Advances in Differential Equations Volume 12, Number 10 (2007), 1081–1114

LIMIT BEHAVIOUR OF A CLASS OF NONLINEAR ELLIPTIC PROBLEMS IN INFINITE CYLINDERS

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(Submitted by: Michel Chipot)

Abstract. We study the asymptotic behaviour of the solution of nonlinear monotone elliptic problem

$$-\operatorname{div}[a(x, Du_l)] = \mu \text{ on } \ (-l, l)^q \times \omega$$

with homogeneous Cauchy-Dirichlet boundary conditions, where ω is a bounded, open, connected subset of $\mathbb{R}^{N-q}$ with second member in $L^1(\omega) + W^{-1,p'}(\omega)$, using the framework of renormalized solutions. Assuming specific dependence of the operator a with respect to the variable $(x_1, x_2) \in (-l, l)^q \times \omega$ and that $\mu = \mu(x_2)$, we show the convergence of u_l, in an appropriate sense, toward the solution of the same problem posed in ω.

1. Introduction

In this paper we study the asymptotic behaviour of solutions of elliptic problems in domains becoming unbounded. This problem was first investigated by Chipot, Rougirel and Xie in [7], [8], [9], [10], [11] and [21] in the variational case. We set $\Omega_l = (-l, l)^q \times \omega$, where ω is a bounded, open, connected subset of $\mathbb{R}^{N-q}$, and we consider the N-dimensional problem of the type

$$\begin{cases} -\operatorname{div}[a(x, Du_l)] &= \mu \quad \text{on } \Omega_l \\ u_l &= 0 \quad \text{on } \partial\Omega_l, \end{cases} \tag{1.1}$$

where $u \mapsto -\operatorname{div}[a(x, Du)]$ is a monotone operator defined on $W_0^{1,p}(\Omega_l)$ with values in $W^{-1,p'}(\Omega_l)$. Setting $x = (x_1, x_2) \in (-l, l)^q \times \omega$, the function a satisfies specific dependence on $(x_1, x_2) \in \Omega_l$ (cf. (2.1) and (2.2)), namely

$$a(x, \xi) = \begin{pmatrix} a_1^1(x, \xi) \\ a_2^1(x, \xi) \end{pmatrix} + \begin{pmatrix} a_1^2(x_2, \xi) \\ a_2^2(x_2, \xi) \end{pmatrix} ,$$

Accepted for publication: June 2007.
AMS Subject Classifications: 35J60, 35B40, 35D10, 35J25.

while the data μ is such that $\mu = \mu(x_2)$. Typically, the model case is the p-Laplace operator defined by

$$\operatorname{div}(a(x, Du)) = \Delta_p u = \operatorname{div}(|Du|^{p-2} Du).$$

We denote by $W^{1,p}_{\Gamma_l}(\Omega_l)$ the space of functions which belong to the Sobolev space $W^{1,p}(\Omega_l)$ and which have null trace on $[-l, l]^q \times \partial\omega$. If $\mu \in W^{-1,p'}(\omega)$ and the function a is strongly monotone, there exists a unique weak solution $u_l \in W^{1,p}_0(\Omega_l)$ of (1.1). In [7], [8], [9], [10], [11] and [21], the authors stated that if the operator a is strongly monotone, u_l converges to v in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$, for any $l_0 > 0$, with rate of convergence, where $v \in W^{1,p}_0(\Omega_l)$ is the unique weak solution of the $N-q$-dimensional problem

$$\begin{cases} -\operatorname{div}[a_2^2(x_2, Dv)] = \mu & \text{on } \omega \\ v = 0 & \text{on } \partial\omega \end{cases} \tag{1.2}$$

The result is numerically interesting because when the cylinder is large in the x_1-direction it is reasonable to approximate u_l by v, the solution of a problem of lower dimension.

Here, we will take the datum μ in $L^1(\omega) + W^{-1,p'}(\omega)$. If $p > N - q$, it is known, by Sobolev imbedding, that $L^1(\omega)$ is a subspace of $W^{-1,p'}(\omega)$, and in this case existence and uniqueness of weak solutions of problems (1.1) and (1.2) is immediate. But, if $1 < p \leq N - q$, we cannot expect to have existence and uniqueness of weak solutions of (1.1) and (1.2) (see [20]). So we use the framework of renormalized solutions introduced by DiPerna and Lions in [13] and [14] for the Boltzmann equations. It was then adapted to the elliptic case by Lions and Murat in [16], [18] and [19] in the case of L^1 datum, and then Boccardo, Gallouët and Orsina in [5] studied the case of $L^1 + W^{-1,p'}$ datum.

In the strongly monotone case, we show a convergence result for $2 \leq p < (2q-1)/(q-1)$. If $p > N-q$, the results of the variational case above apply. In the case $p \leq N - q$, after recalling some estimates met by renormalized solutions in the Lorentz spaces, we adapt the methods developed in [7], [8], [9], [10], [11] and [21] to the renormalized solutions in order to show that Du_l converges to Dv in suitable Lorentz spaces with rate of convergence (see Theorem 3.5) and $T_k(u_l)$ converges to $T_k(v)$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ (see Theorem 3.6) for any $l_0 > 0$, where T_k is the usual truncation at height k.

In the strict monotone case, we show a convergence result under more restrictive assumptions, namely $\mu \geq 0$ and a independent of x_2, using a method based on maximum principle arguments. In the variational case, that is, if the L^1 part of μ is equal to 0 or if $p > N - q$, we show that u_l converges to v in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$, for any $l_0 > 0$ (see Theorem 4.1). These results, in the case where a is the p-Laplace operator, have been proved independently by Chipot and Xie in [10] and [21]. In the unvariational case, that is, $1 < p \le N - q$, we show that $T_k(u_l)$ converges to $T_k(v)$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ (see Theorem 4.3), for any $l_0 > 0$.

The paper is divided as follows. In the next section we set assumptions on the data and definition of renormalized solutions. In Section 3 we deal with the strongly monotone case, and finally in Section 4 we investigate the strict monotone case.

2. Assumptions and definitions

2.1. **Assumptions on the data.** In the whole paper we define $\Omega_l = (-l, l)^q \times\omega$ to be an open subset of $\mathbb{R}^N$ $(N \ge 2)$ where $q \ge 1$, l is a positive real number and ω is a bounded, connected and open subset of $\mathbb{R}^{N-q}$ with Lipschitz boundary $\partial\omega$. We denote by Γ_l the subset of $\partial\Omega_l$ defined by $\Gamma_l = [-l, l]^q \times \partial\omega$. For $x \in \Omega_l$ we will write $x = (x_1, x_2)$ where $x_1 \in (-l, l)^q$ and $x_2 \in \omega$, and D_{x_1}, D_{x_2} the gradients with respect to x_1, x_2.

We denote by $W^{1,p}_{\Gamma_l}(\Omega_l)$ the space of functions belonging to $W^{1,p}(\Omega_l)$ which have a null trace on Γ_l. Since Ω_l is a bounded and connected open subset of $\mathbb{R}^N$ with Lipschitz boundary and since $\mathrm{meas}(\Gamma_l) > 0$, by Poincaré's inequality, we can endow the space $W^{1,p}_{\Gamma_l}(\Omega_l)$ with the norm $\|v\|_{W^{1,p}_{\Gamma_l}(\Omega_l)} = \|Dv\|_{L^p(\Omega_l)}$ (see [22]).

We assume that $a : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Carathéodory function (that is, $a(x, \xi)$ is measurable in $x \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$, for almost every $x \in \mathbb{R}^N$) with the structural condition

$$a(x, \xi) = a^1(x, \xi) + a^2(x_2, \xi) = \begin{pmatrix} a^1_1(x, \xi) \\ a^1_2(x, \xi) \end{pmatrix} + \begin{pmatrix} a^2_1(x_2, \xi) \\ a^2_2(x_2, \xi) \end{pmatrix} , \tag{2.1}$$

where a^1 and a^2 are Carathéodory functions satisfying the following hypotheses:

$$a^1(x, \tbinom{0}{\xi_2}) = 0 \tag{2.2}$$

for almost every $x \in \mathbb{R}^N$ and for every $\xi_2 \in \mathbb{R}^{N-q}$;

$$a(x, \xi) \cdot \xi \ge \alpha|\xi|^p , \tag{2.3}$$

for almost every $x \in \mathbb{R}^N$ and for every $\xi \in \mathbb{R}^N$, where α is a positive constant;

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') > 0\,, \tag{2.4}$$

for almost every $x \in \mathbb{R}^N$ and for every $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$; and

$$|a^1(x,\xi)| \leq \gamma_1 \left[b_1(x) + |\xi|^{p-1}\right]\,,\ |a^2(x_2,\xi)| \leq \gamma_2 \left[b_1(x_2) + |\xi|^{p-1}\right]\,, \tag{2.5}$$

for every $\xi \in \mathbb{R}^N$, for almost every $x \in K \times \omega$ for the first inequality and for almost every $x_2 \in \omega$ for the second inequality, where K is any bounded subset of $\mathbb{R}^q$, b_1 and b_2 are nonnegative functions respectively in $L^{p'}(K)$ and $L^{p'}(\omega)$ and γ_1 and γ_2 are positive constants with γ_1 depending on K.

We assume that

$$f = f(x_2) \in L^1(\omega)\,,\ g = g(x_2) \in \left(L^{p'}(\omega)\right)^{N-q}\,; \tag{2.6}$$

thus, $\mu = f - \operatorname{div}(g)$ can be extended to $\tilde{\mu}$ on $L^1(\Omega_l) + W^{-1,p'}(\Omega_l)$, setting for all $\phi \in W_0^{1,p}(\Omega_l) \cap L^\infty(\Omega_l)$

$$\tilde{\mu}(\phi) = \int_{\Omega_l} f\phi\, dx + \int_{\Omega_l} g \cdot D_{x_2}\phi\, dx\,.$$

Since there is no ambiguity we will denote it also by μ.

Remark 2.1. We recall that if $p > N - q$, by Sobolev injection, $L^1(\omega) \subset W^{-1,p'}(\omega)$. From here, if $\mu \in L^1(\omega)$ then the extension $\mu \in W^{-1,p'}(\Omega_l)$.

Remark 2.2. The structural condition (2.1) and assumptions (2.2)–(2.4) allow us to define problem (1.2) from problem (1.1). Indeed, we identify $a_2^2(x_2, \binom{0}{\xi_2})$ and $a_2^2(x_2,\xi_2)$ and from (2.2) and (2.3), we get

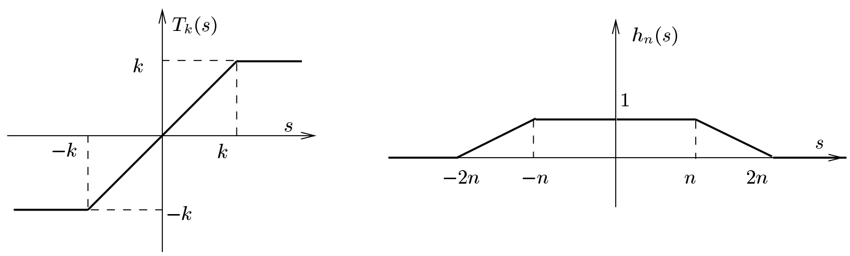
$$a_2^2(x_2,\xi_2) \cdot \xi_2 \geq \alpha|\xi_2|^p\,, \tag{2.7}$$

for almost every $x_2 \in \omega$ and for every $\xi_2 \in \mathbb{R}^{N-q}$; the strict monotonicity of a gives

$$(a_2^2(x_2,\xi_2) - a_2^2(x_2,\xi_2')) \cdot (\xi_2 - \xi_2') > 0\,, \tag{2.8}$$

for almost every $x_2 \in \omega$ and for every $\xi_2, \xi_2' \in \mathbb{R}^{N-q}$, $\xi_2 \neq \xi_2'$.

Remark 2.3. In the linear case, $a(x,Du) = A(x)Du$, conditions (2.1)–(2.5) lead to a matrix A which satisfies the hypotheses of Chipot and Rougirel in [7], [8] and [9]. In the nonlinear case, the p-Laplacian, $-\operatorname{div}(a(x,Du)) = -\operatorname{div}(|Du|^{p-2}Du) = -\Delta_p u$, satisfies conditions (2.1)–(2.5).



2.2. **Definition of renormalized solutions.** In this paper we consider renormalized solutions to the problems (1.1) and (1.2). We recall that for $k > 0$ and $n > 0$ and for $s \in \mathbb{R}$, $T_k(s) = \max(-k, \min(k, s))$ and $h_n(s) = 1 - \frac{|T_{2n}(s) - T_n(s)|}{n}$ (see above).

We introduce the definition of the gradient of a function whose truncates belong to $W_0^{1,p}(\Omega)$, according to [1], where Ω is an open subset of $\mathbb{R}^N$.

Definition 2.4. Let u be a measurable function defined on Ω which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see [1], Lemma 2.1) a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v 1\!\!1_{\{|u| \le k\}} \qquad \text{a.e. in } \Omega, \ \forall\, k > 0.$$

We define the gradient Du of u as this function v, and denote $Du = v$.

Definition 2.5. A measurable function u_l defined on Ω_l and finite almost everywhere on Ω_l is called a renormalized solution of (1.1) if

$$T_k(u_l) \in W_0^{1,p}(\Omega_l), \ \forall k > 0; \tag{2.9}$$

$$\lim_{n \to +\infty} \frac{1}{n} \int_{\{n \le |u_l| \le 2n\}} a(x, Du_l) \cdot Du_l \, dx = 0\,, \tag{2.10}$$

and if for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support,

$$\begin{aligned} -\operatorname{div}[h(u_l) a(x, Du_l)] + h'(u_l) a(x, Du_l) \cdot Du_l &= f h(u_l) \\ &\quad - \operatorname{div}[g h(u_l)] + g \cdot Du_l h'(u_l) \quad \text{in } \mathcal{D}'(\Omega_l). \end{aligned} \tag{2.11}$$

In a similar manner we define a renormalized solution of the problem (1.2).

Definition 2.6. A measurable function v defined on ω and finite almost everywhere on ω is called a renormalized solution of (1.2) if

$$T_k(v) \in W_0^{1,p}(\omega),\ \forall k > 0; \tag{2.9'}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le|v|\le 2n\}} a_2^2(x_2, D_{x_2}v) \cdot D_{x_2}v\, dx = 0\,, \tag{2.10'}$$

and if for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support,

$$\begin{aligned} -\operatorname{div}_{x_2}[h(v)a_2^2(x_2, D_{x_2}v)] + a_2^2(x_2, D_{x_2}v) \cdot D_{x_2}v = fh(v) \\ -\operatorname{div}_{x_2}[gh(v)] + g \cdot D_{x_2}v h'(v) \quad \text{in } \mathcal{D}'(\omega)\,. \end{aligned} \tag{2.11'}$$

We have the following existence and uniqueness result (see [5], [12] and [18]).

Theorem 2.7. *Under assumptions* (2.1)–(2.6) *there exists a unique* u_l (*respectively* v) *a renormalized solution of* (1.1) (*respectively* (1.2)).

Remark 2.8. Taking $h = h_n$ and $\phi = T_k(u_l)$ (respectively $\phi = T_k(v)$) in (2.11) (respectively (2.11′)), using (2.3) and letting n go to $+\infty$, we obtain that u_l and v satisfy the following estimates, for any $k > 0$:

$$\begin{aligned} \int_{\Omega_l} |DT_k(u_l)|^p &\le C\Big(k\|f\|_{L^1(\Omega_l)} + \|g\|^{p'}_{\left(L^{p'}(\Omega_l)\right)^{N-q}}\Big) \\ &= Cl^q\Big(\|f\|_{L^1(\omega)}k + \|g\|^{p'}_{\left(L^{p'}(\omega)\right)^{N-q}}\Big) \end{aligned} \tag{2.12}$$

$$\int_{\omega} |DT_k(v)|^p \le C\Big(\|f\|_{L^1(\omega)}k + \|g\|^{p'}_{\left(L^{p'}(\omega)\right)^{N-q}}\Big), \tag{2.13}$$

where C depends only on α and p.

3. THE STRONG MONOTONE CASE

In this section, with an additional assumption on a and under conditions on p, q and N, we show that, u_l and its truncates converge in suitable Sobolev spaces toward v with rate of convergence. We assume $p \ge 2$ and that a satisfies, in addition to (2.1)–(2.5), a strong monotonicity hypothesis which is

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') \ge c_p|\xi - \xi'|^p\,, \tag{3.1}$$

for almost every $x \in \mathbb{R}^N$, and for every $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$, where c_p is a positive constant. Moreover, we assume a Lipschitz condition on a, that is

$$|a_1^1(x,\xi)| + |a_1^2(x,\xi) - a_1^2(x,\xi')| \le L|\xi - \xi'|(1 + |\xi|^{p-2} + |\xi'|^{p-2}) \tag{3.2}$$

for almost every $x \in \mathbb{R}^N$ and for every $\xi, \xi' \in \mathbb{R}^N$, such that $\xi' = \binom{0}{\xi_2'}$, where L is a positive constant.

We recall, from [7], [8], [9] and [21], a result in the variational case (i.e., $f = 0$). Indeed, in this case, under assumptions (2.1)–(2.6), there exists unique $u_l \in W_0^{1,p}(\Omega_l)$ (respectively $v \in W_0^{1,p}(\omega)$) a weak solution of (1.1) (respectively (1.2)), and with the additional assumptions (3.1) and (3.2), for all $2 \le p < \bar{p}(q)$ (where $\bar{p}(q) = +\infty$ for $q = 1, 2$ and $\bar{p}(q) = \frac{2q}{q-2}$ for $q \ge 3$), for all $l_0 > 0$, we get $u_l \to v$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ with the rates of convergence

$$\|Du_l - Dv\|_{L^p(\Omega_{l_0})} = O(l^\alpha), \quad \forall\, q - \tfrac{2p}{p-2} < \alpha < 0 \text{ if } p > 2 \tag{3.3}$$

$$\|Du_l - Dv\|_{L^2(\Omega_{l_0})} = O(l^\alpha), \quad \forall\, \alpha < 0 \quad \text{ if } p = 2.$$

Remark 3.1. In [21], the result for $p > 2$ above is proved only for the case $N = 2$, but with the same techniques we state this result.

Thus, in the sequel, we restrict ourselves to the case $p \le N - q$ because, by Remark 2.1, in the case $p > N - q$ the variational case applies. Since we consider $L^1 + W^{-1,p'}$ data we cannot expect to have convergence of u_l in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$.

3.1. **Preliminaries about Lorentz spaces.** Let Ω be an open subset of $\mathbb{R}^N$. For $1 < q < \infty$ the Lorentz space $L^{q,1}(\Omega)$ (see [15] and [17]) is the space of measurable functions such that

$$\|f\|_{L^{q,1}(\Omega)} = \int_0^{|\Omega|} [f^*(t) t^{\frac{1}{q}}] \frac{dt}{t} < +\infty, \tag{3.4}$$

endowed with the norm defined by (3.4) and where f^* denotes the decreasing rearrangement of f; i.e.,

$$f^*(t) = \inf\{s \ge 0 \;:\; \text{meas}\{x \in \Omega \;:\; |f(x)| > s\} < t\}, \;\; t \in [0, |\Omega|]\}.$$

For $1 < r < \infty$ the Lorentz space $L^{r,\infty}(\Omega)$ (also known as the Marcinkiewicz space) is the space of measurable functions such that

$$\|f\|_{L^{r,\infty}(\Omega)} = \sup_{t>0} \{t [\text{meas}\{x \in \Omega \;:\; |f(x)| > t\}]^{\frac{1}{r}}\} < +\infty, \tag{3.5}$$

endowed with the norm defined by (3.5). The Lorentz spaces satisfy for every $1 < s < r < \infty$ the following inclusions:

$$L^{r,1}(\Omega) \subset L^{r,r}(\Omega) = L^r(\Omega) \subset L^{r,\infty}(\Omega) \subset L^{s,1}(\Omega).$$

The space $L^{r,\infty}(\Omega)$ is the dual space of $L^{r',1}(\Omega)$ with $\frac{1}{r}+\frac{1}{r'}=1$, and one has the generalized Hölder inequality

$$\int_\Omega |fg| \le \|f\|_{L^{r,\infty}(\Omega)}\|g\|_{L^{r',1}(\Omega)}, \quad \forall f\in L^{r,\infty}(\Omega),\ \forall g\in L^{r',1}(\Omega). \tag{3.6}$$

More generally, if $1<p,q,r<\infty$ are such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$,

$$\|fg\|_{L^{p,1}(\Omega)} \le \|f\|_{L^{q,\infty}(\Omega)}\|g\|_{L^{r,1}(\Omega)}, \quad \forall f\in L^{q,\infty}(\Omega),\ \forall g\in L^{r,1}(\Omega). \tag{3.7}$$

3.2. **Some properties on renormalized solutions.** Finally, we introduce a generalized form of a result stated in [1] and [2].

Lemma 3.2. *Suppose that $1<p\le N$. Let w be a measurable function satisfying $T_k(w)\in W^{1,p}_{\Gamma_l}(\Omega_l)$, for every positive k, and such that*

$$\int_{\Omega_l} |DT_k(w)|^p \le Mk+L, \quad \forall k>0,$$

where M and L are given constants. Then

$$\||w|^{p-1}\|_{L^{s/p,\infty}(\Omega_l)} \le C(M+|\Omega_l|^{1/s}L^{1/p'}),$$

with $s=p*$ (where $p*=Np/(N-p)$) if $p<N$, with $s\in(p,+\infty)$ if $p=N$, where C is a constant depending only on p, N, ω and is independent of l,

$$\||Dw|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \le C(M+|\Omega_l|^{1/r-1/p'}L^{1/p'}),$$

for all $r\in(1,N']$ if $p<N$ (where $N'=N/(N-1)$, for all $r\in(1,N')$ if $p=N$, where C is a constant depending only on p, N and ω and is independent of l.

Proof. We follow the proof of [2] (Lemma A.1). The essential difference is that we take $w\in W^{1,p}_{\Gamma_l}(\Omega_l)$ in place of $w\in W^{1,p}_0(\Omega_l)$ so the Sobolev constant and the Poincaré's constant depend on Ω_l, but, thanks to the geometry of $\Omega_l=(-l,l)^q\times\omega$, these constants are independent of l. We have

$$k^r\,\mathrm{meas}\{x\in\Omega_l:\ |w|>k\} \le \int_{\Omega_l}|T_k(w)|^r. \tag{3.8}$$

If $p<N$, we take $r\in[p,p*]$, so by interpolation

$$\int_{\Omega_l}|T_k(w)|^r \le \|T_k(w)\|^{r\theta}_{L^p(\Omega_l)}\|T_k(w)\|^{r(1-\theta)}_{L^{p*}(\Omega_l)}, \tag{3.9}$$

where θ is such that $\frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{p*}$. By Poincaré's inequality in the x_2 direction, we have

$$\|T_k(w)\|_{L^p(\Omega_l)} \le C(\omega)\|DT_k(w)\|_{L^p(\Omega_l)}, \tag{3.10}$$

where $C(\omega)$ is a positive constant depending only on ω and thus independent of l. By the Sobolev inequality, we have

$$\|T_k(w)\|_{L^{p*}(\Omega_l)} \leq C(\omega, p, N)\|DT_k(w)\|_{L^p(\Omega_l)}, \tag{3.11}$$

where $C(\omega, p, N)$ is a positive constant depending only on ω, p and N, and is independent of l thanks to the geometry of Ω_l. From (3.9), (3.10) and (3.11) it follows that

$$\int_{\Omega_l} |T_k(w)|^r \leq C(p, N, \omega)\|DT_k(w)\|^r_{L^p(\Omega_l)} \leq C(p, N, \omega)(Mk + L)^{r/p}.$$

In the case $p = N$, we take $r \in [p, +\infty)$; the Sobolev inequality gives

$$\|T_k(w)\|_{L^r(\Omega_l)} \leq C(r, N, \omega)\|DT_k(w)\|_{L^p(\Omega_l)},$$

where $C(r, N, \omega)$ is a positive constant depending only on r, N and ω, and is independent of l. It follows that

$$\int_{\Omega_l} |T_k(w)|^r \leq C(r, N, \omega)(Mk + L)^{r/p}.$$

Therefore, from (3.8)

$$k^r \operatorname{meas}\{x \in \Omega_l : |w| > k\} \leq C(r, N, \omega)(Mk + L)^{r/p}.$$

Then, the sequel of the proof is the same as it is in [2]. □

Proposition 3.3. *We can apply this lemma to the inequality of the Remark* 2.8 *in order to obtain the following estimates on* u_l *and* v*:*

$$\||Du_l|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \leq C\Big(l^q\|f\|_{L^1(\omega)} + l^{q/r}\|g\|_{\left(L^{p'}(\omega)\right)^{N-q}}\Big);$$

$$\||Dv|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \leq C\Big(l^q\|f\|_{L^1(\omega)} + l^{q/r}\|g\|_{\left(L^{p'}(\omega)\right)^{N-q}}\Big);$$

where C *is a constant independent of* l*.*

Proof. As $T_k(u_l)$ and $T_k(v)$ belong to $W^{1,p}_{\Gamma_l}(\Omega_l)$, from (2.12) and (2.13) we can take $w = u_l$ or $w = v$ in Lemma 3.2; we get

$$\||Du_l|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \leq C\Big(l^q\|f\|_{L^1(\omega)} + l^{q/p'}\|g\|_{\left(L^{p'}(\omega)\right)^{N-q}}|\Omega_l|^{1/r-1/p'}\Big)$$

and the fact that $|\Omega_l| = l^q|\omega|$ gives the result. □

Remark 3.4. As $r > 1$, $\frac{q}{r} < q$ and l is intended to go to infinity, we can suppose that $l \geq 1$; thus, $l^{q/r} < l^q$ and the estimates above become

$$\||Du_l|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \leq Cl^q\Big(\|f\|_{L^1(\omega)} + \|g\|_{\left(L^{p'}(\omega)\right)^{N-q}}\Big), \tag{3.12}$$

$$\||Dv|^{p-1}\|_{L^{r,\infty}(\Omega_l)} \le Cl^q\Big(\|f\|_{L^1(\omega)} + \|g\|_{\left(L^{p'}(\omega)\right)^{N-q}}\Big). \tag{3.13}$$

3.3. **Behaviour of u_l as l goes to infinity.** According to the regularity of renormalized solutions (see (2.9), (2.9′) and Remark 3.4) we prove in Theorem 3.5 below that Du_l converge to Dv in suitable Lorentz spaces and in Theorem 3.6 that $T_k(u_l)$ converges to $T_k(v)$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$, as l goes to infinity.

Theorem 3.5. *Assume that* (2.1)–(2.6), (3.1) and (3.2) *hold. Let l_0 be a positive real number. If $2 \le p < \frac{2q-1}{q-1}$ (with the convention $\frac{2q-1}{q-1} = +\infty$ for $q = 1$), then we have*

$$\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_0})} = O(l^\alpha), \tag{3.14}$$

for any $1 < r < \bar{r}(p, q, N)$ and for any $\bar{\alpha}(p, q, N, r) < \alpha < 0$, where

$$\bar{r}(p, q, N) = \min\Big(\frac{N}{N-1}, \frac{1}{2 - \frac{1}{q} - \frac{1}{p-1}}\Big), \qquad \text{if } p \le N - q$$

and

$$\bar{\alpha}(p, q, N, r) = \begin{cases} q - \frac{p-1}{p-2} + q\frac{p-1}{p-2}(1 - \frac{1}{r}) & \text{if } \ p > 2, \\ -\infty & \text{if } \ p = 2. \end{cases}$$

Theorem 3.6. *Under assumptions* (2.1)–(2.6), (3.1) and (3.2), *and for any $2 \le p < \frac{2q-1}{q-1}$ (with the convention $\frac{2q-1}{q-1} = +\infty$ for $q = 1$), we have for any $l_0 > 0$*

$$T_k(u_l) \to T_k(v) \ \text{in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}), \quad \forall k \ge 0.$$

Let us mention a few remarks.

Remark 3.7. In the case $p > 2$, the condition $r < \bar{r}(p, q, N)$ is equivalent to $q - \frac{p-1}{p-2} + q\frac{p-1}{p-2}(1 - \frac{1}{r}) < 0$, so the condition on α is valid.

Remark 3.8. In the case $p > 2$, the rate of convergence in (3.14) and the Lorentz space where the convergence takes place depends on r. More precisely, the smaller the Lorentz space where the convergence takes place is, the smaller the rate of convergence is.

Remark 3.9. Thanks to the generalized Hölder's inequality (3.6), Theorem 3.5 implies the convergence of u_l toward v in Sobolev spaces. Indeed, we have for any $l_0 > 0$

$$\|Du_l - Dv\|_{L^s(\Omega_{l_0})} = O(l^{\frac{\alpha}{p-1}})$$

for any $s < (p-1)\frac{N}{N-1}$, with the same conditions on p and α. Thus, by Poincaré's inequality, we get convergence in Sobolev spaces. In particular, we mention a result which will be useful in the proof of Theorem 3.6,

$$\|Du_l - Dv\|_{L^{p-1}(\Omega_{l_0})} \to 0$$

when $l \to +\infty$, for any $2 \le p < \frac{2q-1}{q-1}$.

Remark 3.10. We can give further information on $\bar{r}(p,q,N)$. We suppose $2 \le p < \frac{2q-1}{q-1}$ (with the convention $\frac{2q-1}{q-1} = +\infty$ for $q = 1$). If $p = 2$, then $\bar{r}(2,q,N) = \frac{N}{N-1}$. If $2 < p \le N - q$, then for $q = 1$,

$$\bar{r}(p,1,N) = \tfrac{N}{N-1},$$

and for any $q \ge 2$,

$$\bar{r}(p,q,N) = \tfrac{N}{N-1}, \qquad \text{if } 2 < p \le \frac{1}{1 - \frac{1}{q} + \frac{1}{N}} + 1;$$

$$\bar{r}(p,q,N) = \tfrac{1}{2-\frac{1}{q}-\frac{1}{p-1}}, \qquad \text{if } \tfrac{1}{1-\frac{1}{q}+\frac{1}{N}} + 1 \le p < \tfrac{2q-1}{q-1}.$$

In the case $2 \le p < \frac{1}{1-1/q+1/N} + 1$, r can reach the value $\bar{r}(p,q,N)$.

Proof of Theorem 3.5. The proof is divided into two steps. In Step 1, combining the techniques developed by Chipot and Rougirel (see [7], [8] and [9]) with the framework of renormalized solutions we derive an estimate of $DT_k(u_l - v)$ on Ω_{l_1}, where $l_1 > 0$ (see inequality (3.17) below). Using (3.17), Step 2 is devoted to proving an estimate on $D(u_l - v)$ in an appropriate Lorentz space.

Step 1. In this step, we introduce a smooth function $\rho \in C^1(\mathbb{R}^q)$ such that

$$0 \le \rho \le 1, \ \ \rho = 1 \ \text{ on } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^q, \ \ \rho = 0 \ \text{ on } \mathbb{R}^q \setminus [-1,1]^q \tag{3.15}$$

$$|D_{x_1}\rho| \le \Lambda, \tag{3.16}$$

where Λ is a positive constant, $\Lambda \ge 2$. We prove the following estimate:

$$\int_{\Omega_{l_1/2}} |DT_k(u_l - v)|^p \, dx \le C \frac{k}{l_1} \int_{\Omega_{l_1}} |Du_l - Dv| \left(1 + |Du_l|^{p-2} + |Dv|^{p-2}\right) dx \tag{3.17}$$

for every $0 < l_1 < l$, where C is a positive constant depending only on c_p, L and Λ, where c_p and L are defined in (3.1) and (3.2).

Let $0 < l_1 < l$. Let us use $T_k(T_{2n}(u_l) - T_{2n}(v))h_n(v)\rho(\frac{x_1}{l_1}) \in W_0^{1,p}(\Omega_l)$ as test function in (2.11) with $h = h_n$. Since $\operatorname{supp}(h_n) = [-2n, 2n]$, we have

$$h_n(u_l)h_n(v)T_k(T_{2n}(u_l) - T_{2n}(v)) = h_n(u_l)h_n(v)T_k(u_l - v);$$

then we get

$$\begin{aligned}
&\int_{\Omega_l} a(x, Du_l) \cdot DT_k(u_l - v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) \\
&+ \int_{\Omega_l} a(x, Du_l) \cdot Du_l\, h_n'(u_l)h_n(v)T_k(u_l - v)\rho(\tfrac{x_1}{l_1}) \\
&+ \int_{\Omega_l} a(x, Du_l) \cdot Dv\, h_n'(v)h_n(u_l)T_k(u_l - v)\rho(\tfrac{x_1}{l_1}) \\
&+ \frac{1}{l_1}\int_{\Omega_l} a(x, Du_l) \cdot (D\rho)(\tfrac{x_1}{l_1})h_n(u_l)h_n(v)T_k(u_l - v) \\
&= \int_{\Omega_l} fT_k(u_l - v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) + \int_{\Omega_l} g \cdot D_{x_2}T_k(u_l - v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) \\
&+ \int_{\Omega_l} g \cdot D_{x_2}v\, h_n'(v)h_n(u_l)T_k(u_l - v)\rho(\tfrac{x_1}{l_1}) \\
&+ \int_{\Omega_l} g \cdot D_{x_2}u_l\, h_n'(u_l)h_n(v)T_k(u_l - v)\rho(\tfrac{x_1}{l_1}),
\end{aligned} \tag{3.18}$$

which we write as $I_1 + I_2 + I_3 + I_4 = I_5$. Now by a result of Chipot in [7] (Proposition 3.1), if $\phi \in W_{\Gamma_l}^{1,p}(\Omega_l)$, then it holds that $\phi(x_1, \cdot) \in W_0^{1,p}(\omega)$ for almost every $x_1 \in (-l, l)^q$. Therefore, for almost every $x_1 \in (-l, l)^q$, we can use $T_k(T_{2n}(u_l) - T_{2n}(v))(x_1, \cdot)h_n(u_l(x_1, \cdot) \in W_0^{1,p}(\omega)$ as a test function in (2.11′) with $h = h_n$; we obtain (using again the fact that T_{2n} disappears)

$$\begin{aligned}
&\int_\omega a_2^2(x_2, D_{x_2}v) \cdot D_{x_2}T_k(u_l - v)h_n(u_l)h_n(v) \\
&+ \int_\omega a_2^2(x_2, D_{x_2}v) \cdot D_{x_2}u_l\, h_n'(u_l)h_n(v)T_k(u_l - v) \\
&+ \int_\omega a_2^2(x_2, D_{x_2}v) \cdot D_{x_2}v\, h_n'(v)h_n(u_l)T_k(u_l - v) \\
&= \int_\omega fT_k(u_l - v)h_n(u_l)h_n(v) + \int_\omega g \cdot D_{x_2}T_k(u_l - v)h_n(u_l)h_n(v) \\
&+ \int_\omega g \cdot D_{x_2}v\, h_n'(v)h_n(u_l)T_k(u_l - v) + \int_\omega g \cdot D_{x_2}u_l\, h_n'(u_l)h_n(v)T_k(u_l - v)
\end{aligned}$$

for almost every $x_1 \in (-l,l)^q$. Multiplying the above identity by $\rho(\frac{x_1}{l_1})$ and integrating over $(-l,l)^q$ leads to

$$\begin{aligned}
&\int_{\Omega_l} a_2^2(x_2, D_{x_2}v)\cdot D_{x_2}T_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) \qquad (3.19)\\
&+\int_{\Omega_l} a_2^2(x_2, D_{x_2}v)\cdot D_{x_2}u_l\, h_n'(u_l)h_n(v)T_k(u_l-v)\rho(\tfrac{x_1}{l_1})\\
&+\int_{\Omega_l} a_2^2(x_2, D_{x_2}v)\cdot D_{x_2}v\, h_n'(v)h_n(u_l)T_k(u_l-v)\rho(\tfrac{x_1}{l_1})\\
&=\int_{\Omega_l} fT_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1})+\int_{\Omega_l} g\cdot D_{x_2}T_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1})\\
&+\int_{\Omega_l} g\cdot D_{x_2}v\, h_n'(v)h_n(u_l)T_k(u_l-v)\rho(\tfrac{x_1}{l_1})\\
&+\int_{\Omega_l} g\cdot D_{x_2}u_l\, h_n'(u_l)h_n(v)T_k(u_l-v)\rho(\tfrac{x_1}{l_1}),
\end{aligned}$$

which we write as $J_1+J_2+J_3=J_4$. Since $J_4=I_5$ we have $I_1-J_1+I_2+I_3+I_4-J_2-J_3=0$.

Since v does not depend on x_1, that is, $D_{x_1}v=0$, the structural condition (2.2) implies that

$$a^1(x,Dv)=0, \quad \text{a.e. in } \Omega_l.$$

Then, from the structural condition (2.1),

$$a(x,Dv)\cdot DT_k(u_l-v)=a_1^2(x_2,Dv)\cdot D_{x_1}T_k(u_l-v)+a_2^2(x_2,Dv)\cdot D_{x_2}T_k(u_l-v)$$

almost everywhere in Ω_l. It follows that

$$\begin{aligned}
I_1-J_1=&\int_{\Omega_l}[a(x,Du_l)-a(x,Dv)]\cdot DT_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1})\\
&+\int_{\Omega_l} a_1^2(x_2,Dv)\cdot D_{x_1}T_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}). \qquad (3.20)
\end{aligned}$$

Since v does not depend on x_1, we have

$$a_1^2(x_2,Dv)\cdot D_{x_1}T_k(u_l-v)h_n(v)=D_{x_1}\cdot\big(a_1^2(x_2,Dv)T_k(u_l-v)h_n(v)\big). \qquad (3.21)$$

Then using (3.21) in the last term of (3.20) and applying Green's formula yield that

$$I_1-J_1=\int_{\Omega_l}[a(x,Du_l)-a(x,Dv)]\cdot DT_k(u_l-v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1})$$

$$
\begin{aligned}
&- \int_{\Omega_l} a_1^2(x_2, Dv) \cdot D_{x_1} u_l h_n'(u_l) h_n(v) T_k(u_l - v) \rho(\tfrac{x_1}{l_1}) \\
&- \frac{1}{l_1} \int_{\Omega_l} a_1^2(x_2, Dv) \cdot (D_{x_1}\rho)(\tfrac{x_1}{l_1}) T_k(u_l - v) h_n(u_l) h_n(v). \quad (3.22)
\end{aligned}
$$

Let us note that the integral on the boundary Γ_l deriving from Green's formula is equal to 0 because ρ is 0 there.

We now claim that I_2, I_3, J_2, J_3 and

$$
\int_{\Omega_l} a_1^2(x_2, Dv) \cdot D_{x_1} u_l h_n'(u_l) h_n(v) T_k(u_l - v) \rho(\tfrac{x_1}{l_1})
$$

go to zero as n goes to infinity. Indeed, since $|h_n(v) T_k(u_l - v) \rho(\tfrac{x_1}{l_1})| \le k$ and $|h_n'(s)| = \frac{1}{n} \mathbb{1}_{\{n \le |s| \le 2n\}}$ almost everywhere in $\mathbb{R}$, we have

$$
|I_2| \le \frac{k}{n} \int_{\{n \le |u_l| \le 2n\}} a(x, Du_l) \cdot Du_l \, dx,
$$

so from (2.10) it follows that I_2 goes to 0 as n goes to infinity. Similarly, J_3 goes to 0 as n goes to infinity. For the other terms, by (2.5), Hölder's inequality and the fact $|T_k(u_l - v)\rho(\tfrac{x_1}{l_1})| \le k$, we get

$$
\begin{aligned}
|I_3| \le C \Big[\Big(\int_{\Omega_l} b_1^{p'} \Big)^{1/p'} \Big(\int_{\Omega_l} |Dv|^p h_n'(v) \Big)^{1/p} \\
+ \Big(\frac{1}{n} \int_{\{|u_l| \le 2n\}} |Du_l|^p \Big)^{1/p'} \Big(\frac{1}{n} \int_{\{n \le |v| \le 2n\}} |Dv|^p \Big)^{1/p'} \Big],
\end{aligned}
$$

where C is a constant independent of n. Since by (2.12)

$$
\frac{1}{n} \int_{\{|u_l| \le 2n\}} |Du_l|^p \le C l^q (\|f\|_{L^1(\omega)} + \frac{1}{n} \|g\|_{L^{p'}(\omega)})
$$

and $b_1 \in L^p(\Omega_l)$, we have

$$
|I_3| \le C \Big(\frac{1}{n} \int_{\{n \le |v| \le 2n\}} |Dv|^p \Big)^{1/p'},
$$

where C is a constant independent of n. Therefore the ellipticity condition (2.3) yields that I_3 goes to 0 as n goes to infinity. In a similar manner J_2 and

$$
\int_{\Omega_l} a_1^2(x_2, Dv) \cdot D_{x_1} u_l h_n'(u_l) h_n(v) T_k(u_l - v) \rho(\tfrac{x_1}{l_1})
$$

go to zero as n goes to infinity. Thus, $I_1 - J_1 + I_4$ goes to 0 as n goes to infinity. Recalling the structural condition (2.1) and that $\rho(\frac{x_1}{l_1})$ is independent of x_2, we get

$$a(x, Du_l) \cdot D\rho(\tfrac{x_1}{l_1}) = a_1^1(x, Du_l) \cdot D_{x_1}\rho(\tfrac{x_1}{l_1}) + a_1^2(x, Du_l) \cdot D_{x_1}\rho(\tfrac{x_1}{l_1}).$$

Hence,

$$\begin{aligned} I_1 - J_1 + I_4 = &\int_{\Omega_l} [a(x, Du_l) - a(x, Dv)] \cdot DT_k(u_l - v)h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) \\ &+ \frac{1}{l_1}\int_{\Omega_l} a_1^1(x_2, Du_l) \cdot (D_{x_1}\rho)(\tfrac{x_1}{l_1})T_k(u_l - v)h_n(u_l)h_n(v) \\ &+ \frac{1}{l_1}\int_{\Omega_l} (a_1^2(x_2, Du_l) - a_1^2(x_2, Dv)) \cdot (D_{x_1}\rho)(\tfrac{x_1}{l_1})T_k(u_l - v)h_n(u_l)h_n(v). \end{aligned}$$

Therefore, the strong monotonicity assumption (3.1), the condition (3.2) on a, the fact that $\text{supp}(\rho(\frac{\cdot}{l_1})) = [-l_1, l_1]^q$, $|h_n| \leq 1$ and $Du_l, Dv \in (L^{p-1}(\Omega_l))^N$ imply

$$\begin{aligned} &\int_{\Omega_l} |DT_k(u_l - v)|^p h_n(u_l)h_n(v)\rho(\tfrac{x_1}{l_1}) \\ &\leq \epsilon(n) + C\frac{k}{l_1}\int_{\Omega_{l_1}} |Du_l - Dv| \left(1 + |Du_l|^{p-2} + |Dv|^{p-2}\right), \end{aligned}$$

where $\epsilon(n) \to 0$ as $n \to +\infty$. Finally, since $\rho(\frac{x_1}{l_1}) = 1$ on $(-l_1/2, l_1/2)^q$, letting n go to $+\infty$ and Fatou's lemma imply (3.17).

Step 2. Since $T_k(u_l - v) \in W^{1,p}_{\Gamma_{l_1/2}}(\Omega_{l_1/2})$, we then apply Lemma 3.2 to (3.17) and obtain

$$\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_1/2})} \leq \frac{C}{l_1}\int_{\Omega_{l_1}} |Du_l - Dv| \left(1 + |Du_l|^{p-2} + |Dv|^{p-2}\right),$$

for any $r \in (1, N']$, because $p \leq N - q$ implies $p < N$. By Hölder's inequality in Lorentz spaces (3.7), with

$$\frac{1}{p-1}\frac{1}{r} + \frac{p-2}{p-1}\frac{1}{r}(1 - \tfrac{1}{r}) = 1$$

and using the fact that $\|w^s\|_{L^{p,\infty}(\Omega)} = \|w\|^s_{L^{ps,\infty}(\Omega)}$, we get

$$\begin{aligned} &\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_1/2})} \qquad (3.23) \\ &\leq \frac{C}{l_1}\||Du_l - Dv|^{p-1}\|^{\frac{1}{p-1}}_{L^{r,\infty}(\Omega_{l_1})}\|1 + |Du_l|^{p-1} + |Dv|^{p-1}\|^{\frac{p-2}{p-1}}_{L^{r,\infty}(\Omega_{l_1})}\|1\|_{L^{\frac{r}{r-1},1}(\Omega_{l_1})}. \end{aligned}$$

From (3.12), (3.13), the identity $\|1\|_{L^{s,1}(\Omega)}=|\Omega|^{\frac{1}{s}}$ and the fact that $l_1 \leq l$, it follows that

$$\|1+|Du_l|^{p-1}+|Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_1})}^{\frac{p-2}{p-1}} \leq C(\|1\|_{L^{r,\infty}(\Omega_l)}^{\frac{p-2}{p-1}} + \||Du_l|^{p-1}\|_{L^{r,\infty}(\Omega_l)}^{\frac{p-2}{p-1}} + \||Dv|^{p-1}\|_{L^{r,\infty}(\Omega_l)}^{\frac{p-2}{p-1}}) \leq C\left(l^{\frac{q}{r}\frac{p-2}{p-1}}+l^{q\frac{p-2}{p-1}}\right),$$

where C is a constant independent of l. Thus, using again $\|1\|_{L^{s,1}(\Omega)}=|\Omega|^{\frac{1}{s}}$, (3.23) becomes

$$\||Du_l-Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_1/2})} \leq \frac{Cl^{q\frac{p-2}{p-1}+q(1-\frac{1}{r})}}{l_1}(1+l^{q(\frac{1}{r}-1)\frac{p-2}{p-1}})\||Du_l-Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_1})}^{\frac{1}{p-1}}, \tag{3.24}$$

where C is independent of l and l_1. We are now in position to derive estimate (3.14).

In the case $p=2$, since $1+l^{q(\frac{1}{r}-1)\frac{p-2}{p-1}})=2$, taking $l_1=l/2^{k-1}$ and iterating (3.24), we obtain

$$\|Du_l-Dv\|_{L^{r,\infty}(\Omega_{l/2^k})} \leq Cl^{k\left(q(1-\frac{1}{r})-1\right)}\|Du_l-Dv\|_{L^{r,\infty}(\Omega_l)},$$

where C is a constant independent of l (C actually depends on k). From (3.12) and (3.13) it follows that

$$\|Du_l-Dv\|_{L^{r,\infty}(\Omega_{l/2^k})} \leq Cl^{k\left(q(1-\frac{1}{r})-1\right)+q},$$

where C is a constant independent of l. Therefore, if

$$q(1-\tfrac{1}{r})-1<0, \tag{3.25}$$

taking any $\alpha<0$, and for k large enough, $k(q(1-\frac{1}{r})-1)+q<\alpha$. The condition (3.25) is always satisfied if $q=1$, and for $q \geq 2$ it is equivalent to $r<\frac{1}{1-1/q}$. Then if we take $l_0>0$, for l large and $l_0<l/2^k$, and for r satisfying (3.25), we get

$$\|Du_l-Dv\|_{L^{r,\infty}(\Omega_{l_0})}=O(l^\alpha), \quad \forall\alpha<0.$$

We now turn to the case $p>2$. For $l_1=l/2^{k-1}$, iterating (3.24) gives for $l \geq 1$

$$\||Du_l-Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l/2^k})} \leq Cl^{\left(q\frac{p-2}{p-1}+q(1-\frac{1}{r})-1\right)\left(\frac{1-\frac{1}{(p-1)^k}}{1-\frac{1}{p-1}}\right)}\||Du_l-Dv|^{p-1}\|_{L^{r,\infty}(\Omega_l)}^{\frac{1}{(p-1)^k}}, \tag{3.26}$$

from which it follows together with (3.12) and (3.13) that

$$\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l/2^k})} \le Cl^{q-\frac{p-1}{p-2}+q\frac{p-1}{p-2}(1-\frac{1}{r})+\frac{1}{(p-1)^k}\left(q(\frac{1}{r}-1)\frac{p-1}{p-2}+\frac{p-1}{p-2}\right)}$$

where C is a constant independent of l. The convergence of

$$\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l/2^k})}$$

to zero needs to have

$$q - \frac{p-1}{p-2} + q\frac{p-1}{p-2}\left(1-\tfrac{1}{r}\right) < 0. \tag{3.27}$$

Indeed, if (3.27) holds then taking α such that $q-\frac{p-1}{p-2}+q\frac{p-1}{p-2}\left(1-\frac{1}{r}\right) < \alpha < 0$, and for k large enough, we have

$$q - \frac{p-1}{p-2} + q\frac{p-1}{p-2}\left(1-\frac{1}{r}\right) + \frac{1}{(p-1)^k}\left(q\left(\frac{1}{r}-1\right)\frac{p-1}{p-2}+\frac{p-1}{p-2}\right) - \alpha < 0.$$

For all $q \ge 1$, condition (3.27) is equivalent to $r < \frac{1}{2-1/(p-1)-1/q}$. Recalling that $1 < r < N'$, we get $q\frac{p-2}{p-1}(1-\frac{1}{r}) > 0$; thus, (3.27) holds if and only if the two following conditions,

$$\begin{cases} q - \frac{p-1}{p-2} < 0 \quad \text{i.e.,} \quad p < \frac{2q-1}{q-1} \\ 1 < r < \min\left(N', \frac{1}{2-1/(p-1)-1/q}\right), \end{cases} \tag{3.28}$$

are satisfied. Finally, let us take $l_0 > 0$. Then for l large, $l_0 < l/2^k$; thus, for $p > 2$ and r satisfying (3.27),

$$\||Du_l - Dv|^{p-1}\|_{L^{r,\infty}(\Omega_{l_0})} = O(l^\alpha), \quad \forall\, q - \frac{p-1}{p-2} + q\frac{p-1}{p-2}\left(1-\frac{1}{r}\right) < \alpha < 0.$$

□

Remark 3.11. By Theorem 3.5 and inequality (3.17), we also have the following convergence result: for any $2 \le p < \frac{2q-1}{q-1}$ (with the convention $\frac{2q-1}{q-1} = +\infty$ for $q = 1$) and for any $l_0 > 0$,

$$T_k(u_l - v) \to 0 \text{ in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}), \quad \forall k > 0$$

with the same rates of convergence as those in Theorem 3.5.

Proof of Theorem 3.6. The proof is divided into two steps. In Step 1, we derive an estimate of $T_k(u_l)$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ and we state the weak convergence of the truncates. In Step 2, thanks to the pointwise convergence of the gradients, we show the strong convergence of the truncates.

Step 1. Using $T_k(u_l)\rho(\frac{x_1}{2l_0})$ as test function in (2.11) with $h = h_n$, letting n go to $+\infty$ gives

$$\int_{\Omega_l} a(x, Du_l) \cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0}) + \frac{1}{2l_0}\int_{\Omega_l} a(x, Du_l) \cdot (D\rho)(\tfrac{x_1}{2l_0})T_k(u_l) = \int_{\Omega_l} fT_k(u_l)\rho(\tfrac{x_1}{2l_0}) + \int_{\Omega_l} g \cdot D_{x_2}T_k(u_l)\rho(\tfrac{x_1}{2l_0}). \quad (3.29)$$

Since $\text{supp}(\rho(\frac{x_1}{2l_0})) = [-2l_0, 2l_0]^q$, the integrals must be taken over Ω_{2l_0}. From (2.3), (2.5), (3.15), (3.16) and Young's inequality, we obtain for $\epsilon > 0$

$$\int_{\Omega_{2l_0}} |DT_k(u_l)|^p\rho(\tfrac{x_1}{2l_0}) \leq C(l_0, \Lambda)\Big(k\int_{\Omega_{2l_0}} (b_1 + b_2 + |Du_l|^{p-1}) + k\int_{\Omega_{2l_0}} |f| + C_\epsilon\int_{\Omega_{2l_0}} |g|^{p'} + \epsilon\int_{\Omega_{2l_0}} |DT_k(u_l)|^p\rho(\tfrac{x_1}{2l_0})\Big).$$

Finally, taking ϵ small enough, and using the facts that $\rho(\frac{x_1}{2l_0}) = 1$ on $(-l_0, l_0)^q$ and Du_l is bounded in $(L^{p-1}(\Omega_{2l_0}))^N$ (see Remark (3.9)), we get

$$\int_{\Omega_{l_0}} |DT_k(u_l)|^p \leq C_1k + C_2, \quad (3.30)$$

where C_1 and C_2 are constants independent of l. Since $T_k(u_l)$ is uniformly bounded in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ with respect to l and since u_l converges strongly to v in $W^{1,p-1}_{\Gamma_{l_0}}(\Omega_{l_0})$, we deduce that $T_k(u_l)$ converges to $T_k(v)$ weakly in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$. Moreover, the convergence of Du_l to Dv in $L^{p-1}(\Omega_{l_0})$ implies that up to a subsequence, Du_l converges almost everywhere to Dv. Since $DT_k(u_l) = \mathbb{1}_{\{|u_l|<k\}}Du_l$, by a well-known result in [3] (Lemma 3.2), which claims that for almost every $k > 0$

$$\mathbb{1}_{\{|u_l|<k\}} \to \mathbb{1}_{\{|v|<k\}} \text{ a.e. in } \Omega_{l_0},$$

we deduce that for almost every $k > 0$

$$DT_k(u_l) \to DT_k(v) \quad \text{a.e. in } \Omega_{l_0},$$

and then for almost every $k > 0$

$$a(x, DT_k(u_l)) \to a(x, DT_k(v)) \quad \text{a.e. in } \Omega_{l_0}.$$

Now, by the condition (2.5) on a and the estimate (3.30), we have

$$\int_{\Omega_{l_0}} |a(x, DT_k(u_l))|^{p'} \leq C, \quad (3.31)$$

where C is a constant independent of l. So there exists $\xi_k \in (L^{p'}(\Omega_{l_0}))^N$ such that up to a subsequence

$$a(x, DT_k(u_l)) \rightharpoonup \xi_k \text{ weakly in } (L^{p'}(\Omega_{l_0}))^N. \tag{3.32}$$

Moreover, Vitali's theorem, the estimate (3.31) and the pointwise convergence of $DT_k(u_l)$ imply

$$a(x, DT_k(u_l)) \to a(x, DT_k(v)) \text{ strongly in } \left(L^q(\Omega_{l_0})\right)^N, \quad \forall q < p'.$$

So we can identify $\xi_k = a(x, DT_k(v))$, and by uniqueness of v the whole sequence in (3.32) converges.

Step 2. We assume that $2 \leq p < \frac{2q-1}{q-1}$.

We now pass to the limit as l goes to $+\infty$ in (3.29). By Lemma 3.2 together with estimate (3.30), we have

$$|Du_l|^{p-1} \text{ is bounded in } L^q(\Omega_{2l_0}), \quad \forall q < \tfrac{N}{N-1}.$$

From (2.5) it follows that

$$a(x, Du_l) \text{ is bounded in } \left(L^q(\Omega_{2l_0})\right)^N, \quad \forall q < \tfrac{N}{N-1}. \tag{3.33}$$

Thus, $a(x, Du_l) \cdot D\rho(\frac{x_1}{2l_0})T_k(u_l)$ is equi-integrable in $L^1(\Omega_{2l_0})$. Indeed, taking a $1 < q < \frac{N}{N-1}$, we have that if $\frac{1}{q} + \frac{1}{q'} = 1$, for any $A \subset \Omega_{2l_0}$

$$\begin{aligned} \int_A \left| a(x, Du_l) \cdot D\rho(\tfrac{x_1}{2l_0})T_k(u_l)\right| &\leq C\Big(\int_A |a(x, Du_l)|^q\Big)^{\frac{1}{q}} |A|^{\frac{1}{q'}} \\ &\leq C\, \|a(x, Du_l)\|_{\left(L^q(\Omega_{2l_0})\right)^N} |A|^{\frac{1}{q'}}, \end{aligned} \tag{3.34}$$

where C is a constant depending on Λ, l_0, k and N, and is independent of l. Moreover, from Remark 3.9, up to a subsequence, Du_l converges to Dv almost everywhere in Ω_{2l_0}, which yields

$$a(x, Du_l) \cdot D\rho(\tfrac{x_1}{2l_0})T_k(u_l) \to a(x, Dv) \cdot D\rho(\tfrac{x_1}{2l_0})T_k(v) \quad \text{a.e. in } \Omega_{2l_0}. \tag{3.35}$$

Therefore, by Vitali's theorem $a(x,Du_l) \cdot D\rho(\frac{x_1}{2l_0})T_k(u_l)$ converges to $a(x, Dv) \cdot D\rho(\frac{x_1}{2l_0})T_k(v)$ strongly in $L^1(\Omega_{2l_0})$. Thus, this convergence, the conditions (2.1) and (2.2), and the independence of v with respect to x_1 imply

$$\begin{aligned} &\lim_{l\to+\infty} \int_{\Omega_l} a(x, Du_l) \cdot D\rho(\tfrac{x_1}{2l_0})T_k(u_l) = \int_{\Omega_{2l_0}} a(x, Dv) \cdot D\rho(\tfrac{x_1}{2l_0})T_k(v) \\ &= \int_{\Omega_{2l_0}} a_1^2(x_2, Dv) \cdot D_{x_1}\rho(\tfrac{x_1}{2l_0})T_k(v) = \int_{\Omega_{2l_0}} D_{x_1} \cdot (a_1^2(x_2, Dv)\rho(\tfrac{x_1}{2l_0})T_k(v)) = 0 \end{aligned} \tag{3.36}$$

because we integrate in the x_1 direction first and use the fact that $\rho(\frac{x_1}{2l_0}) = 0$ outside $(-2l_0, 2l_0)^q$. As far as the right-hand side of (3.29) is concerned, since $T_k(u_l)$ converges to $T_k(v)$ weakly in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$, the Lebesgue convergence theorem gives

$$\lim_{l\to+\infty} \Big[\int_{\Omega_l} fT_k(u_l)\rho(\tfrac{x_1}{2l_0}) + \int_{\Omega_l} g\cdot D_{x_2}T_k(u_l)\rho(\tfrac{x_1}{2l_0})\Big] \tag{3.37}$$
$$= \int_{\Omega_{2l_0}} fT_k(v)\rho(\tfrac{x_1}{2l_0}) + \int_{\Omega_{2l_0}} g\cdot D_{x_2}T_k(v)\rho(\tfrac{x_1}{2l_0}).$$

Using $T_k(v)$ as test function in $(2.11')$ with $h = h_n$, letting n go to $+\infty$, then multiplying by $\rho(\frac{x_1}{2l_0})$ and integrating over $(-l, l)^q$ gives

$$\int_{\Omega_l} a_2^2(x_2, D_{x_2}v)\cdot D_{x_2}T_k(v)\rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_l} fT_k(v)\rho(\tfrac{x_1}{2l_0}) + \int_{\Omega_l} g\cdot D_{x_2}T_k(v)\rho(\tfrac{x_1}{2l_0}). \tag{3.38}$$

Thus, by (3.29), (3.36), (3.37) and (3.38) it follows that

$$\lim_{l\to+\infty} \int_{\Omega_{2l_0}} a(x, DT_k(u_l))\cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0}) \tag{3.39}$$
$$= \int_{\Omega_{2l_0}} a_2^2(x_2, D_{x_2}T_k(v))\cdot D_{x_2}T_k(v)\rho(\tfrac{x_1}{2l_0}),$$

where in fact,

$$a_2^2(x_2, D_{x_2}T_k(v))\cdot D_{x_2}T_k(v) = a(x, DT_k(v))\cdot DT_k(v) \quad \text{ in } \Omega_{2l_0}$$

because of the independence of v with respect to x_1 and the structural conditions (2.1) and (2.2). Using the strict monotonicity of a, a standard argument leads to

$$DT_k(u_l) \to DT_k(v) \quad \text{ strongly in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}). \qquad \square$$

4. The strict monotone case

In this section we suppose μ positive; that is,

$$\int_\omega f\phi + \int_\omega g\cdot D_{x_2}\phi \ge 0; \quad \forall\phi\in C_c^\infty(\omega), \quad \phi\ge 0. \tag{4.1}$$

Moreover, we assume that $a^1 = 0$; that is,

$$a(x,\xi) = a^2(x_2,\xi). \tag{4.2}$$

In particular, a is independent of x_1. First of all we will establish the following result in the variational case:

Theorem 4.1. *Under assumptions* (2.1)–(2.6), (4.1), (4.2) *and* $f = 0$, *for all* $1 < p < +\infty$ *and for all* $l_0 > 0$, *we have*

$$u_l \to v \text{ in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}) \text{ as } l \to +\infty.$$

Remark 4.2. These results have been proved independently by Chipot and Xie in [10] and [21] in the case where a is the p-Laplace operator and using the same techniques based on maximum principle arguments.

Thus, in the case $f \neq 0$, we can restrict ourselves to the case $p \leq N - q$, because if $p > N - q$, $L^1(\omega) \subset W^{-1,p'}(\omega)$ and then the previous variational case applies.

Theorem 4.3. *Under assumptions* (2.1)–(2.6), (4.1) and (4.2), *for all* $1 < p \leq N - q$ *and for all* $l_0 > 0$, *we have*

$$T_k(u_l) \to T_k(v) \text{ in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}) \text{ as } l \to +\infty.$$

We start by proving the following results, which use strongly the fact that $\mu \geq 0$ and assumption (4.2).

Proposition 4.4. *Under the assumptions of Theorem* 4.1 *or Theorem* 4.3, *we have for any* $l \leq l'$

$$0 \leq u_l \leq u_{l'} \leq v \text{ a.e. on } \Omega_l.$$

Therefore, up to a subsequence still denoted by u_l, we get

$$u_l \to u_\infty \quad \text{a.e. on } \mathbb{R}^q \times \omega,$$

where u_∞ is finite almost everywhere on $\mathbb{R}^q \times \omega$. The function u_∞ satisfies

Proposition 4.5. *The function* u_∞ *is independent of* x_1.

Proof of Proposition 4.4. We prove this proposition in the case where $f \neq 0$. Indeed, if $f = 0$ then $-\operatorname{div}(g) \in W^{-1,p'}(\omega)$ and the renormalized solution is the variational solution.

- Since $\mu \geq 0$ in $\mathcal{D}'(\Omega_l)$, then $u_l \geq 0$ almost everywhere in Ω_l.
- Let us take $l < l'$. Since $T_{2n}(u_l)$ and $T_{2n}(u_{l'})$ are nonnegative functions belonging to $W^{1,p}_0(\Omega_l)$ and $W^{1,p}_0(\Omega_{l'})$ respectively, we have $(T_{2n}(u_l) - T_{2n}(u_{l'}))^+ \in W^{1,p}_0(\Omega_l)$ (where $x^+ = \max(0, x)$). So we can use $T_k^+(T_{2n}(u_l) - T_{2n}(u_{l'}))h_n(u_l)$ (respectively $T_k^+(T_{2n}(u_l) - T_{2n}(u_{l'}))h_n(u_{l'})$) as test function in (2.11) written in u_l (respectively written in $u_{l'}$), with $h = h_n$. Subtracting the results and using the fact that since $\operatorname{supp}(h_n) = [-2n, 2n]$, we have

$$h_n(u_l)h_n(u_{l'})T_k^+(T_{2n}(u_l) - T_{2n}(u_{l'})) = h_n(u_l)h_n(u_{l'})T_k^+(u_l - u_{l'}),$$

we get

$$\int_{\Omega_l}[a(x_2,Du_l)-a(x_2,Du_{l'})]\cdot DT_k^+(u_l-u_{l'})\,h_n(u_l)h_n(u_{l'})$$
$$+\int_{\Omega_l}[a(x_2,Du_l)-a(x_2,Du_{l'})]\cdot Du_l\,h_n'(u_l)T_k^+(u_l-u_{l'})h_n(u_{l'})$$
$$+\int_{\Omega_l}[a(x_2,Du_l)-a(x_2,Du_{l'})]\cdot Du_{l'}\,h_n'(u_{l'})T_k^+(u_l-u_{l'})h_n(u_l)=0. \quad (4.3)$$

We claim that the second and the third term in (4.3) go to 0 as n goes to ∞. Indeed, for the second term

$$\Big|\int_{\Omega_l}a(x_2,Du_l)\cdot Du_l\,h_n'(u_l)T_k^+(u_l-u_{l'})h_n(u_{l'})\Big|\le\frac{k}{n}\int_{\{n\le|u_l|\le 2n\}}a(x_2,Du_l)\cdot Du_l,$$

which tends to 0 by (2.10). From (2.5) and Hölder's inequality, it follows that

$$\Big|\int_{\Omega_l}a(x_2,Du_{l'})\cdot Du_l\,h_n'(u_l)T_k^+(u_l-u_{l'})h_n(u_{l'})\Big|$$
$$\le C\int_{\Omega_l}(b_2(x_2)+|Du_{l'}|^{p-1})|Du_l||h_n'(u_l)|h_n(u_{l'})\le C\Big(\frac{1}{n}\int_{\{n\le|u_l|\le 2n\}}|Du_l|^p\Big)^{\frac{1}{p}}$$
$$+C\Big(\frac{1}{n}\int_{\{n\le|u_l|\le 2n\}}|Du_l|^p\Big)^{\frac{1}{p}}\Big(\frac{1}{n}\int_{\{|u_{l'}|\le 2n\}}|Du_{l'}|^p\Big)^{\frac{1}{p'}},$$

which tends to 0 by (2.3), (2.10) and (2.12). Similar arguments imply that the third term in (4.3) goes to 0 as n goes to ∞. Therefore, we have

$$\int_{\Omega_l}[a(x_2,Du_l)-a(x_2,Du_{l'}]\cdot DT_k^+(u_l-u_{l'})\,h_n(u_l)h_n(u_{l'})=\epsilon(n),$$

where $\epsilon(n)\to 0$ when $n\to+\infty$. Since by (2.4) the integrand above is positive and since $h_n(u_l)h_n(u_{l'})$ converges to 1 almost everywhere in Ω_l, Fatou's lemma leads to

$$\int_{\Omega_l}[a(x_2,Du_l)-a(x_2,Du_{l'}]\cdot[Du_l-Du_{l'}]\mathbb{1}_{\{0\le u_l-u_{l'}\le k\}}=0.$$

So, the strict monotonicity of a (2.4) implies $Du_l=Du_{l'}$ almost everywhere on $\{0\le\ u_l-u_{l'}\le k\}\cap\Omega_l$ for any $k>0$. Moreover, $(u_l-u_{l'})^+\in W^{1,p-1}_{\Gamma_l}(\Omega_l)$; thus, $(u_l-u_{l'})^+=0$ almost everywhere in Ω_l; i.e., $u_l\le u_{l'}$ on Ω_l.

• In a similar way as above, if we take v in place of $u_{l'}$, we show that

$$u_l\le v\text{ a.e. on }\Omega_l. \qquad\square$$

Proof of Proposition 4.5. We begin by introducing some notations. We fix $l > 0$. For any $-l \le t \le l$, we set $\Omega_{2l}^t = (-2l+t, 2l+t) \times (-2l, 2l)^{q-1} \times \omega$, and we introduce the function v_{2l}^t defined on Ω_{2l}^t by

$$v_{2l}^t(x) = u_{2l}(x - te_1) \text{ a.e. on } \Omega_{2l}^t,$$

where e_1 denotes the unit vector $e_1 = (1, 0, \dots, 0)$. Since $\Omega_l \subset \Omega_{2l}^t$ and after a few computations which use strongly assumption (4.2), by (2.10) and (2.11) we obtain

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n \le |v_{2l}^t| \le 2n\} \cap \Omega_l} a(x_2, Dv_{2l}^t) \cdot Dv_{2l}^t = 0 \tag{4.4}$$

$$\begin{aligned} -\operatorname{div}[h(v_{2l}^t) a(x_2, Dv_{2l}^t)] + h'(v_{2l}^t) a(x_2, Dv_{2l}^t) \cdot Dv_{2l}^t &= fh(v_{2l}^t) \\ - \operatorname{div}[gh(v_{2l}^t)] + g \cdot Dv_{2l}^t h'(v_{2l}^t) \quad &\text{in } \mathcal{D}'(\Omega_l) \end{aligned} \tag{4.5}$$

for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support. Roughly speaking this means that v_{2l}^t is a renormalized solution of

$$\begin{cases} -\operatorname{div}[a(x, Dv_{2l}^t)] = \mu & \text{on } \Omega_l \\ v_{2l}^t = 0 & \text{on } \Gamma_l. \end{cases}$$

By techniques already used in the proof of Proposition 4.4, using (4.4) and (4.5), we show that

$$\int_{\Omega_l} [a(x_2, Du_l) - a(x_2, Dv_{2l}^t)] \cdot [Du_l - Dv_{2l}^t] \mathbb{1}_{\{0 \le u_l - v_{2l}^t \le k\}} = 0.$$

This yields that $u_l \le v_{2l}^t$ almost everywhere on Ω_l. Letting l go to ∞ in this inequality gives

$$u_\infty(x) \le u_\infty(x - te_1) \quad \text{a.e. } \mathbb{R}^q \times \omega; \quad \forall t \in \mathbb{R}.$$

Thus,

$$u_\infty(x) = u_\infty(x - te_1) \quad \text{a.e. } \mathbb{R}^q \times \omega; \quad \forall t \in \mathbb{R}.$$

This identity implies that u_∞ is independent of its first variable. Likewise, we get for all $i = 1, \dots, q$

$$u_\infty(x) = u_\infty(x - te_i) \quad \text{a.e. } \mathbb{R}^q \times \omega; \quad \forall t \in \mathbb{R}.$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. These identities imply that u_∞ is independent of x_1. □

Proof of Theorem 4.1. The proof is divided into three steps. In Step 1, we derive an estimate of u_l in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ which implies that u_l converges weakly to u_∞. Step 2 is devoted to proving the strong convergence. In Step 3, we show that u_∞ and v satisfy the same equation in $\mathbb{R}^q \times \omega$, so we identify u_∞ and v.

Step 1. Using $\rho^p(\frac{x_1}{2l_0})u_l$ as test function in (1.1), by (2.3) and (2.5) together with Proposition 4.4, we get

$$\alpha \int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \le \|g\|_{\left(L^{p'}(\Omega_{2l_0})\right)^{N-q}} \Big(\int_{\Omega_{2l_0}} |Du_l|^p \rho^{p^2}(\tfrac{x_1}{2l_0}) \Big)^{\frac{1}{p}} + \frac{\Lambda}{2l_0} \int_{\Omega_{2l_0}} (b_2 + |Du_l|^{p-1}) v \rho^{p-1}(\tfrac{x_1}{2l_0}).$$

Since $p > 1$, $p^2 > p$ and $\rho^{p^2} \le \rho^p$ because $0 \le \rho \le 1$, and by Hölder's inequality, we also estimate

$$\int_{\Omega_{2l_0}} |Du_l|^{p-1} v \rho^{p-1}(\tfrac{x_1}{2l_0}) \le \Big(\int_{\Omega_{2l_0}} v^p \Big)^{\frac{1}{p}} \Big(\int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \Big)^{\frac{1}{p'}};$$

from here we get

$$\int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \le C\Big(\Big(\int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \Big)^{\frac{1}{p}} + \Big(\int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \Big)^{\frac{1}{p'}} + 1 \Big),$$

where C is a constant independent of l. From here, since $\frac{1}{p} < 1$ and $\frac{1}{p'} < 1$ we obtain

$$\int_{\Omega_{2l_0}} |Du_l|^p \rho^p(\tfrac{x_1}{2l_0}) \le C,$$

where C is a constant independent of l. Since $\rho(\frac{x_1}{2l_0}) = 1$ on $(-l_0, l_0)^q$, it follows that

$$\int_{\Omega_{l_0}} |Du_l|^p \le C,$$

where C is a constant independent of l. Thus, up to a subsequence still denoted u_l, we get

$$u_l \to u_\infty \quad \text{weakly in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0}); \quad \text{strongly in } L^p(\Omega_{l_0}); \quad \text{a.e. in } \Omega_{l_0}. \tag{4.6}$$

Step 2. With (4.6) and the strict monotonicity of the operator a, we can adapt techniques developed in [4] and [6] to show that up to a subsequence, $Du_l \to Du_\infty$ almost everywhere in Ω_{l_0}. Let us use $u_l \rho(\frac{x_1}{2l_0})$ as test function in (1.1); we get

$$\int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot Du_l \, \rho(\tfrac{x_1}{2l_0}) + \frac{1}{2l_0} \int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot D\rho(\tfrac{x_1}{2l_0}) \, u_l \tag{4.7}$$

$$= \int_{\Omega_{2l_0}} g \cdot D_{x_2} u_l \rho(\tfrac{x_1}{2l_0}).$$

By (2.5), $a(x_2, Du_l)$ is bounded in $(L^{p'}(\Omega_{l_0}))^N$; then the almost-everywhere convergence of the gradients implies that

$$a(x_2, Du_l) \to a(x, Du_\infty) \quad \text{weakly in } (L^{p'}(\Omega_{l_0}))^N.$$

Thus,

$$\lim_{l\to+\infty} \int_{\Omega_{2l_0}} a(x_2, Du_l)\cdot(D\rho)(\tfrac{x_1}{2l_0})\, u_l = \int_{\Omega_{2l_0}} a_1^2(x_2, Du_\infty)\cdot(D_{x_1}\rho)(\tfrac{x_1}{2l_0})\, u_\infty = 0. \tag{4.8}$$

For the right-hand side of (4.7), we get

$$\lim_{l\to+\infty} \int_{\Omega_{2l_0}} g \cdot D_{x_2} u_l \rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_{2l_0}} g \cdot D_{x_2} u_\infty \rho(\tfrac{x_1}{2l_0}). \tag{4.9}$$

On the one hand, gathering (4.7), (4.8) and (4.9) gives that

$$\lim_{l\to+\infty} \int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot Du_l\, \rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_{2l_0}} g \cdot D_{x_2} u_\infty \rho(\tfrac{x_1}{2l_0}).$$

On the other hand, using $u_\infty \rho(\frac{x_1}{2l_0})$ as test function in (1.1), we get

$$\int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot Du_\infty\, \rho(\tfrac{x_1}{2l_0}) + \frac{1}{2l_0} \int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot (D\rho)(\tfrac{x_1}{2l_0})\, u_\infty \tag{4.10}$$
$$= \int_{\Omega_{2l_0}} g \cdot D_{x_2} u_\infty \rho(\tfrac{x_1}{2l_0}).$$

Finally, passing to the limit in (4.10) yields that

$$\lim_{l\to+\infty} \int_{\Omega_{2l_0}} a(x_2, Du_l) \cdot Du_l\, \rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_{2l_0}} a(x_2, Du_\infty) \cdot Du_\infty\, \rho(\tfrac{x_1}{2l_0}). \tag{4.11}$$

Therefore, by the convergence (4.11) and since $a(x_2, Du_l) \cdot Du_l \rho(\frac{x_1}{2l_0})$ is a sequence of nonnegative functions that converges almost everywhere to $a(x_2, Du_\infty) \cdot Du_\infty \rho(\frac{x_1}{2l_0})$, a standard application of Fatou's lemma gives

$$a(x_2, Du_l) \cdot Du_l \rho(\tfrac{x_1}{2l_0}) \to a(x_2, Du_\infty) \cdot Du_\infty \rho(\tfrac{x_1}{2l_0}) \quad \text{strongly in } L^1(\Omega_{2l_0}),$$

and the fact that $\rho(\frac{x_1}{2l_0}) = 1$ on Ω_{l_0} implies

$$a(x_2, Du_l) \cdot Du_l \to a(x_2, Du_\infty) \cdot Du_\infty \quad \text{strongly in } L^1(\Omega_{l_0}).$$

This yields that $a(x_2, Du_l) \cdot Du_l$ is equi-integrable in $L^1(\Omega_{l_0})$. Then the ellipticity condition

$$\alpha|Du_l|^p \le a(x_2, Du_l) \cdot Du_l, \quad \text{for a.e. } x_2 \in \omega, \tag{4.12}$$

implies that Du_l is p-equi-integrable. Therefore, by the almost-everywhere convergence of Du_l, the Vitali theorem yields that

$$Du_l \to Du_\infty \quad \text{strongly in } (L^p(\Omega_{l_0}))^N. \tag{4.13}$$

Step 3. Using $\psi \in C_c^\infty(\mathbb{R}^q \times \omega)$ as test function in (1.1) and letting l go to $+\infty$ yields

$$\int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty) \cdot D\psi = \int_{\mathbb{R}^q\times\omega} g \cdot D_{x_2}\psi. \tag{4.14}$$

In fact, for $l > 0$, we can take $\psi \in W_0^{1,p}(\Omega_l)$ in (4.14), extending ψ by 0 to $(\mathbb{R}^q \times \omega) \setminus \Omega_l$. Thus, we take $\phi \in C_c^\infty(\mathbb{R}^q)$, $\phi \ge 0$, and we use $\psi = \phi(u_\infty - v)$ as test function in (4.14) and $u_\infty - v$ as test function in (1.2), we multiply by ϕ and we integrate over $\mathbb{R}^q$; then we subtract these two identities so as to obtain

$$\begin{aligned}\int_{\mathbb{R}^q\times\omega} &(a_1^2(x_2, Du_\infty) - a_1^2(x_2, Dv)) \cdot D_{x_1}\phi(u_\infty - v)\\ &+ \int_{\mathbb{R}^q\times\omega} (a_2^2(x_2, Du_\infty) - a_2^2(x_2, Dv)) \cdot D_{x_2}(u_\infty - v)\phi = 0.\end{aligned} \tag{4.15}$$

Since u_∞ and v are independent of x_1, the divergence theorem implies that the first term of (4.15) is equal to 0. Therefore, we obtain

$$\int_{\mathbb{R}^q\times\omega} (a_2^2(x_2, Du_\infty) - a_2^2(x_2, Dv)) \cdot D_{x_2}(u_\infty - v)\phi = 0.$$

By the strict monotonicity (2.8) of a_2^2, we conclude that $u_\infty = v$ almost everywhere in ω. □

Remark 4.6. It is important to know a fortiori that u_∞ is independent of x_1, because it allows us to treat the first term of (4.15).

Proof of Theorem 4.3. For the simplicity of the proof we will suppose that $g = 0$. The proof is divided into three steps. In Step 1, we derive an estimate of $T_k(u_l)$ in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$, which implies that $T_k(u_l)$ converges weakly to $T_k(u_\infty)$. Step 2 is devoted to proving the strong convergence of the truncates. In Step 3, we show that u_∞ and v satisfy the same equation in $\mathbb{R}^q \times \omega$, so we identify u_∞ and v.

Step 1. From the proof of Theorem 3.6 (see (3.30)), we know that for all $l_0 > 0$ and $k > 0$, $T_k(u_l)$ is bounded in $W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$. Thus, up to a subsequence still denoted u_l, we get

$$T_k(u_l) \to w_k \text{ weakly in } W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0});\ \text{strongly in } L^p(\Omega_{l_0});\ \text{a.e. in } \Omega_{l_0}, \quad (4.16)$$

where $w_k \in W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$. With the fact that $u_l \to u_\infty$ almost everywhere, by continuity of T_k, we obtain $w_k = T_k(u_\infty)$, and then $T_k(u_\infty) \in W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$ with the estimate

$$\int_{\Omega_{l_0}} |DT_k(u_\infty)|^p \leq C_1 k + C_2, \quad (4.17)$$

for any $l_0 > 0$, for any $k > 0$ and where C_1 and C_2 are positive constants independent of l.

Step 2. First of all, with (4.17) and the strict monotonicity of a, we can adapt techniques developed in [4] and [6] to the framework of renormalized solutions to show that Du_l converges almost everywhere to Du_∞. Thus, from the boundedness of $a(x_2, Du_l)$ in $(L^q(\Omega_{l_0}))^N$ (see (3.33)), for every $q < \frac{N}{N-1}$ and from Vitali's theorem it follows that

$$a(x_2, Du_l) \to a(x_2, Du_\infty) \quad \text{in } (L^q(\Omega_{l_0}))^N; \quad \forall q < \tfrac{N}{N-1}. \quad (4.18)$$

Let us use $T_k(u_\infty)\rho(\frac{x_1}{2l_0})$ as test function in (2.11) with $h = h_n$; we have

$$\begin{aligned} &\int_{\Omega_l} a(x_2, Du_l) \cdot DT_k(u_\infty) h_n(u_l) \rho(\tfrac{x_1}{2l_0}) \\ &+ \int_{\Omega_l} a(x_2, Du_l) \cdot Du_l\, h_n'(u_l) T_k(u_\infty) \rho(\tfrac{x_1}{2l_0}) \\ &+ \int_{\Omega_l} a(x_2, Du_l) \cdot D_{x_1}\rho(\tfrac{x_1}{2l_0}) h_n(u_l) T_k(u_\infty) = \int_{\Omega_l} f h_n(u_l) T_k(u_\infty) \rho(\tfrac{x_1}{2l_0}). \end{aligned} \quad (4.19)$$

We now pass to the limit in (4.19) as $l \to +\infty$ and then as $n \to +\infty$.
For the first term of (4.19), on the one hand, from (4.18) and Lebesgue theorem

$$h_n(u_l) a(x_2, Du_l) \to h_n(u_\infty) a(x_2, Du_\infty) \quad \text{in } (L^q(\Omega_{l_0}))^N; \quad \forall q < \tfrac{N}{N-1}.$$

On the other hand, from (2.5) and (3.30) and using the fact that $\text{supp}(h_n) = [-2n, 2n]$, we have

$$h_n(u_l) a(x_2, Du_l) = h_n(u_l) a(x_2, DT_{2n}(u_l))$$

bounded in $(L^{p'}(\Omega_{2l_0}))^N$. Therefore, up to a subsequence,

$$h_n(u_l)a(x_2, Du_l) \rightharpoonup h_n(u_\infty)a(x_2, Du_\infty) \quad \text{weakly in } (L^{p'}(\Omega_{l_0}))^N.$$

This yields

$$\begin{aligned}&\lim_{l\to+\infty}\int_{\Omega_l} a(x_2, Du_l)\cdot DT_k(u_\infty)h_n(u_l)\rho(\tfrac{x_1}{2l_0})\\ &= \int_{\Omega_{2l_0}} a(x_2, DT_{2n}(u_\infty))\cdot DT_k(u_\infty)h_n(u_\infty)\rho(\tfrac{x_1}{2l_0}).\end{aligned}$$

Since for $n \geq \frac{k}{2}$, $a(x_2, DT_{2n}(u_\infty))\cdot DT_k(u_\infty) = a(x_2, DT_k(u_\infty))\cdot DT_k(u_\infty)$, the Lebesgue convergence theorem implies that

$$\begin{aligned}&\lim_{n\to+\infty}\lim_{l\to+\infty}\int_{\Omega_l} a(x_2, Du_l)\cdot DT_k(u_\infty)h_n(u_l)\rho(\tfrac{x_1}{2l_0}) \\ &= \int_{\Omega_{2l_0}} a(x_2, DT_k(u_\infty))\cdot DT_k(u_\infty)\rho(\tfrac{x_1}{2l_0}).\end{aligned} \tag{4.20}$$

By the dominated convergence theorem, since u_∞ is finite almost everywhere, we get

$$\lim_{n\to+\infty}\lim_{l\to+\infty}\int_{\Omega_l} fh_n(u_l)T_k(u_\infty)\rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_{2l_0}} fT_k(u_\infty)\rho(\tfrac{x_1}{2l_0}). \tag{4.21}$$

Next, by (4.18), the independence of u_∞ with respect to x_1 and the divergence theorem,

$$\begin{aligned}&\lim_{l\to+\infty}\int_{\Omega_l} a(x_2, Du_l)\cdot D_{x_1}\rho(\tfrac{x_1}{2l_0})h_n(u_l)T_k(u_\infty)\\ &\qquad= \int_{\Omega_{2l_0}} a(x_2, Du_\infty)\cdot D_{x_1}\rho(\tfrac{x_1}{2l_0})h_n(u_\infty)T_k(u_\infty)\\ &\qquad= \int_{\Omega_{2l_0}} D_{x_1}\cdot\Big[a(x_2, Du_\infty)\rho(\tfrac{x_1}{2l_0})h_n(u_\infty)T_k(u_\infty)\Big] = 0.\end{aligned} \tag{4.22}$$

Finally, as far as the second term in (4.19) is concerned, we claim that

$$\lim_{n\to+\infty}\overline{\lim_{l\to+\infty}}\int_{\Omega_l} a(x_2, Du_l)\cdot Du_l\, h_n'(u_l)\rho(\tfrac{x_1}{2l_0}) = 0. \tag{4.23}$$

Indeed, we take $\frac{1}{n}(T_{2n}(u_l) - T_n(u_l))\rho(\frac{x_1}{2l_0})$ as test function in (2.11) with $h = h_m$, and we let m go to $+\infty$, so that we have

$$\frac{1}{n}\int_{\Omega_l} a(x_2, Du_l)\cdot D(T_{2n}(u_l) - T_n(u_l))\rho(\tfrac{x_1}{2l_0}) \tag{4.24}$$

$$+\int_{\Omega_l} a(x_2,Du_l)\cdot D\rho(\tfrac{x_1}{2l_0})\frac{1}{n}(T_{2n}(u_l)-T_n(u_l))$$
$$=\int_{\Omega_l} f\frac{1}{n}(T_{2n}(u_l)-T_n(u_l))\rho(\tfrac{x_1}{2l_0}).$$

With arguments already used, we get

$$\lim_{l\to+\infty}\int_{\Omega_l} a(x_2,Du_l)\cdot D\rho(\tfrac{x_1}{2l_0})\frac{1}{n}(T_{2n}(u_l)-T_n(u_l))=0. \tag{4.25}$$

Since $u_l\to u_\infty$ almost everywhere in $\mathbb{R}^q\times\omega$, $\frac{1}{n}(T_{2n}(r)-T_n(r))$ converges to 0 and u_∞ is finite almost everywhere, the Lebesgue convergence theorem implies that

$$\lim_{n\to+\infty}\lim_{l\to+\infty}\int_{\Omega_l} f\frac{1}{n}(T_{2n}(u_l)-T_n(u_l))\rho(\tfrac{x_1}{2l_0}) \tag{4.26}$$
$$=\lim_{n\to+\infty}\int_{\Omega_{2l_0}} f\frac{1}{n}(T_{2n}(u_\infty)-T_n(u_\infty))\rho(\tfrac{x_1}{2l_0})=0.$$

Gathering (4.24), (4.25) and (4.26), and using the fact that

$$\frac{1}{n}(T'_{2n}(u_l)-T'_n(u_l))=\mathbb{1}_{\{n\le|u_l|\le 2n\}},$$

we obtain that

$$\lim_{n\to+\infty}\overline{\lim_{l\to+\infty}}\,\frac{1}{n}\int_{\{n\le|u_l|\le 2n\}} a(x_2,Du_l)\cdot Du_l\rho(\tfrac{x_1}{2l_0})=0,$$

and then recalling that $|h'_n|=\frac{1}{n}\mathbb{1}_{\{n\le|u_l|\le 2n\}}$, the convergence (4.23) holds. Gathering (4.19), (4.20), (4.21), (4.22) and (4.23) leads to

$$\int_{\Omega_{2l_0}} a(x_2,DT_k(u_\infty))\cdot DT_k(u_\infty)\rho(\tfrac{x_1}{2l_0})=\int_{\Omega_{2l_0}} fT_k(u_\infty)\rho(\tfrac{x_1}{2l_0}). \tag{4.27}$$

Now, using $T_k(u_l)\rho(\frac{x_1}{2l_0})$ as test function in (2.11), with $h=h_n$, and letting n go to $+\infty$ gives

$$\int_{\Omega_l} a(x,Du_l)\cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0})+\frac{1}{2l_0}\int_{\Omega_l} a(x,Du_l)\cdot(D\rho)(\tfrac{x_1}{2l_0})T_k(u_l)$$
$$=\int_{\Omega_l} fT_k(u_l)\rho(\tfrac{x_1}{2l_0}).$$

By arguments already used it follows that

$$\lim_{l\to+\infty}\int_{\Omega_{2l_0}} a(x_2,DT_k(u_l))\cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0})=\int_{\Omega_{2l_0}} fT_k(u_\infty)\rho(\tfrac{x_1}{2l_0}). \tag{4.28}$$

Then, from (4.27) and (4.28) we get

$$\lim_{l\to+\infty}\int_{\Omega_{2l_0}} a(x_2,DT_k(u_l))\cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0}) = \int_{\Omega_{2l_0}} a(x_2,DT_k(u_\infty))\cdot DT_k(u_\infty)\rho(\tfrac{x_1}{2l_0}). \tag{4.29}$$

Therefore, by the convergence (4.29) and since $a(x_2,DT_k(u_l))\cdot DT_k(u_l)\rho(\frac{x_1}{2l_0})$ is a sequence of nonnegative functions that converges almost everywhere to $a(x_2,DT_k(u_\infty))\cdot DT_k(u_\infty)\rho(\frac{x_1}{2l_0})$, a standard application of Fatou's lemma gives

$$a(x_2,DT_k(u_l))\cdot DT_k(u_l)\rho(\tfrac{x_1}{2l_0}) \to a(x_2,DT_k(u_\infty))\cdot DT_k(u_\infty)\rho(\tfrac{x_1}{2l_0})$$

strongly in $L^1(\Omega_{2l_0})$, and the fact that $\rho(\frac{x_1}{2l_0})=1$ on Ω_{l_0} implies

$$a(x_2,DT_k(u_l))\cdot DT_k(u_l) \to a(x_2,DT_k(u_\infty))\cdot DT_k(u_\infty)$$

strongly in $L^1(\Omega_{l_0})$. This yields that $a(x_2,DT_k(u_l))\cdot DT_k(u_l)$ is equi-integrable in $L^1(\Omega_{l_0})$. Then the ellipticity condition

$$\alpha|DT_k(u_l)|^p \le a(x_2,DT_k(u_l))\cdot DT_k(u_l), \quad \text{for a.e. } x_2\in\omega, \tag{4.30}$$

implies that $DT_k(u_l)$ is p-equi-integrable. Therefore, by the almost-everywhere convergence of $DT_k(u_l)$, the Vitali theorem yields that

$$DT_k(u_l)\to DT_k(u_\infty) \quad \text{strongly in } (L^p(\Omega_{l_0})). \tag{4.31}$$

Step 3. We now prove that u_∞ is roughly speaking a renormalized solution of

$$\begin{cases} -\operatorname{div}[a(x_2,Du_\infty)] &= f & \text{on } \mathbb{R}^q\times\omega \\ u_\infty &= 0 & \text{on } \mathbb{R}^q\times\partial\omega. \end{cases}$$

First, from (4.17), for any $l_0>0$, $T_k(u_\infty)\in W^{1,p}_{\Gamma_{l_0}}(\Omega_{l_0})$. Then, we take $\phi\in C_c^\infty(\mathbb{R}^q\times\omega)$, $h\in W^{1,\infty}(\mathbb{R})$ with compact support included in $[-k,k]$, where $k>0$. Since $\operatorname{supp}(\phi)$ is compact, for l large enough $h(u_\infty)\phi\in W_0^{1,p}(\Omega_l)$, so we can use $h(u_\infty)\phi$ as test function in (2.11), with $h=h_n$, so that we obtain

$$\int_{\Omega_l} a(x_2,Du_l)\cdot Du_\infty\, h'(u_\infty)h_n(u_l)\phi + \int_{\Omega_l} a(x_2,Du_l)\cdot Du_l\, h_n'(u_l)h(u_\infty)\phi + \int_{\Omega_l} a(x_2,Du_l)\cdot D\phi\, h_n(u_l)h(u_\infty) = \int_{\Omega_l} fh(u_\infty)h_n(u_l)\phi. \tag{4.32}$$

For the first term, by the first step we have

$$\begin{aligned}&\int_{\Omega_l} a(x_2, Du_l)\cdot Du_\infty\, h'(u_\infty)h_n(u_l)\phi \\ &= \int_{\Omega_l} a(x_2, DT_{2n}(u_l))\cdot Du_\infty\, h'(u_\infty)h_n(u_l)\phi \\ &\to \int_{\mathbb{R}^q\times\omega} a(x_2, DT_{2n}(u_\infty))\cdot Du_\infty\, h'(u_\infty)h_n(u_\infty)\phi\end{aligned} \tag{4.33}$$

when l goes to $+\infty$. Therefore,

$$\begin{aligned}&\lim_{n\to+\infty}\lim_{l\to+\infty}\int_{\Omega_l} a(x_2, Du_l)\cdot Du_\infty\, h'(u_\infty)h_n(u_l)\phi \\ &= \int_{\mathbb{R}^q\times\omega} a(x_2, DT_k(u_\infty))\cdot Du_\infty\, h'(u_\infty)\phi.\end{aligned} \tag{4.34}$$

By (2.10), the second term of (4.32) converges to 0 as n goes to $+\infty$. By the strong convergence of $a(x, Du_l)$ in $(L^q(\Omega_{l_0}))^N$ for every $q < \frac{N}{N-1}$,

$$\begin{aligned}&\lim_{n\to+\infty}\lim_{l\to+\infty}\int_{\Omega_l} a(x_2, Du_l)\cdot D\phi\, h_n(u_l)h(u_\infty) \\ &= \int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot D\phi\, h(u_\infty).\end{aligned} \tag{4.35}$$

From (4.32), (4.33), (4.34) and (4.35), we get

$$\begin{aligned}&\int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot D\phi\, h(u_\infty) + \int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot Du_\infty\, h'(u_\infty)\phi \\ &= \int_{\mathbb{R}^q\times\omega} fh(u_\infty)\phi\end{aligned}$$

for every $\phi\in C_c^\infty(\mathbb{R}^q\times\omega)$; i.e.,

$$-\operatorname{div}[a(x_2, Du_\infty)h(u_\infty)] + h'(u_\infty)a(x_2, Du_\infty)\cdot Du_\infty = fh(u_\infty) \tag{4.36}$$

in $\mathcal{D}'(\mathbb{R}^q\times\omega)$. Moreover, by (2.5), (4.23), (4.31) and the Vitali theorem, for any $l_0 > 0$ we have

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\Omega_{l_0}\cap\{n\le|u_\infty|\le 2n\}} a(x_2, Du_\infty)\cdot Du_\infty = 0. \tag{4.37}$$

It remains to prove that $u_\infty = v$. Indeed, we take $\phi\in C_c^\infty(\mathbb{R}^q)$, we use $\phi T_k(T_{2n}(u_\infty) - T_{2n}(v))h_n(v)$ as test function in (4.36) with $h = h_n$, and we get

$$\int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot DT_k(u_\infty - v)h_n(v)h_n(u_\infty)\phi$$
$$+\int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot Du_\infty h_n'(u_\infty)h_n(v)T_k(u_\infty - v)\phi$$
$$+\int_{\mathbb{R}^q\times\omega} a(x_2, Du_\infty)\cdot Dv h_n(u_\infty)h_n'(v)T_k(u_\infty - v)\phi$$
$$+\int_{\mathbb{R}^q\times\omega} a_1^2(x_2, Du_\infty)D_{x_1}\phi h_n(u_\infty)h_n(v)T_k(u_\infty - v)$$
$$= \int_{\mathbb{R}^q\times\omega} f\phi T_k(T_{2n}(u_\infty) - T_{2n}(v))h_n(u_\infty)h_n(v). \quad (4.38)$$

Since u_∞ and v do not depend on x_1, the divergence theorem implies that the fourth term of (4.38) is equal to 0. We use $T_k(T_{2n}(u_\infty)-T_{2n}(v))h_n(u_\infty)$ as test function in (2.11′) with $h = h_n$, we multiply by ϕ, and we integrate over $\mathbb{R}^q$; we get

$$\int_{\mathbb{R}^q\times\omega} a(x_2, Dv)\cdot DT_k(u_\infty - v)h_n(v)h_n(u_\infty)\phi$$
$$+\int_{\mathbb{R}^q\times\omega} a(x_2, Dv)\cdot Dv\, h_n(u_\infty)h_n'(v)T_k(u_\infty - v)\phi$$
$$+\int_{\mathbb{R}^q\times\omega} a(x_2, Dv)\cdot Du_\infty\, h_n'(u_\infty)h_n(v)T_k(u_\infty - v)\phi$$
$$= \int_{\mathbb{R}^q\times\omega} f\phi T_k(T_{2n}(u_\infty) - T_{2n}(v))h_n(u_\infty)h_n(v). \quad (4.39)$$

By arguments already used, from (2.10′), (2.13), (4.17) and (4.37), all the terms of (4.38) and (4.39) except the first and the last go to 0 as n goes to $+\infty$. Then, subtracting (4.38) and (4.39) implies

$$\int_{\mathbb{R}^q\times\omega}(a(x_2, Du_\infty)-a(x_2, Dv))\cdot DT_k(u_\infty-v)h_n(v)h_n(u_\infty)\phi = \epsilon(n), \quad (4.40)$$

where $\epsilon(n)$ goes to 0 as n goes to $+\infty$. Therefore, by Fatou's lemma

$$\int_{\mathbb{R}^q\times\omega}(a(x_2, Du_\infty) - a(x_2, Dv))\cdot DT_k(u_\infty - v)\phi = 0.$$

From the strict monotonicity of the operator a, it follows that $u_\infty = v$.

Acknowledgment. The author thanks D. Blanchard and O. Guibé for numerous discussions on the problem under investigation in the present paper.

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