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ASYMPTOTIC STABILITY FOR THE DIFFERENTIAL SYSTEM $u'' + \sigma(t)|u|^\alpha|u'|^\beta u' + f(u) = 0$

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1. Introduction. We shall be concerned with the asymptotic stability of the rest state of the quasi-variational system

$$(\nabla\mathcal{L}(t,u,u'))' - \nabla_u\mathcal{L}(t,u,u') = Q(t,u,u'), \qquad J = [T,\infty), \tag{1.1}$$

where $u : J \to \mathbb{R}^N$ and $\mathcal{L}(t,u,p) = G(u,p) - F(t,u)$ and where

$$G \in C^1(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}), \quad F \in C^1(J \times \mathbb{R}^N;\mathbb{R}), \quad Q \in C(J \times \mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}^N).$$

We assume here that the nonlinear damping magnitude $|Q|$ is controlled from below by a function of the natural form $\sigma(t)\,\phi(u,p)$. The case when

$$\phi(u,p) > 0 \qquad \text{for} \;\; p \neq 0 \tag{1.2}$$

has been treated extensively in the literature; see for example the work of Arstein and Infante, Burton, Levin and Nohel, Pucci and Serrin, Salvadori, Smith, Thurston and Wong.

In this paper we study the more general situation in which

$$\phi(u,p) > 0 \quad \text{for} \;\; u \neq 0, \;\; p \neq 0. \tag{1.3}$$

This is a relatively new case. When $G(p) = |p|^2/2$, and *only in the scalar case*, Yoshizawa ([17]) obtained some results when the function σ is bounded from zero and from above, while Ballieu and Peiffer in [2] studied the case $\phi(u,p) = \tau(u)|p|^\beta$ and $\sigma(t) \equiv 1$. More recently Pucci and Serrin ([13]) have extended and generalized these studies to quasi-variational *systems* of the general form (1.1). In their work the function σ is assumed to satisfy a mean integral condition of the type introduced by Hatvani ([4]). Although this is a rather mild assumption, since it allows σ to have the behavior $\liminf_{t\to\infty}\sigma(t) = 0$, it is not satisfied, however, by functions σ for which $\lim_{t\to\infty}\sigma(t) = 0$. Also, in the vectorial case their condition (V_1), namely

(V_1) *For all $U > 0$ and $p_0 > 0$ there is a nonnegative measurable function $h \notin L^1(J)$ such that*

$$|(Q(t,u,p),p)| \geq h(t) \quad \textit{for } t \in J,\ |u| \leq U,\ |p| \geq p_0,$$

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does not allow the possibility $Q(t,0,p)=0$. Thus nonlinear damping terms Q of the natural form $Q(t,u,p)=-\sigma(t)|u|^{\alpha}|p|^{\beta}p$, as in the title of the paper, or, more generally, of the form

$$Q(t,u,p)=-\sigma(t)\tau(u)\phi(p), \quad \text{where } \tau(0)=0, \tag{1.4}$$

are not covered in [13], either in the vectorial case (when $N>1$), or when $\sigma(t)\to 0$ as $t\to\infty$; see the discussion at the end of Section 2.

The purpose of this paper is to study nonlinear damping terms Q for which possibly $Q(t,0,p)=0$, as in (1.4), and for which the Hatvani condition on σ fails. The set of hypotheses and the techniques employed here are closely related to those of [12] rather than [13], since all the demonstrations in [13] strongly rely on the Hatvani condition and on assumption (V_1), which are not used here (except in Theorem 6).

We will show that all the results obtained in [12] for the case (1.2) can be extended to the new case (1.3), under some additional *natural* hypotheses on the functions G and F, which are always satisfied when G is independent of u and F independent of t.

Our assumptions are easy to verify in applications. Moreover the function σ is now neither controlled from below nor from above, and the distinction between the scalar and vectorial case no longer holds.

In the second part of this paper we study the case when the function G has a more specialized structure, of the type possessed for example by $G(p)=|p|^m/m$, $m>1$. In this context we obtain some results which are closely related to Theorems 2 and 3 of [13]. In particular, in Theorem 6 we use the Hatvani condition on σ but neither assumption (V_1) of [13] nor condition (V_2), which appears slightly complicated although it is almost always satisfied in applications.

To describe our results, we consider first the important special case in which

$$\mathcal{L}(t,u,p)=|p|^2/2-F(u) \quad \text{and} \quad Q(t,u,p)=-A(t,u,p)p,$$

where A is a continuous $N\times N$ nonnegative definite matrix. Then (1.1) takes the simple form

$$u''+A(t,u,u')u'+f(u)=0. \tag{1.5}$$

Here $f=\nabla_u F$ and

$$(f(u),u)>0 \quad \text{for } u\neq 0.$$

For this system we suppose that for every $U>0$ there exist two measurable functions σ, $\delta: J\to[0,\infty)$ and a continuous function $\phi:\mathbb{R}^N\times\mathbb{R}^N\to[0,\infty)$, with $\phi(u,p)>0$ for $u\neq 0$, $p\neq 0$, such that

$$(A(t,u,p)p,p)\geq \text{Pos. Const.}\,|A(t,u,p)p|\cdot|p|$$

and

$$\sigma(t)\phi(u,p)\leq|A(t,u,p)p|\leq\delta(t)|p|$$

for all $t\in J$, $|u|\leq U$ and $p\in\mathbb{R}^N$.

The following conclusions then hold for the system (1.5) as immediate consequences of our main results.

Theorem A. (i) *Suppose $\sigma\delta$ is bounded on J and that $\sigma \notin L^1(J)$. If σ is absolutely continuous and of bounded variation or if $\log\sigma$ is uniformly Lipschitz continuous, then the rest state of* (1.5) *is a global attractor.*

(ii) *Suppose that $1/\sigma\delta$ is bounded on J and that $1/\delta \notin L^1(J)$. If $1/\delta$ is absolutely continuous and of bounded variation, or if $\phi(u,p) = |u|\cdot|p|$ and $|(1/\delta)'| \le$ Const. $\sqrt{\sigma/\delta}$, then the rest state of* (1.5) *is a global attractor.*

Theorem B. *The rest state of* (1.5) *is a global attractor if one of the following conditions* (a), (b), (c) *is satisfied.*

(a) *There exist a positive number c and a number $a \in [0,1]$ such that*

$$t^a\,\sigma(t) \ge c, \qquad \int_T^t \frac{\delta(s)}{s^{2a}}\,ds \le \text{Const.}\begin{cases} t^{1-a} & \textit{for } a<1 \\ \log t & \textit{for } a=1. \end{cases}$$

(b) *There exists a nonnegative absolutely continuous function $k = k(t)$ of bounded variation on J such that*

$$k \notin L^1(J), \qquad k(t) \le \text{Const.}\,\sigma(t) \quad \textit{in } J, \tag{1.6}$$

and

$$(i) \quad \delta\,k \in L^\infty(J) \qquad or \qquad (ii) \quad \delta\,k^2 \in L^1(J).$$

(c) $\phi(u,p) = |u|\cdot|p|$. *There exists a nonnegative bounded absolutely continuous function $k = k(t)$ on J satisfying* (1.6) *and (i) or (ii) such that*

$$|k'| \le \text{Const.}\sqrt{k\,\sigma} \quad \textit{a.e. in } J.$$

Theorem A follows from Theorem B with $k = \sigma$ in the first part and $k = 1/\delta$ in the second. Theorem B(a)–(b) comes from Theorem 4 in Section 8, with $\mu = 1$ (and $k(t) = 1/t^a$ in part (a)), while Theorem B(c) follows from Theorem 5 in Section 8 with $\lambda = 1/2$, $m = 2$ and $\alpha = \nu = 1$.

Theorems A and B extend the analogous theorems in Section 1 of [12], where $\phi(u,p) = |p|$.

The conditions obtained in Theorems A and B are rather sharp. A simple example is given by the scalar equation

$$u'' + \delta(t)\,|u|\,u' + u = 0, \qquad \delta(t) \ge 0. \tag{1.7}$$

It is easy to see that any of the conditions (a)–(c) of Theorem B is satisfied if

$$\sigma(t) = \text{Pos. Const.} \le \delta(t) \le \text{Const.}\,t \quad \text{and} \quad k(t) = 1/t,$$

but they fail when $\delta(t) \ge t^{1+\epsilon}$ for some $\epsilon > 0$; while there are equations with $\delta(t) \ge t^{1+\epsilon}$ for which the rest state is not stable. Indeed when $T > 0$ and

$$\delta(t) = \epsilon^{-1}t^{1+\epsilon} + (\epsilon+1)\left(t + t^{1-\epsilon}\right)^{-1}$$

then equation (1.7) admits the solution $u(t) = 1 + t^{-\varepsilon}$ which is bounded but does not tend to zero. Thus if δ is sufficiently large, due to overdamping, the rest state loses asymptotic stability.

On the other hand if δ is small, say $\delta \in L^1(J)$, then again there are solutions of (1.7) which are bounded but do not tend to zero as $t \to \infty$. To see this assume, without loss of generality, that

$$\int_T^\infty \delta(s)\, ds \leq \tfrac{1}{4} \tag{1.8}$$

(if this is not the case we can make the change of variable $v = u/4||\delta||_{L^1(J)}$), and then consider the initial value problem $|u'(T)|^2 + |u(T)|^2 = 1$. If we now multiply (1.7) by u' and integrate we obtain

$$|u'(t)|^2 + |u(t)|^2 = -2\int_T^t \delta(s)\,|u(s)| \cdot |u'(s)|^2 ds + 1,$$

from which we immediately derive, also by (1.8), that

$$\tfrac{1}{2} \leq |u'(t)|^2 + |u(t)|^2 \leq 1 \quad \text{for all } t \geq T.$$

The following section contains our basic assumptions. In Section 3 we present the main conclusions of the paper and in Sections 4–7 their proofs. Finally in Section 8 some generalizations of the previous results are given.

2. Principal hypotheses. We consider vector solutions $u = (u_1, \dots, u_N)$ of the quasi-variational ordinary differential system

$$(\nabla G(u, u'))' - \nabla_u G(u, u') + f(t, u) = Q(t, u, u'), \quad J = [T, \infty), \tag{2.1}$$

where ∇ denotes the gradient operator with respect to the variable p and

$$f(t, u) = \nabla_u F(t, u).$$

Throughout the paper we suppose that

$$G \in C^1(\mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}), \quad F \in C^1(J \times \mathbb{R}^N; \mathbb{R}), \quad Q \in C(J \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N),$$

and also that the following conditions hold:

(H_1) $G(u, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $u \in \mathbb{R}^N$, with $G(u, 0) = 0$ and $\nabla G(u, 0) = 0$. There exist u_1, $p_1 > 0$ and $b \in (0, 1)$ such that

$$-(\nabla_u G(u, p), u) \leq b\,(\nabla G(u, p), p) \quad \text{for all } |u| \leq u_1 \text{ and } |p| \geq p_1. \tag{2.2}$$

(H_2) $F(t, 0) = 0$. For all u_0, U, with $0 < u_0 \leq U$, there exist a nonnegative function $\psi \in L^1(J)$ and a constant $\kappa_0 > 0$ such that

$$F_t(t, u) \leq \psi(t) \quad \text{when } t \in J \text{ (a.e.)} \quad \text{and} \quad |u| \leq U. \tag{2.3}$$

$$(f(t, u), u) \geq \kappa_0 \quad \text{when } t \in J \quad \text{and} \quad |u| \in [u_0, U]. \tag{2.4}$$

(H_3) $\lim_{u \to 0}(f(t, u), u) = 0$ uniformly for all $t \in J$.

(H_4) $(Q(t, u, p), p) \leq 0$ for all $t \in J$, $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^N$.

Condition (2.2) is satisfied automatically in the important case when G is independent of u. It also holds for functions G of the form $G(u,p)=\varphi(u)\,\bar{G}(p)$, where $\varphi(u)>0$ in $\mathbb{R}^N$ and $\bar{G}$ satisfies the convexity requirements of (H_1).

Property (H_3) is implied by the *natural* condition

$$|f(t,u)| \le \text{Const.} \quad \text{for all } t\in J \text{ and all sufficiently small } u\in\mathbb{R}^N. \tag{2.5}$$

When F is of the form (see [12])

$$F(t,u)=\ell(t)\,h(u), \quad \text{with } \ell : J\to(0,\infty) \text{ and } h:\mathbb{R}^N\to[0,\infty), \tag{2.6}$$

then (2.3) is equivalent to $(\ell')^+\in L^1(J)$, while (2.4) is satisfied if and only if

$$\ell(t)\ge \text{Const.}>0 \qquad \text{and} \qquad (\nabla_u h(u),u)>0 \quad \text{for } u\ne 0.$$

Since

$$0<\ell(t)\le \ell(T)+\int_T^\infty (\ell')^+(s)\,ds<\infty$$

property (H_3) follows.

Conditions (2.4) and (H_4) imply that $f(t,0)=Q(t,u,0)=0$. Since $\nabla_u G(u,0)=\nabla G(u,0)=0$ by (H_1), it is clear that the rest state $u\equiv 0$ is a solution of (2.1). We say that this state is a *global attractor* if for any *bounded* solution u we have

$$u(t),\ u'(t)\to\ 0 \quad \text{as } t\to\infty.$$

The following conditions on the nonlinear damping term Q are also assumed.

(A_1) For every $U>0$ there exist a nonnegative damping function $\sigma : J\to\mathbb{R}$ and a continuous function $\phi:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ such that

$$|(Q(t,u,p),p)|\ge\sigma(t)\phi(u,p)|p| \quad \text{for } t\in J,\ |u|\le U \text{ and } p\in\mathbb{R}^N. \tag{2.7}$$

Moreover we assume that the function ϕ satisfies the property

$$\inf\{\phi(u,p) : |u|\in[u_0,U],\ |p|\ge p_0\}>0 \tag{2.8}$$

for all u_0 with $0<u_0\le U$ and all $p_0>0$.

(A_2) For all $U>0$ there exist a measurable control set $I\subset J$, a nonnegative damping function $\delta : I\to\mathbb{R}$ and a continuous weight function $\rho : [0,\infty)\to[0,\infty)$, with $\rho(0)=0$, such that

$$(Q(t,u,p),u)\le\delta(t)\rho(|p|)$$

for $t\in I$, $|u|\le U$ and all sufficiently small $p\in\mathbb{R}^N$.

Finally, we assume a *tameness* condition on Q; that is,

(A_3) For every $U>0$, there exists a positive constant $\gamma\ge 1$ such that

$$|Q(t,u,p)|\cdot|p|\le\gamma\left|(Q(t,u,p),p)\right|$$

when $t\in I$, $|u|\le U$ and $p\in\mathbb{R}^N$.

When the function ϕ has the form $\phi(u,p) = \tau(u)\phi_1(|p|)$, with $\tau : \mathbb{R}^N \to [0,\infty)$ and $\phi_1 : [0,\infty) \to [0,\infty)$, then condition (2.8) is satisfied if we assume that $\tau(u) > 0$ for $u \neq 0$ and that ϕ_1 is increasing. In the particular case when $\tau(u) \equiv 1$ conditions (A_1)–(A_3) are due to Pucci and Serrin ([12]).

Condition (A_3) holds automatically when $N = 1$ or whenever Q and $-p$ have the same direction.

The conditions (A_1)–(A_3) are easily verified for the canonical example in the title, namely

$$u'' + \sigma(t)|u|^\alpha |u'|^\beta u' + f(u) = 0, \quad \text{with } \alpha > 0, \quad \beta > -1,$$

since here $Q(t,u,p) = -\sigma(t)|u|^\alpha |p|^\beta p$ and we can take $\phi(u,p) = |u|^\alpha |p|^{\beta+1}$, $\delta(t) = \sigma(t)$, $\rho(s) = U^{\alpha+1} s^{\beta+1}$ and $\gamma = 1$.

3. Main results. In this section we present our principal stability results. From now on we assume that conditions (H_1)–(H_4) and (A_1)–(A_3) hold. In stating the theorems we agree that the function $\delta\, k$ is extended to all of J by the definition $\delta(t)\, k(t) = 0$ for $t \in J \backslash I$. Also we denote by $H(u,\cdot)$ the Legendre transform of $G(u,\cdot)$, namely

$$H(u,p) = (\nabla G(u,p), p) - G(u,p). \tag{3.1}$$

Theorem 1. *Suppose that for every $U > 0$ there exists a bounded absolutely continuous function k on J such that*

$$k \notin L^1(J), \qquad k = 0 \quad \textit{on } J \backslash I, \tag{3.2}$$

$$k \in \mathrm{BV}(J) \qquad \textit{or} \quad \log k \in \mathrm{Lip}\,(J) \tag{3.3}$$

and

$$0 \le k(t) \le \theta\, \sigma(t) \qquad \textit{in } J, \tag{3.4}$$

for some positive constant θ. Assume also that

$$\liminf_{t\to\infty} \int_T^t \delta(s)\, k(s)\, ds \Big/ \int_T^t k(s)\, ds < \infty. \tag{3.5}$$

Finally we suppose that for every $U > 0$ there exist two nonnegative constants b_1, p_1 such that

$$(\nabla_u G(u,p), u) \le b_1\, |\nabla G(u,p)| \cdot |p| \quad \textit{for all } |u| \le U \textit{ and } |p| \ge p_1. \tag{3.6}$$

Then the rest state of (2.1) *is a global attractor.*

A condition of the type (3.6) first appears in [12]. Note that it is satisfied automatically in the important case when G is independent of u, and in particular for the canonical actions $G(p) = |p|^m/m$, $m > 1$, and $G(p) = \sqrt{1+|p|^2} - 1$. Condition (3.6) also holds for functions G of the form $G(u,p) = \varphi(u)\, \bar{G}(p)$, where $\varphi(u) > 0$ in $\mathbb{R}^N$ and $\bar{G}$ satisfies the convexity requirements of (H_1).

Condition (3.5) is satisfied if δ is bounded, while it fails if $\delta(t) \to \infty$ as $t \to \infty$. On the other hand there are equations with δ tending to ∞ as $t \to \infty$ for which the rest state is not a global attractor. Consider for example the equation

$$u'' + \delta(t)\,\rho(u') + u = 0, \quad J = [1,\infty),$$

where

$$\delta(t) = \log t^2\big(1 + \frac{1}{t} + \frac{2}{t^3}\big) \quad \text{and} \quad \rho(p) = \begin{cases} -p/|p|\log|p| & \text{for } 0 < |p| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This equation admits the solution $u(t) = 1 + t^{-1}$ which is bounded but does not tend to zero as $t \to \infty$.

Theorem 2. *Theorem* 1 *remains true if in place of condition* (3.6) *we assume that*

$$H(u,p) \to \infty \qquad \text{as } |p| \to \infty \text{ uniformly for } u \text{ in compact sets.} \tag{3.7}$$

Under the hypotheses of Theorem 2 condition (2.8) can be weakened to

$$\phi(u,p) > 0 \quad \text{for } u \neq 0,\, p \neq 0, \tag{2.8$'$}$$

while (2.2) can be dropped. See the remark at the end of Lemma 5.1 below.

Theorem 3. *Suppose that for every* $U > 0$ *there is a number* $D > 0$ *such that*

$$|f(t,u)| \le D \qquad \text{for all } t \in J \text{ and } |u| \le U. \tag{3.8}$$

Then Theorem 1 *continues to hold if we drop condition* (3.6) *and in place of* (3.3) *and* (3.5) *we assume that*

$$\liminf_{t\to\infty} \int_T^t \{\delta(s)\,k(s) + |k'(s)|\}\,ds \Big/ \int_T^t k(s)\,ds < \infty. \tag{3.9}$$

Remarks. Condition (3.8) is stronger than (2.5) and immediately implies (H_3). When F has the form (2.6) then (3.8) is automatically satisfied since we have already seen in Section 2 that ℓ is bounded.

When $k \in BV(J)$ then $k' \in L^1(J)$, while when $\log k \in \mathrm{Lip}(J)$ then $|k'| \le \mathrm{Const.}\,k$. Therefore in both cases (using also $(3.2)_1$) condition (3.9) is equivalent to (3.5).

Note that if (3.5) holds, condition (3.9) is implied by the easier condition

$$|k'| \le \mathrm{Const.}\ \delta\,k \qquad \text{in } I, \tag{3.10}$$

which is weaker then $(3.3)_2$.

Theorem 1 is closely related to Theorems 3.1 and 3.2 of [12], where (1.2) was assumed rather than the weaker condition (1.3), while on the other hand (2.2) and (H_3) were not used. Theorems 2 and 3 are new.

The corollaries in Section 3 of [12] continue to hold in the present context.

4. Preliminary lemmas. In the sequel we shall use the following lemmas, which stem from conditions (H_1)–(H_3), (A_1)–(A_3) and from the hypotheses of Theorems 1–3.

Lemma 4.1 (cf. Lemma 4.1 of [13]). *For every $U > 0$ and $p_0 > 0$*

$$\inf\{(\nabla G(u,p),p)/|p| : |u| \le U,\ |p| \ge p_0\} = g_0(U,p_0) > 0. \tag{4.1}$$

Lemma 4.2. *Let $\mathcal{G}(t,u,p) = (\nabla_u G(u,p),u) + (\nabla G(u,p),p) - (f(t,u),u)$. Then*

(i) *for every $p_0 > 0$ there exist two positive numbers κ_1, u_2 such that*

$$\mathcal{G}(t,u,p) \ge \kappa_1 \qquad \text{for } t \in J,\ |u| \le u_2 \text{ and } |p| \ge p_0;$$

(ii) *for every u_0, U with $0 < u_0 \le U$ there exist two positive numbers κ_2, p_1 such that*

$$\mathcal{G}(t,u,p) \le -\kappa_2 \qquad \text{for } t \in J,\ |u| \in [u_0,U] \text{ and } |p| \le p_1;$$

(iii) *if (3.6) holds, then for every $U > 0$ and $p_0 > 0$ there exists a positive constant κ_3 such that*

$$\mathcal{G}(t,u,p) \le \kappa_3 |\nabla G(u,p)| \cdot |p| \qquad \text{for } t \in J,\ |u| \le U \text{ and } |p| \ge p_0.$$

Proof. (i) By the previous lemma there is a constant $g_0 = g_0(p_0) > 0$ such that

$$(\nabla G(u,p),p) \ge p_0 g_0 \qquad \text{when } |u| \le 1 \text{ and } |p| \ge p_0, \tag{4.2}$$

while by (2.2) there exist three numbers b, $u_1 \in (0,1)$ and $p_1 > 0$, such that

$$(\nabla_u G(u,p),u) \ge -b\,(\nabla G(u,p),p) \quad \text{when } |u| \le u_1 \text{ and } |p| \ge p_1. \tag{4.3}$$

Without loss of generality we can assume $p_1 > p_0$. Let

$$K_1 = \max\{|\nabla_u G(u,p)| : |u| \le 1,\ |p| \in [p_0,p_1]\}.$$

If we take $u_2 = \min\{u_1, p_0 g_0 b/K_1\}$ then by (4.2) and (4.3) we conclude

$$(\nabla_u G(u,p),u) \ge -b\,(\nabla G(u,p),p) \quad \text{when } |u| \le u_2 \text{ and } |p| \ge p_0. \tag{4.4}$$

Taking u_2 even smaller, if necessary, and using (H_3) gives

$$-(f(t,u),u) \ge -\tfrac{1}{2}(1-b)\,p_0 g_0 \quad \text{when } t \in J \text{ and } |u| \le u_2. \tag{4.5}$$

Combining (4.2), (4.4)–(4.5) and taking $\kappa_1 = \frac{1}{2}(1-b)\,p_0 g_0$ yields the desired conclusion.

(ii) By (2.4) there is a constant $\kappa = \kappa(u_0,U) > 0$ such that

$$-(f(t,u),u) \le -\kappa \quad \text{when } t \in J \text{ and } |u| \in [u_0,U]. \tag{4.6}$$

Since $\nabla_u G(u,0) = \nabla G(u,0) = 0$, it follows that if $|p|$ is small enough, say $|p| \le p_1$, then

$$(\nabla_u G(u,p), u) + (\nabla G(u,p), p) \le \tfrac{1}{2}\,\kappa \qquad \text{when } |u| \le U \text{ and } |p| \le p_1. \tag{4.7}$$

(ii) now follows from (4.6)–(4.7) with $\kappa_2 = \frac{1}{2}\,\kappa$.

(iii) By (3.6) there exist two nonnegative constants b_1, p_1 such that

$$(\nabla_u G(u,p), u) \le b_1 |\nabla G(u,p)| \cdot |p| \quad \text{for all } |u| \le U \text{ and } |p| \ge p_1. \tag{4.8}$$

Let $p_0 > 0$ be given. We can assume $p_1 > p_0$, for otherwise (iii) is immediate. Let

$$K_2 = \max\left\{(\nabla_u G(u,p), u) : |u| \le U,\ |p| \in [p_0, p_1]\right\}.$$

Then clearly, also by (4.2),

$$(\nabla_u G(u,p), u) \le K_2 \le \hat{b}|\nabla G(u,p)| \cdot |p| \quad \text{for all } |u| \le U \text{ and } |p| \in [p_0, p_1], \tag{4.9}$$

where $\hat{b} = \max\{b_1, K_2/p_0 g_0\}$. It now follows from (4.8)–(4.9) that when $|u| \le U$ and $|p| \ge p_0$

$$(\nabla_u G(u,p), u) + (\nabla G(u,p), p) \le (1 + \hat{b})\,|\nabla G(u,p)| \cdot |p|.$$

Since $(f(t,u), u) \ge 0$ by (2.4), we can take $\kappa_3 = 1 + \hat{b}$.

Lemma 4.3. *Let $E = E_{u_0,p_0} = \{(u,p) \in \mathbb{R}^N \times \mathbb{R}^N : |u| \ge u_0,\ |p| \ge p_0\}$ for any fixed $u_0 \in (0, U]$ and $p_0 > 0$. Then*

$$k(t)\,|p| \le \text{Const.}\ |(Q(t,u,p), p)| \quad \textit{for } (t,u,p) \in J \times E,$$

where the constant depends on u_0, p_0 and U.

Proof. By (2.8)

$$\phi_0 = \inf\{\phi(u,p) : |u| \in [u_0, U], |p| \ge p_0\} > 0.$$

Therefore by (3.4) and (2.7) we have

$$k(t)\,|p| \le \frac{\sigma(t)}{\theta} \cdot \frac{\phi(u,p)}{\phi_0}\,|p| \quad \le (\theta\,\phi_0)^{-1} |(Q(t,u,p), p)| \quad \text{for } (t,u,p) \in J \times E.$$

5. Preliminary lemmas, II. In what follows, we consider (weak) solutions of (2.1) on J, namely vector functions $u : J \to \mathbb{R}^N$ of class C^1 for which

$$\nabla G(u(t), u'(t)) \in C^1(J; \mathbb{R}^N),$$

and which satisfy the system (2.1). From now on we denote by U (> 0) any fixed upper bound for the set $\{|u(t)| : t \in J\}$.

Lemma 5.1. (i) *There exists $\ell \geq 0$ such that*

$$H(u,u') + F(t,u) \to \ell \qquad as \quad t \to \infty;$$

(ii) $|\nabla G(u,u')| \leq C$ *for* $t \in J$ *and for some positive constant* C*;*
(iii) $(Q(t,u,u'),u') \in L^1(J)$.

Remark. In the case of Theorem 2, it follows that $H(u(t),u'(t))$ is bounded on J, by virtue of Lemma 5.1(i) and the fact that $F(t,u(t)) \geq 0$ by (H_2). Then by condition (3.7) this implies that u' is bounded on J, say $|u'(t)| \leq K$ for $t \in J$. Therefore, in this case *one can restrict consideration to compact subsets of vectors* (u,p) *in* $\mathbb{R}^N \times \mathbb{R}^N$. In particular, it is enough to assume the validity of (2.7) and (A_3) only for (u,p) in compact sets, while conditions (2.2) and (3.6) become irrelevant. Thus Theorem 2 can be treated as a special case of Theorem 1. Finally for Theorem 2, the set E in Lemma 4.3 reduces to

$$E = \{(u,p) \in \mathbb{R}^N \times \mathbb{R}^N : |u| \in [u_0, U],\ |p| \in [p_0, K]\}$$

and thus we can use condition $(2.8)'$ instead of (2.8) to conclude that

$$\phi_0 = \min\{\phi(u,p) : |u| \in [u_0,U],\ |p| \in [p_0,K]\} > 0.$$

Lemma 5.2. (i) *If* $\ell = 0$ *in Lemma* 5.1(i), *then* $u(t),\ u'(t) \to 0$ *as* $t \to \infty$*;*
(ii) *if* $\ell > 0$ *in Lemma* 5.1(i), *then there exist* $T_1 \geq T$ *and a positive number* $q = q(\ell)$ *such that*

$$|u(t)| + |u'(t)| \geq 2q \qquad \textit{for all } t \in J_1 = [T_1, \infty).$$

We omit the proofs of Lemmas 5.1, 5.2(i) since they can be found in [12, 13] and are easily derived from (H_1)–(H_2) and (H_4). In particular Lemma 5.1(i) is based on the primary identity

$$\{H(u,u') + F(t,u)\}' = (Q(t,u,u'),u') + F_t(t,u) \quad \text{in } J;$$

which is easily verified if $u \in C^2(J)$, but holds equally under the weaker smoothness conditions imposed here, namely that G is of class C^1; see [11].

Proof of Lemma 5.2(ii). Since $H(u,0) = 0$ for all $u \in \mathbb{R}^N$ by (H_1), we can find a number $q_1 = q_1(U,\ell)$ such that

$$H(u,p) \leq \frac{\ell}{4} \qquad \text{for all } |u| \leq U \text{ and } |p| \leq q_1.$$

By Lemma 5.1(i) there exists $T^* = T^*(\ell) \geq T$ such that

$$H(u,u') + F(t,u) \geq \frac{\ell}{2} \qquad \text{for all } t \geq T^*.$$

There are now two cases. For those $t \geq T^*$ such that $|u'(t)| > q_1$ there is nothing to prove.

For those $t \geq T^*$ such that $|u'(t)| \leq q_1$ we have

$$F(t,u) \geq \frac{\ell}{2} - H(u,u') \geq \frac{\ell}{4}.$$

We can now apply Lemma 4.2(ii) of [13], which requires only (H_2), to obtain two numbers $T_1 \geq T^*$ and $u_3 > 0$ (depending only on U and ℓ) such that

$$|u(t)| \geq u_3 \quad \text{whenever } t \geq T_1 \text{ and } F(t,u) \geq \frac{\ell}{4}.$$

To complete the proof it is sufficient to take $q = \frac{1}{2}\min\{q_1, u_3\}$.

Our proofs in the following sections depend on the construction of appropriate differential inequalities, based on the general theory of variational identities introduced in [9]. In particular, in view of (2.1) the following identity holds in J along any (weak) solution $u = u(t)$ of (2.1), and for any scalar function $h \in C^1(J)$:

$$\begin{aligned}\left\{k\,h\,(\nabla G(u,u'),u)\right\}' &= k\,h\,\mathcal{G}(t,u,u') + k\,h\,(Q(t,u,u'),u)\\ &\quad + k'\,h\,(\nabla G(u,u'),u) + k\,h'\,(\nabla G(u,u'),u)\,,\end{aligned} \tag{5.1}$$

where

$$\mathcal{G}(t,u,p) = (\nabla_u G(u,p),u) + (\nabla G(u,p),p) - (f(t,u),u),$$

as in Section 4.

6. Proof of Theorems 1–2. As in the previous section we denote by $u = u(t)$ any bounded solution of (2.1), by $u' = u'(t)$ its derivative, and by $U > 0$ an upper bound for the set $\{|u(t)| : t \in J\}$. Turning now to the demonstration of Theorems 1–2, note that by Lemma 5.2(i) if $\ell = 0$ in Lemma 5.1(i) then the rest state of (2.1) is a global attractor, that is Theorems 1–2 hold. Thus in what follows we assume that $\ell > 0$.

Lemma 6.1. *Let u be a solution of (2.1) with $\ell > 0$. Under the hypotheses of Theorems 1–2 there exist a function $\psi_1 \in L^1(J)$ and a positive constant κ (depending on ℓ) such that*

$$\left\{k\,h\,(\nabla G(u,u'),u)\right\}' \leq \psi_1 - \kappa\,k + \frac{\kappa}{M}\,\delta\,k \quad \text{for } t \in J_1, \tag{6.1}$$

where $h(t) = g(|u(t)|)$ and $g : [0,\infty) \to [-1,1]$ is an increasing function of class C^1 such that

$$g(s) = \begin{cases} -1 & \text{if } 0 \leq s \leq \varepsilon \\ 0 & \text{if } s = 2\varepsilon \\ 1 & \text{if } s \geq 3\varepsilon, \end{cases} \tag{6.2}$$

for a sufficiently small constant $\varepsilon > 0$ (depending on ℓ). Here M is any fixed constant strictly greater than the lim inf *given in (3.5).*

Proof. By the remark at the end of Lemma 5.1 one can treat Theorem 2 as a special case of Theorem 1. Hence we only need to prove Theorem 1. By Lemma 5.2(ii) there exists q depending on ℓ such that

$$|u(t)| + |u'(t)| \geq 2q \quad \text{for all } t \in J_1 = [T_1, \infty), \tag{6.3}$$

while by Lemma 4.2(i)–(ii) there exist positive numbers u_2, p_1, and $\kappa = \min\{\kappa_1, \kappa_2\}$ such that

$$\begin{aligned} &\mathcal{G}(t,u,p) \geq \kappa \qquad \text{when } t \in J, \quad |u| \leq u_2 \quad \text{ and } |p| \geq q; \\ &\mathcal{G}(t,u,p) \leq -\kappa \qquad \text{when } t \in J, \quad |u| \in [q,U] \text{ and } |p| \leq p_1 \end{aligned} \tag{6.4}$$

(clearly κ, u_2, p_1 also depend on ℓ). We now take $h(t) = g(|u(t)|)$ in (5.1), where ε remains to be chosen, and estimate the various terms on the right side of that identity in the case $k \in BV(J)$. We consider the case $\log k \in \mathrm{Lip}\,(J)$ later.

Step 1: Estimation of $k\,h\,\mathcal{G}(t,u,u')$. We claim that

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa\,k + \text{Const. } |(Q(t,u,u'),u')| \quad \text{for all } t \in J_1, \tag{6.5}$$

provided that ε is sufficiently small. To see this, take $I_1 = \{t \in J_1 : |u| \leq 2\varepsilon\}$ and let $\varepsilon \leq \frac{1}{2}\min\{q, u_2\}$. By (6.3) we get

$$|u'| \geq 2q - |u| \geq 2q - 2\varepsilon \geq q \quad \text{for } t \in I_1.$$

Since $h(t) \leq 0$ for $t \in I_1$ by $(6.2)_2$ and the fact that g is increasing, it follows from $(6.4)_1$ that

$$k\,h\,\mathcal{G}(t,u,u') \leq \kappa\,k\,h \qquad \text{for } t \in I_1. \tag{6.6}$$

There are now two cases. For those $t \in I_1$ such that $|u| \leq \varepsilon$ we have $h(t) = -1$ by $(6.2)_1$ and thus

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa\,k.$$

For those $t \in I_1$ such that $\varepsilon < |u| \leq 2\varepsilon$ we obtain

$$k\,h\,\mathcal{G}(t,u,u') \leq 0 \leq -\kappa\,k + \kappa\,k\,|u'|/q \leq -\kappa\,k + \text{Const. } |(Q(t,u,u'),u')|,$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = q$.

If $t \in J_1 \backslash I_1$ then $|u| > 2\varepsilon$. Again there are two cases. For those $t \in J_1 \backslash I_1$ such that $|u'| \leq 2\varepsilon$, by (6.3) we have

$$|u| \geq 2q - |u'| \geq 2q - 2\varepsilon \geq q.$$

Hence $h(t) = 1$ by $(6.2)_3$, provided $\varepsilon \leq q/3$. It now follows from $(6.4)_2$ that

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa\,k,$$

provided we take $\varepsilon \leq p_1/2$.

For those $t \in J_1 \backslash I_1$ such that $|u'| > 2\varepsilon$ we have $0 \leq h(t) \leq 1$ again by $(6.2)_2$ and the fact that g is increasing. Hence also by Lemma 4.2(iii) there is a positive constant $\kappa_3 = \kappa_3(\varepsilon)$ such that

$$\begin{aligned} k\,h\,\mathcal{G}(t,u,u') &\leq k\,\kappa_3 |\nabla G(u,u')| \cdot |u'| = -\kappa\,k + k(\kappa + \kappa_3 |\nabla G(u,u')| \cdot |u'|) \\ &\leq -\kappa\,k + (\kappa\,(2\varepsilon)^{-1} + \kappa_3 C) k\,|u'| \quad \text{by Lemma 5.1(ii)} \\ &\leq -\kappa\,k + \text{Const. } |(Q(t,u,u'),u')|, \end{aligned} \tag{6.7}$$

by Lemma 4.3 with $u_0 = 2\varepsilon$ and $p_0 = 2\varepsilon$.

In summary, with the choice

$$\varepsilon \le \min\{q/3, u_2/2, p_1/2\} \tag{6.8}$$

the above estimates for the sets I_1 and $J_1\backslash I_1$ are simultaneously valid. Combining these gives (6.5) and the claim is proved.

Step 2: Estimation of $k\,h\;(Q(t,u,u'),u)$. Let $\alpha_1 \le q/\,(\sup_{t\in J} k)$. If $|u'| \le \alpha_1 k$ then $|u| \ge q \ge 3\varepsilon$ by (6.3) and the fact that $\varepsilon \le q/3$. Therefore $h(t) = 1$ by $(6.2)_3$ and thus

$$k\,h\;(Q(t,u,u'),u) \le k\,\delta\,\rho(|u'|) \le k\,\delta\,\rho\,(\alpha_1 \sup_J k) \le \frac{\kappa}{M}\,\delta\,k,$$

where α_1 is taken so small that (A_2) can applied and also so that $\rho\,(\alpha_1 \sup_J k) \le \kappa/M$ (recall that $\rho(s) \to 0$ as $s \to 0$ by (A_2)). Here M is the constant given in (6.1). The previous estimate obviously holds also for those t such that $k(t) = 0$.

If $|u'| > \alpha_1 k > 0$, then by (A_3) and the fact that $|h(t)| \le 1$ we obtain

$$k\,h\;(Q(t,u,u'),u) \le U\,k\,|Q(t,u,u')| \cdot \frac{|u'|}{\alpha_1 k} \le \frac{\gamma\,U}{\alpha_1}\,|(Q(t,u,u'),u')|\,.$$

Step 3: $k'\,h\;(\nabla G(u,u'),u) \le C\,U\,|k'|$ by Lemma 5.1(ii) and the fact that $|h(t)| \le 1$.

Step 4: Estimation of $k\,h'\;(\nabla G(u,u'),u)$. Observe that $g'(s) \equiv 0$ whenever $0 \le s \le \varepsilon$ or $s \ge 3\varepsilon$. Consequently

$$\|g'\|_\infty = \sup\{|g'(s)| : s \in [0,\infty)\} = \max\{|g'(s)| : \varepsilon < s < 3\varepsilon\} < \infty.$$

Moreover

$$h'(t) = 0 \qquad \text{if either } |u| \le \varepsilon \text{ or } |u| \ge 3\varepsilon.$$

In the remaining set we have $\varepsilon < |u| < 3\varepsilon$ so that $|u'| > q$ by (6.3) and the fact that $\varepsilon \le q/3$. By (6.2) and Lemma 5.1(ii) we have

$$\begin{aligned} k\,h'\;(\nabla G(u,u'),u) &= k\,g'(|u|)\,(u/|u|,u') \cdot (\nabla G(u,u'),u) \\ &\le ||g'||_\infty C\,U\,k\,|u'| \le \text{Const.}\;|(Q(t,u,u'),u')| \end{aligned}$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = q$.

Combining the previous *Steps 1–4* gives

$$\big\{k\,h\;(\nabla G(u,u'),u)\big\}' \le \psi_1 - \kappa\,k + \frac{\kappa}{M}\,\delta\,k \quad \text{in } J_1, \tag{6.9}$$

where $\psi_1 = \text{Const.}\;(|(Q,u')| + |k'|)$ is in $L^1(J)$ by Lemma 5.1(iii) and the fact that k is of bounded variation.

When $\log k \in \mathrm{Lip}\,(J)$ then clearly $|k'| \le B\,k$ almost everywhere in J. The previous proof now applies word-for-word, except for Step 3. For this step, since $\nabla G(u,0) = 0$, we have $|\nabla G(u,p)| \le \kappa/2BU$ for $|u| \le U$ and $|p|$ sufficiently small, say $|p| \le p_3$. Therefore, by Lemma 5.1(ii) we obtain the alternative estimate

$$k'\,h\,(\nabla G(u,u'),u) \le \tfrac{1}{2}\,\kappa\,k \quad \text{ if } |u| \le \kappa/2BC \text{ or } |u'| \le p_3.$$

In the remaining set $|u| > \kappa/2BC$ and $|u'| > p_3$. Hence by Lemma 5.1(ii) and the fact that $|h(t)| \le 1$ we get

$$k'\,h\,(\nabla G(u,u'),u) \le BCU\,k\,|u'|/p_3 \le \text{Const. } |(Q(t,u,u'),u')|$$

by Lemma 4.3 with $u_0 = \kappa/2BC$ and $p_0 = p_3$. The constant κ should now be replaced by $\kappa/2$ in the last two terms of the right side of (6.9), while the function ψ_1 reduces to $\psi_1 = \text{Const. } |(Q,u')|$. This completes the proof the Lemma.

We now turn to the proof of Theorems 1–2. By Lemma 5.2 it is enough to consider the case $\ell > 0$. By (6.1) we obtain

$$\begin{aligned} k\,h\,(\nabla G(u,u'),u) &\le \int_T^t \psi_1(s)\,ds - \kappa \int_T^t k\,ds + \frac{\kappa}{M}\int_T^t \delta\,k\,ds + \text{Const.}(T_1) \\ &\le \text{Const.} + \frac{\kappa}{M}\int_T^t k(s)\,ds \left\{ \int_T^t \delta(s)\,k(s)\,ds \Big/ \int_T^t k(s)\,ds - M \right\}, \end{aligned}$$

where by $(3.2)_1$ we can take t be so large that $\int_T^t k(s)\,ds > 0$ and we have used the fact that $\psi_1 \in L^1(J)$, see Lemma 6.1. By $(3.2)_1$, (3.5) and the fact that M is strictly greater than the liminf in (3.5) it follows that

$$\liminf_{t\to\infty} k\,h\,(\nabla G(u,u'),u) = -\infty,$$

which is a contradiction by Lemma 5.1(ii) and the boundedness of the functions k, u, g. Hence $\ell > 0$ is impossible, and the proof is complete.

7. Proof of Theorem 3. Under the hypotheses of Theorem 3, Lemma 6.1 should be replaced by the following:

Lemma 7.1. *Let u be a solution of* (2.1) *with $\ell > 0$. Under the hypotheses of Theorem 3 there exist a function $\psi_1 \in L^1(J)$ and a positive constant κ (depending on ℓ) such that*

$$\big\{k\,h\,(\nabla G(u,u'),u)\big\}' \le \psi_1 - \kappa\,k + \frac{\kappa}{M}\,\{\delta\,k + |k'|\} \quad \textit{for } t \in J_1, \tag{7.1}$$

where

$$h(t) = g_1(|u(t)|) - g_1\left(|\nabla G(u(t),u'(t))|\right) \tag{7.2}$$

and $g_1 : [0,\infty) \to [-1,0]$ is an increasing function of class C^1 such that

$$g_1(s) = \begin{cases} -1 & \textit{if } 0 \le s \le \varepsilon \\ 0 & \textit{if } s \ge 2\varepsilon, \end{cases} \tag{7.3}$$

for a sufficiently small constant $\varepsilon > 0$ (*depending on* ℓ). *Here* M *is any fixed constant strictly greater than the liminf given in* (3.9).

Proof. The proof of this lemma is very similar to that of Lemma 6.1. We start from the identity (5.1), with the function h now taken as in (7.2). Corresponding to Step 1 of Lemma 6.1, take $I_1 = \{t \in J_1 : |u| \le 2\varepsilon\}$, as before. By Lemma 4.1

$$d_1 \stackrel{\text{def}}{=} \inf\{|\nabla G(u,p)| : |u| \le U,\, |p| \ge q\} > 0. \tag{7.4}$$

Let $\varepsilon \le \frac{1}{2}\min\{q, d_1, u_2\}$, where u_2 is the number in $(6.4)_1$. Then $|u'| \ge q$ by (6.3) so that $|\nabla G(u,u')| \ge d_1 \ge 2\varepsilon$ by (7.4). Therefore $g_1(|\nabla G(u,u')|) = 0$ by $(7.3)_2$ and consequently by (7.2) and $(6.4)_1$

$$k\,h\,\mathcal{G}(t,u,u') \le \kappa\,k\,g_1(|u|) \quad \text{for } t \in I_1.$$

There are now two cases. For those $t \in I_1$ such that $|u| \le \varepsilon$ we have $g_1(|u(t)|) = -1$ by $(7.3)_1$ and thus

$$k\,h\,\mathcal{G}(t,u,u') \le -\kappa\,k.$$

For those $t \in I_1$ such that $\varepsilon < |u| \le 2\varepsilon$ we obtain

$$k\,h\,\mathcal{G}(t,u,u') \le 0 \le -\kappa\,k + \kappa\,k\,|u'|/q \le -\kappa\,k + \text{Const.}\ |(Q(t,u,u'),u')|, \tag{7.5}$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = q$. Here we have used the fact that $g_1 \le 0$. It is clear from the previous estimates that (7.5) actually holds for *all* $t \in I_1$.

In the remaining set $t \in J_1 \backslash I_1$ we have $|u| > 2\varepsilon$, so that $g_1(|u|) = 0$ by $(7.3)_2$. Let

$$I_2 = \{t \in J_1 \backslash I_1 : |\nabla G(u,u')| < 2\varepsilon\}.$$

Recalling that $\varepsilon \le d_1/2$, we obtain $|u'(t)| < q$ in I_2, since if it were $|u'| \ge q$ we would have $|\nabla G(u,u')| \ge d_1$ by (7.4), which is not the case. Therefore $|u| > q$ by (6.3). By Lemma 4.1

$$d_2 \stackrel{\text{def}}{=} \inf\{|\nabla G(u,p)| : |u| \le U,\, |p| \ge p_1\} > 0, \tag{7.6}$$

where p_1 is the number given in $(6.4)_2$. Letting $\varepsilon < d_2/2$ and reasoning as before, we can conclude from (7.6) that $|u'| < p_1$. By applying $(6.4)_2$ we obtain

$$k\,h\,\mathcal{G}(t,u,u') \le \kappa\,k\,g_1(|\nabla G(u,u')|) \quad \text{for } t \in I_2.$$

For those $t \in I_2$ such that $|\nabla G(u,u')| \le \varepsilon$ we have $g_1(|\nabla G(u,u')|) = -1$ by $(7.3)_1$ so that

$$k\,h\,\mathcal{G}(t,u,u') \le -\kappa\,k.$$

For those $t \in I_2$ such that $\varepsilon < |\nabla G(u,u')| < 2\varepsilon$ we have

$$k\,h\,\mathcal{G}(t,u,u') \le 0. \tag{7.7}$$

The previous estimate obviously holds also for those $t \in J_1 \backslash I_1$ such that $|\nabla G(u,u')| \ge 2\varepsilon$, since $h(t) = 0$ by (7.2) and $(7.3)_2$.

Since by (H_1) there is a constant $p_4 = p_4(U,\varepsilon) > 0$ such that

$$|u'| \geq p_4 \qquad \text{whenever } |\nabla G(u,u')| \geq \varepsilon, \tag{7.8}$$

arguing as in (7.5) we can conclude from (7.7) that

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa\,k + \text{Const.}\ |(Q(t,u,u'),u')|. \tag{7.9}$$

In summary, with the choice

$$\varepsilon < \tfrac{1}{2}\min\{q,u_2,d_1,d_2\} \tag{7.10}$$

the estimates (7.5) and (7.9) for the sets I_1 and $J_1\backslash I_1$ are simultaneously valid. Combining these gives

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa\,k + \text{Const.}\ |(Q(t,u,u'),u')| \quad \text{for all } t \in J_1,$$

as before.

For Step 2, the only change is in the set where $|u'| \leq \alpha_1 k$. In this set $|u| \geq q \geq 2\varepsilon$ by (6.3) and (7.10). Therefore $g_1(|u|) = 0$ by $(7.3)_2$ and in turn $h(t) \geq 0$ by (7.2). We can now proceed as before.

For Step 3, with the constant ε in (7.3) chosen to satisfy $2\varepsilon \leq \kappa/(M\max\{U,C\})$, as well as (7.10), and by using Lemma 5.1(ii), we obtain the alternative estimate

$$k'\,h\,(\nabla G(u,u'),u) \leq \frac{\kappa}{M}\,|k'| \qquad \text{if either } |u| \leq 2\varepsilon \text{ or } |\nabla G(u,u')| \leq 2\varepsilon.$$

In the remaining set $k'\,h\,(\nabla G(u,u'),u) \equiv 0$ by $(7.3)_2$.

For Step 4, it follows from (7.2) and (7.3) that

$$h'(t) = 0 \quad \text{if } |u| \leq \varepsilon, \quad \text{or} \quad |\nabla G(u,u')| \leq \varepsilon, \quad \text{or} \quad |u|,\ |\nabla G(u,u')| \geq 2\varepsilon.$$

If $\varepsilon < |u| < 2\varepsilon$ then $|u'| > q$ by (6.3) and (7.10). Hence $|\nabla G(u,u')| \geq d_1 \geq 2\varepsilon$ by (7.4) and in turn $g_1'\,(|\nabla G(u,u')|) = 0$ by $(7.3)_2$. Therefore by (7.2), (7.3) and Lemma 5.1(ii) we have

$$\begin{aligned} k\,h'\ (\nabla G(u,u'),u) &= k\,g_1'(|u|)\,(u/|u|,u')\cdot(\nabla G(u,u'),u) \\ &\leq ||g_1'||_\infty C\,U\,k\,|u'| \leq \text{Const.}\ |(Q(t,u,u'),u')|, \end{aligned}$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = q$.

If $\varepsilon < |\nabla G(u,u')| < 2\varepsilon$ then $|u'| < q$ by (7.4) and (7.10). This implies as before that $|u| > q \geq 2\varepsilon$ by (6.3) and (7.10). Hence $g_1'(|u|) = 0$ by $(7.3)_2$. Let

$$D_1 = \max\{|\nabla_u G(u,p)| : |u| \leq U,\ |p| \leq q\}. \tag{7.11}$$

By (7.2), (7.3), (2.1), (7.11), (3.8), Lemma 5.1(ii), (7.8), Lemma 4.3 and (A_3)

$$\begin{aligned} k\,h'\ (\nabla G(u,u'),u) &= -k\,g_1'(|\nabla G|)\,(\nabla G/|\nabla G|,\nabla_u G + f + Q)\cdot(\nabla G,u) \\ &\leq ||g_1'||_\infty(D_1+D)C\,U\,k\,|u'|/p_4 + (\sup_J k)\ \|g_1'\|_\infty C\,U\,|Q|\cdot|u'|/p_4 \\ &\leq \text{Const.}\ |(Q,u')| + (\sup_J k)\ \|g_1'\|_\infty C\,U\,\gamma\,|(Q,u')|/p_4. \end{aligned}$$

This concludes the proof of the lemma.

Now, arguing as in the conclusion of Section 6, but using (7.1) and (3.9) rather than (6.1) and (3.5), gives a contradiction as before and proves Theorem 3.

8. Additional results. In this section we analyze the same asymptotic problem when G and Q have a more specialized structure (see [12]).

Theorem 4. *Assume that condition* (A_2) *is satisfied with the specific function*

$$\rho(s) = |s|^{\mu}, \quad \mu > 0. \tag{8.1}$$

Then condition (3.5) *in Theorems* 1–2 *can be weakened to*

$$\liminf_{t\to\infty} \int_T^t \delta(s)\, k^{\mu+1}(s)\, ds \Big/ \int_T^t k(s)\, ds < \infty, \tag{8.2}$$

while condition (3.9) *in Theorem* 3 *can be weakened to*

$$\liminf_{t\to\infty} \int_T^t \{\delta(s)\, k^{\mu+1}(s) + |k'(s)|\}\, ds \Big/ \int_T^t k(s)\, ds < \infty. \tag{8.3}$$

Proof. The only real change is in Step 2 of Lemmas 6.1 and 7.2 in the set where $|u'(t)| \le \alpha_1 k(t)$. Here by (A_2), (8.1) and the fact that $h(t) \ge 0$ we have

$$k\, h\,(Q(t,u,u'),u) \le k\,\delta\,\rho(\alpha_1 k(t)) = \alpha_1^{\mu}\,\delta\, k^{\mu+1} \le \frac{\kappa}{M}\,\delta\, k^{\mu+1},$$

provided α_1 is taken so small that (A_2) can be applied and also so that $\alpha_1 \le (\kappa/M)^{1/\mu}$.

Theorem 4 first appears in [12]. We have given here a simpler proof. The analog of condition (3.10) for Theorem 4 is

$$|k'| \le \text{Const.}\ \{\delta\, k^{\mu} + 1\} k \qquad \text{in } I.$$

Condition (8.2) can actually be weakened to

$$\liminf_{t\to\infty} \int_T^t \delta(s)\, k^{\mu+1}(s)\, ds \Big/ \left(\int_T^t k(s)\, ds\right)^{\mu+1} < \infty. \tag{8.2$'$}$$

This condition has been discovered by Pucci and Serrin in [10], where they treat damped wave systems. Their proof carries out to the present case with only minor modifications. We refer to [10] for further details. Moreover, in [10] the authors also show that, when the function ϕ in (A_1) has the form $\phi(u,p) = \text{Const.}\ |p|$ for $|p| \ge 1$, then condition (3.4) can be avoided if in place of $(8.2)'$ we assume

$$\liminf_{t\to\infty} \int_T^t (\delta(s) + \sigma^{-\mu}(s))\, k^{\mu+1}(s)\, ds \Big/ \left(\int_T^t k(s)\, ds\right)^{\mu+1} < \infty, \tag{8.2$''$}$$

where we suppose $\sigma \in L^{\infty}_{\text{loc}}(J)$.

Theorems 1–2 can be further improved if the functions G and Q have additional structures. In particular, we introduce the following conditions, which sharpen (H_1) and (A_1):

There exist a positive constant Θ *and an exponent* $m > 1$ *such that*

$$|\nabla G(u,p)| \le \Theta |p|^{m-1} \quad \textit{when } |u| \le U \textit{ and } |p| \le 1. \tag{8.4}$$

Assumption (A_1) *is satisfied with the specific function*

$$\phi(u,p) = |u|^{\alpha} \cdot \min\{1, |p|^{\nu}\}, \qquad \alpha,\ \nu > 0. \tag{8.5}$$

Theorem 5. *Let* (8.4) *and* (8.5) *hold. Then Theorems* 1–2 *and* 4 *continue to hold if in place of* (3.3) *we assume that*

$$|k'| \le B\sigma^\lambda k^{1-\lambda} \qquad \text{a.e. on } J, \tag{8.6}$$

where B is a positive constant and

$$\lambda = \begin{cases} 1, & \text{if } \nu \le m-2 \quad \text{and} \quad \alpha \le 1 \\ \min\left\{\frac{1}{\alpha}, \frac{m-1}{\nu+1}\right\}, & \text{if } \nu > m-2 \quad \text{or} \quad \alpha > 1. \end{cases} \tag{8.7}$$

It goes almost without saying that in the case $\nu \le m-2$ in (8.7) we must have $m > 2$. When (1.2) rather than (1.3) is satisfied, then we can take $\alpha = 0$. In this case condition (8.7) reduces to condition (4.9) of Theorem 4.2 in [12].

Proof. The proof of all the theorems above apply essentially word-for-word, except for Step 3 in Lemmas 6.1 and 7.1. Here, in J_1 we see by (8.6), Lemma 5.2(ii) and $|h(t)| \le 1$ that

$$\begin{aligned} k'\, h\,(\nabla G(u,u'),u) &\le B\,\sigma^\lambda k^{1-\lambda}|\nabla G|\cdot|u| \\ &\le B\,\sigma \begin{cases} \theta^{1-\lambda} C\, U & \text{if } |u|\ge q \text{ and } |u'| \ge q \\ C(k/\sigma)^{1-\lambda}|u| & \text{if } |u| < q \text{ and } |u'| \ge q \\ \Theta U (k/\sigma)^{1-\lambda}|u'|^{m-1} & \text{if } |u| \ge q \text{ and } |u'| < q, \end{cases} \end{aligned} \tag{8.8}$$

by (3.4), Lemma 5.1(ii) and (8.4).

First, if $|u(t)| \ge q$ and $|u'(t)| \ge q$ then by $(8.8)_1$ we have

$$k'\, h\,(\nabla G(u,u'),u) \le \text{Const. } \sigma \le \text{Const. } |(Q(t,u,u'),u')|$$

by (2.7) and (8.5).

Next, we let

$$I_3 = \{t \in J_1 : |u(t)| < q,\ |u'(t)| \ge q\}, \qquad I_4 = \{t \in J_1 : |u(t)| \ge q,\ |u'(t)| < q\}.$$

Without loss of generality we can assume that $q < 1$ in Lemma 5.2(ii). Therefore if $\lambda = 1$ in (8.7) then we have $|u| \le |u|^\alpha$ in I_3, since $\alpha \le 1$. Thus by $(8.8)_2$, (2.7) and (8.5)

$$k'\, h\,(\nabla G,u) \le \text{Const. } \sigma\,|u| \le \text{Const. } \sigma\,|u|^\alpha \le \text{Const. } |(Q,u')| \quad \text{in } I_3.$$

If $\lambda < 1$ in (8.7) set

$$I_3' = \left\{t \in I_3 : (k/\sigma)^{1-\lambda}\alpha_2 \le |u|^{\lambda'}\right\}, \qquad I_3'' = \left\{t \in I_3 : |u|^\lambda < (k/\sigma)^{1-\lambda}\alpha_2\right\},$$

where α_2 is a positive constant and $\lambda' = (1-\lambda)/\lambda$. For $t \in I_3'$ by $(8.8)_2$, (2.7) and (8.5)

$$k'\, h\,(\nabla G,u) \le \text{Const. } \sigma\,(k/\sigma)^{1-\lambda}|u| \le \text{Const. } \sigma\,|u|^{1+\lambda'} \le \text{Const. } |(Q,u')|,$$

where we have used the fact that $|u|^{1+\lambda'} \le |u|^\alpha$ in I_3, since $q < 1$ and $1+\lambda' = 1/\lambda \ge \alpha$.

By the second inequality in (8.8) and Lemma 5.1(ii), for $t \in I_3''$

$$k'\, h\,(\nabla G, u) \le \text{Const. } \sigma^\lambda k^{1-\lambda}[(k/\sigma)^{1-\lambda}\alpha_2]^{\lambda/(1-\lambda)} = \text{ Const. } \alpha_2^{\lambda/(1-\lambda)} k.$$

If we now take $\alpha_2^{\lambda/(1-\lambda)} \le \kappa/2$ Const., also by the previous estimate in I_3', we conclude that

$$k'\, h\,(\nabla G(u,u'), u) \le \tfrac{1}{2}\,\kappa\, k + \text{Const. } |(Q(t,u,u'), u')| \qquad \text{in } I_3.$$

The same estimate can be obtained in the set I_4. We omit the proof since it is quite similar to the previous one and it can be essentially found in [12, Theorem 4.2]. The rest of the proof is now the same as before.

In what follows we assume that condition (2.3) holds in the *stronger* form

$$|F_t(t,u)| \le \psi(t) \qquad \text{when } t \in J \text{ (a.e.)} \quad \text{and} \quad |u| \le U. \tag{2.3$'$}$$

When F has the form (2.6) then (2.3) and (2.3)$'$ are equivalent. Indeed since $\ell(t) \ge 0$ we get

$$\int_T^\infty (\ell')^-(s)\, ds \le \ell(T) + \int_T^\infty (\ell')^+(s)\, ds < \infty.$$

Theorem 6. *Theorem* 3 *remains true if in place of condition* (3.8) *we suppose that the function* σ *in* (A_1) *satisfies the positive mean value criterion*

$$\int_{I_1} \sigma(s)\, ds \ge a(|I_1|) > 0 \quad \textit{for all intervals } I_1 \subset J \textit{ such that } |I_1| < 1, \tag{8.9}$$

where $a : (0,1) \to (0,\infty)$ *is a nondecreasing function.*

In this case Lemmas 5.1–5.2 continue to hold, while Lemma 7.1 should be replaced by

Lemma 8.1. *Under the hypotheses of Theorem* 6 *we have*

$$\liminf_{t\to\infty} |u'(t)| = 0.$$

Proof. Assume for contradiction that

$$\liminf_{t\to\infty} |u'(t)| > 0.$$

Then there exist two numbers $p_5 > 0$ and $T_3 \ge T_1$ such that

$$|u'(t)| \ge p_5 \qquad \text{for } t \in J_3 = [T_3, \infty). \tag{8.10}$$

We now use a modified version of Lemma 6.1. Take

$$h(t) = g_1(|u(t)|) \tag{8.11}$$

in (5.1), where g_1 is the function given in Lemma 7.1. By Lemma 4.2 there exist two positive numbers κ_4 and u_4 such that

$$\mathcal{G}(t,u,p) \geq \kappa_4 \qquad \text{when } t \in J,\ |u| \leq u_4 \text{ and } |p| \geq p_5; \tag{8.12}$$

Step 1: Let $2\varepsilon \leq u_4$. If $|u| \leq \varepsilon$ then $h(t) = -1$ by $(7.3)_1$. Therefore by (8.10) and (8.12)

$$k\,h\,\mathcal{G}(t,u,u') \leq -\kappa_4\,k.$$

If $\varepsilon < |u| < 2\varepsilon$ then by (8.10) and the fact that $h(t) \leq 0$

$$k\,h\,\mathcal{G}(t,u,u') \leq 0 \leq -\kappa_4\,k + \kappa_4\,k\,|u'|/p_5 \leq -\kappa_4\,k + \text{Const.}\ |(Q(t,u,u'),u')|$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = p_5$. This estimate continues to hold when $|u| \geq 2\varepsilon$ since in this case $h(t) = 0$ by $(7.3)_2$.

Step 2: By (8.10), (A_3) and the fact that $|h(t)| \leq 1$ we obtain

$$k\,h\ (Q(t,u,u'),u) \leq (\sup\nolimits_J k)\,U\,|Q(t,u,u')| \cdot \frac{|u'|}{p_5} \leq (\sup\nolimits_J k)\,\frac{\gamma\,U}{p_5}\,|(Q(t,u,u'),u')|\,.$$

Step 3: By Lemma 5.2(ii) we have

$$k'\,h\ (\nabla G(u,u'),u) \leq 2C\varepsilon\,|k'| \leq \frac{\kappa_4}{M}\,|k'| \quad \text{when } |u| < 2\varepsilon,$$

provided $\varepsilon \leq \kappa_4/2CM$, where M is any fixed constant strictly greater than the lim inf given in (3.9); while

$$k'\,h\ (\nabla G(u,u'),u) = 0 \quad \text{when } |u| \geq 2\varepsilon$$

by $(7.3)_2$.

Step 4: Clearly $h'(t) = 0$ if either $|u| \leq \varepsilon$ or $|u| \geq 2\varepsilon$. In the remaining set $\varepsilon < |u| < 2\varepsilon$. By (7.3) and Lemma 5.1(ii) we have

$$\begin{aligned} k\,h'\ (\nabla G(u,u'),u) &= k\,g_1'(|u|)\,(u/|u|,u') \cdot (\nabla G(u,u'),u) \\ &\leq \|g_1'\|_\infty C\,U\,k\,|u'| \leq \text{Const.}\ |(Q(t,u,u'),u')| \end{aligned}$$

by Lemma 4.3 with $u_0 = \varepsilon$ and $p_0 = p_5$.

Combining the previous steps gives

$$\big\{k\,h\ (\nabla G(u,u'),u)\big\}' \leq \psi_2 - \kappa_4\,k + \frac{\kappa_4}{M}\,|k'| \quad \text{in } J_1,$$

where $\psi_2 = \text{Const.}\ |(Q,u')|$ is in $L^1(J)$ by Lemma 5.1(iii). Arguing as in Section 6 but using (3.9) rather than (3.5) gives the required contradiction and proves the lemma.

Lemma 8.2. *Under the hypotheses of Theorem* 6 *we have*

$$\left\{k\ (\nabla G(u,u'),u)\right\}' \le \psi_1 - \kappa\, k + \frac{\kappa}{M}\{\delta\, k + |k'|\} \quad \textit{for } t \textit{ large},$$

where $\psi_1 \in L^1(J)$, κ *is a positive constant and* M *is any fixed constant strictly greater than the* lim inf *given in* (3.9).

Proof. By Lemma 8.1 we have $\liminf_{t\to\infty}|u'(t)| = 0$. Consequently by the second part of the proof of Lemma 5.3 of [13] we obtain that $u'(t) \to 0$ as $t \to \infty$. The conclusion now follows from Lemma 1 in [5], the only exception being the term $k'\ (\nabla G(u,u'),u)$ for which we have the alternative estimate

$$|k'\ (\nabla G(u,u'),u)| \le \varepsilon_4(t)\,|k'| \le \frac{\kappa}{M}\,|k'| \qquad \text{for } t \text{ large},$$

thanks to the fact that $\varepsilon_4(t) \to 0$ as $t \to \infty$, since $\nabla G(u,0) = 0$ and $u'(t) \to 0$ as $t \to \infty$.

The proof of Theorem 6 is now essentially the same as that of Theorem 3, using Lemma 8.2 instead of Lemma 6.1.

Theorem 6 is closely related to a result of Pucci and Serrin (cf. [13, Theorem 2]). Here their conditions (V_1), (V_2) and (3.4) are replaced by the natural assumptions (2.2), (H_3) and (3.4). Note that (2.2) and (H_3) are always satisfied when G is independent of u and F of t. Moreover, in the important case when $G(p) = |p|^m/m$, $m > 1$, and indeed whenever (3.7) is satisfied, condition (3.4) can be dropped. We shall not dwell on this here. Thus for the special system (1.5) none of the conditions (2.2), (H_3) and (3.4) is actually needed.

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