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SCATTERING AND MODIFIED SCATTERING FOR ABSTRACT WAVE EQUATIONS WITH TIME-DEPENDENT DISSIPATION

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Abstract. We consider the initial-value problem of abstract wave equations with weak dissipation. We show that under conditions on the dissipation coefficient and its derivative the solutions to the abstract dissipative equation are closely related to solutions of the free problem multiplied by a decay function. This paper gives the counterpart to a recent paper of T.Yamazaki [Adv. Differential Equ., 11(4):419–456, 2006], where effective dissipation terms and the relation to the corresponding abstract parabolic problem are considered.

1. Introduction

Let H be a separable Hilbert space with norm $\|\cdot\|$. Let further A be a closed and self-adjoint non-negative operator with domain $\mathcal{D}(A)\subseteq H$. Note that the domain of A becomes itself a Hilbert space if we endow it with the graph norm $\|u\|_A^2=\|u\|^2+\|Au\|^2\sim\|\langle A\rangle u\|^2$. To simplify notation we set for the following $\Lambda=\sqrt{A}$.

We consider for a positive-valued C^1-function $b=b(t)$ on $[0,\infty)$ the abstract dissipative wave equation

$$\begin{cases} u''+2b(t)u'+Au=0\\ u(0)=u_1\in\mathcal{D}(\Lambda),\ u'(0)=u_2\in H \end{cases} \tag{1.1}$$

and compare its solutions to the corresponding free problem

$$\begin{cases} v''+Av=0\\ v(0)=v_1\in\mathcal{D}(\Lambda),\ v'(0)=v_2\in H \end{cases} \tag{1.2}$$

in the abstract energy space associated to Λ under decay assumptions on the coefficient function b.

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The first result is as follows. If we assume that $b \in L^1[0,\infty)$, then the solutions of (1.1) are asymptotically free in the energy norm. This means, if we set E to be the closure of $(\mathcal{D}(\Lambda) \ominus \mathrm{Ker}\Lambda) \times H$ with respect to the norm

$$\|(u_1,u_2)\|_E = \|(\Lambda u_1, u_2)\|_{H\times H}, \tag{1.3}$$

the following theorem is valid:

Theorem 1.1. *Assume $b \in L^1[0,\infty)$. Then there exists an invertible operator $W_+ \in \mathcal{L}(E)$ of the abstract energy space, such that for $(u_1,u_2) \in E$ and $(v_1,v_2) = W_+(u_1,u_2)$ the corresponding solutions to* (1.1) *and* (1.2) *satisfy*

$$\|(u,u') - (v,v')\|_E \le C\|(u_1,u_2)\|_E \int_t^\infty b(\tau)\mathrm{d}\tau \to 0 \tag{1.4}$$

as $t \to \infty$.

Remark 1. Note that the solutions to the free problem have the conservation of energy property; i.e., it holds for all $t \in \mathbb{R}$ that $\|(v,v')\|_E = \|(v_1,v_2)\|_E$. Thus, the statement of Theorem 1.1 implies the non-decay to zero of the energy of solutions to the perturbed problem (1.1) together with the existence of asymptotically equivalent solutions to both problems.

Remark 2. Note that the operator W_+ in this theorem is the inverse of the Møller wave operator. In this respect our notation differs from [3]. This inverse operator is more appropriate to describe asymptotic properties of solutions to problems with perturbed coefficients.

The aim of this article is to generalize this result to the case of so-called non-effective weak dissipation terms $b = b(t)$ with

(A1) $b(t) \ge 0$ for all $t \in [0,\infty)$,
(A2) $|b(t)| \le C_1\langle t\rangle^{-1}$ and $|b'(t)| \le C_2\langle t\rangle^{-2}$,

including non-integrable coefficients $b \notin L^1[0,\infty)$. In this case the abstract energy $\|(u,u')\|_E$ of the solution decays to zero. We will show that if 0 is not an eigenvalue of A then the solutions can be written asymptotically as the product of a free solution and a time-dependent function describing the energy decay rate. To be more precise, let us formulate a theorem (which is contained in Theorem 3.1).

Theorem 1.2. *Assume* (A1) *and* (A2), *together with $\limsup_{t\to\infty} tb(t) < \frac{1}{2}$ and $\mathrm{KerA} = \{0\}$. Then to $u_1 \in \mathcal{D}(\Lambda)$ and $u_2 \in H$ there exist corresponding data $(v_1,v_2) = W_+(u_1,u_2) \in E$ such that the corresponding solutions to* (1.1) *and* (1.2) *satisfy*

$$\|\lambda(t)(u,u') - (v,v')\|_E \to 0 \tag{1.5}$$

as $t \to \infty$, where the auxiliary function $\lambda(t)$ is given by

$$\lambda(t) = \exp\Big(\int_0^t b(\tau)\mathrm{d}\tau\Big).$$

Remark 3. If $b \notin L^1[0,\infty)$ the integral in equation (1.4) becomes infinite. In order to obtain a relation between the solutions u and v we have to pay. This can be seen from the differences of the statements. The operator W_+ maps $\mathcal{D}(\Lambda) \times H$ continuously to E; it is in general not bounded from E to E. Furthermore, we cannot give a uniform rate of asymptotic equivalence. The auxiliary function $\lambda(t)$ describes the decay of energy. If $\mathrm{KerA} = \{0\}$, we get the two-sided estimate

$$\|(u,u')\|_E \sim \frac{1}{\lambda(t)}. \tag{1.6}$$

Since the convergence rate in Theorem 1.2 is not uniform in the data, the constants in this estimate depend in a (non-linear) way on the data (u_1,u_2) and do not yield a two-sided norm estimate for a corresponding solution operator.

Example 1. One typical class of coefficient functions $b = b(t)$ for our approach is

$$b(t) = \frac{\mu}{(1+t)\log(e+t)\cdots\log^{[n]}(e^{[n]}+t)}, \qquad \mu \geq 0, \quad n = 0,1,\dots \tag{1.7}$$

with iterated logarithms $\log^{[1]} = \log$, $\log^{[n+1]} = \log \circ \log^{[n]}$ and corresponding iterated exponentials $e^{[0]} = 1$ and $e^{[n+1]} = e^{(e^{[n]})}$. In this case

$$\lambda(t) \sim \big(\log^{[n]}(e^{[n]}+t)\big)^{\mu}.$$

The decay rate for the abstract energy becomes arbitrarily small in the scale of iterated logarithms. □

Recently, T. Yamazaki, in [11] and [12], considered the related problem in the case of effective weak dissipation, i.e., if $tb(t) \to \infty$ as $t \to \infty$ together with $b(t) \to 0$ and suitable conditions on the first derivative. In this case, a relation to the free problem (1.2) does not occur. Instead, there arises a close relation to the abstract parabolic equation $b(t)w' + Aw = 0$. While the purpose of [11] and [12] was to demonstrate the close relationship of abstract wave equations with effective dissipation and the corresponding abstract parabolic problem, our purpose is to show what happens if the influence of the dissipation becomes less strong and how the abstract parabolic-type asymptotics changes to an abstract wave-type asymptotics. The methods used are based on [2].

This paper is organized as follows. First we will sketch one proof of Theorem 1.1 and give some remarks on possible generalizations. Then in Section 2 we will summarize the construction of representations of solutions to (1.1) by means of a spectral resolution of the operator A and discuss its consequences for the decay rates of the energy. In Section 3 we state the main result of this paper, and in Section 4 we discuss some possible applications of the abstract results in concrete settings.

The main part of this paper follows the author's treatment from [9] combined with ideas from [2]. In that paper precise L^p–L^q decay estimates are proven for solutions to the Cauchy problem of a wave equation with non-effective time-dependent dissipation. The approach was based on a diagonalization scheme applied to the full symbol of the operator. Here we formulate the Cauchy problem in an abstract setting and base our representation of solutions on a spectral theorem for the operator A. Besides this, there are two essential differences to the treatment in [9]. On the one hand the derivation of dispersive estimates is based on stronger assumptions on the coefficient function (we need estimates for $n+2$ derivatives of the coefficient function for dispersive estimates) and requires more steps of diagonalization, while in the Hilbert space setting estimates for b and its first derivative are sufficient. The second difference from [9] is that we obtain the modified scattering result not only for $\limsup_{t\to\infty} tb(t) < 1/2$. We use a new idea to impose conditions on the data in order to obtain a similar result for $\liminf_{t\to\infty} tb(t) > 1/2$.

1.1. **Outline of the proof of the classical scattering result.** Theorem 1.1 can be reduced to abstract results already known from the literature. Using a substitution of the time variable $t = t(\tau) = \int_0^\tau a(s)\mathrm{d}s$ such that $\log\lambda(t) = \int_0^t b(s)\mathrm{d}s = \log a(\tau)$ reduces (1.1) to the abstract problem $w'' + a^2(\tau)Aw = 0$. Because we have $a^2(\tau) \to \int_0^\infty b(t)\mathrm{d}t < \infty$ and $\int_0^\infty (a(\tau) - a(\infty))\mathrm{d}\tau = \int_0^\infty \tau(t)b(t)\mathrm{d}t < \infty$, the technique of A. Arosio, [1], can be applied. Transformation back yields Theorem 1.1.

For completeness and because the idea of the proof is basic for our treatment in Sections 2 and 3 we give an independent proof of Theorem 1.1.

Proof of Theorem 1.1. We use the canonical identification $J : E \xrightarrow{\simeq} (H \ominus \mathrm{Ker}\Lambda) \times \mathrm{H}$ of the energy space with $\mathrm{clR}(\Lambda) \times \mathrm{H}$ and write the Cauchy problems in system form. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Lambda u\\ u'\end{pmatrix} = \begin{pmatrix}\Lambda u'\\ u''\end{pmatrix} = \begin{pmatrix} & \Lambda\\ -\Lambda & -2b(t)\end{pmatrix}\begin{pmatrix}\Lambda u\\ u'\end{pmatrix}. \tag{1.8}$$

We denote by $\mathcal{E}(t,s)$ and $\mathcal{E}_0(t-s)$ the corresponding semigroups of operators,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0(t) = \begin{pmatrix} & \Lambda \\ -\Lambda & \end{pmatrix} \mathcal{E}_0(t), \qquad \mathcal{E}_0(0) = I \in \mathcal{L}(H\times H), \tag{1.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,s) = \begin{pmatrix} & \Lambda \\ -\Lambda & -2b(t) \end{pmatrix} \mathcal{E}(t,s), \qquad \mathcal{E}(s,s) = I \in \mathcal{L}(H\times H). \tag{1.10}$$

The free semigroup $\mathcal{E}_0(t)$ is unitary (because its generator is skew), while under assumption (A1) the semigroup $\mathcal{E}(t,s)$ is contractive.[1] If we consider the operator

$$\mathcal{Q}(t,s) = \mathcal{E}_0(s-t)\mathcal{E}(t,s) \in \mathcal{L}(H\times H), \tag{1.11}$$

we obtain the abstract differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}(t,s) &= -\mathcal{E}_0(s-t)\begin{pmatrix} & \Lambda \\ -\Lambda & \end{pmatrix}\mathcal{E}(t,s) + \mathcal{E}_0(s-t)\begin{pmatrix} & \Lambda \\ -\Lambda & -2b(t) \end{pmatrix}\mathcal{E}(t,s) \\ &= -2b(t)\mathcal{E}_0(s-t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{E}_0(t-s)\mathcal{Q}(t,s) = \mathcal{R}(t,s)\mathcal{Q}(t,s)\end{aligned} \tag{1.12}$$

with initial data $\mathcal{Q}(s,s) = I \in \mathcal{L}(H\times H)$. Using that the operator norm satisfies $|||\mathcal{R}(t,s)||| \le 2b(t) \in L^1[0,\infty)$ we see that $\mathcal{Q}(t,s)$ can be represented by the abstract Peano-Baker formula

$$\mathcal{Q}(t,s) = I + \sum_{k=1}^{\infty}\int_s^t \mathcal{R}(t_1,s)\int_s^{t_1}\mathcal{R}(t_2,s)\cdots\int_s^{t_{k-1}}\mathcal{R}(t_k,s)\mathrm{d}t_k\cdots\mathrm{d}t_1 \tag{1.13}$$

in terms of Bochner integrals. The L^1-bound on the norm of $\mathcal{R}$ implies a uniform bound on $\mathcal{Q}(t,s)$ and that

$$\lim_{t\to\infty}\mathcal{Q}(t,s) = \mathcal{Q}(\infty,s) \tag{1.14}$$

exists in $\mathcal{L}(H\times H)$ and satisfies

$$|||\mathcal{Q}(\infty,s) - \mathcal{Q}(t,s)||| \le 2\int_t^\infty b(\tau)\mathrm{d}\tau\ \exp\left(2\int_0^\infty b(\theta)\mathrm{d}\theta\right). \tag{1.15}$$

Furthermore,

$$|||\mathcal{Q}(\infty,t) - I||| \le 2\int_t^\infty b(\tau)\mathrm{d}\tau\ \exp\left(2\int_0^\infty b(\theta)\mathrm{d}\theta\right) \to 0 \tag{1.16}$$

[1] For a treatment without (A1) we construct $\mathcal{E}(t,s) = \mathcal{E}_0(t-s)\mathcal{Q}(t,s)$ directly with the aid of formula (1.13)

as $t \to \infty$ implies invertibility of $\mathcal{Q}(\infty, t)$ for sufficiently large t, while the propagation property $\mathcal{Q}(\infty, s) = \mathcal{E}_0(s-t)\mathcal{Q}(\infty, t)\mathcal{E}(t, s)$ extends this result to arbitrary s.

Now the statement of Theorem 1.1 follows with

$$JW_+J^* = \lim_{t\to\infty} \mathcal{E}_0(-t)\mathcal{E}(t,0) = \mathcal{Q}(\infty, 0)$$

together with the fact that $\mathcal{E}_0(t)$ is unitary. □

Remark 4. Because it was not necessary to use (A1), we can reverse the time direction in this statement and obtain the existence of a corresponding wave operator

$$W_- = \lim_{t\to-\infty} J^*\mathcal{E}_0(-t)\mathcal{E}(t,0)J \tag{1.17}$$

when $b \in L^1(\mathbb{R})$ and construct the scattering operator $S = W_+W_-^{-1}$. This operator maps incoming free waves (from $t = -\infty$) to outgoing free waves (to $t = \infty$), both parameterized by corresponding Cauchy data at time $t = 0$, which are connected by one solution of the dissipative equation.

Remark 5. Theorem 1.1 remains true if we replace (1.1) by

$$u'' + B(t)u' + Au = 0, \tag{1.1'}$$

where $B = B(t) \in L^1([0,\infty), \mathcal{L}(H))$; cf. [6].

2. Representation of solutions

If $b \notin L^1[0,\infty)$ we will not apply (1.13) directly to represent the solutions of system (1.8) (although it is still valid). Our approach is based on a transformation of the system (1.8) into a diagonal-dominated form.

To make the calculations more transparent we use the spectral resolution of the operator A (or Λ, respectively) and reduce the analytic properties of the operator to algebraic properties of a corresponding function. Following [5, Theorem VIII.4, p. 260] there exists a unitary operator $\Xi : H \to L^2(X, \mathrm{d}\nu)$, $(X, \mathrm{d}\nu)$ a suitable measure space, such that the operator $A : H \supseteq \mathcal{D}(A) \to H$ is unitarily equivalent to

$$(Au)(\xi) = A(\xi)u(\xi), \qquad \xi \in X, \quad u \in \mathcal{D}(A), \tag{2.1}$$

where $A(\xi)$ is a non-negative and ν-measurable function. Furthermore,

$$\mathcal{D}(A^\gamma) \simeq L^2(X, (1 + A^{2\gamma}(\xi))\mathrm{d}\nu), \qquad \gamma \geq 0. \tag{2.2}$$

For the following the square root of A is of particular importance. We denote $\Lambda(\xi) = \sqrt{A(\xi)}$. Then equation (1.1) is equivalent to

$$\partial_t^2 u(t,\xi) + 2b(t)\partial_t u(t,\xi) + \Lambda^2(\xi)u(t,\xi) = 0, \tag{2.3}$$

and system (1.8) reads as

$$\partial_t \begin{pmatrix} \Lambda(\xi)u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} & \Lambda(\xi) \\ -\Lambda(\xi) & -2b(t) \end{pmatrix} \begin{pmatrix} \Lambda(\xi)u \\ \partial_t u \end{pmatrix}. \tag{2.4}$$

Estimates in the energy space E correspond via $\Xi \circ J$ to estimates for this system in $L^2(X_+, \mathrm{d}\nu) \times L^2(X, \mathrm{d}\nu)$, where $X_+ = \{\Lambda(\xi) > 0\}$.

We will solve this system following [8] and [9]. For a (later to be determined) number $N > 0$ we decompose the extended phase space $[0, \infty) \times X$ in two sets, the *dissipative zone*

$$Z_{diss}(N) := \big\{(t, \xi) \in [0, \infty) \times X : \Lambda(\xi)(1 + t) < N \big\} \tag{2.5}$$

and the *hyperbolic zone*

$$Z_{hyp}(N) := \big\{(t, \xi) \in [0, \infty) \times X : \Lambda(\xi)(1 + t) \geq N \big\}. \tag{2.6}$$

Let further $t_\xi : X \supseteq \{\Lambda(\xi) \leq N\} \to [0, \infty)$ be implicitly defined by $\Lambda(\xi)(1 + t_\xi) = N$.

In the first one we reformulate (2.4) as system of Volterra integral equations and solve mainly by brute force, while in the second one we apply two steps of a diagonalization procedure to extract the main terms of the representation of solutions and to get sharp estimates for the remainder terms.

2.1. **Consideration in the dissipative zone.** We will prove estimates for the fundamental solution $\mathcal{E}(t, s, \xi)$ to (2.4), i.e., the matrix-valued solution to that system with initial condition $\mathcal{E}(s, s, \xi) = I \in \mathbb{C}^{2\times 2}$. Let for this

$$\lambda(t) = \exp\left(\int_0^t b(\tau)\mathrm{d}\tau\right) \tag{2.7}$$

denote an auxiliary function related to the dissipation term $b(t)$. We distinguish different cases related to the asymptotic behaviour of $\lambda(t)$.

Case (C1): $\overline{\mu} := \limsup_{t\to\infty} tb(t) < 1/2$**.**

We denote by $v(t, \xi)$ and $w(t, \xi)$ the entries of one of the rows of $\mathcal{E}(t, 0, \xi)$. Then these functions satisfy

$$v(t, \xi) = \eta_1 + \Lambda(\xi) \int_0^t w(\tau, \xi)\mathrm{d}\tau, \tag{2.8a}$$

$$w(t, \xi) = \frac{1}{\lambda^2(t)}\eta_2 - \Lambda(\xi)\frac{1}{\lambda^2(t)} \int_0^t \lambda^2(\tau)v(\tau, \xi)\mathrm{d}\tau, \tag{2.8b}$$

where $\eta = (\eta_1, \eta_2) = (1, 0)$ or $\eta = (0, 1)$ for the first and second column, respectively.

Lemma 2.1. *Assume* (A1) *and* (A2). *Then in the case* (C1) *the fundamental solution* $\mathcal{E}(t,0,\xi)$ *to* (2.4) *satisfies for all* $(t,\xi)\in Z_{diss}(N)$ *the pointwise estimate*[2]

$$|\mathcal{E}(t,0,\xi)| \lesssim \frac{1}{\lambda^2(t)}\begin{pmatrix}\Lambda^{-\gamma}(\xi) & 1\\ \Lambda^{-\gamma}(\xi) & 1\end{pmatrix} \tag{2.9}$$

for any γ *such that* $\Lambda^\gamma(\xi)\lambda^2(t)\lesssim 1$.

Example 2. If we consider the case $b(t)=\mu/(1+t)$ with $\mu\in[0,\frac{1}{2})$ we obtain $\lambda(t)\sim t^\mu$, and thus we can choose $\gamma=2\mu$ in the previous statement.

Example 3. In general we can calculate the constant γ as follows: Let $\overline{\mu}=\limsup_{t\to\infty} tb(t)$ and assume $\overline{\mu}<\frac{1}{2}$. Then the conditions of the above lemma are satisfied and $\lambda^2(t)\lesssim t^{2\overline{\mu}+\epsilon}$ for any $\epsilon>0$. Hence, we choose $\gamma>2\overline{\mu}$. In particular, if $tb(t)\to 0$ as $t\to\infty$ we have the above statement for any $\gamma>0$. Furthermore, if $b\in L^1[0,\infty)$ the choice $\gamma=0$ is possible.[3]

Proof of Lemma 2.1. We start by estimating the first column, i.e., $\eta=(1,0)$. Plugging the second integral equation into the first one yields

$$\begin{aligned} v(t,\xi) &= 1-\Lambda^2(\xi)\int_0^t \frac{1}{\lambda^2(\tau)}\int_0^\tau \lambda^2(\theta)v(\theta,\xi)\mathrm{d}\theta\mathrm{d}\tau\\ &= 1-\Lambda^2(\xi)\int_0^t \lambda^2(\theta)v(\theta,\xi)\int_\theta^t\frac{1}{\lambda^2(\tau)}\mathrm{d}\tau\mathrm{d}\theta, \end{aligned} \tag{2.10}$$

such that $\lambda^2(t)\Lambda^\gamma(\xi)v(t,\xi)$ satisfies an Volterra integral equation with kernel $k(t,\theta,\xi)=-\Lambda^2(\xi)\lambda^2(t)\int_\theta^t \mathrm{d}\tau/\lambda^2(\tau)$ and source term $h(t,\xi)=\Lambda^\gamma(\xi)\lambda^2(t)\lesssim 1$. We represent its solution by a Neumann series

$$\begin{aligned} \lambda^2(t)\Lambda^\gamma(\xi)v(t,\xi) &= h(t,\xi)+\sum_{\ell=1}^\infty\int_0^t k(t,t_1,\xi)\cdots\\ &\quad\times\int_0^{t_{k-1}} k(t_{k-1},t_k,\xi)h(t_k,\xi)\mathrm{d}t_k\cdots\mathrm{d}t_1. \end{aligned} \tag{2.11}$$

From (C1) we know that $t/\lambda^2(t)$ is monotone increasing for large t. Hence, we conclude the kernel estimate

$$\sup_{(t,\xi)\in Z_{diss}(N)}\int_0^t \sup_{0\le\tilde{t}\le t_\xi}|k(\tilde{t},\theta,\xi)|\mathrm{d}\theta \le \sup_{(t,\xi)\in Z_{diss}(N)}\Lambda^2(\xi)\lambda^2(t_\xi)\int_0^t\int_\theta^{t_\xi}\frac{\mathrm{d}\tau}{\lambda^2(\tau)}\mathrm{d}\theta$$

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[2] For a matrix A we denote by $|A|$ the matrix consisting of the absolute values of the entries of A.

[3] Otherwise, the choice $\gamma=2\overline{\mu}$ is not always possible, as the example $b(t)=\frac{1}{4(e+t)}+\frac{1}{(e+t)\log(e+t)}$ shows.

$$= \sup_{(t,\xi)\in Z_{diss}(N)} \Lambda^2(\xi)\lambda^2(t_\xi)\int_0^{t_\xi} \frac{\tau}{\lambda^2(\tau)}\mathrm{d}\tau \lesssim \Lambda^2(\xi)t_\xi^2 \lesssim 1,$$

which implies by standard arguments that $|\lambda^2(t)\Lambda^\gamma(\xi)v(t,\xi)| \lesssim 1$ uniformly on $Z_{diss}(N)$. Then, the second integral equation implies

$$|w(t,\xi)| \le \frac{\Lambda^{-\gamma}(\xi)}{\lambda^2(t)}\Lambda(\xi)\int_0^t |\lambda^2(\tau)\Lambda^\gamma(\xi)v(\tau,\xi)|\mathrm{d}\tau \lesssim \frac{\Lambda^{-\gamma}(\xi)}{\lambda^2(t)}\Lambda(\xi)t \lesssim \frac{\Lambda^{-\gamma}(\xi)}{\lambda^2(t)} \tag{2.12}$$

by the definition of the zone.

For the second column we use the same idea, plugging the second integral equation into the first one yields the new source term $\int_0^t \Lambda(\xi)/\lambda^2(\tau)\mathrm{d}\tau \lesssim 1/\lambda^2(t)$, such that $\lambda^2(t)v(t,\xi)$ is uniformly bounded on $Z_{diss}(N)$.

Case (C2): $\underline{\mu} := \liminf_{t\to\infty} tb(t) > 1/2$.

In this case we obtain the decay rate $1/t$ (which seems to be natural from the point of view of effective dissipation; cf. [10] and [11]).

Lemma 2.2. *Assume* (A1) *and* (A2). *Then in the case* (C2) *the fundamental solution* $\mathcal{E}(t,0,\xi)$ *to* (2.4) *satisfies for all* $(t,\xi)\in Z_{diss}(N)$ *the pointwise estimate*

$$|\mathcal{E}(t,0,\xi)| \lesssim \frac{1}{1+t}\begin{pmatrix}\Lambda^{-1}(\xi) & 1\\ \Lambda^{-1}(\xi) & 1\end{pmatrix}. \tag{2.13}$$

Proof. We proceed in a way similar to that of the proof of Lemma 2.1. Plugging the second integral equation into the first one gives

$$\begin{aligned}(1+t)v(t,\xi) &= (1+t)\eta_1 + (1+t)\Lambda(\xi)\eta_2\int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\\ &\quad - \Lambda^2(\xi)\int_0^t (1+\theta)v(\theta,\xi)\int_\theta^t \frac{1+t}{1+\theta}\frac{\lambda^2(\theta)}{\lambda^2(\tau)}\mathrm{d}\tau\mathrm{d}\theta.\end{aligned} \tag{2.14}$$

Under condition (C2) we know that $1/\lambda^2(t) \in L^1(\mathbb{R}_+)$ and $\lambda^2(t)/t$ is monotone increasing for large t. This implies again a kernel estimate for this Volterra integral equation

$$\begin{aligned}&\sup_{(t,\xi)\in Z_{diss}(N)} \Lambda^2(\xi)\int_0^t \sup_{0\le\tilde{t}\le t_\xi}\int_\theta^{\tilde{t}} \frac{1+\tilde{t}}{1+\theta}\frac{\lambda^2(\theta)}{\lambda^2(\tau)}\mathrm{d}\tau\mathrm{d}\theta\\ &\qquad\le \sup_{(t,\xi)\in Z_{diss}(N)} \Lambda^2(\xi)\int_0^{t_\xi}\int_0^\tau \frac{1+t_\xi}{1+\theta}\frac{\lambda^2(\theta)}{\lambda^2(\tau)}\mathrm{d}\theta\mathrm{d}\tau \lesssim 1\end{aligned}$$

and we obtain for the first column that $|\Lambda(\xi)(1+t)v(t,\xi)| \lesssim 1$ is uniformly bounded on $Z_{diss}(N)$ while for the second one $|(1+t)v(t,\xi)| \lesssim 1$ on $Z_{diss}(N)$.

The second integral equation yields corresponding bounds for $w(t,\xi)$ from

$$\Lambda(\xi)\frac{1+t}{\lambda^2(t)}\int_0^t \frac{\lambda^2(\tau)}{1+\tau}\mathrm{d}\tau \lesssim (1+t)\Lambda(\xi) \lesssim 1. \tag{2.15}$$

□

Remark 6. There remains a gap between the cases (C1) and (C2). In [7] the special case $2b(t) = \mu/(1+t)$ for $A = -\Delta$ on $L^2(\mathbb{R}^n)$ was studied. The above exceptional value $\mu = 1$ is related to the occurrence of logarithmic terms in the representation of solutions.

2.2. **Consideration in the hyperbolic zone.** In $Z_{hyp}(N)$ we apply two steps of diagonalization to system (2.4). In a first step we use

$$M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \qquad M^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \tag{2.16}$$

to obtain

$$M^{-1}\begin{pmatrix} & \Lambda(\xi) \\ -\Lambda(\xi) & -2b(t) \end{pmatrix} M = \begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix} - b(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.17}$$

In a second step we want to transform the second matrix without changing the structure of the first one. For this we set

$$N^{(1)}(t,\xi) = i\begin{pmatrix} & -\frac{b(t)}{2\Lambda(\xi)} \\ \frac{b(t)}{2\Lambda(\xi)} & \end{pmatrix}, \tag{2.18}$$

$$B^{(1)}(t,\xi) = \partial_t N^{(1)}(t,\xi) + b(t)N^{(1)}. \tag{2.19}$$

Then we have by construction

$$\begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix} N^{(1)}(t,\xi) - N^{(1)}(t,\xi)\begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix} = -b(t)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \tag{2.20}$$

such that with $N_1(t,\xi) = I - N^{(1)}(t,\xi)$ the operator identity

$$\left(\partial_t - \begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix} + b(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) N_1(t,\xi)$$
$$= \partial_t - N_1(t,\xi)\begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix} + b(t)N_1(t,\xi) - B^{(1)}(t,\xi)$$

holds. If the zone constant N is chosen sufficiently large, we have that $\det N_1(t,\xi) = 1 - b^2(t)/(4\Lambda^2(\xi)) \geq c > 0$ is uniformly bounded away from zero on $Z_{hyp}(N)$. Then $N_1^{-1}(t,\xi)$ exists and $N_1(t,\xi)$ and $N_1^{-1}(t,\xi)$ are both uniformly bounded on $Z_{hyp}(N)$.

Setting $R_1(t,\xi) = -N_1^{-1}(t,\xi)B^{(1)}(t,\xi)$ we obtain the operator identity

$$N_1^{-1}(t,\xi)\left(\partial_t - \begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi)\end{pmatrix} + b(t)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)N_1(t,\xi)$$
$$= \partial_t - \begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi)\end{pmatrix} + b(t)I + R_1(t,\xi) \quad (2.21)$$

with remainder term $R_1(t,\xi)$ subject to the pointwise estimate

$$\|R_1(t,\xi)\| \lesssim \frac{1}{\Lambda(\xi)(1+t)^2} \quad (2.22)$$

following directly from Assumption (A2). Note that we did not use Assumption (A1) for the treatment of the hyperbolic zone.

We are now in a position to derive the main result of this section.

Lemma 2.3. *Assume* (A2). *Then the fundamental solution* $\mathcal{E}(t,s,\xi)$ *of* (2.4) *can be represented as*

$$\mathcal{E}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}M^{-1}N_1^{-1}(t,\xi)\widetilde{\mathcal{E}}_0(t,s,\xi)\mathcal{Q}_1(t,s,\xi)N_1(s,\xi)M \quad (2.23)$$

for $t \geq s$ *and* $(s,\xi) \in Z_{hyp}(N)$, *where*

- *the function* $\lambda(t) = \exp\left(\int_0^t b(\tau)\mathrm{d}\tau\right)$ *describes the main influence of the dissipation* $b = b(t)$,
- *the matrices* $N_1(t,\xi)$ *and* $N_1^{-1}(t,\xi)$ *are uniformly bounded on* $Z_{hyp}(N)$ *tending on* $\{\Lambda(\xi) \geq \epsilon\} \subseteq X$ *uniformly to the identity matrix* I,
- *the matrix* $\widetilde{\mathcal{E}}_0(t,s,\xi)$ *is given by*

$$\widetilde{\mathcal{E}}_0(t,s,\xi) = \begin{pmatrix} e^{-i\Lambda(\xi)(t-s)} & \\ & e^{i\Lambda(\xi)(t-s)}\end{pmatrix}, \quad (2.24)$$

- *and the matrix* $\mathcal{Q}_1(t,s,\xi)$ *is uniformly bounded and invertible on* $Z_{hyp}(N)$ *tending on* $\{\Lambda(\xi) \geq \epsilon\} \subseteq X$ *uniformly to the invertible matrix* $\mathcal{Q}_1(\infty,s,\xi)$.

Remark 7. The free propagator can be represented as

$$\mathcal{E}_0(t-s,\xi) = M^{-1}\widetilde{\mathcal{E}}_0(t,s,\xi)M.$$

Thus, Lemma 2.3 gives a relation between the free propagator $\mathcal{E}_0(t-s,\xi)$ and $\mathcal{E}(t,s,\xi)$.

Remark 8. If the operator A is boundedly invertible, the function $\Lambda(\xi)$ is bounded from below, and hence the matrices $N_1(t,\xi)$ and $\mathcal{Q}_1(t,s,\xi)$ converge uniformly on X as $t \to \infty$. Furthermore, the dissipative zone can be skipped in this case (because it is contained in a finite time strip). Thus, Lemma 2.3 describes all essential properties of the fundamental solution in this case.

Proof of Lemma 2.3. The construction of the representation of solutions will be done in two steps. First, note that

$$\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\widetilde{\mathcal{E}}_0(t,s,\xi) \tag{2.25}$$

is the fundamental solution to the diagonal main part,

$$\partial_t\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi) = \begin{pmatrix} -i\Lambda(\xi) & \\ & i\Lambda(\xi) \end{pmatrix}\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi)+b(t)\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi), \qquad \tilde{\tilde{\mathcal{E}}}_0(s,s,\xi) = I. \tag{2.26}$$

Thus, making the ansatz $\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi)\mathcal{Q}_1(t,s,\xi)$ for the fundamental solution to the transformed operator $\partial_t - \operatorname{diag}(-i\Lambda(\xi), i\Lambda(\xi)) + b(t)I + R_1(t,\xi)$, we obtain a system for $\mathcal{Q}_1(t,s,\xi)$,

$$\partial_t\mathcal{Q}_1(t,s,\xi) = \mathcal{R}(t,s,\xi)\mathcal{Q}_1(t,s,\xi), \qquad \mathcal{Q}_1(s,s,\xi) = I \tag{2.27}$$

with coefficient matrix $\mathcal{R}(t,s,\xi) = \widetilde{\mathcal{E}}_0(s,t,\xi)R_1(t,\xi)\widetilde{\mathcal{E}}_0(t,s,\xi)$. Using that $\widetilde{\mathcal{E}}_0(t,s,\xi)$ is unitary, we see that $\mathcal{R}(t,s,\xi)$ satisfies the same estimates as $R_1(t,\xi)$.

This allows us to estimate in a second step the solution $\mathcal{Q}_1(t,s,\xi)$ directly from the representation as Peano-Baker series

$$\mathcal{Q}_1(t,s,\xi)=I+\sum_{k=1}^{\infty}\int_s^t \mathcal{R}(t_1,s,\xi)\int_s^{t_1}\mathcal{R}(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{R}(t_k,s,\xi)\mathrm{d}t_k\cdots\mathrm{d}t_1. \tag{2.28}$$

Thus,

$$\|\mathcal{Q}_1(t,s,\xi)\| \le \exp\left(\int_s^t \|R_1(\tau,\xi)\|\mathrm{d}\tau\right) \le \exp\left(\frac{c}{\Lambda(\xi)}\int_s^t \frac{\mathrm{d}\tau}{(1+\tau)^2}\right) \lesssim 1 \tag{2.29}$$

uniformly in $t \ge s$ and $(s,\xi) \in Z_{hyp}(N)$. Note that the inverse matrix satisfies a similar differential equation and therefore an analogue to series (2.28).

It remains to show that $\mathcal{Q}_1(t,s,\xi)$ converges uniformly on $\{\,\Lambda(\xi) \ge \epsilon\,\} \subseteq X$ as $t \to \infty$. This follows by the Cauchy criterion applied to the series (2.28) or by the estimate

$$\|\mathcal{Q}_1(\infty,s,\xi) - \mathcal{Q}_1(t,s,\xi)\| \le \int_t^\infty \|R_1(\tau,\xi)\|\mathrm{d}\tau \exp\left(\int_s^t \|R_1(\tau,\xi)\|\mathrm{d}\tau\right)$$

$$\lesssim \frac{1}{\Lambda(\xi)} \int_t^\infty \frac{\mathrm{d}\tau}{(1+\tau)^2} \lesssim \frac{1}{\epsilon(1+t)}. \quad (2.30)$$

The theorem is proven. ☐

2.3. **Combination of results.** We combine the previously constructed representations of $\mathcal{E}(t,s,\xi)$ to get a representation for $\mathcal{E}(t,0,\xi)$ and the corresponding operator $\mathcal{E}(t,0)$. We will obtain $J^*\mathcal{E}(t,0)J \in \mathcal{L}(E)$ and

$$|||J^*\mathcal{E}(t,0)J|||_{\tilde{E}\to E} \lesssim \frac{1}{\lambda(t)} \quad (2.31)$$

by the right choice of a smaller space $\tilde{E} \subset E$. Again J denotes the canonic identification $E \simeq (H \ominus \mathrm{Ker}\Lambda) \times \mathrm{H}$.

If $\Lambda(\xi) > N$ for all $\xi \in X$, we have to consider only the hyperbolic zone and $\mathcal{E}(t,0,\xi)$ is given by Lemma 2.3, hence (2.31) holds true with $\tilde{E} = E$. We focus on the remaining case and consider ξ with $\Lambda(\xi) \leq N$ here. There it holds

$$\mathcal{E}(t,0,\xi) = \mathcal{E}(t,t_\xi,\xi)\mathcal{E}(t_\xi,0,\xi) \quad (2.32)$$

and we obtain corresponding estimates by Lemmata 2.1 to 2.3. We distinguish some cases.

Case (C1): Using Lemmata 2.1 and 2.3 we conclude

$$|\mathcal{E}(t,0,\xi)| \lesssim \frac{\lambda(t_\xi)}{\lambda(t)} \frac{1}{\lambda^2(t_\xi)} \begin{pmatrix} \Lambda^{-\gamma}(\xi) & 1 \\ \Lambda^{-\gamma}(\xi) & 1 \end{pmatrix} \lesssim \frac{1}{\lambda(t)} \begin{pmatrix} \Lambda^{-\gamma}(\xi) & 1 \\ \Lambda^{-\gamma}(\xi) & 1 \end{pmatrix} \quad (2.33)$$

for $\Lambda(\xi) \leq N$, $t \geq t_\xi$ and γ with $\Lambda^\gamma(\xi)\lambda^2(t_\xi) \lesssim 1$. For $t \leq t_\xi$ the same estimate follows from Lemma 2.1.

To simplify notation we introduce $[\Lambda(\xi)] = \min(\Lambda(\xi), N)$. If we choose $\gamma = 1$, we need for the data an estimate of the form $\|[\Lambda(\xi)]^{-1}\Lambda(\xi)u_1(\xi)\|_2 \sim \|\langle\Lambda(\xi)\rangle u_1(\xi)\|_2 = \|u_1\|_{\mathcal{D}(\Lambda)}$. Thus we obtain that the operator $J^*\mathcal{E}(t,0)J$ restricted to $\mathcal{D}(\Lambda) \times H \to E$ satisfies the bound (2.31).

Case (C2): Lemma 2.2 implies only the weaker decay rate $1/(1+t)$. This can be compensated for by assumptions on the data, which are valid near $\{\Lambda(\xi) = 0\} \subseteq X$. Using

$$\frac{\Lambda^\gamma(\xi)\lambda(t_\xi)}{1+t_\xi} \sim \frac{\lambda(t_\xi)}{(1+t_\xi)^{\gamma+1}} \lesssim 1 \quad (2.34)$$

for $\gamma > \overline{\mu} - 1$, $\overline{\mu} = \limsup_{t\to\infty} tb(t)$, we conclude that

$$|\mathcal{E}(t,0,\xi)| \lesssim \frac{\lambda(t_\xi)}{\lambda(t)} \frac{1}{1+t_\xi} \begin{pmatrix} \Lambda^{-1}(\xi) & 1 \\ \Lambda^{-1}(\xi) & 1 \end{pmatrix} \lesssim \frac{1}{\lambda(t)} \begin{pmatrix} \Lambda^{-1}(\xi) & 1 \\ \Lambda^{-1}(\xi) & 1 \end{pmatrix} \Lambda^{-\gamma}(\xi) \quad (2.35)$$

for all $\Lambda(\xi) \le N$ and $t \ge t_\xi$. For $t \le t_\xi$ Lemma 2.2 implies the same bound using $\Lambda^\gamma(\xi)\lambda(t)/(1+t) \lesssim \lambda(t)/(1+t)^{\gamma+1} \lesssim 1$.

Thus, if we define for $\gamma \ge 0$ the *modified energy space* $E^{(\gamma)}$ to be the closure of $(\mathcal{D}(\Lambda) \cap R(\Lambda^\gamma) \ominus \mathrm{Ker}\Lambda) \times (\mathrm{R}(\Lambda^\gamma) \ominus \mathrm{Ker}\Lambda)$ with respect to the norm

$$\|(u_1,u_2)\|^2_{E^{(\gamma)}} = \big\|[\Lambda(\xi)]^{-\gamma-1}\Lambda(\xi)u_1(\xi)\big\|^2_{L^2(X,\mathrm{d}\nu)} + \|[\Lambda(\xi)]^{-\gamma}u_2(\xi)\|^2_{L^2(X,\mathrm{d}\nu)}, \tag{2.36}$$

the operator $J^*\mathcal{E}(t,0)J$ restricted to $E^{(\gamma)} \to E$ satisfies the bound (2.31).

Remark 9. Using $[\Lambda(\xi)]^{-1}\Lambda(\xi) \sim \langle\Lambda(\xi)\rangle$ we obtain in particular

$$E^{(0)} = (\mathcal{D}(\Lambda) \times H) \ominus (\mathrm{Ker}\Lambda)^2.$$

So in the case (C1) we may use $E^{(0)}$ for the data, while in (C2) we use $E^{(\gamma)}$ with $\gamma = \max(\overline{\mu}^+ - 1, 0)$, where $\overline{\mu}^+$ denotes any number larger than $\overline{\mu}$.

Theorem 2.4. *Let u be the solution to* (1.1) *for data $(u_1,u_2) \in E^{(\gamma)}$ for $\gamma = \max(\overline{\mu}^+ - 1, 0)$. Then the estimate*

$$\|(u,u')\|_E \lesssim \frac{1}{\lambda(t)}\|(u_1,u_2)\|_{E^{(\gamma)}} \tag{2.37}$$

is valid under assumptions (A1) *and* (A2) *in the cases* (C1) *and* (C2).

Remark 10. We used conditions on the data related to $\{\Lambda(\xi) = 0\}$. In principle there arise two different cases. On the one hand, if $\nu\{\Lambda(\xi) = 0\} > 0$ it follows that $0 \in \sigma_P(A)$ and 0 is an eigenvalue of A. In this case we cut out the null-space of A and $R(A)$ is dense in the ortho-complement of KerA. Hence $E^{(\gamma)}$ is dense in $E^{(0)} = (\mathcal{D}(\Lambda) \ominus \mathrm{Ker}\Lambda) \times (\mathrm{H} \ominus \mathrm{Ker}\Lambda)$, which is dense in the ortho-complement of $\{0\} \times \mathrm{Ker}\Lambda$ in E.

If $\nu\{\Lambda(\xi) = 0\} = 0$ is a null-set, $R(A)$ is dense in H and the result of Theorem 2.4 holds on a dense subset of $E^{(0)} = \mathcal{D}(\Lambda) \times H$.

3. Scattering-type theorems

The main purpose of this note is to explain in which sense Theorem 2.4 is sharp. This sharpness is formulated as a generalization of Theorem 1.1 with an additional energy decay function.

Theorem 3.1. *Under the assumptions of Theorem* 2.4 *there exists a bounded operator $W_+ : E^{(\gamma)} \to E$, $\gamma = \max(\overline{\mu}^+ - 1, 0)$ such that for Cauchy data $(u_1,u_2) \in E^{(\gamma)}$ of* (1.1) *and associated data $(v_1,v_2) = W_+(u_1,u_2)$ to* (1.2) *the corresponding solutions $u = u(t)$ and $v = v(t)$ satisfy*

$$\|\lambda(t)(u,u') - (v,v')\|_E \to 0 \tag{3.1}$$

as $t \to \infty$*. Furthermore, the operator* W_+ *satisfies* $\mathrm{KerW}_+ = \{0\}$*.*

Proof. The proof is based on an explicit representation of the (modified) inverse wave operator W_+. From Lemma 2.3 we know that the limit

$$\mathcal{Q}_1(\infty, t_\xi, \xi) = \lim_{t\to\infty} \mathcal{Q}_1(t, t_\xi, \xi)$$

exists uniformly in $\{\Lambda(\xi) \geq \epsilon\}$ for any $\epsilon > 0$. Hence, if we consider $\mathcal{E}_0(-t,\xi)\mathcal{E}(t,0,\xi)$ on $\{\Lambda(\xi) \geq \epsilon\}$ we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \lambda(t)\mathcal{E}_0(-t,\xi)\mathcal{E}(t,0,\xi)\\
&= \lim_{t\to\infty} \lambda(t_\xi)\mathcal{E}_0(-t,\xi)M^{-1}N_1^{-1}(t,\xi)\widetilde{\mathcal{E}}_0(t,t_\xi,\xi)\mathcal{Q}_1(t,t_\xi,\xi)N_1(t_\xi,\xi)M\mathcal{E}(t_\xi,0,\xi)\\
&= \lim_{t\to\infty} \lambda(t_\xi)\mathcal{E}_0(-t,\xi)\mathcal{E}_0(t-t_\xi,\xi)M^{-1}\mathcal{Q}_1(t,t_\xi,\xi)N_1(t_\xi,\xi)M\mathcal{E}(t_\xi,0,\xi)\\
&=\lambda(t_\xi)\mathcal{E}_0(-t_\xi,\xi)M^{-1}\mathcal{Q}_1(\infty,t_\xi,\xi)N_1(t_\xi,\xi)M\mathcal{E}(t_\xi,0,\xi)
\end{aligned}$$

using the fact that $N_1^{-1}(t,\xi) \to I$ uniformly on $\{\Lambda(\xi) \geq \epsilon\}$. Denoting this limit by $\tilde{W}_+(\xi)$ we see that

$$|\tilde{W}_+(\xi)| \lesssim \lambda(t_\xi)|\mathcal{E}(t_\xi, 0, \xi)|. \tag{3.2}$$

Now the estimates (2.33) and (2.35) on $\mathcal{E}(t_\xi, 0, \xi)$ imply that $\tilde{W}_+(\xi)$ defines on $\{\Lambda(\xi) > 0\} = X \setminus \{\Lambda(\xi) = 0\}$ a multiplication operator with estimate

$$|\tilde{W}_+(\xi)| \lesssim \begin{pmatrix} \Lambda^{-1}(\xi) & 1 \\ \Lambda^{-1}(\xi) & 1 \end{pmatrix} [\Lambda(\xi)]^{-\gamma} \tag{3.3}$$

for $\gamma = \max(\overline{\mu}^+ - 1, 0)$. This implies that the corresponding operator $W_+ = J^*\tilde{W}_+J$ restricted to $E^{(\gamma)} \to E$ is bounded. Furthermore, we obtain from the fact that the free propagator $\mathcal{E}_0(t)$ is unitary for all t

$$\begin{aligned}
\|(u_1,u_2)\|_{E^{(\gamma)}} &\gtrsim \|\lambda(t)\mathcal{E}(t,0)(u_1,u_2) - \mathcal{E}_0(t)W_+(u_1,u_2)\|_E\\
&= \|\lambda(t)\mathcal{E}_0(-t)\mathcal{E}(t,0)(u_1,u_2) - W_+(u_1,u_2)\|_E \to 0
\end{aligned} \tag{3.4}$$

for all (u_1, u_2) with $\mathrm{supp}_\xi(u_1, u_2) \subseteq \{\Lambda(\xi) > 0\}$. But this is a dense subset of $E^{(\gamma)}$ and by the Banach-Steinhaus theorem this gives convergence for all (u_1, u_2).

On $\{\Lambda(\xi) > 0\}$ the inverse wave operator W_+ is represented as the product of invertible matrices. Since $\{\Lambda(\xi) = 0\}$ is excluded by the definition of $E^{(\gamma)}$, the null-space KerW_+ of $W_+ : E^{(\gamma)} \to E$ is trivial and the theorem is proven. □

If we combine the previous theorem with the conservation of abstract energy for the free problem (1.2), $\|(v,v')\|_E = \|(v_1,v_2)\|_E$, we immediately obtain

Corollary 3.2. *For all data* $(u_1,u_2)\in E^{(\gamma)}$ *with* $\gamma=\max\{\overline{\mu}^+-1,0\}$ *the abstract energy of the solution to* (1.1) *satisfies the two-sided estimate*

$$\|(u,u')\|_E \sim \frac{1}{\lambda(t)} \tag{3.5}$$

for all $t\geq 0$.

Note that the definition of $E^{(\gamma)}$ (formula (2.36)) implies $E^{(\gamma)}\perp \mathrm{KerA}$, such that corresponding eigenfunctions of A are excluded in the previous corollary. If the data belong to KerA, the energy decays like $1/\lambda^2(t)$.

If the operator A has a bounded inverse, i.e., $\Lambda(\xi)\geq c_0>0$ uniformly on X, the dissipative zone $Z_{diss}(N)$ can be shrunken to $Z_{diss}(N)\cap\{\Lambda(\xi)\geq c_0\}$, such that $1+t\leq Nc_0^{-1}$ holds uniformly. In this case, the estimate of $\mathcal{E}(t,s,\xi)$ in the dissipative zone can be replaced by the trivial estimate $\|\mathcal{E}(t,s,\xi)\|\lesssim 1$ following directly from system (2.4) using the boundedness of $|b(t)|$ and $\Lambda(\xi)$. Thus, in this case we can revert the time variable and consider the limits as $t\to-\infty$ under (A2) as well. Furthermore, $E=E^{(\gamma)}=\mathcal{D}(\Lambda)\times H$ for all choices of γ and Theorem 3.1 follows directly from Lemma 2.3 without relying on the Banach-Steinhaus theorem.

Theorem 3.3. *Assume* $(Aw,w)\geq c_0\|w\|$ *for all* $w\in H$. *Assume further that the coefficient* $b=b(t)$ *satisfies* (A2). *Then there exist bounded and invertible operators* $W_+ : E\to E$ *and* $W_- : E\to E$*, such that for all Cauchy data* $(u_1,u_2)\in E$ *and corresponding data* $(v_1^\pm,v_2^\pm)=W_\pm(u_1,u_2)$ *the estimate*

$$\|\lambda(t)(u,u')-(v,v')\|_E\leq C\langle t\rangle^{-1}\|(u_1,u_2)\|_E \tag{3.6}$$

holds true. The constant C *depends on the coefficient* $b(t)$ *and* c_0.

Proof. The statement follows directly from Lemma 2.3 (for positive times and, if we replace $b(t)$ by $-b(t)$, also for negative times) if we recall the following additional estimates (written for $t\geq 0$):

$$\|N_1^{-1}(t,\xi)-I\|+\|\mathcal{Q}_1(t,s,\xi)-\mathcal{Q}_1(\infty,s,\xi)\|\lesssim\frac{1}{\Lambda(\xi)(1+t)}\lesssim\frac{1}{1+t}. \tag{3.7}$$

The first one follows from the properties of $N^{(1)}(t,\xi)$, the last one restates (2.30).

The operator W_+ is defined via the limit $\tilde{W}_+(\xi)$ as in the proof of Theorem 3.1, but the lower bound on A allows us to skip the dissipative zone and to use $t_\xi = 0$ instead. Then the above estimates imply for the difference

$$\begin{aligned}&\|\lambda(t)\mathcal{E}(t,0,\xi) - \mathcal{E}_0(t)W_+(\xi)\|\\ &= \|\underbrace{M^{-1}N_1^{-1}(t,\xi)M}_{=I+\mathcal{O}(t^{-1})}\mathcal{E}_0(t,\xi)\underbrace{M^{-1}\mathcal{Q}_1(t,0,\xi)N_1(0,\xi)M}_{=(*)+\mathcal{O}(t^{-1})}\\ &\qquad\qquad - \mathcal{E}_0(t,\xi)\underbrace{M^{-1}\mathcal{Q}_1(\infty,0,\xi)N_1(0,\xi)M}_{=(*)}\| \lesssim \frac{1}{1+t}\end{aligned}$$

and the desired statement follows. □

Remark 11. In contrast to Theorem 3.1 we can use Theorem 3.3 to obtain two-sided norm estimates for $J^*\mathcal{E}(t,0)J$. It follows that

$$|||J^*\mathcal{E}(t,0)J|||_{E\to E} \sim \frac{1}{\lambda(t)} \quad \text{for all } t. \tag{3.8}$$

4. Applications

We will sketch several applications to the previously given abstract results. The list is not complete, but it is intended to give an overview of the different cases and types of results.

Example 4. As first example we consider $A = -\Delta$ on $L^2(\mathbb{R}^n)$. Then the abstract equation (1.1) reads as a wave equation with time-dependent dissipation. This case was treated in [8] and [9] by similar techniques. The abstract spaces reduce to Sobolev scale; we have $\mathcal{D}(\Lambda) = H^1(\mathbb{R}^n)$ and $E = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In this special case $\mathrm{Ker}\Delta = \{0\}$. Thus, as a special result, we obtain that for arbitrary data $u_1 \in H^1(\mathbb{R}^n)$ and $u_2 \in L^2(\mathbb{R}^n)$ the hyperbolic energy of the corresponding solution $u = u(t,x)$ satisfies

$$\|\nabla u(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_2^2 \sim \frac{1}{\lambda^2(t)} \tag{4.1}$$

as a two-sided energy estimate.

The same result is true for $A = -\Delta$, the Dirichlet- or Neumann-Laplacian on $L^2(\Omega)$ for an exterior domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary.

Example 5. As a second example we consider the Klein-Gordon-type equation, where $A = -\Delta + 1$ on $L^2(\mathbb{R}^n)$. Then $\sigma(A) = [1,\infty)$ and A is boundedly invertible. Furthermore, $\mathcal{D}(\Lambda) = H^1(\mathbb{R}^n)$ and $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $E^{(\gamma)} = E$ for all $\gamma \geq 0$.

In this case it is a simple consequence of the representation of $\mathcal{Q}_1(t,s,\xi)$ that $W_+ \in \mathcal{L}(E)$ is invertible in $\mathcal{L}(E)$. In particular, (4.1) is valid.

The same result holds true in the case of $A = -\Delta$ the Dirichlet-Laplacian on $L^2(\Omega)$, Ω an interior domain with smooth boundary.

Example 6. Contrary to this for $A = -\Delta$ the Neumann-Laplacian on $L^2(\Omega)$, Ω bounded domain with smooth boundary, A has the non-trivial null-space $\mathrm{KerA} = \mathbb{C}$ consisting of constant functions. In this case the two-sided estimate (4.1) does not hold. Solving the corresponding ordinary differential equation we see that solutions to constant data behave like $1/\lambda^2(t)$ or $b(t)/\lambda^2(t)$.

Example 7. Let $(a_{ij}(x))_{i,j=1,\dots n} \in L^\infty(\mathbb{R}^n, \mathbb{C}^{n\times n})$ be a positive and self-adjoint matrix. Then we can consider the second-order elliptic operator

$$A = -\sum_{i,j=1,\dots n} \partial_i a_{ij}(x) \partial_j \tag{4.2}$$

on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^n)$. The corresponding abstract energy space has norm

$$\|(u_1,u_2)\|_E^2 = \int_{\mathbb{R}^n} \left(|T(x)\nabla u|^2 + |u_t|^2\right) \mathrm{d}x \tag{4.3}$$

where $T(x) = \sqrt{(a_{ij}(x))}$ in the sense of positive matrices.

Example 8. The approach is not restricted to second-order operators. We can also consider for example the damped plate equation with $A = \Delta^2$. In this case the abstract energy space is $E = \dot{H}^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

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