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BLOW UP FOR $u_t - \triangle u = g(u)$ REVISITED

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1. Introduction. In this paper we are concerned with the relations between the existence of global, classical solutions of the evolution equation

$$\begin{cases} u_t - \triangle u = g(u) & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

and the existence of weak solutions of the stationary problem

$$\begin{cases} -\triangle u = g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Here, and throughout the paper, $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and $g : [0,\infty) \to [0,\infty)$ is a C^1 convex, nondecreasing function. For *some* results, we will also assume that there exists $x_0 \geq 0$ such that $g(x_0) > 0$ and

$$\int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty. \tag{3}$$

Solutions u of (1) and (2) are always assumed to be nonnegative. The initial condition u_0 is always assumed to be in $L^\infty(\Omega)$ and $u_0 \geq 0$, so that a classical solution of (1) exists on a maximal interval $(0, T_m)$.

By a weak solution of (2), we mean the following.

Definition 1. A weak solution of (2) is a function $u \in L^1(\Omega)$, $u \geq 0$, such that

$$g(u)\delta \in L^1(\Omega), \tag{4}$$

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where δ denotes the function distance to the boundary,

$$\delta(x) = \operatorname{dist}(x, \partial\Omega), \tag{5}$$

and

$$-\int_\Omega u \triangle \zeta = \int_\Omega g(u)\zeta, \tag{6}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. (Note that the second integral makes sense since $|\zeta(x)| \le C\delta(x)$ for all $x \in \Omega$.)

Our first result is the following.

Theorem 1. *Assume* (3). *If there exists a global, classical solution of* (1) *for some* $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, *then there exists a weak solution of* (2).

Remark 1. Theorem 1 is quite surprising since we do not assume any bound (as $t \to \infty$) for the global solution u.

Remark 2. The existence of a global solution of (1) does not, in general, imply the existence of a **classical** solution of (2). In many examples, the existence of a weak solution of (2) implies the existence of a classical solution of (2). However, there are situations where the stationary problem admits no classical solution, and still there exists a global, classical solution of the evolution equation. See Theorem 2 and Remark 5.

An obvious consequence of Theorem 1 is the following:

Corollary 1. *Assume* (3). *If there is no weak solution of* (2), *then for any initial value* $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, *the solution of* (1) *blows up in finite time.*

Remark 3. There are very sharp results concerning the existence or nonexistence of weak solutions of (2). See properties a) and d) below and Corollary 2.

There is a converse of Theorem 1, which does not require assumption (3).

Theorem 2. *If there exists a weak solution* w *of* (2), *then for any* $u_0 \in L^\infty(\Omega)$ *with* $0 \le u_0 \le w$, *the solution* u *of* (1) *with* $u(0) = u_0$ *is global.*

Remark 4. If w is a classical solution of (2), then the existence of a global solution of (1) follows immediately from the maximum principle. On the other hand, if $w \notin L^\infty(\Omega)$, then the conclusion is far from obvious. Indeed, suppose that the solution blows up in finite time T_m. Clearly $u(t,x) \le w(x)$ on $(0,T_m) \times \Omega$, but this estimate in itself does not prevent $\|u(t)\|_{L^\infty}$ from blowing up in finite time. It is well known that $u(t,x)$ can converge to a blow-up profile $u(T_m,x)$, which may be finite everywhere except at one point (see e.g. Weissler [16]).

A basic ingredient in the proof of Theorem 2 consists in proving that some "perturbations" of (2) have classical solutions if (2) has a weak solution. A typical result in that direction is the following:

Theorem 3. *If there exists a* **weak** *solution w of* (2)*, then, for every $\varepsilon \in (0,1)$, there exists a* **classical** *solution w_ε of*

$$\begin{cases} -\triangle w_\varepsilon = (1-\varepsilon) g(w_\varepsilon) & \textit{in} \quad \Omega, \\ \quad w_\varepsilon = 0 & \textit{on} \quad \partial\Omega. \end{cases} \tag{7}$$

Theorem 3 allows us to sharpen some well-known results concerning the problem

$$\begin{cases} -\triangle u = \lambda g(u) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

Here we assume in addition that

$$g(0) > 0 \quad \text{and} \quad g \not\equiv g(0). \tag{9}$$

We recall that there exists $0 < \lambda^* < \infty$ such that:

a) For every $0 < \lambda < \lambda^*$ equation (8) has a minimal, positive classical solution $u(\lambda)$, which is the unique stable solution of (8); stability means that

$$\lambda_1(-\triangle - \lambda g'(u(\lambda))) > 0.$$

(There may exist, for some values of $\lambda \in (0, \lambda^*)$, one or many other solutions, which are all unstable.)

b) The map $\lambda \mapsto u(\lambda)$ is increasing.

c) For $\lambda > \lambda^*$, there is *no classical solution* of (2).

d) For $\lambda = \lambda^*$, and if

$$\frac{g(u)}{u} \underset{u\to\infty}{\longrightarrow} \infty, \tag{10}$$

then there is a weak solution $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u(\lambda)$ of (8).

For all these results, we refer to I.M. Gel'fand ([7]), H.B. Keller and D.S. Cohen ([10]), H.B. Keller and J. Keener ([11]), M.G. Crandall and P.H. Rabinowitz ([3]), H. Brezis and L. Nirenberg ([2]).

Property d) is not absolutely standard; see Lemma 5.

Remark 5. The solution u^* is sometimes a classical solution. For example when $g(u) = e^u$ and $N \le 9$ or when $g(u) = (1+u)^p$ and $N \le 10$ (see F. Mignot and J.-P. Puel, [14]). However, there are important cases where there is **no classical solution** at $\lambda = \lambda^*$—for example when Ω is the unit ball of $\mathbb{R}^N$ with $N \ge 10$ and $g(u) = e^u$; in this case $\lambda^* = 2(N-2)$ and $u^*(x) = \log(\frac{1}{|x|^2})$ (see D.D. Joseph and T.S. Lundgren [8]).

The main novelty is:

Corollary 2. *Assume* (9). *If* $\lambda > \lambda^*$, *then there is* **no weak solution** *of* (8).

This is an obvious consequence of Theorem 3 applied to the function λg, and the characterization of λ^*.

Remark 6. A result similar to Corollary 2 was obtained by Gallouët, Mignot and Puel ([6]) in the case $g(u) = e^u$ (and for a stronger notion of weak solution).

Putting together Theorems 1, 2 and 3, we can now state the following.

Corollary 3. *Assume* (3) *and* (9), *and consider the* (*classical*) *solution* u *of*

$$\begin{cases} u_t - \triangle u = \lambda g(u) & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \\ u(0) = 0 & in \quad \Omega. \end{cases} \tag{11}$$

If $\lambda \le \lambda^*$, *then* u *is global. If* $\lambda > \lambda^*$, *then* u *blows up in finite time.*

Remark 7. It is somewhat surprising that one finds the same dividing line λ^* in the stationary problem and in the evolution problem.

Starting with the celebrated papers of H. Fujita ([4, 5]), dealing with the case $g(u) = e^u$, a number of authors have investigated the question of blow up in finite time or the existence of a global solution for (11). A. Lacey ([12]) had established that the solution of (11) blows up in finite time for $\lambda > \lambda^*$ under some additional assumption: either $u^* \in L^\infty(\Omega)$ or Ω is a ball. H. Bellout ([1]) had reached the same conclusion, with the additional assumption that $(\frac{g}{g'})'' \le 0$. On the other hand, A Lacey and D. Tzanetis ([13]) proved that for $\lambda = \lambda^*$ the solution of (11) is global when Ω is a ball and $u_0 \le u^*$, $u_0 \in L^\infty(\Omega)$ and u_0 is spherically symmetric (and also for general domains but under various restrictive conditions).

2. Proof of Theorem 3. We begin with a lemma concerning the linear Laplace equation.

Lemma 1. *Given* $f \in L^1(\Omega, \delta(x)dx)$, *there exists a unique* $v \in L^1(\Omega)$ *which is a weak solution of*

$$\begin{cases} -\triangle v = f \quad in \quad \Omega, \\ v_{|\partial\Omega} = 0, \end{cases} \tag{12}$$

in the sense that

$$-\int_\Omega v\triangle\zeta = \int_\Omega f\zeta, \tag{13}$$

for all $\zeta \in C^2(\overline{\Omega})$ *with* $\zeta = 0$ *on* $\partial\Omega$. *Moreover,*

$$\|v\|_{L^1} \le C\|f\|_{L^1(\Omega,\delta(x)dx)}, \tag{14}$$

for some constant C *independent of* f. *In addition, if* $f \ge 0$ *almost everywhere in* Ω, *then* $v \ge 0$ *almost everywhere in* Ω.

Proof. The uniqueness is clear. Indeed, let v_1 and v_2 be two solutions of (12). Then $v = v_1 - v_2$ satisfies

$$\int_\Omega v \triangle \zeta = 0,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Given any $\varphi \in \mathcal{D}(\Omega)$ let ζ be the solution of

$$\begin{cases} \triangle \zeta = \varphi & \text{in } \Omega, \\ \zeta_{|\partial\Omega} = 0. \end{cases}$$

It follows that

$$\int_\Omega v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.

For the existence, we may assume that $f \geq 0$ (otherwise we write $f = f_+ - f_-$). Given an integer $k \geq 0$ set $f_k(x) = \min\{f(x), k\}$, so that $f_k \underset{k\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let v_k be the solution of

$$\begin{cases} -\triangle v_k = f_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

The sequence $(v_k)_{k\geq 0}$ is clearly monotone nondecreasing. It is also a Cauchy sequence in $L^1(\Omega)$ since

$$\int_\Omega (v_k - v_\ell) = \int_\Omega (f_k - f_\ell)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} -\triangle \zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Hence

$$\int_\Omega |v_k - v_\ell| \leq C \int_\Omega |f_k - f_\ell| \delta(x)\, dx.$$

Passing to the limit in (15) (after multiplication by ζ), we obtain (13). Finally, taking $\zeta = \zeta_0$ in (13), we obtain

$$\|v\|_{L^1} = \int_\Omega v = \int_\Omega f\zeta_0 \leq C\|f\|_{L^1(\Omega,\delta(x)dx)},$$

and (14) follows. □

Our next lemma is a variant of Kato's inequality (see [9]).

Lemma 2. *Let $f \in L^1(\Omega, \delta(x)dx)$, and let $u \in L^1(\Omega)$ be the weak solution of* (12). *Let $\Phi \in C^2(\mathbb{R})$ be concave, with Φ' bounded and $\Phi(0) = 0$. Then*

$$-\triangle\Phi(u) \geq \Phi'(u)f,$$

in the sense that

$$-\int_\Omega \Phi(u)\triangle\zeta \geq \int_\Omega \Phi'(u)f\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$, such that $\zeta = 0$ on $\partial\Omega$.

Proof. Consider $(f_n)_{n\geq 0} \subset \mathcal{D}(\Omega)$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let u_n be the solution of

$$\begin{cases} -\triangle u_n = f_n & \text{in} \quad \Omega, \\ \quad u_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It follows from Lemma 1 that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^1(\Omega)$. On the other hand we have

$$\triangle\Phi(u_n) = \Phi'(u_n)\triangle u_n + \Phi''(u_n)|\nabla u_n|^2 \leq \Phi'(u_n)\triangle u_n = -\Phi'(u_n)f_n.$$

Therefore,

$$-\int_\Omega \Phi(u_n)\triangle\zeta \geq \int_\Omega \Phi'(u_n)f_n\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ such that $\zeta = 0$ on $\partial\Omega$; and so the result follows easily by letting $n \to \infty$. $\square$

Lemma 3. *Let $\overline{w}$ be a weak super-solution of* (2), *in the sense that $\overline{w} \in L^1(\Omega)$, $\overline{w} \geq 0$, $g(\overline{w})\delta \in L^1(\Omega)$, where δ is given by* (5), *and*

$$-\int_\Omega \overline{w}\triangle\zeta \geq \int_\Omega g(\overline{w})\zeta, \tag{17}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Then there exists a weak solution w of (2) *with $0 \leq w \leq \overline{w}$.*

Proof. We use a standard monotone iteration argument: define the sequence $(w_n)_{n\geq 1}$ by

$$\begin{cases} -\triangle w_{n+1} = g(w_n) & \text{in} \quad \Omega, \\ \quad w_{n+1} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for $n \geq 1$, starting with $w_1 = \overline{w}$. It is easy to check that $\overline{w} = w_1 \geq w_2 \geq \cdots \geq 0$. Indeed, it suffices to prove that $w_1 \geq w_2 \geq 0$, and the rest follows by induction, using Lemma 1. We have

$$\int_\Omega (w_1 - w_2)(-\triangle\zeta) \geq 0, \tag{18}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Given $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, let ζ_φ be the solution of

$$\begin{cases} -\triangle\zeta_\varphi = \varphi & \text{in} \quad \Omega, \\ \zeta_\varphi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Taking $\zeta = \zeta_\varphi$ in (18), we obtain

$$\int_\Omega (w_1 - w_2)\varphi \geq 0.$$

Since $\varphi \geq 0$ is arbitrary, we deduce that $w_2 \leq w_1$ almost everywhere in Ω. On the other hand, it follows from Lemma 1 that $w_2 \geq 0$.

Since the sequence $(w_n)_{n\geq 1}$ is nonincreasing, it converges to a limit $u \in L^1(\Omega)$, which is clearly a weak solution of (2). $\square$

An essential ingredient in the proof of Theorem 3 is the following.

Lemma 4. *Assume $g(0) > 0$ and set*

$$h(u) = \int_0^u \frac{ds}{g(s)},$$

for all $u \geq 0$. Let $\widetilde{g}$ be a C^1 positive function on $[0,\infty)$ such that $\widetilde{g} \leq g$ and $\widetilde{g}' \leq g'$. Set

$$\widetilde{h}(u) = \int_0^u \frac{ds}{\widetilde{g}(s)},$$

and

$$\Phi(u) = \widetilde{h}^{-1}(h(u)),$$

for all $u \geq 0$. Then

(i) *$\Phi(0) = 0$ and $0 \leq \Phi(u) \leq u$ for all $u \geq 0$.*

(ii) *Φ is increasing, concave and $\Phi'(u) \leq 1$ for all $u \geq 0$.*

(iii) *If $h(\infty) < \infty$ and $\widetilde{g} \not\equiv g$, then $\Phi(\infty) < \infty$.*

Proof. Properties (i) and (iii) are clear. We have

$$\Phi'(u) = \frac{\widetilde{g}(\Phi(u))}{g(u)} > 0,$$

and

$$\begin{aligned} \Phi''(u) &= \frac{g(u)\widetilde{g}'(\Phi(u))\Phi'(u) - \widetilde{g}(\Phi(u))g'(u)}{g(u)^2} \\ &= \frac{\widetilde{g}(\Phi(u))(\widetilde{g}'(\Phi(u)) - g'(u))}{g(u)^2}. \end{aligned}$$

Since $\widetilde{g}'(\Phi(u)) \le g'(\Phi(u)) \le g'(u)$, it follows that Φ is concave. Hence (ii). □

Proof of Theorem 3. If $g(0) = 0$, then 0 is a weak solution of (7), so we assume $g(0) > 0$. We consider two cases.

Case 1. Suppose

$$\int_0^\infty \frac{ds}{g(s)} < \infty.$$

Let $v = \Phi(w)$, with the notation of Lemma 4, where $\widetilde{g} = (1-\varepsilon)g$. It follows from Lemmas 2 and 4 that $v \in L^\infty(\Omega)$ is a super-solution of (7). The result follows from Lemma 3.

Case 2. Suppose

$$\int_0^\infty \frac{ds}{g(s)} = \infty.$$

Let $\widetilde{g} = (1-\varepsilon)g$, and consider the function Φ introduced in Lemma 4. Set

$$v_1 = \Phi(w).$$

We have $0 \le v_1 \le w$. We observe that by concavity of the function $h(u) = \int_0^u \frac{ds}{g(s)}$,

$$h(w) \le h(v_1) + (w - v_1)h'(v_1) = h(v_1) + \frac{w - v_1}{g(v_1)}.$$

Since $h(v_1) = (1-\varepsilon)h(w)$, we deduce that

$$\varepsilon g(v_1) \le \frac{w - v_1}{h(w)} \le \frac{w}{h(w)} \le C(1 + w),$$

so that in particular, $g(v_1) \in L^1(\Omega)$. Now, we observe that by Lemma 2, v_1 is a weak super-solution of the equation

$$\begin{cases} -\triangle u_1 = (1-\varepsilon)g(u_1) & \text{in} \quad \Omega, \\ \quad u_1 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{19}$$

Therefore, it follows from Lemma 3 that there exists a weak solution u_1 of (19) such that $0 \le u_1 \le v_1$. In particular, we have $0 \le g(u_1) \le g(v_1) \in L^1(\Omega)$, so that $u_1 \in L^p(\Omega)$, for all $p \ge 1$ such that (see e.g. Stampacchia [15])

$$p < \frac{N}{N-2} \quad (p \le \infty \;\text{ if }\; N = 1,\; p < \infty \;\text{ if }\; N = 2). \tag{20}$$

By the same construction, we find a solution u_2 of the equation

$$\begin{cases} -\triangle u_2 = (1-\varepsilon)^2 g(u_2) & \text{in} \quad \Omega, \\ \quad u_2 = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

such that $0 \le u_2 \le u_1$ and $g(u_2) \le C(1+u_1)$. In particular, $g(u_2) \in L^p(\Omega)$, for all $p \ge 1$ satisfying (20). This implies that $u_2 \in L^r(\Omega)$, for all $r \ge 1$ such that $r < \frac{N}{N-4}$ ($r \le \infty$ if $N = 1,2,3$, $r < \infty$ if $N = 4$). By iteration, we find that if $k(N) = [N/2]+1$, then the solution u_k of the equation

$$\begin{cases} -\triangle u_k = (1-\varepsilon)^k g(u_k) & \text{in} \quad \Omega, \\ \quad u_k = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

belongs to $L^\infty(\Omega)$. Since $\varepsilon \in (0,1)$ is arbitrary, this completes the proof. □

3. Proof of Theorem 1. We assume $g(0) > 0$, for otherwise $w \equiv 0$ is a weak solution of (2). Furthermore, we may also assume that $u_0 = 0$, so that $u \ge 0$ and $u_t \ge 0$ for all $t \ge 0$.

Next, observe that $g'(u) \underset{u\to\infty}{\longrightarrow} +\infty$ by (3), so that there exists a constant $M > 0$ such that

$$g(s) - \lambda_1 s \ge \frac{1}{2} g(s) \quad \text{for} \quad s \ge M, \tag{21}$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. Let $\varphi \in C^2(\overline{\Omega})$ with $\varphi_{|\partial\Omega} = 0$. It follows from (1) that

$$\frac{d}{dt}\int_\Omega u(t)\varphi + \int_\Omega u(t)(-\triangle\varphi) = \int_\Omega g(u(t))\varphi. \tag{22}$$

We first claim that

$$\sup_{t\ge 0}\int_\Omega g(u)\varphi_1 \le (1+\lambda_1)M, \tag{23}$$

where M is as in (22) and φ_1 is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$ such that $\int_\Omega \varphi_1 = 1$. Indeed, taking $\varphi = \varphi_1$ in (22), we find

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 + \lambda_1\int_\Omega u(t)\varphi_1 = \int_\Omega g(u(t))\varphi_1 \ge g\left(\int_\Omega u(t)\varphi_1\right), \tag{24}$$

by Jensen's inequality. If there exists $t_0 \ge 0$ such that $\int_\Omega u(t_0)\varphi_1 > M$, then it follows from (24) and (21) that

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \ge \frac{1}{2} g\left(\int_\Omega u(t)\varphi_1\right),$$

for $t \ge t_0$, which is absurd by (3); and so

$$\int_\Omega u(t)\varphi_1 \le M,$$

for all $t \geq 0$. Integrating (24) on $(t, t+1)$ and since $u_t \geq 0$, we find

$$\int_\Omega g(u(t))\varphi_1 \leq \int_t^{t+1} \int_\Omega g(u)\varphi_1 \leq \int_\Omega u(t+1)\varphi_1 + \lambda_1 \int_t^{t+1} \int_\Omega u\varphi_1 \\ \leq (1+\lambda_1)M,$$

hence (23).

We next claim that there exists K such that

$$\sup_{t \geq 0} \|u(t)\|_{L^1} \leq K. \tag{25}$$

Indeed, let ζ_0 be the solution of (16). Taking $\varphi = \zeta_0$ in (22) and integrating on $(t, t+1)$, we find

$$\int_\Omega u(t) \leq \int_t^{t+1} \int_\Omega u = \int_\Omega u(t)\zeta_0 - \int_\Omega u(t+1)\zeta_0 + \int_t^{t+1} \int_\Omega g(u)\zeta_0,$$

and (25) follows by applying (23).

By monotone convergence, it follows from (25) and (23) that $u(t)$ has a limit w in $L^1(\Omega)$ and that $g(u)$ converges to $g(w)$ in $L^1(\Omega, \delta(x)dx)$, as $t \to \infty$. Let $\varphi \in C^2(\overline{\Omega})$, $\varphi_{|\partial\Omega} = 0$. Integrating (22) on $(t, t+1)$, we obtain

$$\left[\int_\Omega u\varphi\right]_t^{t+1} + \int_t^{t+1} \int_\Omega u(t)(-\triangle\varphi) = \int_t^{t+1} \int_\Omega g(u(t))\varphi.$$

Letting $t \to \infty$, we find

$$\int_\Omega w(-\triangle\varphi) = \int_\Omega g(w)\varphi.$$

Therefore, w is a weak solution of (2). $\square$

We now give an alternative proof of Theorem 1 in the spirit of the proof of Theorem 3. It makes use of the following lemma.

Lemma 5. *Assume* (9), *and let λ^* be the supremum of all $\lambda > 0$ such that* (8) *has a minimal, positive, classical solution $u(\lambda)$. Then $\lambda^* < \infty$. If furthermore* (10) *holds, then $\lim_{\lambda \uparrow \lambda^*} u(\lambda) = u^*$ is a weak solution of* (8) *with $\lambda = \lambda^*$.*

Proof. We first observe that by (9) and convexity of g, there exists $\varepsilon > 0$ such that $g(u) \geq \varepsilon u$, for all $u \geq 0$; and so

$$-\triangle u(\lambda) \geq \lambda\varepsilon u(\lambda). \tag{26}$$

Let λ_1 be the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, and let φ_1 be a corresponding eigenvector. Multiplying (26) by φ_1, we see that $\lambda\varepsilon \leq \lambda_1$; and so, $\lambda^* \leq \frac{\lambda_1}{\varepsilon}$. If (10) holds, then there exists C such that $g(u) \geq \frac{2\lambda_1}{\lambda^*}u - C$, for all $u \geq 0$. Multiplying (8) by φ_1, we obtain

$$\lambda \int_\Omega g(u(\lambda))\varphi_1 = \lambda_1 \int_\Omega u(\lambda)\varphi_1 \leq \frac{\lambda^*}{2} \int_\Omega (g(u(\lambda)) + C)\varphi_1.$$

Letting $\lambda \uparrow \lambda^*$, we deduce that

$$\lim_{\lambda \uparrow \lambda^*} \int_\Omega g(u(\lambda))\varphi_1 < \infty. \tag{27}$$

Multiplying now (8) by the solution ζ_0 of (16), we obtain

$$\int_\Omega u(\lambda) = \lambda \int_\Omega g(u(\lambda))\zeta_0 \leq C\lambda \int_\Omega g(u(\lambda))\varphi_1,$$

so that $u(\lambda)$ is bounded in $L^1(\Omega)$ by (27). Since $u(\lambda)$ is increasing in λ, it follows that $u(\lambda)$ has a limit $u^* \in L^1(\Omega)$ and that $g(u(\lambda))$ converges to $g(u^*)$ in $L^1(\Omega, \delta(x)\,dx)$. It follows easily that u^* is a weak solution of (8) with $\lambda = \lambda^*$. $\square$

Alternative proof of Theorem 1. We may assume as above that $g(0) > 0$ and $u_0 = 0$. Given $0 < \varepsilon < 1$, let $\widetilde{g} = (1-\varepsilon)g$ and let Φ be as in Lemma 4. Set $v_\varepsilon(t) = \Phi(u(t))$, for all $t \geq 0$. It follows from Lemma 4 that there exists $M_\varepsilon < \infty$ such that

$$0 \leq v_\varepsilon \leq M_\varepsilon. \tag{28}$$

Furthermore, it follows from Lemmas 2 and 4 that

$$-\triangle v_\varepsilon \geq \Phi'(u)(-\triangle u) = \Phi'(u)(g(u) - u_t) = (1-\varepsilon)g(v_\varepsilon) - (v_\varepsilon)_t,$$

so that v_ε is a super-solution of the equation

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \triangle u_\varepsilon = (1-\varepsilon)g(u_\varepsilon), \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \\ u_\varepsilon(0) = 0. \end{cases} \tag{29}$$

It now follows from (28) that the solution u_ε of (29) is global and bounded by M_ε. As above, we deduce that $w_\varepsilon = \lim_{t\to\infty} u_\varepsilon(t)$ a (classical) solution of the equation

$$\begin{cases} -\triangle w_\varepsilon = (1-\varepsilon)g(w_\varepsilon) & \text{in} \quad \Omega, \\ w_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It follows (see property c) in the introduction) that $\lambda^* \geq 1$. By Lemma 5, (2) has a weak solution. $\square$

4. Proof of Theorem 2. Since the hypotheses of Theorem 2 allow g to vanish at the origin, we need a variant of Lemma 4 that applies to the case $g(0) = 0$.

Lemma 6. *Assume* (3). *There exist constants* $K \geq 0$ *and* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, *there is a function* $\Phi_\varepsilon \in C^2([0,\infty))$, *concave, increasing, with*

$$\Phi_\varepsilon(0) = 0, \tag{30}$$

$$0 < \Phi_\varepsilon(x) \leq x \quad \text{for} \quad x > 0, \tag{31}$$

$$1 \geq \Phi_\varepsilon'(x) \geq \frac{(g(\Phi_\varepsilon(x)) - \varepsilon K)^+}{g(x)} \quad \text{for} \quad x \geq 0. \tag{32}$$

Moreover, $\sup_{x\geq 0} \Phi_\varepsilon(x) < \infty$.

Proof. If $g(0) > 0$ we apply Lemma 4 with $\widetilde{g}(u) = g(u) - \varepsilon$ and the conclusions follow with $\varepsilon_0 = g(0)$ and $K = 1$.

Therefore we may assume that $g(0) = 0$. Let $a > 0$ be the unique solution of $g(a) = 1$. Set

$$H(x) = a + \int_a^x \frac{ds}{g(s)} \text{for} \quad x \geq a.$$

Since $g(a) = 1$, there exists $0 < \varepsilon_0 < 1$ such that $0 < \varepsilon < g((1-\varepsilon)a)$, for $0 < \varepsilon < \varepsilon_0$. For such an ε, let

$$H_\varepsilon(x) = a + \int_{(1-\varepsilon)a}^x \frac{ds}{g(s) - \varepsilon} \quad \text{for} \quad x \geq (1-\varepsilon)a.$$

Note that $H_\varepsilon((1-\varepsilon)a) = a = H(a)$. Moreover,

$$\lim_{x\to\infty} H_\varepsilon(x) > a + \int_{(1-\varepsilon)a}^\infty \frac{ds}{g(s)} \geq \lim_{x\to\infty} H(x).$$

Thus, $\Psi_\varepsilon(x) = H_\varepsilon^{-1}(H(x))$, is well defined for $x \geq a$, $\Psi_\varepsilon(a) = (1-\varepsilon)a$ and $\sup_{x\geq a} \Psi_\varepsilon(x) < \infty$. Furthermore, for $x \geq a$,

$$\Psi_\varepsilon'(x) = \frac{g(\Psi_\varepsilon(x)) - \varepsilon}{g(x)}. \tag{33}$$

In addition, for $x \geq a$ we have

$$\begin{aligned}\Psi_\varepsilon''(x) &= \frac{g(x)g'(\Psi_\varepsilon(x))\Psi_\varepsilon'(x) - (g(\Psi_\varepsilon(x)) - \varepsilon)g'(x)}{g(x)^2} \\ &= \frac{(g(\Psi_\varepsilon(x)) - \varepsilon)(g'(\Psi_\varepsilon(x)) - g'(x))}{g(x)^2} \leq 0,\end{aligned}$$

since $\Psi_\varepsilon(x) \leq x$ thus $g'(\Psi_\varepsilon(x)) \leq g'(x)$. We finally consider a concave function $\Phi_\varepsilon \in C^2([0,\infty))$ such that $\Phi_\varepsilon(x) = \Psi_\varepsilon(x)$ for $x \geq a$, $\Phi_\varepsilon(0) = 0$, and $\Phi_\varepsilon'(x) \leq 1$ for all $x \geq 0$. Such a function exists since

$$\Psi_\varepsilon'(a) \leq \frac{\Psi_\varepsilon(a)}{a} \leq 1.$$

Clearly Φ_ε satisfies (30) and (31). We claim that (32) holds with $K = 1 + ag'(a)$. Indeed, it follows from (33) that for $x \geq a$

$$\Phi'_\varepsilon(x) \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)},$$

so that (32) holds for $x \geq a$ (since $\Phi'_\varepsilon \geq 0$). For $x \leq a$, we have

$$\Phi'_\varepsilon(x) \geq \Phi'_\varepsilon(a) = g((1-\varepsilon)a) - \varepsilon.$$

Furthermore, by convexity,

$$g((1-\varepsilon)a) \geq g(a) - \varepsilon a g'(a) = 1 - \varepsilon(K-1);$$

and so, for $x \leq a$,

$$\begin{aligned} \Phi'_\varepsilon(x) &\geq 1 - \varepsilon K = 1 - \frac{\varepsilon K}{g(a)} \geq 1 - \frac{\varepsilon K}{g(x)} \\ &= \frac{g(x) - \varepsilon K}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)}. \end{aligned}$$

It follows that (32) is satisfied for $x \leq a$, which completes the proof. □

Lemma 7. *Let δ be given by (5). For every $0 < T < \infty$, there exists $\varepsilon_1(T) > 0$ such that if $0 < \varepsilon \leq \varepsilon_1$, then the solution Z of the equation*

$$\begin{cases} Z_t - \triangle Z = -\varepsilon & \text{in} \quad (0,\infty) \times \Omega, \\ Z = 0 & \text{on} \quad (0,\infty) \times \partial\Omega, \\ Z(0) = \delta, \end{cases}$$

satisfies $Z \geq 0$ on $[0,T] \times \overline{\Omega}$.

Proof. Let $(T(t))_{t\geq 0}$ be the heat semigroup with Dirichlet boundary condition, and consider the solution ζ_0 of (16). We have

$$\zeta_0 = T(t)\zeta_0 + \int_0^t T(s) 1_\Omega \, ds,$$

for all $t \geq 0$. Since $T(t)\zeta_0 \geq 0$, it follows that

$$\int_0^t T(s) 1_\Omega \, ds \leq \zeta_0 \leq C\delta, \tag{35}$$

for all $t \geq 0$. On the other hand, we have

$$Z(t) = T(t)\delta - \varepsilon \int_0^t T(s) 1_\Omega \, ds;$$

and so,

$$Z(t) \geq T(t)\delta - \varepsilon C\delta.$$

Consider now $c_0, c_1 > 0$ such that $c_0\varphi_1 \leq \delta \leq c_1\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$, associated to the eigenvalue λ_1. We have

$$T(t)\delta \geq c_0 T(t)\varphi_1 = c_0 e^{-\lambda_1 t}\varphi_1 \geq \frac{c_0}{c_1} e^{-\lambda_1 t}\delta.$$

Therefore,

$$Z(t) \geq \big(\frac{c_0}{c_1} e^{-\lambda_1 t} - \varepsilon C\big)\delta.$$

It follows that $Z(t) \geq 0$ on $[0, T]$, provided $\varepsilon \leq \dfrac{c_0}{c_1 C} e^{-\lambda_1 T}$. $\square$

Proof of Theorem 2. If (3) fails, then the solution of

$$\theta' = g(\theta), \qquad \theta(0) = \|u_0\|_{L^\infty},$$

is global. Since $\theta(t)$ is a super-solution of (1) and 0 is a sub-solution, it follows that all the solutions of (1) are global.

We now assume that (3) holds. Furthermore, we may assume

$$w \notin L^\infty(\Omega), \tag{36}$$

since otherwise $u(t) \leq w$ by the maximum principle, and so u is global. We denote by $[0, T_m)$ the maximal interval of existence of u, and we now proceed in five steps.

Step 1. We have $u(t) \leq w$ for all $t \in [0, T_m)$. (Note that if w were a smooth solution of (2), this would follow from the maximum principle.) Fix $T < T_m$. Let $h(t, x) \in \mathcal{D}((0, T) \times \Omega)$, $h \geq 0$, and let ζ be the solution of

$$-\zeta_t - \triangle\zeta = h, \quad \zeta_{|\partial\Omega} = 0, \quad \zeta(T) = 0.$$

We have in particular $\zeta \in C([0, T], C^2(\overline{\Omega}) \cap C_0(\Omega))$. Multiplying (1) by ζ and integrating on $(0, T) \times \Omega$, we find

$$-\int_\Omega u_0\zeta(0) + \int_0^T \int_\Omega uh = \int_0^T \int_\Omega g(u)\zeta.$$

On the other hand,

$$-\int_0^T \int_\Omega w\zeta_t - \int_\Omega w\zeta(0) = 0,$$

and

$$-\int_0^T \int_\Omega w\triangle\zeta = \int_0^T \int_\Omega g(w)\zeta.$$

Therefore,

$$-\int_\Omega (u_0 - w)\zeta(0) + \int_0^T \int_\Omega (u-w)h = \int_0^T \int_\Omega (g(u) - g(w))\zeta.$$

Since $\zeta \geq 0$ and $u_0 - w \leq 0$, this yields

$$\int_0^T \int_\Omega (u-w)h \leq \int_0^T \int_{\{u \geq w\}} (g(u) - g(w))\zeta \leq C \int_0^T \int_\Omega (u-w)^+ \zeta.$$

(Note that $\|u\|_{L^\infty((0,T)\times\Omega)} < \infty$, so that g is Lipschitz on $[0, \|u\|_{L^\infty((0,T)\times\Omega)}]$.) Therefore,

$$\int_0^T \int_\Omega (u-w)h \leq C \left(\int_0^T \int_\Omega [(u-w)^+]^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega \zeta^2 \right)^{\frac{1}{2}}.$$

On the other hand,

$$\zeta(t) = \int_t^T T(s-t)h(s)\, ds,$$

where $(T(t))_{t \geq 0}$ is the heat semigroup with Dirichlet boundary condition, thus

$$\|\zeta(t)\|_{L^2}^2 \leq \left(\int_t^T \|h(s)\|_{L^2}\, ds \right)^2 \leq (T-t) \int_0^T \int_\Omega h^2.$$

Therefore,

$$\int_0^T \int_\Omega \zeta^2 \leq \frac{T^2}{2} \int_0^T \int_\Omega h^2;$$

and so,

$$\int_0^T \int_\Omega (u-w)h \leq \frac{CT}{\sqrt{2}} \left(\int_0^T \int_\Omega [(u-w)^+]^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega h^2 \right)^{\frac{1}{2}}.$$

Now we observe that $(u-w)^+ \in L^\infty((0,T)\times\Omega)$, and we let h converge to $(u-w)^+$ in $L^2((0,T)\times\Omega)$ and be bounded in $L^\infty((0,T)\times\Omega)$. Since $u - w \in L^1(\Omega)$, we obtain

$$\int_0^T \int_\Omega [(u-w)^+]^2 \leq \frac{CT}{\sqrt{2}} \int_0^T \int_\Omega [(u-w)^+]^2.$$

It follows that $u \leq w$ provided $C^2T^2 < 2$. The result follows by iteration.

Step 2. There exist $0 < \tau < T_m$ and $C_0, c_0 > 0$ such that

$$u(\tau) \leq C_0 \delta, \tag{37}$$

and

$$u(\tau) \leq w - c_0 \delta. \tag{38}$$

Set $v_0 = \min\{w, 1+u_0\}$. We have $v_0 \geq u_0$ and $v_0 \not\equiv u_0$ by (36). In particular, there exists a function $\gamma : [0,\infty) \to \mathbb{R}$ such that $\gamma(t) > 0$ for $t > 0$ and

$$T(t)(v_0 - u_0) \geq \gamma(t)\delta, \tag{39}$$

where δ is defined by (5) and $(T(t))_{t\geq 0}$ is the heat semigroup with Dirichlet boundary condition. Let v be the solution of (1) with the initial value $v(0) = v_0$, and let $[0, \overline{T})$ be the maximal interval of existence of v. We have $v \geq 0$, and by Step 1, $v \leq w$. Let $z(t) = u(t) + T(t)(v_0 - u_0)$ for $0 \leq t < \overline{T}$. We have

$$\begin{cases} z_t - \triangle z = g(u) \leq g(z) & \text{in} \quad (0,\overline{T}) \times \Omega, \\ z = 0 & \text{on} \quad \partial\Omega, \\ z(0) = v_0 & \text{in} \quad \Omega, \end{cases}$$

so that $z \leq v$ by the maximum principle. Therefore,

$$u(t) \leq v(t) - T(t)(v_0 - u_0) \leq w - T(t)(v_0 - u_0) \leq w - \gamma(t)\delta, \tag{40}$$

for $0 \leq t < \overline{T}$ by (39). Fix $0 < T < \min\{\overline{T}, T_m\}$. u is bounded by some constant M on $[0,T] \times \overline{\Omega}$, so that

$$u(t) \leq MT(t)1_\Omega + g(M)\int_0^t T(s)1_\Omega \, ds.$$

There exists a function $\overline{C} : (0,\infty) \to \mathbb{R}$ such that $T(t)1_\Omega \leq C(t)\delta$ for $t > 0$, so that we deduce from (35) that

$$u(t) \leq MC(t)\delta + g(M)C\delta, \tag{41}$$

for $0 < t \leq T$. (37) and (38) now follow from (40) and (41).

Step 3. We may assume without loss of generality that

$$u_0 \leq C_0\delta, \tag{42}$$

and

$$u_0 \leq w - c_0\delta, \tag{43}$$

where C_0, c_0 are as in Step 2. Indeed, we need only consider $u(\cdot + \tau)$ instead of $u(\cdot)$.

Step 4. Let ε_0 and Φ_ε be as in Lemma 6, and set $w_\varepsilon = \Phi_\varepsilon(w)$ for $0 < \varepsilon < \varepsilon_0$. Then

$$w_\varepsilon \in L^\infty(\Omega), \tag{44}$$

and

$$\int_\Omega \zeta(-\triangle w_\varepsilon) \geq \int_\Omega (g(w_\varepsilon) - \varepsilon K)\zeta, \tag{45}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ on Ω and $\zeta_{|\partial\Omega} = 0$. Moreover, there exists $0 < \varepsilon_1 \leq \varepsilon_0$ such that

$$u_0 \leq w_\varepsilon - \frac{c_0}{2}\delta, \tag{46}$$

for $0 < \varepsilon < \varepsilon_1$, where c_0 is as in (43). Indeed, (44) and (45) follow from Lemmas 2 and 6. In order to prove (46), set

$$\eta = \min\{w, (C_0 + c_0)\delta\}, \quad \text{and} \quad \eta_\varepsilon = \Phi_\varepsilon(\eta).$$

Here, δ is given by (5) and C_0 is as in (42). It follows from (42) and (43) that

$$u_0 \le \eta - c_0\delta. \tag{47}$$

We claim that

$$\eta \le \eta_\varepsilon + \frac{c_0}{2}\delta, \tag{48}$$

for $\varepsilon > 0$ small enough. Note that it follows from (47) and (48) that $u_0 \le \eta_\varepsilon - \frac{c_0}{2}\delta$, and (46) follows since $\eta_\varepsilon \le w_\varepsilon$ (since Φ_ε is nondecreasing). Thus we need only prove (48). Note that $\eta_\varepsilon \le \eta \le M$, where $M = (C_0 + c_0)\|\delta\|_{L^\infty}$, and that $\Phi'_\varepsilon(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} 1$, uniformly on $[0, M]$ by Lemma 6. Therefore,

$$\eta - \eta_\varepsilon \le \eta \sup_{0\le x\le M}(1 - \Phi'_\varepsilon(x)) \le (C_0 + c_0)\delta \sup_{0\le x\le M}(1 - \Phi'_\varepsilon(x)) \le \frac{c_0}{2}\delta,$$

for ε small enough, and (48) follows.

Step 5. Conclusion. Assume for the sake of contradiction that $T_m < \infty$. Let $\varepsilon > 0$ be small enough so that

$$u_0 \le w_\varepsilon - \frac{c_0}{2}\delta$$

(see Step 4), and so that the solution Z of the equation

$$\begin{cases} Z_t - \triangle Z = -\varepsilon K & \text{in} \quad (0, T_m) \times \Omega, \\ Z = 0 & \text{on} \quad \partial\Omega, \\ Z(0) = \dfrac{c_0}{2}\delta & \text{in} \quad \Omega, \end{cases}$$

is nonnegative on $[0, T_m] \times \overline{\Omega}$ (see Lemma 7; here, K is given by Lemma 6). Let v be the solution of

$$\begin{cases} v_t - \triangle v = g(|v|) - \varepsilon K & \text{in} \quad (0, T) \times \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \\ v(0) = w_\varepsilon & \text{in} \quad \Omega. \end{cases}$$

Let $[0, S_m)$ be the maximal interval of existence of v. Set $z(t) = Z(t) + u(t)$ for $0 \le t < T_m$. We have $z \ge u \ge 0$ and

$$\begin{cases} z_t - \triangle z = g(u) - \varepsilon K \le g(z) - \varepsilon K & \text{on} \quad (0, T_m) \times \Omega, \\ z_{|\partial\Omega} = 0, & \\ z(0) = u_0 + \dfrac{c_0}{2}\delta \le w_\varepsilon & \text{in} \quad \Omega. \end{cases}$$

By the maximum principle, we have $z \le v$ on $[0, \min\{T_m, S_m\})$. In particular, $v \ge 0$ on $[0, \min\{T_m, S_m\})$; by the maximum principle and (45), $v \le w_\varepsilon$. Since $w_\varepsilon \in L^\infty(\Omega)$, this implies that $T_m < S_m = +\infty$. Therefore, $u \le z \le v \le w_\varepsilon$ on $[0, T_m)$, which is absurd. □

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REFERENCES

[1] H. Bellout, *A criterion for blow-up of solutions to semi-linear heat equations,* SIAM J. Math. Anal., 18 (1987), 722–737.

[2] H. Brezis and L. Nirenberg, "Nonlinear Functional Analysis and Applications, Part 1," to appear.

[3] M.G. Crandall and P.H Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems,* Arch. Rational Mech. Anal., 58 (1975), 207–218.

[4] H. Fujita, *On the nonlinear equations* $\triangle u + e^u = 0$ *and* $\frac{\partial u}{\partial t} = \triangle u + e^u$*,* Bull. Amer. Math. Soc., 75 (1969), 132–135.

[5] H. Fujita, *On the asymptotic stability of solutions of the equation* $v_t = \triangle v + e^v$*,* in "Proc. Internat. Conf. on Functional Analysis and Related Topics" (Tokyo, 1970), Univ. of Tokyo Press, Tokyo, 1970, 252–259.

[6] T. Gallouët, F. Mignot and J.-P. Puel, *Quelques résultats sur le problème* $-\triangle u = \lambda e^u$*,* C. R. Acad. Sci. Paris, 307 (1988), 289–292.

[7] I.M. Gel'fand, *Some problems in the theory of quasilinear equations,* Amer. Math. Soc. Transl., 29 (1963), 295–381.

[8] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources,* Arch. Rational Mech. Anal., 49 (1973), 241–269.

[9] T. Kato, *Schrödinger operators with singular potentials,* Israel J. Math., 13 (1972), 135–148.

[10] H.B. Keller and D.S. Cohen, *Some positone problems suggested by nonlinear heat generation,* J. Math. Mech., 16 (1967), 1361–1376.

[11] H.B. Keller and J. Keener, *Positive solutions of convex nonlinear eigenvalue problems,* J. Differ. Eq., 16 (1974), 103–125.

[12] A. Lacey, *Mathematical analysis of thermal runaway for spatially inhomogeneous reactions,* SIAM J. Appl. Anal., 43 (1983), 1350–1366.

[13] A. Lacey and D. Tzanetis, *Global existence and convergence to a singular steady state for a semilinear heat equation,* Proc. Royal Soc. Edinburgh Sect. A, 105 (1987), 289–305.

[14] F. Mignot and J.-P. Puel, *Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe,* Comm. PDE., 5 (1980), 791–836.

[15] G. Stampacchia, "Équations elliptiques du second ordre à coefficients discontinus," Les Presses de l'Université de Montréal, Montréal, 1966.

[16] F.B. Weissler, *Single point blow-up for a semilinear initial value problem,* J. Diff. Eq., 55 (1984), 204–224.