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BOUNDARY BLOW UP FOR SEMILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR GRADIENT TERMS*

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Abstract. The paper deals with the equation $\Delta u \pm |\nabla u|^q = f(u)$ in $\Omega \subset \mathbf{R}^n$, where u blows up at the boundary $\partial\Omega$ and Ω is a bounded domain, which satisfies an interior and an exterior sphere condition. The existence and the asymptotic behaviour of u near the boundary are investigated, showing how the nonlinear gradient term affects the results.

1. Introduction. Let $D \subset \mathbf{R}^N$ be a bounded domain satisfying an inner and outer sphere condition. In this paper we discuss problems of the following type:

$$\begin{cases} \Delta u \pm |\nabla u|^q = f(u) & \text{in } D \\ u(x) \to \infty & \text{as } x \to \partial D\,, \end{cases} \tag{P}^{\pm}$$

where $q > 0$ is an arbitrary fixed number and f is a positive, increasing function. The main questions which will be addressed are the existence of solutions and their asymptotic behaviour near the boundary. The corresponding semilinear problem

$$\Delta u = f(u) \quad \text{in } D, \quad u(x) \to \infty \;\text{ as }\; x \to \partial D \tag{P_0}$$

has been studied by many authors and in different contexts ([4, 5, 9, 11, 12, 14]). A rather complete theory is available for (P_0) and some of its generalizations ([3, 4, 5, 9, 11]).

We shall deal with classical solutions belonging to $C^{2,\alpha}(D')$ for every compact subdomain $D' \subset D$.

For the classical models $f(t) = t^p$ and $f(t) = e^t$, Problem (P_0) has a solution provided $p > 1$. It turns out that in both cases the solution is unique and satisfies in the first case ([3, 4, 11])

$$\begin{cases} u(x) - \phi(\delta(x)) \to 0 & \text{as } x \to \partial D, \quad \text{if } p > 3 \\ u(x)/\phi(\delta(x)) \to 1 & \text{as } x \to \partial D, \quad \text{if } p > 1. \end{cases} \tag{1.1}$$

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Here $\phi(t) = [\sqrt{2(p+1)}/(p-1)]^{2/(p-1)} t^{-2/(p-1)}$ and $\delta(x) = \operatorname{dist}(x, \partial D)$. If $f(t) = e^t$, then

$$u(x) + 2\log\delta(x) \to \log 2 \quad \text{as} \quad x \to \partial D. \tag{1.2}$$

In these cases the asymptotic behaviour of $\nabla u(x)$ can also be expressed in a simple way, namely,

$$\begin{aligned} \tfrac{1}{2}(p+1)|\nabla u|^2/u^{p+1} \to 1 \quad &\text{as} \quad x \to \partial D, \quad \text{if} \quad f(t) = t^p, \\ |\nabla u(x)|\delta(x) \to 2 \quad &\text{as} \quad x \to \partial D, \quad \text{if} \quad f(t) = e^t. \end{aligned} \tag{1.3}$$

The most surprising fact is that in the first approximation, the behaviour of $u(x)$ near the boundary is independent of the geometry.

The aim of this paper is to study the influence of the nonlinear gradient term. It turns out that the presence of this term can have a significant influence on the existence of a solution as well as on its asymptotic behaviour.

The simplest case is $q = 2$, which can be reduced to a problem without gradient term (cf. also [13]). Indeed the function $v = e^{\pm u}$, where u is a solution of $(P)^{\pm}$, satisfies

$$\pm\Delta v = v f(\pm \log v), \quad v \to \infty(0) \quad \text{on} \quad \partial D. \tag{1.4}$$

We shall therefore mainly consider the cases where $q \neq 2$.

As an illustration we state our results for the two particular cases $f(t) = t^p$ and $f(t) = e^t$. The corresponding theorems for more general nonlinearities f are given in Sections 4 and 5.

Theorem 1.1. *Let $f(t) = t^p$, $p > 0$, and let $\phi(t)$ be as in* (1.1).

(i) *If $p > 1$ and $q < \frac{2p}{p+1} (< 2)$, then $(P)^{\pm}$ possesses at least one solution. Every solution of $(P)^{\pm}$ satisfies $u(x)/\phi(\delta(x)) \to 1$ as $\delta(x) \to 0$.*

(ii) *The same statement for $(P)^+$ is true if $\frac{2p}{p+1} < q < p$ except that in this case*

$$u(x)\Big\{\frac{p-q}{q}\delta(x)\Big\}^{q/(p-q)} \to 1 \quad \textit{as} \quad \delta(x) \to 0.$$

(iii) *If $\max\{\frac{2p}{p+1}, 1\} < q < 2$, then $(P)^-$ possesses a solution. Each solution of $(P)^-$ satisfies $u(x)(2-q)[(q-1)\delta(x)]^{\frac{2-q}{q-1}} \to 1$ as $\delta(x) \to 0$.*

(iv) *If $q = 2$, $(P)^-$ has a solution for all $p > 0$ which satisfies*

$$\lim_{x \to \partial D} u(x)/\log\delta(x) = 1.$$

It is interesting to note that the asymptotic behaviour described in (iii) and (iv) of the theorem above is the same as the one obtained in [10] for a problem which is similar to $(P)^-$, except that the right-hand side is $f(x) + \lambda u$, $f(x)$ bounded and $\lambda > 0$ (see Theorem II.1 in [10]).

The results concerning $f(t) = e^t$ are even simpler to describe, namely,

Theorem 1.2.

(i) *If $q < 2$, then $(P)^{\pm}$ has a solution. All solutions satisfy $u(x)/\log\delta^{-1}(x) \to 2$ as $x \to \partial D$.*

(ii) *If $q \geq 2$, $(P)^{+}$ has a solution. All solutions of $(P)^{+}$ satisfy $u(x)/\log\delta^{-1}(x) \to q$ as $x \to \partial D$.*

(iii) *If $q = 2$, $(P)^{-}$ is solvable. The asymptotic behaviour of the solutions is the same as in* (ii).

Problems of the type $(P)^{-}$ appear in stochastic control theory and have been studied in a substantial paper by Lasry and Lions ([10]). The corresponding parabolic equation was considered in [15]. The main objective is to study the effect on the damping term to the blow up or to the long-time behaviour of the solutions. The bounds can also be used in parabolic problems exactly in the same way as in [2].

Note that by the maximum principle a solution of $(P)^{\pm}$ provides an upper bound for every solution of

$$\Delta u = g(u, \nabla u) \quad \text{in} \quad D, \tag{1.5}$$

where $g(t, \xi) \geq f(t) \mp |\xi|^q \ \forall\, t \in \mathbf{R}$ and $\xi \in \mathbf{R}^N$.

The paper is organized as follows. In Section 2 we discuss the one-dimensional problem by means of a phase plane analysis. Those results are then extended in Section 3 to balls and annuli. Using comparison methods we study in Section 4 the asymptotic behaviour in general domains. The main tool is the *monotonicity lemma* which states that *if $f' \geq 0$, $D_1 \subset D_2$ and u_1, u_2 are solutions of $(P)^{\pm}$ in D_1 and D_2, then $u_1(x) \geq u_2(x)$ in D_1*. Section 5 is concerned with existence of solutions. For this purpose we use the method of upper and lower solutions ([1, 8]).

2. The one-dimensional case. In this section we shall discuss problems of the form

$$\begin{aligned} &u''(x) \pm |u'(x)|^q = f(u(x)) \quad \text{in} \quad (0, R) \\ &u(x) \to \infty \quad \text{as} \quad x \to R. \end{aligned} \tag{2.1$^{\pm}$}$$

Here $q > 0$ is any given number and $f : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies

$$f'(t) > 0 \ \text{ for } \ t > 0 \quad \text{and} \quad \lim_{t\to\infty} f(t) = \infty. \tag{$F-1$}$$

It is easily seen that u cannot have a local maximum. Hence if $u'(x_0) > 0$ for some $x_0 \geq 0$, then $u'(x)$ stays positive for all $x \geq x_0$.

For reasons which will become more transparent in the next section, we will only study those solutions for which $u(0) \geq 0$, $u'(0) \geq 0$ and $(u(0), u'(0)) \neq (0, 0)$. In this case we have $u'(x) > 0$ for all $x > 0$, and Problem $(2.1)^{\pm}$ can thus be transformed into

$$\frac{dv}{du} \pm 2v^{q/2}(u) = 2f(u), \tag{2.2$^{\pm}$}$$

where $v(u) = {u'}^2(x(u))$.

The trajectories Γ corresponding to the above solutions lie in the quarter plane $Q^+ = \{(u,v) : u > 0, v > 0\}$. For each $P \in \partial Q^+ - \{O\}$ there is exactly one trajectory $\Gamma(P)$ emanating from P. This is a consequence of the classical theory on ordinary differential equations when $q \geq 2$.

Otherwise we can argue as follows. Suppose that v_1, v_2 are two different solutions such that $v_1 > v_2$ in $(u_0, u_0 + \varepsilon)$ and $v_1(u_0) = v_2(u_0)$. Put $\delta = v_1 - v_2$. Then for the problem $(2.2)^+$

$$\delta(u_0) = 0 \quad \text{and} \quad \delta'(u) < 0 \quad \text{in} \quad (u_0, u_0 + \varepsilon)$$

in contrast with the assumption.

As regards to the Problem $(2.2)^-$ we have since $q < 2$

$$\delta' \leq q v_2^{(q-2)/2} \delta, \quad u_0 < u < u_0 + \varepsilon.$$

Thus

$$\delta' \delta^{-1} \leq q v_2^{(q-2)/2} \quad \text{in} \;\; (u_0,\, u_0 + \varepsilon).$$

If $v_2(u_0) = 0$ then $v_2'(u_0) = f(u_0) > 0$. Hence $v_2^{(q-2)/2}(u)$ is for $q \in (0,2)$ integrable at $u = u_0$. The above inequality thus implies that

$$\delta(u) \leq \delta(u_0) \exp \int_{u_0}^{u} q v_2^{(q-2)/2} \, ds = 0.$$

This contradicts our assumption.

A phase plane analysis yields the following picture of the trajectories Γ:

(i) Problem $(2.2)^+$. Denote by $D_{0(1)}$ the region $\{(u,v) \in Q^+ : v^{q/2} < f(u)(v^{q/2} > f(u))\}$ and let $\gamma = \bar{D}_0 \cap \bar{D}_1$ be their common boundary. By $(F-1)$ a trajectory $\Gamma(u_0, 0)$ emanating from the point $P = (u_0, 0)$ or the point $P = (0, v_0)$, where $0 < v_0 \leq f(0)$, is monotone and stays in D_0, whereas if Γ starts from a point $P = (0, v_0)$ and $v_0 > f(0)$, Γ first decreases until it reaches γ and then increases and remains in D_0.

Clearly v satisfies

$$v' \leq 2f(u) \tag{2.3}$$

and hence $v(u) \leq 2F(u) + c$, where $F(t) = \int_0^t f(s)\, ds$. Consequently $v(u)$ exists for all $u \geq u_0$. Such a solution will be called *global*. It is clear that if $(2.1)^{\pm}$ has a solution the corresponding v is global.

(ii) Problem $(2.2)^-$. In this case all trajectories are monotone and stay in Q^+.

The solutions are not necessarily global, because of the inequality

$$v' > 2v^{q/2}. \tag{2.4}$$

For $q \neq 2$ integration yields

$$v^{1-q/2}(s)/(1-q/2)|_{\bar{u}}^{u} \geq 2(u - \bar{u}),$$

where $u > \bar{u}$ and $v(\bar{u}) > 0$. This implies that no global solutions exist if $q > 2$.

Let $q \leq 2$. From the equation we derive

$$v(u) \geq 2[F(u) - F(u_0)] + v(u_0).$$

Integrating $(2.2)^-$ we find using the lower bound for v

$$v(u) \leq v(u_0) + 2\int_{u_0}^{u} f(s)\,ds + 2\int_{u_0}^{u} v(s)w(s)\,ds,$$

where $w(s) = \{2[F(s) - F(u_0)] + v(u_0)\}^{\frac{q}{2}-1}$. Note that w is integrable at $s = u_0$. Hence by Gronwall's inequality

$$v(u) \leq v(u_0)\exp\Big(\int_{u_0}^{u} 2w(s)\,ds\Big) + 2\int_{u_0}^{u} f(s)\exp\Big(2\int_{s}^{u} w(t)\,dt\Big)\,ds$$

which shows that v is global.

Observe that for the solutions of $(2.1)^{\pm}$

$$\int_{u(0)}^{\infty} \frac{ds}{\sqrt{v(s)}} = R. \tag{2.5}$$

In a next step we study under what conditions the integral (2.5) is finite. For this purpose we shall determine the asymptotic behaviour of $v(u)$ as $u \to \infty$.

(a) Problem $(2.2)^+$. From (2.3) we have $v(u) \leq 2F(u) + c$. As observed at the beginning, Γ lies in D_0 for sufficiently large u. Hence $v'(u) = 2[f(u) - v^{q/2}] \geq 0$ for $u \geq \bar{u}$, and thus $v \leq f^{2/q}(u)$. Consequently, v satisfies, for u sufficiently large,

$$v(u) \leq \min\{2F(u) + c,\ f^{2/q}(u)\}. \tag{2.6}$$

Necessary conditions for the integral (2.5) to be finite are in view of (2.6)

$$\int^{\infty} \frac{du}{\sqrt{F(u)}} < \infty \tag{$F-2$}$$

and

$$\int^{\infty} \frac{du}{f^{1/q}(u)} < \infty. \tag{$F-3$}$$

Next we shall show that under additional assumptions, they are also sufficient. Assume

$$F(u)/f^{2/q}(u) \to 0 \quad \text{as} \quad u \to \infty. \tag{$F-4$}$$

Remark. $(F-2)$ and $(F-4)$ imply $(F-3)$.

Lemma 2.1.

(i) *Under the conditions* $(F-1)$ *and* $(F-4)$*, every solution of* $(2.2)^+$ *is global and satisfies* $v(u)/2F(u) \to 1$ *as* $u \to \infty$.

(ii) *Assume* $(F-1)$ *and* $(F-4)$. $(F-2)$ *is a necessary and sufficient condition for* R *in formula* (2.3) *to be finite.*

Proof. (i) In view of (2.6) it suffices to construct a lower bound. Let v be an arbitrary solution of $(2.2)^+$. Given any $\varepsilon > 0$, there exists, by $(F-4)$, a number u_1 such that

$$[2F(u)]^{q/2} < \varepsilon f(u) \quad \forall\, u \geq u_1.$$

Let $z = 2(1-\varepsilon)F(u) + c$, where $c \leq 0$ is such that $z(u_1) \leq v(u_1)$. For $u \geq u_1$

$$z' + 2z^{q/2} = 2(1-\varepsilon)f(u) + 2[2(1-\varepsilon)F(u) + c]^{q/2} < 2f(u).$$

Hence $v' - z' + 2(v^{q/2} - z^{q/2}) > 0$. A simple argument shows that $v(u) > z(u)\ \forall\, u \geq u_1$. Thus $v(u) > 2(1-\varepsilon)F(u) + c$. The assertion is now obvious.

(ii) As already observed, it follows from (2.6) that $(F-2)$ is necessary. The previous statement (i) implies that it is also sufficient.

Examples. (1) If $f(t) = t^p$, then $(F-2)$ implies $p > 1$, whereas $(F-4)$ implies $q < \frac{2p}{p+1} < 2$.

(2) If $q \leq 1$, then $(F-4)$ holds for any f satisfying $(F-1)$ and $(F-2)$.

Proof. Since f is increasing, $F(u) \leq uf(u)$. Then for u sufficiently large

$$F(u)/f^{2/q}(u) \leq uf^{1-2/q}(u).$$

It was shown in [5] that under the above conditions, $f(u)/u \to \infty$ as $u \to \infty$. This proves $(F-4)$.

Suppose now that, in contrast to $(F-4)$, the condition

$$F(u)/f^{2/q} \to \infty \quad \text{as} \quad u \to \infty \qquad (F-5)$$

holds. We shall consider the slightly stronger condition

$$f(u)^{2-2/q}/f'(u) \to \infty \quad \text{as} \quad u \to \infty. \qquad (F-5)'$$

It is easily seen by the rule of Bernoulli-L'Hospital that $(F-5)'$ implies $(F-5)$.

Lemma 2.2.

(i) *Assume* $(F-1)$ *and* $(F-5)'$. *Then all solutions of* $(2.2)^+$ *satisfy*

$$v(u)/f^{2/q}(u) \to 1 \quad \textit{as} \quad u \to \infty.$$

(ii) *If* $(F-1)$ *and* $(F-5)'$ *hold, then* $(F-3)$ *is a necessary and sufficient condition for* R *to be finite.*

Proof. (i) Because of (2.6) we need a lower bound for v. Let $\varepsilon > 0$ be given and choose u_1 so large that

$$f^{2/q-1}f'/q < \{1-(1-\varepsilon)^{q/2}\}f \quad \text{for} \quad u \geq u_1.$$

Put $z = (1-\varepsilon)f^{2/q}(u) + c$, where $c \leq 0$ is such that $z(u_1) \leq v(u_1)$. Then

$$z' + 2z^{q/2} = \frac{2}{q}(1-\varepsilon)f^{2/q-1}f' + 2[(1-\varepsilon)f^{2/q} + c]^{q/2} < 2f(u).$$

By comparison $z(u) \leq v(u)$ for $u \geq u_1$. The first assertion is now immediate.

(ii) The proof of the second one is as for Lemma 2.1 (ii).

Remarks. (1) If $f(t) = t^p$ and $\frac{2p}{p+1} < q < p$, $(F-3)$ and $(F-5)$ are satisfied.

(2) In the boarderline case $f(t) = t^p$ and $q = 2p/(p+1)$ neither $(F-4)$ nor $(F-5)'$ is satisfied. In this case a particular solution is known, namely $v(u) = bu^{p+1}$, b being the unique root of $b(p+1) = 2(1-b^{q/2})$. The asymptotic behaviour neither corresponds to the one in Lemma 2.1 nor in Lemma 2.2.

(3) $(F-5)$ and $(F-3)$ imply $(F-2)$.

(b) Problem $(2.2)^-$. As we have already seen, global solutions exist only if $q \leq 2$. They satisfy $v(u) \geq 2F(u) + c$.

Lemma 2.3. *Assume* $(F-1)$, $(F-4)$ *and* $q < 2$.

(i) *Every solution of* $(2.2)^-$ *satisfies* $v(u)/2F(u) \to 1$ *as* $u \to \infty$.
(ii) R *is finite if and only if* $(F-2)$ *holds.*

Proof. The reasoning is similar to that in Lemma 2.1. Let $\varepsilon > 0$ be fixed and put $z = 2(1+\varepsilon)F(u)$. Then

$$z' - 2z^{q/2} = 2(1+\varepsilon)f(u) - 2\cdot 2^{q/2}(1+\varepsilon)^{q/2}F^{q/2}(u).$$

For u_1 sufficiently large, we have by $(F-4)$

$$z' - 2z^{q/2} > 2f(u) \forall\, u \geq u_1.$$

By comparison $v(u) \leq z(u)$ for all solutions v such that $v(u_1) \leq z(u_1)$. Let $\tilde{v}$ be a solution with $\tilde{v}(u_1) > z(u_1)$. Let v be a solution with $v(u_1) \leq z(u_1)$. Then the difference $d(u) = \tilde{v}(u) - v(u)$ is positive for all $u \geq u_1$ and satisfies in view of our assumptions

$$d' = 2(\tilde{v}^{q/2} - v^{q/2}) \leq qv^{q/2-1}d \leq q\frac{(2F+c)^{q/2}}{2F+c}d \leq \frac{2\varepsilon f}{2F+c}d \,\forall\, u \geq k(\varepsilon).$$

Thus $\log\frac{d(u)}{d(k)} \leq \varepsilon \log\frac{2F(u)+c}{2F(k)+c}$ and $d(u) \leq (2F(u)+c)^{\varepsilon}\gamma$, γ independent of u. Hence

$$0 < \tilde{v}(u)/2F(u) - v(u)/2F(u) \leq \gamma\frac{(2F(u)+c)^{\varepsilon}}{2F(u)}.$$

Choosing $\varepsilon < 1$ we obtain by virtue of the previous considerations,

$$\lim_{u\to\infty} v(u)/2F(u) = 1,$$

whence $\lim_{u\to\infty} \tilde{v}(u)/2F(u) = 1$. This establishes (i). The second statement is a consequence of (i).

Suppose now $q < 2$ and that $(F-5)$ holds. By $(2.2)^-$ we have $v' \geq 2v^{q/2}$ and hence, keeping in mind that $q < 2$, $v^{(2-q)/2} \geq (2-q)u + c$. Inserting this inequality into $(2.2)^-$ we get

$$v' \leq 2f\Big(\frac{v^{(2-q)/2} - c}{2-q}\Big) + 2v^{q/2}.$$

Since f is increasing we have $F(t) \leq tf(t)$, which together with $(F-5)$ implies that $t/f^{\frac{2-q}{q}}(t) \to \infty$ as $t \to \infty$. Thus for given $\varepsilon > 0$ there exists v_1 sufficiently large such that

$$f\Big(\frac{v^{(2-q)/q} - c}{2-q}\Big) < \varepsilon v^{q/2} \quad \text{for all} \quad v \geq v_1.$$

Consequently,

$$v' \leq 2(1+\varepsilon)v^{q/2} \quad \text{for all} \quad v \geq v_1.$$

We have established the following result.

Lemma 2.4. *Assume $(F-1)$ and $(F-5)$ and $q < 2$. Then*

$$v(u)/[(2-q)u]^{2/(2-q)} \to 1 \quad \textit{as} \quad u \to \infty.$$

Moreover R is finite if and only if $q > 1$.

Remark. For $f(t) = t^p$ the assumptions of Lemma 2.4 are satisfied if $p > 0$ and $\max\{1, \frac{2p}{p+1}\} < q < 2$.

Special cases. 1) If $q = 2$, $(2.2)^-$ is a linear equation which can be integrated. Its general solution is of the form

$$v(u) = ce^{2u} + 2\int_{u_0}^{u} e^{2(u-s)} f(s)\, ds.$$

2) If $f = t^p$ and $q = 2p/(p+1)$, then $(F-4)$ and $(F-5)$ are violated. A particular solution is $v(u) = au^{p+1}$, where a is the unique root of $a(p+1) = 2(1+a^{q/2})$. In this case Lemma 2.3 (i) and Lemma 2.4 are obviously wrong.

This completes the discussion of the trajectories in the phase plane.

Let us return to the initial problem focussing on two particular cases, namely,

$$\begin{cases} u'' \pm |u'|^q = f(u) & \text{in} \quad (0,R) \\ u(0) = 0, \quad u(x) \to \infty & \text{as} \quad x \to R \end{cases} \tag{2.1$^\pm_D$}$$

and $(2.1)^{\pm}_{N}$ where the Dirichlet condition $u(0)=0$ is replaced by the Neumann condition $u'(0)=0$.

The trajectories Γ in the (u,v)-plane, corresponding to $(2.1)^{\pm}_{D}$, emanate at a point $(0,v_0)$. Those corresponding to $(2.1)^{\pm}_{N}$ emanate at a point $(u_0,0)$. We shall denote them by $v(u;v_0)$ or $v(u;u_0)$, respectively.

The solutions of $(2.1)^{\pm}_{D,N}$ are given implicitly by

$$x=\int_0^u \frac{ds}{\sqrt{v(s;v_0)}} \quad \text{or} \quad x=\int_{u_0}^u \frac{ds}{\sqrt{v(s;u_0)}}. \tag{2.7}$$

Once we know the behaviour of v, the problem now reduces to finding $v_0(u_0)$ such that

$$R=\int_0^\infty \frac{ds}{\sqrt{v(s;v_0)}} \quad \text{or} \quad R=\int_{u_0}^\infty \frac{ds}{\sqrt{v(s;u_0)}}. \tag{2.8}$$

Lemma 2.5. *$R(u_0)$ and $R(v_0)$ are continuous and decreasing functions of u_0 or v_0, respectively.*

Proof. The first property is a consequence of the continuous dependence of $v(u)$ on the initial data. In order to establish the second statement, we consider the function $w(u)=v(u-\alpha;u_0)$. It satisfies $w(u_0+\alpha)=0$, and by $(F-1)$

$$w'=2[f(u-\alpha)\mp w^{q/2}]<2[f(u)\mp w^{q/2}].$$

By comparison, $w(u)\le v(u;u_0+\alpha)$. Hence

$$R(u_0)=\int_{u_0}^\infty \frac{ds}{v^{1/2}(s;u_0)}=\int_{u_0+\alpha}^\infty \frac{ds}{w^{1/2}(s)}\ge\int_{u_0+\alpha}^\infty \frac{ds}{v^{1/2}(s;u_0+\alpha)}=R(u_0+\alpha).$$

The monotonicity of $R(v_0)$ follows immediately from the fact that $v(u;v_0)\ge v(u;v_1)$ $\forall\, v_0\ge v_1$. For the next result we need the condition

$$f^{2/q}(s)/s \quad \text{is increasing} \ \ \forall\, s>\bar{s}. \tag{$F-6$}$$

$(F-6)$ implies that

$$f^{2/q}(u_0+s)-f^{2/q}(u_0)\ge f^{2/q}(s) \quad \forall\, u_0>\bar{s} \quad \text{and} \quad \forall\, s>\bar{s} \tag{2.9}$$

and also

$$f^{2/q}(u_0+s)-f^{2/q}(u_0)\ge \frac{f^{2/q}(u_0)}{u_0}s \quad \forall\, s>0 \ \text{ and } \ \forall\, u_0>\bar{s}. \tag{2.10}$$

Lemma 2.6. *We have* $R(u_0) \to 0$ *as* $u_0 \to \infty$ *and* $R(v_0) \to 0$ *as* $v_0 \to \infty$ *in the case of* $(2.2)^+$ *when either the conditions* $(F-1)$, $(F-2)$, $(F-4)$ *or* $(F-1)$, $(F-3)$, $(F-5)'$, $(F-6)$ *hold. The same is true for Problem* $(2.2)^-$ *if either* $q \leq 2$ *and* $(F-1)$, $(F-2)$, *or* $1 < q < 2$ *and* $(F-1)$ *are satisfied.*

Proof. By our assumptions there exists a number u_1 sufficiently large depending only on $\varepsilon > 0$ and f such that the solutions of $(2.2)^+$ satisfy either

$$v(u;u_0) \geq 2(1-\varepsilon)[F(u) - F(u_0)] \quad \forall\, u_0 \geq u_1 \tag{2.11}$$

or

$$v(u;u_0) \geq (1-\varepsilon)[f^{2/q}(u) - f^{2/q}(u_0)] \quad \forall\, u_0 \geq u_1. \tag{2.12}$$

This follows from the arguments given in the proofs of the Lemmas 2.1 and 2.2.

Suppose that $(F-1)$, $(F-2)$, $(F-4)$ hold. Then by $(F-2)$ there exists for any $\varepsilon > 0$ a number $M = M(\varepsilon) > 0$ such that

$$\int_M^\infty \frac{ds}{\sqrt{F(s)}} < \varepsilon.$$

By the convexity of F we have $F(u) - F(u_0) \geq F(u - u_0)$,

$$\begin{aligned} R(u_0) &\leq \frac{1}{\sqrt{1-\varepsilon}} \int_{u_0}^{M+u_0} \frac{du}{\sqrt{2(F(u)-F(u_0))}} + \frac{1}{\sqrt{1-\varepsilon}} \int_{M+u_0}^{\infty} \frac{du}{\sqrt{2(F(u)-F(u_0))}} \\ &\leq \frac{1}{\sqrt{1-\varepsilon}} \int_{u_0}^{M+u_0} \frac{du}{\sqrt{2f(u_0)(u-u_0)}} + \frac{1}{\sqrt{1-\varepsilon}} \int_{M+u_0}^{\infty} \frac{du}{\sqrt{2F(u-u_0)}} \\ &\leq \frac{1}{\sqrt{1-\varepsilon}} \Big[\frac{\sqrt{2}M^{1/2}}{\sqrt{f(u_0)}} + \varepsilon\Big]. \end{aligned}$$

This completes the proof of the first case. The reasoning is the same if $(F-1)$, $(F-3)$, $(F-5)'$, $(F-6)$ hold. We have only to replace (2.11) by (2.12) and use (2.9) and (2.10).

For Problem $(2.2)^-$ we use the estimates $v(u;u_0) \geq 2[F(u) - F(u_0)]$ if $(F-2)$ holds and also $v(u;u_0) \geq \{(2-q)(u-u_0)\}^{2/(2-q)}$ if $1 < q < 2$, and then we proceed as before.

Let us now consider $R(v_0)$. If $v(u;v_0)$ solves $(2.2)^-$, we have $v(u;v_0) \geq v_0 + 2F(u)$ and $v(u;v_0) \geq \{(2-q)u + v_0^{(2-q)/2}\}^{2/(2-q)}$. Inserting the first inequality into (2.8) if the conditions $(F-1)$, $(F-2)$ hold and the second if $1 < q < 2$, we get $R(v_0) \to 0$ as $v_0 \to \infty$.

If $v(u,\, v_0)$ solves $(2.2)^+$, we choose $u_0 > u_1$ defined as in (2.11) and (2.12). Consider the trajectory Γ through $(u_0, f^{2/q}(u_0))$. It intersects the v-axis at $(0, v_0)$. We have

$$\begin{aligned} v(u;v_0) &\geq f^{2/q}(u_0) \quad \text{in} \;\; (0, u_0) \\ v(u;v_0) &\geq v(u;u_0) \quad\; \text{for} \;\; u \geq u_0. \end{aligned}$$

Hence
$$R(v_0) = \int_0^{u_0} \frac{du}{\sqrt{v(u,v_0)}} + \int_{u_0}^{\infty} \frac{du}{\sqrt{v(u,v_0)}} \le \frac{u_0}{f^{1/q}(u_0)} + R(u_0).$$

By $(F-3)$ there is a sequence $u_0^{(k)} \to \infty$ such that $\frac{u_0^{(k)}}{f^{1/q}(u_0^{(k)})} \to 0$ as $k \to \infty$. Consequently $\lim_{k\to\infty} R(v_0^{(k)}) = 0$ for some sequence $\{v_0^{(k)}\}_{k=1}^{\infty}$ and by the monotonicity $R(v_0) \to 0$ as $v_0 \to \infty$.

Let us put the previous results in that form which will be needed in Section 3.

Theorem 2.7. *Assume* $(F-1)$ *and* $q > 0$.

(i) *If, in addition,* $(F-2)$ *and* $(F-4)$ *hold, then the Problems* $(2.1)_D^+$ *and* $(2.1)_N^+$ *possess a solution for all* $R \in (0, R')$. *They satisfy* ${u'}^2(x)/2F(u(x)) \to 1$ *as* $x \to R$.

(ii) *If* $(F-3)$, $(F-5)'$ *and* $(F-6)$ *are satisfied, the Problems* $(2.1)_D^+$ *and* $(2.1)_N^+$ *possess a solution for all* $R \in (0, R')$. *Every solution satisfies*
$${u'}^2(x)/f^{2/q}(u(x)) \to 1 \quad \textit{as} \quad x \to R.$$

(iii) *Under the additional assumptions* $q < 2$, $(F-2)$ *and* $(F-4)$, *the Problems* $(2.1)_D^-$ *and* $(2.1)_N^-$ *have a solution for all* $R \in (0, R')$; *each of them satisfies* ${u'}^2(x)/2F(u(x)) \to 1$, *as* $x \to R$.

(iv) *If* $1 < q < 2$ *and* $(F-5)$ *holds, statement* (iii) *is true except that in this case*
$${u'}^2(x)/\{(2-q)u\}^{2/(2-q)} \to 1 \quad \textit{as} \quad x \to R.$$

Example. If $f = t^p$, $p > 0$, then (i) and (iii) hold if $q < \frac{2p}{p+1}$ and $p > 1$, (ii) is true if $\frac{2p}{p+1} < q < p$ and (iv) is satisfied if $\max\{1, \frac{2p}{p+1}\} < q < 2$.

From the asymptotic behaviour of u' we can recover, under additional conditions on f, the asymptotic behaviour of u. Namely, if for $u \ge u_1$, $(1-\varepsilon)g(u) \le v^{1/2}(u) \le (1+\varepsilon)g(u)$, then integration yields, for $R - x \le \delta(\varepsilon)$,
$$(1-\varepsilon)(R-x) \le \int_u^{\infty} \frac{ds}{g(s)} \le (1+\varepsilon)(R-x). \tag{2.13}$$

In our case $g(s) = \sqrt{2F(s)}$ or $g(s) = f^{1/q}(s)$, or $g(s) = [(2-q)s]^{1/(2-q)}$.

Denote
$$\psi(u) = \int_u^{\infty} \frac{ds}{g(s)} \quad \text{and} \quad \phi = \psi^{-1}.$$

Note that for $g(s) = [(2-q)s]^{1/(2-q)}$
$$\phi(s) = \frac{1}{(2-q)}[(q-1)s]^{(2-q)/(1-q)}.$$

By (2.13)

$$\phi((1-\varepsilon)(R-x)) \geq u(x) \geq \phi((1+\varepsilon)(R-x)). \tag{2.14}$$

Under the assumption

$$\lim_{t\to\infty} \inf \psi(\beta t)/\psi(t) > 1 \quad \forall \beta \in (0,1) \tag{$F-7$}$$

(cf. [3]), we get

$$u(x)/\phi(R-x) \to 1 \quad \text{as} \quad x \to R. \tag{2.15}$$

3. Radial solutions in balls and annuli. The radial solutions of $\Delta u \pm |\nabla u|^q = f(u)$ in a ball satisfy

$$\begin{aligned} &u''(r) + \frac{N-1}{r}u'(r) \pm |u'(r)|^q = f(u(r)), \quad r > 0 \\ &u(0) = u_0, \quad u'(0) = 0. \end{aligned} \tag{$3.1^\pm$}$$

Throughout this section we assume $(F-1)$.

We first show that under some conditions already imposed in Section 2 all solutions of $(3.1)^\pm$ blow up at some $R > 0$. Note that u cannot have a local maximum. Hence $u' > 0$.

Let us start with $(3.1)^+$. In this case $u'' \leq f(u)$ which implies

$$u'(r) \leq \sqrt{2F(u) + c}, \quad c = -2F(u_0). \tag{3.2}$$

Inserting this inequality into $(3.1)^+$ we get

$$u'' + u'^q \geq f(u) - \frac{N-1}{r}\sqrt{2F(u) + c}. \tag{3.3}$$

It is clear from the equation that $u(r)$ cannot tend to a finite number as r increases. Hence $u(r) \to \infty$ as $r \to R$ (possibly $R = \infty$).

It follows from (3.2) that R cannot be finite unless $(F-2)$ holds. If this is the case we have ([5])

$$\lim_{u\to\infty} \sqrt{F(u)}/f(u) = 0. \tag{3.4}$$

Thus, given $\varepsilon > 0$ there exists u_1 such that u satisfies

$$u'' + u'^q > (1-\varepsilon)f(u) \quad \text{for} \;\; r \geq r_1. \tag{3.5}$$

The solution of the one-dimensional problem $w'' + {w'}^q = (1-\varepsilon)f(w)$ for $x \geq r_1$, $w(r_1) = u(r_1)$, $w'(r_1) = u'(r_1)$ is a lower bound for u. Consequently

Lemma 3.1. *Assume* $(F-1)$ *and* $(F-2)$. *All the solutions of* $(3.1)^+$ *blow up in the sense that* $u(r) \to \infty$ *at* R, *for some finite* R, *whenever the same is true for all the solutions of the corresponding one-dimensional Problem* $(2.1)^+$.

Consider now $(3.1)^-$. Since u is increasing, we have

$$u'' + \frac{N-1}{r} u' \geq \max\{f(u)\,, u'^{\,q}\}. \tag{3.6}$$

By well-known results ([4]), u blows up if f satisfies $(F-1)$ and $(F-2)$. If $q > 1$ holds instead of $(F-2)$, we argue as follows. From (3.6) we deduce that $u'(r) \geq k_0 r - k_1$ for $r \geq \rho$. Hence there exists r_1 such that $u'' \geq (1-\varepsilon)u'^q$ for $r \geq r_1$. Comparison with Lemma 2.4 shows that u blows up for some finite R. Note that in contrast to Problem $(3.1)^+$ where a bound for u yields a bound for u' [cf. (3.2)], u' may blow up even if u remains bounded. A discussion similar to that in Section 2 shows that for $q \leq 2$, u' is bounded whenever u is.

In summary we have

Lemma 3.2. *Assume* $(F-1)$ *and* $q \leq 2$. *If either* $(F-2)$ *or* $1 < q < 2$ *holds, then the solutions of* $(3.1)^-$ *blow up for some finite* R.

The asymptotic behaviour of the blow-up solutions is as in the one-dimensional case. Since $u' \geq 0$, $u'' \pm u'^q \leq f(u)$. If we make the transformation $u'^2 = v(u)$, we find

$$v' \pm 2v^{q/2} \leq 2f(u), \quad v(u_0) = 0. \tag{3.7}$$

By comparison with Problem $(2.2)^\pm_N$ we get an upper bound for $v(u)$.

A lower bound is obtained from (3.5) and (3.6). Combining the upper and lower bounds, we get

Theorem 3.3. *The statements of Theorem* 2.7 *remain valid for Problem* $(3.1)^\pm$.

Now consider the radial solutions of $\Delta u \pm |\nabla u|^q = f(u)$ in an annulus $\{r_0 - R < r < r_0\}$ which satisfy

$$u''(r) + \frac{N-1}{r} u'(r) \pm |u'(r)|^q = f(u(r)), \quad u(r_0) = 0, \quad u(r) \to \infty \text{ as } r \searrow r_0 - R. \tag{3.8$^\pm$}$$

In the following theorem we prove existence and illustrate the asymptotic behaviour of the solutions $u(r)$ of $(3.8)^\pm$.

Theorem 3.4. *Under the same assumptions as in Theorem* 2.7, *the Problems* $(3.8)^\pm$ *are solvable for all* $R \leq R'$. *The asymptotic behaviour of their solutions is the same as described in Theorem* 2.7, *except that in this case* $r \searrow r_0 - R$.

Proof. Let $w(r)$ be the solution of the one-dimensional problem

$$\begin{aligned} &w''(r) \pm |w'|^q = f(w), \quad r > 0 \\ &w(0) = 0, \quad w'(0) = v_0 > 0. \end{aligned}$$

It was shown in Section 2 that if R' is sufficiently small, it is possible to fix v_0 large enough so that $w(r) \to \infty$ as $r \to R'$. Let $R' < r_0$, and let $u(r)$, $r < r_0$, be the solution of the equation in $(3.8)^+$ $[(3.8)^-]$ which satisfies $u(r_0) = 0$, $u'(r_0) = -v_0$. Then $u'(r) < 0$ for $r < r_0$. Now set $\bar{u}(r) = u(r_0 - r)$, and observe that, if $r < R'$,

$$\bar{u}'' \pm |\bar{u}'(r)|^q > f(\bar{u}), \quad \bar{u}(0) = 0, \quad \bar{u}'(0) = v_0.$$

Hence by comparison

$$w(r) \leq \bar{u}(r). \tag{3.9}$$

Notice that if $u(r)$ satisfies the equation $(3.8)^+$ we obtain, multiplying both sides of the equation by $-2r^{2n-2}u'$,

$$-(r^{2n-2}{u'}^2)' \leq -2r^{2n-2}f(u)u'.$$

Integration between r and r_0 gives

$$|u'(r)| \leq \Big(\frac{r_0}{r}\Big)^{n-1}\sqrt{2F(u(r)) + v_0^2}. \tag{3.10}$$

Hence u' cannot blow up at a point where u is finite. The same is true if $u(r)$ satisfies the equations in $(3.8)^-$, as can be seen by a calculation similar to that at the beginning of Section 2. Hence by (3.9) $\bar{u}(r)$ blows up at some $R \leq R' < r_0$, whence $u(r) \to \infty$ as $r \searrow r_0 - R$. Moreover, $R \to 0$ as $v_0 \to \infty$ by Lemma 2.6.

To study the asymptotic behaviour we use the transformation $v(u) = {u'}^2(r(u))$. From the inequality

$$v'(u) \pm 2v^{q/2} > 2f(u), \quad u \geq 0$$

we easily get a lower bound for $v(u)$, by a comparison argument.

To complete the proof we need an upper bound for $v(u)$. Suppose first that $u(r)$ satisfies $(3.8)^+$. Then, by (3.4) and (3.10) we obtain, for $\varepsilon > 0$,

$$v'(u) + 2v^{q/2} < 2f(u)(1+\varepsilon), \quad u \geq u_\varepsilon$$

and so, by comparison, $v(u) \leq z(u)$, $u \geq u_\varepsilon$, where $z(u)$ satisfies

$$z'(u) + 2z^{q/2} = 2(1+\varepsilon)f(u), \quad z(u_\varepsilon) = v(u_\varepsilon).$$

This completes the study of $(3.8)^+$. If $u(r)$ is a solution of $(3.8)^-$ we can obtain the required upper bound for $v(u)$ by an argument essentially similar to that in the proofs of Lemma 2.3 and Lemma 2.4.

4. Asymptotic behaviour in general domains. In this section we shall study Problem $(P)^\pm$ in a general domain D as described in Section 1.

We describe the asymptotic behaviour of a solution, assuming that it exists. This follows immediately from the maximum principle and the results of the last section.

Indeed let $x \in D$ be close to the boundary, $P \in \partial D$ be the nearest point from x to the boundary and let B be a ball contained in D with $P \in \partial B$. If u_B is a solution of $(P)^{\pm}$ in B, then

$$u(x) \leq u_B(x). \tag{4.1}$$

On the other hand if $u_A(x)$ is a solution of $(3.8)^{\pm}$ in the annulus A whose inner boundary lies outside of D and contains P and which is so thin that $u_A \leq u$ on $\partial(A \cap D)$, then

$$u_A(x) \leq u(x). \tag{4.2}$$

(4.1) and (4.2) together with Theorem 2.7, Theorem 3.3 and Theorem 3.4 will provide the exact asymptotic behaviour of $u(x)$ as x tends to the boundary. For the clarity of the exposition we shall restate the results, distinguishing between Problem $(P)^+$ and $(P)^-$.

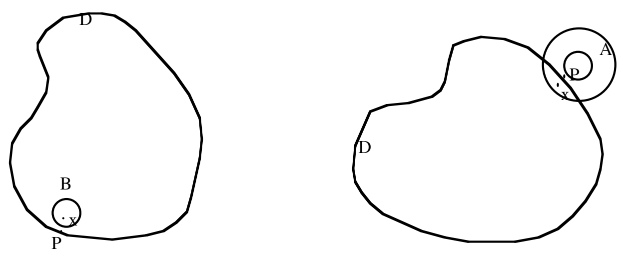

Figure 1

In the sequel we shall always assume $(F-1)$.

(i) Problem $(P)^+$. Denote by $\mathcal{A}$ the set of assumptions $\{(F-2), (F-4)\}$ and, correspondingly, let $\mathcal{B} = \{(F-3), (F-5)', (F-6)\}$. Put

$$\psi_0(u) = \int_u^\infty \frac{ds}{\sqrt{2F(s)}} \quad \text{and} \quad \phi_0 = \psi_0^{-1}.$$

Note that ϕ_0 is a solution to

$$\phi_0'' = f(\phi_0) \quad \text{in} \quad (0, x_0), \quad \phi_0(x) \to \infty \quad \text{as} \quad x \to 0. \tag{4.3}$$

Let

$$\psi_1(u) = \int_u^\infty \frac{ds}{f^{1/q}(s)} \quad \text{and} \quad \phi_1 = \psi_1^{-1},$$

where ϕ_1 is a solution of

$$\begin{aligned} &\phi_1' = -f^{1/q}(\phi_1) \quad \text{in} \quad (0, x_1), \\ &\phi_1(x) \to \infty \quad \text{as} \quad x \to 0. \end{aligned} \tag{4.4}$$

ϕ_0 (ϕ_1) exists if and only if $(F-2)$ ($(F-3)$) holds.

With this notation we can express the asymptotic behaviour of u as follows.

Theorem 4.1.

(i) *Assume* $\mathcal{A}$ *and let* ψ_0 *satisfy* $(F-7)$. *Then any solution of* $(P)^+$ *satisfies* $u(x)/\phi_0(\delta(x)) \to 1$ *uniformly as* $x \to \partial D$.

(ii) *Assume* $\mathcal{B}$ *and let* ψ_1 *satisfy* $(F-7)$. *Then we have* $u(x)/\phi_1(\delta(x)) \to 1$ *uniformly as* $x \to \partial D$.

(ii) Problem $(P)^-$. Denote by $\mathcal{C}$ the set of assumptions $\{(F-5),\ q > 1\}$. The results of Theorem 4.1 are also valid for Problem $(P)^-$ in a slightly modified version, given in

Theorem 4.2. *Let* $u(x)$ *be a solution of* $(P)^-$ *with* $q < 2$.

(i) *If* $\mathcal{A}$ *holds and if* ψ_0 *satisfies* $(F-7)$, *then* $u(x)/\phi_0(\delta(x)) \to 1$ *uniformly as* $x \to \partial D$.

(ii) *Under the conditions* $\mathcal{C}$ *we have* $(2-q)u(x)\{(q-1)\delta(x)\}^{(2-q)/(q-1)} \to 1$ *uniformly as* $x \to \partial D$.

5. Existence. In this section we shall establish the existence of a solution of $(P)^\pm$ by means of a classical Perron process. For this purpose consider the Dirichlet problem

$$\begin{aligned} \Delta u_n \pm |\nabla u_n|^q &= f(u_n) \quad \text{on } D \\ u_n &= n \quad \text{on } \partial D,\ n \in \mathbf{N}. \end{aligned} \tag{5.1$^\pm$}$$

If $q \le 2$ the existence of a solution u_n can be proved by means of the method of upper and lower solutions ([1, 8]). Indeed $\overline{u}_n = n$ is an upper solution. If we impose on f the additional condition

$$\begin{aligned} &\lim_{t \to t_0} f(t) = 0 \quad \text{for } t_0 \in [-\infty, 0], \\ &f(t) \text{ and } f'(t) \text{ are positive and continuous for } t > t_0, \end{aligned} \tag{$F-8$}$$

then $\underline{u}_n = ar^2 - M$ is for suitable a and M a lower solution. The constant M can always be chosen so large that $\underline{u}_n \le \overline{u}_n$. Hence $(5.1)^\pm$ has a solution $\underline{u}_n \le u_n \le \overline{u}_n$. By $(F-8)$ it is unique. Moreover,

$$u_n(x) \le u_{n+1}(x). \tag{5.2}$$

Suppose that for all $n \in \mathbf{N}$, $u_n(x) \le C_0$ in $\overline{\Omega} \subset D$. Then by [8, Theorem 6.5] $u_n \in W^{2,m}(\Omega)$ for $m > N$. Actually $u_n \in C^{2,\alpha}(\Omega) \cap W^{2,m}(\Omega)$ because of the classical regularity results for elliptic equations. By standard arguments if $q < 2$, $\lim_{n\to\infty} u_n(x) = U(x)$ in Ω, where $U(x)$ is a solution of $\Delta U \pm |\nabla U|^q = f(U)$ in Ω. If this holds for any compact subdomain $\overline{\Omega} \subset D$, $U(x)$ is a solution of $(P)^\pm$.

Remark. The solution $U(x)$ is the smallest solution of $(P)^\pm$.

Theorem 5.1. *Assume* $0 < q < 2$ *and* $(F-8)$.

(i) *If* $\mathcal{A}$ *holds, then* $(P)^+$ *has a solution.*

(ii) *If* $\mathcal{B}$ *holds, then* $(P)^+$ *has a solution.*

(iii) *Under the assumption* $(F-2)$ *or* $q > 1$, $(P)^-$ *has a solution.*

Proof. In view of the previous observations it suffices to construct an upper bound near the boundary. This is done with the solutions of $(P)^{\pm}$ in the balls contained in D.

Let us now consider the cases where $q \geq 2$. If $q > 2$, it was shown in Section 1 that, in the one-dimensional case, Problem $(P)^{-}$ can never have a solution for positive f. However it is readily shown that it is true in general. We therefore restrict ourselves to Problem $(P)^{+}$. Let us first establish the existence of a solution to $(5.1)^{+}$ for large n. For this purpose let us consider the perturbed problem

$$\begin{aligned} \Delta\tilde{u}_n + \chi(|\nabla\tilde{u}_n|^2) &= f(\tilde{u}_n) \quad \text{in } D \\ \tilde{u}_n &= n \qquad\quad\; \text{in } \partial D, \end{aligned} \tag{$\widetilde{5.1}$}$$

where $\chi(s)$ is a positive, continuously differentiable, increasing function such that $\chi(s) = s^{q/2}$ for $s \leq M$ and $\overline{\lim}_{s\to\infty}\chi(s)/s = \gamma$. M is a large number which will be determined later. As for $q \leq 2$, we can show that under the condition $(F-8)$ there exists a solution $\tilde{u}_n$.

Lemma 5.2. *$|\nabla\tilde{u}_n|^2$ assumes its maximum at the boundary ∂D.*

Proof. Let $\phi = |\nabla\tilde{u}_n|^2$. A straightforward calculation yields

$$\Delta\phi = 2\sum_{i,k}\Big(\frac{\partial^2\tilde{u}_n}{\partial x_i\partial x_k}\Big)^2 + 2f'(\tilde{u}_n)\phi - 2\chi'\cdot(\nabla\phi,\nabla\tilde{u}_n).$$

The assertion now follows from the maximum principle.

Let $P \in \partial D$ be a point where $|\nabla\tilde{u}_n|$ attains its maximum. Keep in mind that also $\tilde{u}_n$ is maximal on ∂D. Consider now an annulus A whose inner boundary lies outside of D and touches ∂D at P.

Let v be a solution of $\Delta v + |\nabla v|^q = f(v)$ in A, $v = 0$ on the outer and $v = n$ on the inner boundary. By Theorem 3.4 we have on the inner boundary

$$|\nabla v|^2 \leq 2(1+\varepsilon)F(n)$$

for $n \geq n(\varepsilon)$. Hence v is a solution of $\Delta v + \chi(|\nabla v|^2) = f(v)$ if $M > 2(1+\varepsilon)F(n)$.

By comparison $|\nabla\tilde{u}_n|^2 \leq |\nabla v|^2 < M$ at P. Whence this estimate together with Lemma 5.2 implies that $\tilde{u}_n$ is the unique solution of $(5.1)^{+}$.

An upper bound for all $\tilde{u}_n$ in compact subdomains is given by the solution of $(P)^{+}$ in balls contained in D. As in Theorem 5.1 we can prove the existence of a solution to $(P)^{+}$.

Theorem 5.3. *The assertions* (i) and (ii) *of Theorem* 5.2 *are valid also for $q > 2$.*

Proof. As observed before there exists for sufficiently large $n \in \mathbf{N}$ a solution u_n of $(5.1)^{+}$. For all $\overline{\Omega} \subset D$, $\{u_n\}$ is uniformly bounded. Let h_n be the solution of $\Delta h_n = f(u_n)$ in Ω, $h_n = u_n$ on $\partial\Omega$. Then $\Delta(u_n - h_n) + |\nabla u_n|^q = 0$ in Ω. By the maximum principle $u_n \geq h_n$ in Ω. Let $\Omega \subset\subset \Omega^* \subset\subset D$ and let k_n be defined as h_n with Ω replaced by $\Omega^* \setminus \Omega$. By the maximum principle $u_n \geq k_n$. By Lemma

5.2, $|\nabla u_n|$ takes its maximum on $\partial\Omega$, say at P. Hence either $|\nabla u_n(P)| \leq |\nabla h_n(P)|$ or $|\nabla u_n(P)| \leq |\nabla k_n(P)|$. The classical regularity results imply that $|\nabla h_n| \leq C$ in $\Omega' \subset\subset \Omega$ for all n. Here C depends on the bound for u_n in Ω and on dist (Ω', Ω). A similar result holds for k_n. From here we deduce that $u = \lim_{n\to\infty} u_n$ belongs to $W^{1,r}(\Omega')$ for all $r > 0$. The standard arguments yield the result.

Remark. In order to discuss the case $q = 2$ which is not included in either of the Theorems 5.1 and 5.3 we suggest making the transformation given in Section 1 and studying Problem (1.4) (cf. also [13]).

6. Additional remarks. 1) Many assumptions can be weakened. Since we are mainly interested in the behaviour of the solutions for large values it suffices that $f(t)$ is increasing for large values of t.

2) The Laplacian can be replaced by a uniformly second-order elliptic operator ([5]).

3) In some special cases the techniques of [3] and [5] can be used to determine the asymptotic behaviour of the derivatives.

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