of Theorem 1.4 for the case $N=2$. □

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