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DEGENERATE PARABOLIC PDEs WITH DISCONTINUITIES AND GENERALIZED EVOLUTIONS OF SURFACES

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Abstract. We establish comparison theorems for solutions of degenerate parabolic partial differential equations with discontinuities (or singularities). The results are used to define the generalized evolutions of sets generated by geometric partial differential equations with discontinuities. These results generalize recent results obtained by Ohnuma-Sato ([13]) and Gurtin-Soner-Souganidis ([11]).

1. Introduction. In this paper we shall be concerned with the degenerate parabolic partial differential equation (PDE in short)

$$u_t(x,t) + F(Du(x,t), D^2u(x,t)) = 0 \quad \text{in} \ \ Q_T \equiv \Omega \times (0,T). \tag{1.1}$$

Here Ω is a domain of $\mathbf{R}^N$, $T > 0$ is a given constant, u represents the real unknown function on $\Omega \times (0,T)$ and F is a real function on $\mathbf{R}^N \times \mathbf{S}^N$, where $\mathbf{S}^N$ denotes the set of real $N \times N$ symmetric matrices.

Recent developments have revealed that equations (1.1) with discontinuous or singular F are important in the study of generalized evolutions of surfaces, especially, in the level set approach.

Chen, Giga and Goto ([4]) and Evans and Spruck ([6]) initiated the level set approach, on a firm mathematical basis, to evolutions of surfaces driven by their mean curvature or some other geometric quantities alike. In their approach there appear PDEs (1.1) with $F(p,X)$ discontinuous for $p = 0$. More recently Ohnuma and Sato ([13]) and Gurtin, Soner and Souganidis ([11]) studied PDE (1.1) with $F(p,X)$ discontinuous in a finite number of directions of p's. In this connection, see also [1], [2] and [9].

In this paper we study PDE (1.1) with $F(p,X)$ having discontinuities in a continuum of directions of p's. Comparison and existence results for solutions of (1.1) will be established. Following [4] and [6], we shall define the generalized evolution of sets generated by (1.1) via the level set approach. The assumptions and main results in this direction will be discussed in Sections 2 and 3. The proof of those results are provided in Section 5. The assumptions are examined in two examples of PDEs in Section 4. Section 6 provides a proof of lemmas which are needed in our proof of main results.

Concerning the singularity of $F(p,X)$ for $p = 0$, we restrict here our study to that studied by [4] and [6]. However, by using the ideas in [10] and [12], PDE (1.1) with higher singularities for $p = 0$ can be treated as well by our method.

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An interesting degenerate palabolic PDE with singularities arises in [7], where the author studies the Cauchy problem for the heat equation with multivalued functions as initial data. The singular PDE suggests that our assumption (A2) below is somehow optimal for solutions of the Cauchy problem to be continuous.

2. A comparison theorem. We begin with assumptions on F. We will use the notation:

$$S^{N-1} = \{x \in \mathbf{R}^N : |x| = 1\} \quad \text{and} \quad \mathbf{R}_+ = (0, \infty).$$

We shall thus write $\overline{\mathbf{R}}_+ = [0, \infty)$ and, for any $M \subset \mathbf{R}^N$, $\overline{\mathbf{R}}_+ M = \{tx : t \geq 0, x \in M\}$.

(A1) There is a C^2 submanifold M without boundary of S^{N-1} of dimension $d \in \{0, \ldots, N-2\}$ such that

$$F \in C\big((\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M) \times \mathbf{S}^N\big).$$

Our next assumption is a kind of continuity requirement on F on the set $(\overline{\mathbf{R}}_+ M) \times \mathbf{S}^N$, which is weaker than the usual continuity.

To explain this, let us introduce the notation. Let V be a linear subspace of $\mathbf{R}^N$. We denote by π_V the matrix representation of the orthogonal projection of $\mathbf{R}^N$ onto V. We define the subspace $\mathbf{S}^N_V$ of $\mathbf{S}^N$ by

$$\mathbf{S}^N_V = \{X \in \mathbf{S}^N : \pi_V X \pi_V = X\}.$$

For $p \in M$ let $T_p M$ denote the tangent space of M at p, which we will regard as a subspace of $\mathbf{R}^N$ and define $\mathbf{S}^N(p)$ by

$$\mathbf{S}^N(p) = \mathbf{S}^N_V, \qquad \text{with} \ \ V = T_p M \oplus \operatorname{span}\{p\}.$$

We set $\mathbf{S}^N(p) = \mathbf{S}^N(p/|p|)$ for $p \in \mathbf{R}_+ M$ and define $\mathbf{S}^N(0)$ as the subset $\{0\}$ of $\mathbf{S}^N$.

(A2) $F^*(p, X) = F_*(p, X) \in \mathbf{R}$ for all $p \in \overline{\mathbf{R}}_+ M$ and $X \in \mathbf{S}^N(p)$, where F^* and F_* are defined by

$$F^*(\xi) = \lim_{\varepsilon \downarrow 0} \sup \{F(\eta) \mid \eta \in (\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M) \times \mathbf{S}^N, \|\eta - \xi\| < \varepsilon\},$$

and $F_* = -(-F)^*$, respectively. Here $\|\cdot\|$ denotes an appropriate norm in the space $\mathbf{R}^N \times \mathbf{S}^N$.

The next assumption is concerned with the degenerate ellipticity of F.

(A3) For any $p \in \mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M$ and $X, Y \in \mathbf{S}^N$, if $X \leq Y$, then $F(p, X) \geq F(p, Y)$.

We remark that the values of F on $(\overline{\mathbf{R}}_+ M) \times \mathbf{S}^N$ are irrelevant to our discussions below. Thus, we may regard F as a function on $(\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M) \times \mathbf{S}^N$.

We are now in a position to state the main comparison theorem formulated for bounded domains Ω. Let us introduce the notation:

$$\partial_p Q_T = (\partial\Omega \times [0, T)) \cup (\overline{\Omega} \times \{0\}) \quad \text{and} \quad R_T = \overline{\Omega} \times [0, T) \equiv Q_T \cup \partial_p Q_T.$$

Theorem 1. *Let* (A1), (A2) *and* (A3) *hold. Assume that* Ω *is bounded. Let* $u \in USC(\overline{\Omega} \times [0,T))$ *and* $v \in LSC(\overline{\Omega} \times [0,T))$ *be a viscosity subsolution and a viscosity supersolution of* (1.1), *respectively. Assume that* $u \le v$ *on* $\partial_p Q_T$. *Then* $u \le v$ *in* Q_T.

In the above and in what follows $USC(V)$ and $LSC(V)$ denote the sets of all real upper semicontinuous functions on V and of all real lower semicontinuous functions, respectively.

We refer for the definition of viscosity solutions, the notation related to viscosity solutions and a general review on the theory of viscosity solutions to Crandall, Ishii and Lions ([5]). However, since F has discontinuities or singularities on the set $\overline{\mathbf{R}}_+ M \times \mathbf{S}^N$, we present here the definition of viscosity subsolutions, supersolutions and solutions of (1.1).

We call a function $u : Q_T \to \mathbf{R}$ a viscosity subsolution (respectively, supersolution) of (1.1) if u is locally bounded above (respectively, below) and if

$$a + F_*(p,X) \le 0 \quad \forall (x,t) \in Q_T, \forall (p,a,X) \in \mathcal{P}^{2,+}u^*(x,t)$$

$$\big(\text{respectively,} \quad a + F^*(p,X) \ge 0 \quad \forall (x,t) \in Q_T, \forall (p,a,X) \in \mathcal{P}^{2,-}u_*(x,t)\ \big),$$

where $\mathcal{P}^{2,+}$ and $\mathcal{P}^{2,-}$ denote the parabolic semijets and u^* and u_* denote the usual upper semicontinuous and lower semicontinuous envelopes of u, respectively. We call a function u a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1).

Note that this definition differs from that of [5] in two points. Indeed, in [5] viscosity subsolutions and supersolutions are assumed to be semicontinuous and here we do not require the semicontinuities. The other one is in that our definition of semicontinuous envelopes of F depends on the choice of M. In this sense, it may be more appropriate to write $F^{*,M}$ and $F_{*,M}$ for F^* and F_*, respectively. The set M will be referred to as the set of discontinuities of F.

We shall often need to make coordinate changes in what follows. It is therefore convenient to see here that each of conditions (A1)–(A3) is invariant under orthogonal changes of variables.

For a given matrix A we shall denote by ${}^tA$ the transpose of A.

To see the invariance, let U be an orthogonal matrix of order N and u be a function on Q_T. Define a function v on $Q_{T,U}$, where $Q_{T,U} = U(\Omega)\times(0,T)$, by setting $v(y,t) = u(x,t)$ if $y = Ux$.

Observe first that for $(x,t) \in Q_T$ and $(p,a,X) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{S}^N$, we have

$$u(\xi,s) \le u(x,t) + \langle p, \xi - x\rangle + a(s-t) + \tfrac{1}{2}\langle X(\xi - x), \xi - x\rangle + o(|\xi - x|^2 + |s-t|)$$

as $(\xi,s) \to (x,t)$ if and only if

$$v(\eta,s) \le v(y,t) + \langle Up, \eta - y\rangle + a(s-t) + \tfrac{1}{2}\langle UX{}^tU(\eta - y), \eta - y\rangle + o(|\eta - y|^2 + |s-t|)$$

as $(\eta,s) \to (y,t)$ for $y = Ux$. This shows that for $(x,t) \in Q_T$ and $(p,a,X) \in \mathbf{R}^{N+1}\times\mathbf{S}^N$, we have $(p,a,X) \in \mathcal{P}^{2,+}u(x,t)$ if and only if $(Up,a,UX{}^tU) \in \mathcal{P}^{2,+}v(Ux,t)$. Therefore, we see that u is a viscosity subsolution (respectively, supersolution, solution) of (1.1) if and only if v is a viscosity subsolution (respectively, supersolution, solution) of

$$v_t + G(Dv, D^2v) = 0 \quad \text{in} \quad Q_{T,U}, \tag{2.1}$$

where G is the function on $(\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ U(M)) \times \mathbf{S}^N$ defined by

$$G(q, Y) = F({}^tUq, {}^tUYU).$$

Note that $U(M)$ is the set of discontinuities of G.

Observe that F satisfies (A1) (respectively, (A3)) if and only if G satisfies (A1) (respectively, (A3)) with M replaced by $U(M)$.

Let us examine that if F satisfies (A2), then G satisfies (A2) with M replaced by $U(M)$. To do this, let $q \in U(M)$. Put $p = {}^tUq$. It is clear that $p \in M$. Set

$$V_p = T_pM \oplus \operatorname{span}\{p\} \quad \text{and} \quad V_q = T_qU(M) \oplus \operatorname{span}\{q\},$$

and observe that $U(V_p) = V_q$ and hence $\pi_{V_q} = U\pi_{V_p}{}^tU$.

For completeness, let us check this latter observation. Let $f_1, \dots, f_d$ form an orthonormal basis of T_pM and set $g_i = Uf_i$ for $i = 1, \dots, d$, so that the set of $g_1, \dots, g_d$ is an orthonormal basis of $T_qU(M)$. We then have

$$\begin{aligned}\pi_{V_q}\xi =& \langle \xi, g_1\rangle g_1 + \cdots + \langle \xi, g_d\rangle g_d \\ =& U\big(\langle {}^tU\xi, f_1\rangle f_1 + \cdots + \langle {}^tU\xi, f_d\rangle f_d\big) = U\pi_{V_p}{}^tU\xi\end{aligned}$$

for all $\xi \in \mathbf{R}^N$, and hence, $\pi_{V_q} = U\pi_{V_p}{}^tU$.

It is now clear that for $Y \in \mathbf{S}^N$, we have $\pi_{V_q}Y\pi_{V_q} = Y$ if and only if $\pi_{V_p}X\pi_{V_p} = X$ for $X = {}^tUYU$. Therefore, for $Y \in \mathbf{S}^N$, we have $Y \in \mathbf{S}^N(q)$ if and only if ${}^tUYU \in \mathbf{S}^N(p)$. Let F satisfy (A2). Then, using the above observation, we have

$$G^*(q, Y) = F^*(p, {}^tUYU) = F_*(p, {}^tUYU) = G_*(q, Y).$$

It is obvious that $G^*(0, 0) = G_*(0, 0)$. Hence, we see that G satisfies (A2) with $U(M)$ replacing M. Thus we see that F satisfies (A2) if and only if G satisfies (A2) with $U(M)$ replacing M. We thus conclude that each of assumptions (A1)–(A3) is invariant under orthogonal changes of variables.

Let $p \in M$. Some comments regarding the set $\mathbf{S}^N(p)$ may be in order.

Note that $X \in \mathbf{S}^N(p)$ if and only if $(I - \pi_V)X = X(I - \pi_V) = 0$, with $V = T_pM \oplus \operatorname{span}\{p\}$. Moreover, note that if $X \in \mathbf{S}^N$, $A \in \mathbf{S}^N(p)$ and $-A \le X \le A$, then $X \in \mathbf{S}^N(p)$. Indeed, if we write $V = T_pM \oplus \operatorname{span}\{p\}$, then the inequality $-A \le X \le A$ yields

$$\begin{aligned}&\big|\langle \pi_V X\pi_V\xi, \xi\rangle + 2t\langle \pi_V X(I - \pi_V)\eta, \xi\rangle + t^2\langle (I - \pi_V)X(I - \pi_V)\eta, \eta\rangle\big| \\ &\quad = |\langle X(\pi_V\xi + t(I - \pi_V)\eta), \pi_V\xi + t(I - \pi_V)\eta\rangle| \\ &\quad \le \langle \pi_V A\pi_V\xi, \xi\rangle \quad \text{for all} \quad \xi, \eta \in \mathbf{R}^N,\ t \in \mathbf{R}.\end{aligned}$$

From this we deduce that $(I-\pi_V)X(I-\pi_V)=0$ and $\pi_V X(I-\pi_V)=0$, and therefore, that $\pi_V X \pi_V = X$.

Let $e_1,\ldots,e_N$ denote the standard basis of $\mathbf{R}^N$, and suppose that $e_N \in M$ and $T_{e_N}M = \operatorname{span}\{e_1,\ldots,e_d\}$. Let $X=(x_{ij})_{1\le i,j\le N} \in \mathbf{S}^N$. Then we have $X \in \mathbf{S}^N(e_N)$ if and only if whenever $d<i<N$ or $d<j<N$, then $x_{ij}=0$.

3. Generalized evolution of sets. In this section we will define the generalized evolution of surfaces (or, simply, sets) generated by (1.1), following [6], [4], [3], [12], etc.

To this end, we introduce another assumption on F.

(A4) $F(\lambda p, \lambda X + \mu p\otimes p) = \lambda F(p,X)$
$\forall (p,X) \in \big(\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M\big) \times \mathbf{S}^N, \forall \lambda > 0, \forall \mu \in \mathbf{R}$.

PDE (1.1) or the function F is called *geometric* if this condition is satisfied.

The following proposition is of a particular importance in the level-set approach which we follow.

Proposition 1. *Assume that* (A1)–(A4) *hold. Let* $\theta \in C(\mathbf{R})$ *be a nondecreasing function. If* u *is a viscosity subsolution* (*respectively, supersolution*) *of* (1.1), *then so is* $\theta\circ u$.

For a proof of this proposition we refer the reader to [6], [4] or [3].

We need a comparison theorem similar to Theorem 1 which is valid for unbounded Ω.

Theorem 2. *Assume that* (A1)–(A4) *hold. Let* $u \in USC(R_T)$ *and* $v \in LSC(R_T)$ *be a viscosity subsolution and a viscosity supersolution of* (1.1), *respectively. Assume that* u *and* v *are bounded on* R_T *and that*

$$\lim_{\varepsilon\downarrow 0}\sup\{u(z)-v(\zeta) : (z,\zeta) \in (R_T\times\partial_p Q_T)\cup(\partial_p Q_T\times R_T), |z-\zeta|\le\varepsilon\} \le 0.$$

Then $u \le v$ *on* Q_T.

Now, we let $\Omega = \mathbf{R}^N$ and consider the Cauchy problem

$$u_t + F(Du, D^2u) = 0 \quad \text{in } \ Q_T, \tag{3.1}$$

with initial data

$$u(\cdot,0) = g \qquad \text{on } \ \mathbf{R}^N, \tag{3.2}$$

where g is a given function on $\mathbf{R}^N$.

Theorem 3. *Let* $g \in BUC(\mathbf{R}^N)$. *Assume that* (A1)–(A4) *hold. Then there is a unique viscosity solution* $u \in BUC(R_T)$ *of* (3.1) *saitsfying* (3.2).

The following observation is fundamental in the level set approach as observed in [4], [6], [12], etc.

Theorem 4. *Assume that* (A1)–(A4) *hold. Let* $g_1, g_2 \in BUC(\mathbf{R}^N)$ *satisfy*

$$\{x \in \mathbf{R}^N : g_1(x) > 0\} \subset \{x \in \mathbf{R}^N : g_2(x) > 0\}$$

$$(\textit{respectively,} \quad \{x \in \mathbf{R}^N : g_1(x) < 0\} \subset \{x \in \mathbf{R}^N : g_2(x) < 0\}\ ).$$

Let $u_1, u_2 \in BUC(R_T)$ *be the viscosity solutions of* (3.1)–(3.2), *with* g *replaced by* g_1 *and* g_2, *respectively. Then*

$$\{z \in Q_T : u_1(z) > 0\} \subset \{z \in Q_T : u_2(z) > 0\}$$

$$(\textit{respectively,} \quad \{z \in Q_T : u_1(z) < 0\} \subset \{z \in Q_T : u_2(z) < 0\}\ ).$$

Now, let $\mathcal{E}$ denote the collection of triples (Γ, D^+, D^-) consisting of (possibly empty) subsets of $\mathbf{R}^N$ such that Γ is a closed set, D^+ and D^- are open sets, and $\mathbf{R}^N = \Gamma \cup D^+ \cup D^-$ (a disjoint decomposition). It is easily checked that for any three subsets Γ, D^+ and D^- of $\mathbf{R}^N$, we have $(\Gamma, D^+, D^-) \in \mathcal{E}$ if and only if there is a function $g \in BUC(\mathbf{R}^N)$ such that

$$\Gamma = \{x \in \mathbf{R}^N : g(x) = 0\}, \quad D^+ = \{x \in \mathbf{R}^N : g(x) > 0\}, \quad D^- = \{x \in \mathbf{R}^N : g(x) < 0\}. \tag{3.3}$$

We are now ready to define the generalized evolution $\{E_t\}_{t \geq 0}$ of a subset of $\mathbf{R}^N$ generated by (3.1). As will be seen soon, the evolution is actually not an evolution of a set but a triple of sets. Fix an initial $(\Gamma_0, D_0^+, D_0^-) \in \mathcal{E}$ and choose a function $g \in BUC(\mathbf{R}^N)$ satisfying (3.3) with $\Gamma = \Gamma_0$ and $D^\pm = D_0^\pm$. Then, in view of Theorem 3, let $u \in C(R_T)$ be the function which is a unique viscosity solution of (3.1) in Q_T satisfying (3.2) and the condition, $u \in BUC(R_T)$, for any $T > 0$. Moreover set

$$\Gamma_t = \{x \in \mathbf{R}^N : u(x,t) = 0\} \quad \text{and} \quad D_t^\pm = \{x \in \mathbf{R}^N : \pm u(x,t) > 0\}.$$

It is not hard to infer from Theorem 4 that the set $(\Gamma_t, D_t^+, D_t^-) \in \mathcal{E}$ is determined by $(\Gamma_0, D_0^+, D_0^-) \in \mathcal{E}$ but independent of the choice of g. Now, the map $E_t : \mathcal{E} \to \mathcal{E}$, with $t \geq 0$, is defined by

$$E_t(\Gamma_0, D_0^+, D_0^-) = (\Gamma_t, D_t^+, D_t^-). \tag{3.4}$$

We will call the collection $\{E_t\}_{t \geq 0}$ the *generalized evolution* of sets *generated* by (3.1). Note that

$$E_0 = \mathrm{id}_{\mathcal{E}} \quad \text{and} \quad E_t \circ E_s = E_{t+s} \quad \forall t, s \geq 0.$$

4. Examples. In this section we will consider two examples of PDEs with discontinuities which arise in the level set approach to evolutions of surfaces.

Here we shall throughout assume that $\Omega = \mathbf{R}^N$. Let us first consider the PDE

$$u_t - \mathrm{tr}\,\big(I - \frac{Du \otimes Du}{|Du|^2}\big) D^2 u = 0 \quad \text{in} \ \ Q_T. \tag{4.1}$$

This is the basic equation for the level set approach to motion by mean curvature. Indeed, the generalized evolution by mean curvature is defined with this equation via the method of the previous section. See [4] and [6].

We now view the space $\mathbf{R}^N$ as a subspace of $\mathbf{R}^{N+m}$, where m is a positive integer. That is, for a given real function u on Q_T, we introduce a new function v on $\widetilde{Q}_T \equiv \mathbf{R}^N \times \mathbf{R}^m \times (0,T)$ by $v(x,y,t) = u(x,t)$. Then, it is easily seen that u is a viscosity subsolution (respectively, supersolution, solution) of (4.1) if and only if so is v of the PDE

$$v_t - \mathrm{tr}\,\big(I - \frac{D_x v \otimes D_x v}{|D_x v|^2}\big) D_x^2 v = 0 \quad \text{in } \widetilde{Q}_T, \tag{4.2}$$

where I denotes as above the unit matrix of order N and $D_x v$ and $D_x^2 v$ denote the gradient $(v_{x_1}, \ldots, v_{x_N})$ and the Hessian matrix $(v_{x_i x_j})$ in the variable x, respectively.

Let F and $\widetilde{F}$ denote the functions on $(\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N$ and on $\big(\mathbf{R}^{N+m} \setminus (\{0\} \times \mathbf{R}^m)\big) \times \mathbf{S}^{N+m}$, respectively, defined by

$$F(p,X) = -\mathrm{tr}\,\big(I - \frac{p \otimes p}{|p|^2}\big) X,$$

and

$$\widetilde{F}(p,q,\widetilde{X}) = -\mathrm{tr}\,\big(I - \frac{p \otimes p}{|p|^2}\big) X.$$

In the latter X denotes the $N \times N$ minor matrix of $\widetilde{X}$ determined by the first N columns and rows.

Note that F is continuous on $(\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N$ but it can not be extended for $p = 0$ as a continuous function. Similarly, $\widetilde{F}$ is continuous on $\big(\mathbf{R}^{N+m} \setminus (\{0\} \times \mathbf{R}^m)\big) \times \mathbf{S}^{N+m}$ but can not be extended for $p = 0$ as a continuous function. Note further that the subspace $\{0\} \times \mathbf{R}^m$ of $\mathbf{R}^{N+m}$ is m dimensional while the subspace $\{0\} \subset \mathbf{R}^N$ is zero dimensional.

The theory developed in [4] or [6] applies to (4.1) but not (4.2). However, our theory applies to (4.2). That is, $\widetilde{F}$ satisfies (A1)–(A3) with $M = (\{0\} \times \mathbf{R}^m) \cap S^{N+m-1}$.

It is easily seen that $\widetilde{F}$ satisfies (A1) and (A3) with this choice of M, and so we do not give here the proof.

Let us check that $\widetilde{F}$ satisfies (A2) with the above choice of M. It is easily seen (e.g., [4]) that $\widetilde{F}^*(0,0) = \widetilde{F}_*(0,0)$.

It remains to check condition (A2) for $(p,q) \in \mathbf{R}_+ M$. To see this, fix any $(p_0,q_0) \in \mathbf{R}_+ M$. By making a change of variables in the y coordinates, we may assume that $(p_0,q_0) = t e_{N+m}$ for some $t > 0$, where $e_1, \ldots, e_{N+m}$ denote, as usual, the standard basis of $\mathbf{R}^{N+m}$, and M is still $(\{0\} \times \mathbf{R}^m) \cap S^{N+m-1}$. By the homogeneity of $\widetilde{F}$ in the (p,q) variables, we may assume that $t = 1$; i.e., $(p_0,q_0) = e_{N+m}$. Observe then that

$$T_{e_{N+m}} M = \mathrm{span}\,\{e_{N+1}, \ldots, e_{N+m-1}\},$$

$$\mathbf{S}^{N+m}(e_{N+m}) = \{(x_{ij}) \in \mathbf{S}^{N+m} : x_{ij} = 0 \ \text{ if either } \ 1 \le i \le N \text{ or } 1 \le j \le N\}.$$

Fix $\widetilde{X}_0 \in \mathbf{S}^{N+m}(e_{N+m})$ and $\varepsilon > 0$. Fix $\widetilde{X} \in \mathbf{S}^{N+m}$ so that $\|\widetilde{X} - \widetilde{X}_0\| \le \varepsilon$, where $\|\cdot\|$ denotes an appropriate norm for matrices. Let us write $\widetilde{X} = \widetilde{X}_0 + \widetilde{X}_1$. Let X_0 and X_1

denote the $N \times N$ minor matrices of $\widetilde{X}_0$ and $\widetilde{X}_1$, respectively, determined by the first N columns and rows. Let $p \in \mathbf{R}^N \setminus \{0\}$, and observe that $X_0 = 0$ and that

$$\big\|\big(I - \frac{p \otimes p}{|p|^2}\big)(X_0 + X_1)\big\| = \big\|\big(I - \frac{p \otimes p}{|p|^2}\big)X_1\big\| \le C\big\|X_1\big\| \le C\varepsilon.$$

Here and henceforth the letter C is used to denote a generic positive constant. This shows that

$$|\widetilde{F}(p, q, \widetilde{X})| \le C\varepsilon \quad \forall p \in \mathbf{R}^N \setminus \{0\}, \forall q \in \mathbf{R}^m.$$

Hence, by the arbitrariness of $\varepsilon > 0$, we have

$$\widetilde{F}^*(e_{N+m}, \widetilde{X}_0) = \widetilde{F}_*(e_{N+m}, \widetilde{X}_0) = 0,$$

from which we conclude that (A2) holds.

We now know that equation (4.1) defines the generalized evolution of sets via the level-set approach as described in the previous section.

Now we turn to the example, which is a higher dimensional analogue of the PDEs studied in [13] and [11].

Let H^1 and H^2 be real, continuous, positively homogeneous functions of degree one on $\mathbf{R}^N$ such that $H^k \in C^2(\mathbf{R}^N \setminus \{0\})$ for $k = 0, 1$. Let V be an open subset of $\mathbf{R}^N$ such that $V = tV$ for all $t > 0$. Define the function H on $\mathbf{R}^N$ by

$$H(p) = \begin{cases} H^1(p) & \text{for } p \in V, \\ H^2(p) & \text{for } p \in \mathbf{R}^N \setminus V. \end{cases}$$

We assume that

$$H \in C(\mathbf{R}^N) \cap C^1(\mathbf{R}^N \setminus \{0\}). \tag{4.3}$$

In other words, we assume that

$$H^1(p) = H^2(p) \quad \text{and} \quad DH^1(p) = DH^2(p) \quad \forall p \in \partial V \setminus \{0\}. \tag{4.4}$$

Set

$$M = \partial V \setminus S^{N-1}.$$

It is obvious that $\partial V = \overline{\mathbf{R}}_+ M$. We assume in addition that

$$M \text{ is a } C^2 \text{ submanifold without boundary of } S^{N-1} \text{ of dimension } N-2. \tag{4.5}$$

Now, we consider the PDE

$$u_t - |Du| \operatorname{tr}\big(D^2 H(Du) D^2 u\big) = 0 \quad \text{in } Q_T. \tag{4.6}$$

Note that PDE (4.1) corresponds to the case when $H(p) = |p|$. We define the function F on $(\mathbf{R}^N \setminus \partial V) \times \mathbf{S}^N$ by

$$F(p, X) = -|p| \operatorname{tr}\big(D^2 H(p) X\big).$$

We wish to discuss if F satisfies (A1)–(A3). Since $H^k \in C^2(\mathbf{R}^N \setminus \{0\})$ and is positively homogeneous of degree one, it is easily seen that $F \in C((\mathbf{R}^N \setminus \partial V) \times \mathbf{S}^N)$. That is, F satisfies (A1). By the same reasoning as above, we see that $F^*(0,0) = F_*(0,0)$.

Next, let us check condition (A2) for $p \in \mathbf{R}_+ M$. Let $p_0 \in \mathbf{R}_+ M$. By the homogeneity of F and by using a change of coordinates if necessary, we may assume that $p_0 = e_{N-1}$ and

$$T_{e_{N-1}} M = \operatorname{span}\{e_1, \dots, e_{N-2}\},$$

where $e_1, \dots, e_N$ denote the standard basis of $\mathbf{R}^N$. Then we can choose a neighborhood U of $(0, \dots, 0, 1) \in \mathbf{R}^{N-1}$ and a C^2 function Φ on U satisfying $\Phi(0, \dots, 0, 1) = 0$ and $D\Phi(0, \dots, 0, 1) = 0$ such that for any $p = (p_1, \dots, p_N) \in U \times \mathbf{R}$,

$$p \in \partial V \iff p_N = \Phi(p_1, \dots, p_{N-1}).$$

From (4.4), we have

$$DH^1(p_1, \dots, p_{N-1}, \Phi(p_1, \dots, p_{N-1})) = DH^2(p_1, \dots, p_{N-1}, \Phi(p_1, \dots, p_{N-1}))$$

for all $(p_1, \dots, p_{N-1}) \in U$. Differentiating this, we have

$$H^1_{p_i p_j}(e_{N-1}) = H^2_{p_i p_j}(e_{N-1})$$

if either $1 \le i \le N-1$ or $1 \le j \le N-1$. This implies that if either $1 \le i \le N-1$ or $1 \le j \le N-1$, then $H_{p_i p_j}$ is continuous at $p = e_{N-1}$.

Observe that

$$\mathbf{S}^N(e_{N-1}) = \{(x_{ij}) \in \mathbf{S}^N : x_{ij} = 0 \ \text{ if either } i = N \text{ or } j = N\}.$$

Fix $X^0 \in \mathbf{S}^N(e_{N-1})$ and $\varepsilon > 0$. Fix $p \in \mathbf{R}^N \setminus \partial V$ and $X \in \mathbf{S}^N$ such that $\|X - X^0\| \le \varepsilon$. Let us write

$$X^0 = \begin{pmatrix} X^0_{11} & X^0_{12} \\ {}^t X^0_{12} & X^0_{22} \end{pmatrix} \quad \text{and} \quad |p| D^2 H(p) = A(p) = \begin{pmatrix} A_{11}(p) & A_{12}(p) \\ {}^t A_{12}(p) & A_{22}(p) \end{pmatrix},$$

where X^0_{11} and $A_{11}(p)$ are $(N-1) \times (N-1)$ matrices, X^0_{12} and $A_{12}(p)$ are $N-1$ column vectors and X^0_{22} and $A_{22}(p)$ are real numbers. Note that $X^0_{12} = 0$ and $X^0_{22} = 0$. We also write

$$X = \begin{pmatrix} X^0_{11} & 0 \\ 0 & 0 \end{pmatrix} + Y \quad \text{and} \quad A(p) = \begin{pmatrix} A_{11}(0) & A_{12}(p) \\ {}^t A_{12}(p) & A_{22}(p) \end{pmatrix} + B(p).$$

We have $\|Y\| = \|X - X^0\| \le \varepsilon$. Since the function $A_{11}(p)$ is continuous at $p = e_{N-1}$, there is a real increasing function ω on $[0, \infty)$ satisfying $\omega(+0) = 0$ such that $\|B(p)\| \le \omega(\varepsilon)$. Observe that since $H^k \in C^2(\mathbf{R}^N \setminus \{0\})$ and are positively homogeneous of degree one for $k = 1, 2$, the function $A(p)$ is bounded on $\mathbf{R}^N \setminus \partial V$.

Now, we compute that

$$\begin{aligned}\operatorname{tr} A(p)X &= \operatorname{tr}\Big\{ \begin{pmatrix} X_{11}^0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11}(0) & A_{12}(p) \\ {}^tA_{12}(p) & A_{22}(p) \end{pmatrix} + \begin{pmatrix} X_{11}^0 & 0 \\ 0 & 0 \end{pmatrix} B(p) + YA(p) \Big\} \\ &= \operatorname{tr} X_{11}^0 A_{11}(0) + \operatorname{tr} \begin{pmatrix} X_{11}^0 & 0 \\ 0 & 0 \end{pmatrix} B(p) + \operatorname{tr} YA(p).\end{aligned}$$

Hence we have

$$|\operatorname{tr} A(p)X - \operatorname{tr} X_{11}^0 A_{11}(0)| \le C(\|X_{11}^0\|\omega(\varepsilon) + \varepsilon\|A(p)\|)$$

for some constant $C > 0$. Thus we see that

$$F^*(e_{N-1}, X^0) = F_*(e_{N-1}, X^0) = -\operatorname{tr} X_{11}^0 A_{11}(0),$$

which guarantees that (A2) holds.

To continue, we need to assume that

$$H \text{ is convex on } \mathbf{R}^N. \tag{4.7}$$

Then, we see that $D^2H(p) \ge 0$ for all $p \in \mathbf{R}^N \setminus \partial V$ and hence that (A3) holds.

Finally, we note that the PDE (4.6) is geometric and that the method described in the previous section now applies to (4.6) in order to define the generalized evolution of sets generated by (4.6). This generalized evolution of sets generated by (4.6) or the PDE (4.6) itself in the case $N = 2$ has been studied in [13] and [11]. The motivation for studying (4.6) or the generalized evolution of sets generated by (4.6) lies in its application to the evolution of phase boundaries driven by anisotropic interfacial energies. See for this [1], [2], [13] and [11].

5. Proof of theorems. We shall need the following two lemmas, which will be proved in the next section.

Lemma 1. *Let $u, v \in USC(V)$, where V is an open subset of $\mathbf{R}^m$, and define $w \in USC(V \times V)$ by $w(x,y) = u(x) + v(y)$. Let $x, y \in V$, $p, q \in \mathbf{R}^N$ and $A \in \mathbf{S}^N$ satisfy*

$$\Big(p, q, \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}\Big) \in J^{2,+}w(x,y) \quad \text{and} \quad A \ge 0.$$

Then there are $X, Y \in \mathbf{S}^m$ such that

$$(p, X) \in \overline{J}^{2,+}u(x), \ (q, Y) \in \overline{J}^{2,+}v(y),$$

$$-3\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.$$

Lemma 2. *Under assumption* (A1), *there is a function* $\psi \in C(\mathbf{R}^N) \cap C^{1,1}(\mathbf{R}^N \setminus \{0\})$ *such that*

(i) ψ *is convex and positively homogeneous of degree* 1 *on* $\mathbf{R}^N$,
(ii) $\psi(x) > 0$ *for* $x \neq 0$ *and* $\psi(0) = 0$,
(iii) ψ *is twice continuously differentiable in a neighborhood of* $\mathbf{R}_+ M$,
(iv) $x \in \mathbf{R}_+ M$ *if and only if* $D\psi(x) \in \mathbf{R}_+ M$,
(v) *if* $D\psi(x) \in \mathbf{R}_+ M$, *then* $D^2\psi(x) \in \mathbf{S}^N\big(D\psi(x)\big)$.

With the above lemmas at hand, the proof of Theorem 1 is a kind of repetition of the standard argument in the theory of viscosity solutions (see, e.g., [8], [3] or [12]).

Proof of Theorem 1. Let ψ be a function on $\mathbf{R}^N$ having the properties listed in Lemma 2. Define $\varphi \in C(\mathbf{R}^N)$ by $\varphi(x) = \psi(x)^4$. Observe that for $x \in \mathbf{R}_+ M$,

$$D^2\varphi(x) = 4\psi(x)^3 D^2\psi(x) + 12\psi(x)^2 D\psi(x) \otimes D\psi(x).$$

Observe also that the matrix

$$\frac{D\psi(x) \otimes D\psi(x)}{|D\psi(x)|^2},$$

with $x \in \mathbf{R}_+ M$, represents the orthogonal projection of $\mathbf{R}^N$ onto span$\{D\psi(x)\}$. We thus see that for $x \in \mathbf{R}_+ M$, we have $D^2\varphi(x) \in \mathbf{S}^N\big(D\varphi(x)\big)$ and moreover that the function φ has the same properties as ψ listed in Lemma 2 except that φ is now in $C^{1,1}(\mathbf{R}^N)$ and positively homogeneous of degree 4 on $\mathbf{R}^N$.

Let $\varepsilon > 0$, and consider the function

$$\Phi(x,t,y,s) = u(x,t) - v(y,s) - \frac{1}{\varepsilon}\varphi(x-y) - \frac{1}{\varepsilon}(t-s)^2 - \frac{\varepsilon}{T-t} - \frac{\varepsilon}{T-s}$$

on $R_T \times R_T$. Replacing T by any positive number smaller than T if necessary, we may assume that the function $u(x,t) - v(y,s)$ is bounded on R_T^2.

Suppose that $\sup_{Q_T}(u-v) > 0$. Let $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ be a maximum point of Φ. It is easily checked that if ε is small enough, then $(\hat{x}, \hat{t}), (\hat{y}, \hat{s}) \in Q_T$.

Fix such an $\varepsilon > 0$ and set

$$u_\varepsilon(x,t) = u(x,t) - \frac{\varepsilon}{T-t}, \quad v_\varepsilon(x,t) = v(x,t) + \frac{\varepsilon}{T-t},$$
$$w_\varepsilon(x,t,y,s) = u_\varepsilon(x,t) - v_\varepsilon(y,s)$$

for $(x,t), (y,s) \in Q_T$. Set

$$p = \frac{1}{\varepsilon} D\varphi(\hat{x} - \hat{y}) \quad \text{and} \quad a = \frac{2}{\varepsilon}(\hat{t} - \hat{s}).$$

Let U be an open neighborhood of $\mathbf{R}_+ M$ such that $\varphi \in C^2(U)$. Choose a constant $C > 0$ such that $(D\varphi(x), C|x|^2 I) \in J^{2,+}\varphi(x)$ for all $x \in \mathbf{R}^N$. This is possible because $\varphi \in C^{1,1}(\mathbf{R}^N)$ and φ is positively homogeneous of degree 4. Set

$$A = \begin{cases} \frac{1}{\varepsilon} D^2\varphi(\hat{x} - \hat{y}) & \text{if } \hat{x} - \hat{y} \in U, \\ \frac{1}{\varepsilon} C|\hat{x} - \hat{y}|^2 I & \text{otherwise,} \end{cases}$$

and

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{2}{\varepsilon} \end{pmatrix} \in \mathbf{S}^{N+1}.$$

Note that

$$\Big(p, a, -p, -a, \begin{pmatrix} \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{pmatrix}\Big) \in J^{2,+} w_\varepsilon(\hat{x}, \hat{t}, \hat{y}, \hat{s}).$$

By virtue of Lemma 1, we find $X, Y \in \mathbf{S}^N$ such that

$$\big(p, a, X\big) \in \overline{\mathcal{P}}^{2,+} u_\varepsilon(\hat{x}, \hat{t}), \quad \big(p, a, -Y\big) \in \overline{\mathcal{P}}^{2,-} v_\varepsilon(\hat{y}, \hat{s}),$$

$$-3\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.$$

In particular, we have

$$-3A \le X \le 3A, \quad -3A \le Y \le 3A \quad \text{and} \quad X + Y \le 0.$$

Now, observe that if $p \in \mathbf{R}_+ M$, then $A \in \mathbf{S}^N(p)$ by Lemma 2 and that if $p = 0$, then $A = 0$. Therefore, the first two inequalities above guarantee that if $p \in \overline{\mathbf{R}}_+ M$, then $X, Y \in \mathbf{S}^N(p)$. Thus, by using assumptions (A2) and (A3), we have

$$F_*(p, X) = F^*(p, X) \ge F^*(p, -Y). \tag{5.1}$$

Since u and v are viscosity sub- and supersolutions of (1.1), respectively, we obtain

$$\frac{\varepsilon}{(T-\hat{t})^2} + a + F_*(p, X) \le 0 \le -\frac{\varepsilon}{(T-\hat{s})^2} + a + F^*(p, -Y). \tag{5.2}$$

Now, (5.1) and (5.2) together give a contradiction. □

The proof of Theorem 2 parallels that of Theorem 1, and so we just give its outline.

Outline of proof of Theorem 2. The proof goes as that of Theorem 1. We thus suppose that $\sup_{Q_T}(u - v) > 0$ and will get a contradiction.

Choose a function $\varphi \in C^{1,1}(\mathbf{R}^N)$ as in the proof of Theorem 1. Let $\varepsilon > 0$ and $\delta > 0$. Consider the function

$$\Phi(x, t, y, s) = u(x, t) - v(y, s) - \frac{1}{\varepsilon}\varphi(x - y) - \frac{1}{\varepsilon}(t - s)^2 - \frac{\varepsilon}{T - t} - \frac{\varepsilon}{T - s} + \delta|x|^2$$

on R_T^2. This function clearly attains a maximum. Let $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ be a maximum point of Φ. We can and do choose ε so that $(\hat{x}, \hat{t}), (\hat{y}, \hat{s}) \in Q_T$ for all $\delta > 0$. Define u_ε, v_ε and w_ε, and choose an open subset U of $\mathbf{R}^N$ and a constant $C > 0$ as in the proof of Theorem 1. Define functions $u_{\varepsilon,\delta}$ and $w_{\varepsilon,\delta}$ on R_T and R_T^2, respectively, by $u_{\varepsilon,\delta}(x, t) = u_\varepsilon(x, t) - \delta|x|^2$ and $w_{\varepsilon,\delta}(x, t, y, s) = w_\varepsilon(x, t, y, s) - \delta|x|^2$.

Taking the dependence of Φ on δ into account, we set

$$\xi_\delta = \hat{x} - \hat{y}, \quad p_\delta = \frac{1}{\varepsilon}D\varphi(\xi_\delta), \quad a_\delta = \frac{2}{\varepsilon}(\hat{t} - \hat{s}),$$

$$A_\delta = \begin{cases} \frac{1}{\varepsilon} D^2\varphi(\xi_\delta) & \text{if } \xi_\delta \in U, \\ \frac{1}{\varepsilon} C|\xi_\delta|^2 I & \text{otherwise,} \end{cases} \tag{5.3}$$

and

$$\tilde{A}_\delta = \begin{pmatrix} A_\delta & 0 \\ 0 & \frac{2}{\varepsilon} \end{pmatrix} \in \mathbf{S}^{N+1}.$$

Then,

$$\left(p_\delta, a_\delta, -p_\delta, -a_\delta, \begin{pmatrix} \tilde{A}_\delta & -\tilde{A}_\delta \\ -\tilde{A}_\delta & \tilde{A}_\delta \end{pmatrix}\right) \in J^{2,+} w_{\varepsilon,\delta}(\hat{x}, \hat{t}, \hat{y}, \hat{s}).$$

Lemma 1 guarantees that there are $X_\delta, Y_\delta \in \mathbf{S}^N$ such that

$$\big(p_\delta, a_\delta, X_\delta\big) \in \overline{\mathcal{P}}^{2,+} u_{\varepsilon,\delta}(\hat{x}, \hat{t}), \quad \big(p_\delta, a_\delta, -Y_\delta\big) \in \overline{\mathcal{P}}^{2,-} v_\varepsilon(\hat{y}, \hat{s}),$$

$$-3 \begin{pmatrix} A_\delta & 0 \\ 0 & A_\delta \end{pmatrix} \le \begin{pmatrix} X_\delta & 0 \\ 0 & Y_\delta \end{pmatrix} \le 3 \begin{pmatrix} A_\delta & -A_\delta \\ -A_\delta & A_\delta \end{pmatrix}. \tag{5.4}$$

Fix such $X_\delta, Y_\delta \in \mathbf{S}^N$. Since u and v are a viscosity subsolution and a viscosity supersolution of (1.1), respectively, we obtain

$$\frac{\varepsilon}{(T-\hat{t})^2} + a_\delta + F_*(p_\delta + 2\delta\hat{x}, X_\delta + 2\delta I) \le 0 \le -\frac{\varepsilon}{(T-\hat{s})^2} + a_\delta + F^*(p_\delta, -Y_\delta). \tag{5.5}$$

For any $x \in \Omega$, we have

$$\Phi(x, 0, x, 0) \le \Phi(\hat{x}, \hat{t}, \hat{y}, \hat{s}),$$

and hence,

$$\frac{1}{\varepsilon}\varphi(\xi_\delta) + \delta|\hat{x}|^2 \le u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{s}) + \frac{2\varepsilon}{T} + \delta|x|^2.$$

Therefore,

$$\frac{1}{\varepsilon}\varphi(\xi_\delta) + \delta|\hat{x}|^2 \le \sup_{Q_T} u - \inf_{Q_T} v + \frac{2\varepsilon}{T} + \delta\big(\operatorname{dist}(0, \Omega)\big)^2.$$

This shows that $\{\xi_\delta\}$ is bounded in $\mathbf{R}^N$ and $\delta x_\delta \to 0$ as $\delta \downarrow 0$. Moreover, in view of (5.3), $\{A_\delta\}$ is bounded in $\mathbf{S}^N$ and therefore, by (5.4), $\{X_\delta\}$ and $\{Y_\delta\}$ are bounded as $\delta \downarrow 0$. Hence we can choose a sequence $\{\delta_n\} \subset (0, 1)$ converging to zero such that

$$\xi_{\delta_n} \to \xi \quad \text{and} \quad A_{\delta_n} \to A$$

for some $\xi \in \mathbf{R}^N$ and $A \in \mathbf{S}^N$ as $n \to \infty$. Note that $p_{\delta_n} \to p \equiv \frac{1}{\varepsilon} D\varphi(\xi)$ as $n \to \infty$.

In the limit, from (5.3), (5.4) and (5.5) we have

$$A = \frac{1}{\varepsilon} D^2\varphi(\xi) \quad \text{if } \xi \in U,$$
$$A \frac{1}{\varepsilon} \le C|\xi|^2 I \quad \text{otherwise,}$$

$$-3\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}$$

and

$$\frac{\varepsilon}{T^2} + F_*(p, X) \le -\frac{\varepsilon}{T^2} + F^*(p, -Y).$$

From these, we get a contradiction as in the proof of Theorem 1. □

Proof of Theorem 3. The uniqueness follows from Theorem 2.

In view of Perron's method (see, e.g., Theorem 4.1 in [5]), we need only to show that there are a viscosity supersolution $w^+ \in USC(R_T)$ and a viscosity subsolution $w^- \in LSC(R_T)$ such that $w^- \le w^+$ on R_T, $w^-(\cdot, 0) = w^+(\cdot, 0) = g$ on $\mathbf{R}^N$,

$$\lim_{\varepsilon \downarrow 0} \sup \{w^+(z) - w^-(\zeta) : (z, \zeta) \in (R_T \times \partial_p Q_T) \cup (\partial_p Q_T \times R_T), |z - \zeta| \le \varepsilon\} \le 0,$$

and w^+, w^- are bounded on R_T. Note that in the current case, $\partial_p Q_T = \mathbf{R}^N \times \{0\}$.

To see the existence of such sub- and supersolutions, let us first observe that F_* and F^* satisfy (A3) with $M = \emptyset$. In view of (A2), there are constants $C > 0$ and $\delta > 0$ such that for all $(p, X) \in (\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M) \times \mathbf{S}^N$, we have $|F(p, X)| \le C$ if $|p| + \|X\| \le \delta$. From (A4) we deduce that $|F(p, X)| \le (C/\delta)(|p| + \|X\|)$ for all $(p, X) \in (\mathbf{R}^N \setminus \overline{\mathbf{R}}_+ M) \times \mathbf{S}^N$.

Fix $\varepsilon > 0$, and choose $A(\varepsilon) > 0$ so that $|g(x) - g(y)| \le \varepsilon + A(\varepsilon)|x - y|^2$ for all $x, y \in \mathbf{R}^N$. Fix any $y \in \mathbf{R}^N$, and define a function v on R_T by

$$v(x, t) = \varepsilon + A(\varepsilon)|x - y|^2.$$

Choose a constant $B(\varepsilon) > 0$ so that

$$2(CA(\varepsilon)/\delta)(1 + r) \le B(\varepsilon)(\varepsilon + A(\varepsilon)r^2) \quad \text{for all } r \ge 0.$$

Then,

$$\begin{aligned} v_t(x, t) + F^*\big(Dv(x, y), D^2 v(x, t)\big) &= F^*\big(2A(\varepsilon)(x - y), 2A(\varepsilon)I\big) \\ &\ge -2(CA(\varepsilon)/\delta)(1 + |x - y|) \ge -B(\varepsilon)v(x, t) \quad \text{in } Q_T \end{aligned}$$

in the classical sense. Define a function w on R_T, which will be denoted also by $w(\cdot; \varepsilon, y)$, by $w(x, t) = g(y) + v(x, t)e^{B(\varepsilon)t}$. Finally, define the function w^+ on R_T by

$$w^+(x, t) = \inf\{w(x, t; \varepsilon, y) : \varepsilon \in (0, 1), y \in \mathbf{R}^N\}.$$

Note that this w^+ is clearly a viscosity supersolution of (3.1) and upper semicontinuous on R_T. Define a fuction w^- on R_T by

$$w^-(x, t) = \sup\{g(y) - (\varepsilon + A(\varepsilon)|x - y|^2)e^{B(\varepsilon)t} : \varepsilon \in (0, 1), y \in \mathbf{R}^N\}.$$

Then, the functions w^+ and w^- are a viscosity supersolution and a viscosity subsolution of (3.1), respectively, having all the required properties. □

We give here a proof of Theorem 4 for completeness, but it is an easy adaptation of the arguments in [12] for the proof of Theorem 1.9.

Lemma 3. *Let* (A1)–(A4) *be satisfied. Let* $g, g_n \in BUC(\mathbf{R}^N)$ *satisfy*

$$g_n(x) \uparrow g(x) \quad \forall x \in \mathbf{R}^N \qquad \text{as } n \to \infty.$$

Let $u, u_n \in BUC(R_T)$ *be the viscosity solutions of* (3.1) *with initial data* g *and* g_n*, respectively. Then*

$$u_n(z) \uparrow u(z) \quad \forall z \in R_T \qquad \text{as } n \to \infty.$$

Proof. By comparison (in other words, by using Theorem 2), we see that

$$u_n \le u_{n+1} \le u \quad \text{ in } R_T,\ \forall n \in \mathbf{N}. \tag{5.6}$$

Set

$$v(z) = \sup_{n \in \mathbf{N}} u_n(z) \equiv \lim_{n\to\infty} u_n(z) \quad \text{for } z \in R_T.$$

Then,

$$u_n(z) \le \lim_{r\downarrow 0} \inf\{u_k(\zeta) : \zeta \in R_T, |\zeta - z| \le r, k > \frac{1}{r}\} \le v(z) \quad \forall z \in R_T, \forall n \in \mathbf{N}.$$

Therefore,

$$v(z) = \lim_{r\downarrow 0} \inf\{u_k(\zeta) : \zeta \in R_T, |\zeta - z| \le r, k > \frac{1}{r}\} \quad \forall z \in R_T.$$

This guarantees (see, e.g., Lemma 6.1 in [5]) that v is a viscosity supersolution of (3.1).

Fix $y \in \mathbf{R}^N$ and $\varepsilon > 0$. In view of the proof of Theorem 3, there are constants $A = A(\varepsilon) > 0$ and $B = B(\varepsilon) > 0$ such that if we set

$$w(x,t) = g(y) - \varepsilon - (\varepsilon + A|x-y|^2)e^{Bt} \qquad \text{for all } z \in R_T,$$

then w is a viscosity subsolution of (3.1) and $g(x) \ge \varepsilon + w(x,0)$ for all $x \in \mathbf{R}^N$.

Dini's theorem asserts that $g_n(x) \to g(x)$ locally uniformly in $\mathbf{R}^N$ as $n \to \infty$. Noting that $w(x,0) \to -\infty$ as $|x| \to \infty$, we find an $l \in \mathbf{N}$ such that $g_l \ge w(\cdot, 0)$ in $\mathbf{R}^N$. By comparison, we have $u_n \ge w$ on R_T for all $n \ge l$ and hence, $v \ge w$ on R_T. Therefore,

$$u(x,t) - v(y,s) \le u(x,t) - w(y,s) \le u(x,t) - g(y) + \varepsilon(1 + e^{Bt})$$

for all $(x,t) \in R_T$ and $0 \le s < T$. Because of the arbitrariness of $y \in \mathbf{R}^N$ and $\varepsilon > 0$, we see that

$$\lim_{\varepsilon\downarrow 0} \sup\{u(z) - v(\zeta) : (z,\zeta) \in (R_T \times \partial_p Q_T) \cup (\partial_p Q_T \times R_T), |z - \zeta| \le \varepsilon\} \le 0.$$

Hence, by comparison, we have $u \le v$ in R_T. On the other hand, from (5.6) we have $u \ge v$ on R_T. We thus conclude that $u = v$ on R_T. $\square$

Proof of Theorem 4. We will only prove that if

$$\{x \in \mathbf{R}^N : g_1(x) > 0\} \subset \{x \in \mathbf{R}^N : g_2(x) > 0\}, \tag{5.7}$$

then

$$D_1^+ \equiv \{z \in R_T : u_1(z) > 0\} \subset D_2^+ \equiv \{z \in R_T : u_2(z) > 0\}. \tag{5.8}$$

Assume that (5.7) is satisfied. Let θ be the function on $\mathbf{R}$ defined by $\theta(r) = \max\{r, 0\}$. Set $u_i^+ = \theta \circ u_i$ for $i = 1, 2$. Observe by using Proposition 1 that u_i^+ is a viscosity solution of (3.1), with initial data $g = \theta \circ g_i$ for $i = 1, 2$, respectively.

For $n \in \mathbf{N}$, we define $h_n \in BUC(\mathbf{R}^N)$ by

$$h_n(x) = \min\{\theta \circ g_1(x), n\theta \circ g_2(x)\}.$$

and let $v_n \in BUC(R_T)$ be the viscosity solution of (3.1) with initial data h_n. Observe that $h_n(x) \uparrow \theta \circ g_1(x)$ for all $x \in \mathbf{R}^N$ as $n \to \infty$. Lemma 3 guarantees that $v_n(z) \uparrow u_1^+(z)$ for all $z \in R_T$ as $n \to \infty$.

Now, let $z \in D_1^+$. Then we have $u_1^+(z) > 0$ and therefore $v_n(z) > 0$ for some $n \in \mathbf{N}$. Fix such an $n \in \mathbf{N}$. Since $v_n(\cdot, 0) = h_n \le n\theta \circ g_2 \le nu_2^+(\cdot, 0)$ on $\mathbf{R}^N$ and the function $nu_2^+ = (n\theta) \circ u_2$ is a viscosity solution of (3.1) with initial data $n\theta \circ g_2$, we have $v_n \le nu_2^+$ on R_T by comparison. Hence, we see that $u_2(z) > 0$, from which follows (5.8). □

6. Proof of Lemmas 1 and 2. We begin with the proof of Lemma 1.

Proof of Lemma 1. Recall that for any function $f : \mathbf{R}^n \to \mathbf{R}$ and any nonsingular $n \times n$ matrix Q, if we set $g(x) = f(Qx)$ then

$$(p, X) \in J^{2,+} f(Qx) \iff ({}^tQp, {}^tQXQ) \in J^{2,+} g(x).$$

Now, let $p, q \in \mathbf{R}^m$, $A \in \mathbf{S}^m$ and $\hat{x}, \hat{y} \in V$ satisfy

$$\left(p, q, \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}\right) \in J^{2,+} w(\hat{x}, \hat{y}) \quad \text{and} \quad A \ge 0.$$

Fix $\delta > 0$, and set

$$A_\delta = A + \delta I, \ S_\delta = A_\delta^{1/2}, \quad \tilde{u}(x) = u\big(S_\delta^{-1}x\big), \quad \tilde{v}(x) = v\big(S_\delta^{-1}x\big),$$

and $\tilde{w}(x, y) = \tilde{u}(x) + \tilde{v}(y)$. Since

$$\delta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \ge 0,$$

we have

$$\left(p, q, \begin{pmatrix} A_\delta & -A_\delta \\ -A_\delta & A_\delta \end{pmatrix}\right) \in J^{2,+} w(\hat{x}, \hat{y}).$$

Observing that

$$\begin{pmatrix} S_\delta^{-1} & 0 \\ 0 & S_\delta^{-1} \end{pmatrix} \begin{pmatrix} A_\delta & -A_\delta \\ -A_\delta & A_\delta \end{pmatrix} \begin{pmatrix} S_\delta^{-1} & 0 \\ 0 & S_\delta^{-1} \end{pmatrix} = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

we find that

$$\left(S_\delta^{-1}p,\ S_\delta^{-1}q,\ \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right) \in J^{2,+}\tilde{w}\big(S_\delta\hat{x}, S_\delta\hat{y}\big).$$

By the standard result analogous to Lemma 1 (see, e.g., Theorem 3.2 in [5]), there are $X_\delta, Y_\delta \in \mathbf{S}^m$ such that

$$\big(S_\delta^{-1}p, X_\delta\big) \in \overline{J}^{2,+}\tilde{u}(S_\delta\hat{x}), \quad \big(S_\delta^{-1}q, Y_\delta\big) \in \overline{J}^{2,+}\tilde{v}(S_\delta\hat{y}),$$

and

$$-3\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\delta & 0 \\ 0 & Y_\delta \end{pmatrix} \le 3\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Converting things into the original coordinates, we obtain

$$\big(p, S_\delta X_\delta S_\delta\big) \in \overline{J}^{2,+}u(\hat{x}) \quad \text{and} \quad \big(q, S_\delta Y_\delta S_\delta\big) \in \overline{J}^{2,+}v(\hat{y}).$$

Also, we have

$$-3\begin{pmatrix} A_\delta & 0 \\ 0 & A_\delta \end{pmatrix} \le \begin{pmatrix} S_\delta X_\delta S_\delta & 0 \\ 0 & S_\delta Y_\delta S_\delta \end{pmatrix} \le 3\begin{pmatrix} A_\delta & -A_\delta \\ -A_\delta & A_\delta \end{pmatrix}.$$

Sending $\delta \downarrow 0$ along a suitable sequence, we find the limits $X, Y \in \mathbf{S}^m$ of $S_\delta X_\delta S_\delta$ and $S_\delta Y_\delta S_\delta$, respectively, for which we have

$$(p, X) \in \overline{J}^{2,+}u(\hat{x}),\ (q, Y) \in \overline{J}^{2,+}v(\hat{y}),$$

and

$$-3\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le 3\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.$$ □

Now, we turn to the proof of Lemma 2. This is the hardest part of this paper, although the underlying idea is very simple.

The idea may be explained as follows. Let $M \subset S^{N-1}$ be a C^2 submanifold as in Lemma 2. Fix any direction $q \in S^{N-1}$ and a smooth, strictly convex body K so that K contains the unit ball $B(0,1)$ and so that $q \in \partial K$ and all the principal curvatures of ∂K at q vanish. Then, for each $p \in M$ we define K_p as the convex set obtained by rotating K around the origin so that the new position of q is at p. The function ψ is defined as the Minkowski functional of the ε-neighborhood of the set

$$\cap\{K_p : p \in M\}, \quad \text{with } \varepsilon > 0.$$

We carry out the above program with minor modifications in what follows.

Proof of Lemma 2. The arguments that follow will be divided roughly into three steps.

Step I. We define $f : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ by

$$f(x,\xi) = \max\{|x|^2 - 4, (\langle x,\xi\rangle + 1)^2 + |x - \langle x,\xi\rangle\xi|^4 - 4\}.$$

We shall be interested in the level sets $\{x \in \mathbf{R}^N : f(x,\xi) = 0\}$ with $\xi \in S^{N-1}$. It is obvious that for every $\xi \in S^{N-1}$,

$$f(\xi,\xi) = 0, \tag{6.1}$$

$$f(x,\xi) < 0 \quad \text{if} \;\; |x| \le 1 \;\; \text{and} \;\; x \ne \xi, \tag{6.2}$$

$$f(x,\xi) > 0 \quad \text{if} \;\; |x| > 2, \tag{6.3}$$

and

$$f(x,\xi) = (\langle x,\xi\rangle + 1)^2 + |x - \langle x,\xi\rangle\xi|^4 - 4 \quad \text{for all } x \in B(\xi,\delta) \tag{6.4}$$

and for some $\delta > 0$. It should be noted that if $Q : \mathbf{R}^N \to \mathbf{R}^N$ is an orthogonal matrix, then $f(Qx, Q\xi) = f(x,\xi)$ for all $x, \xi \in \mathbf{R}^N$. Henceforth we shall denote by $e_1, \dots, e_N$ the standard basis of $\mathbf{R}^N$. If we write

$$g(x,\xi) = \big(\langle x,\xi\rangle + 1\big)^2 + |x - \langle x,\xi\rangle\xi|^4 - 4,$$

then

$$g(x,e_N) = (x_N + 1)^2 + (x_1^2 + \cdots + x_{N-1}^2)^2 - 4,$$

and hence the function $g(x,e_N)$ is strictly convex on $\mathbf{R}^N$. Therefore, for each $\xi \in S^{N-1}$ the function $f(x,\xi)$ is strictly convex on $\mathbf{R}^N$. By computation, we have

$$f_{x_i}(e_N,e_N) = f_{\xi_i}(e_N,e_N) = 4\delta_{iN},$$

$$f_{x_i x_j}(e_N,e_N) = f_{\xi_i\xi_j}(e_N,e_N) = 2\delta_{iN}\delta_{jN}, \quad f_{x_i\xi_j}(e_N,e_N) = 4\delta_{ij} + 2\delta_{iN}\delta_{jN},$$

for all $i,j = 1,\dots,N$. That is, for any $\xi \in S^{N-1}$,

$$D_x f(\xi,\xi) = D_\xi f(\xi,\xi) = 4\xi, \tag{6.5}$$

$$D_x^2 f(\xi,\xi) = D_\xi^2 f(\xi,\xi) = 2\xi \otimes \xi, \tag{6.6}$$

$$D_x D_\xi f(\xi,\xi) = 4I + 2\xi \otimes \xi. \tag{6.7}$$

We define $v : \mathbf{R}^N \to \mathbf{R}$ by

$$v(x) = \sup\{f(x,\xi) : \xi \in M\} = \max\{f(x,\xi) : \xi \in \overline{M}\}.$$

In view of properties (6.2), (6.3) and (6.1), we see that

$$v(x) < 0 \quad \text{if} \;\; |x| < 1, \qquad v(x) > 0 \quad \text{if} \;\; |x| > 2,$$

and that for any $x \in S^{N-1}$,

$$v(x) = 0 \iff x \in \overline{M}.$$

Also, we note that v is strictly convex on $\mathbf{R}^N$ since it is a pointwise maximum of a family of strictly convex functions. Furthermore, using (6.2), we easily check that for any $z \in M$ and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$v(x) = \max\{f(x,\xi) \mid \xi \in \overline{M} \cap B(z,\varepsilon)\} \quad \text{for all } x \in B(z,\delta). \tag{6.8}$$

Now, we want to prove that there is a neighborhood V of M such that $v \in C^2(V)$. Fix any $z \in M$. If $d = 0$, then (6.4) guarantees that $v(x) = f(x,z) = g(x,z)$ in a neighborhood of z and therefore, v is a C^2 function there. Moreover, by (6.5) and (6.6) we have $Dv(z) = 4z$ and $D^2v(z) = 2z \otimes z \in \mathbf{S}^N\big(Dv(z)\big)$. Next, we assume that $d \geq 1$. We may assume that $e_N = z$ and $T_zM = \operatorname{span}\{e_1, \dots, e_d\}$. Otherwise, we choose an orthonormal basis $a_1, \dots, a_N$ of $\mathbf{R}^N$ so that $a_N = z$ and $T_zM = \operatorname{span}\{a_1, \dots, a_d\}$. We define the $N \times N$ orthogonal matrix Q by $Q = (a_1 \cdots a_N)$, and set $\tilde{v}(x) = v(Qx)$ and $\widetilde{M} = QM \equiv \{Q\xi \mid \xi \in M\}$. Then we have: $Qe_i = a_i$ for all $i = 1, \dots, N$, $\widetilde{M}$ is a C^2 submanifold of dimension d, $T_{e_N}\widetilde{M} = \operatorname{span}\{e_1, \cdots, e_d\}$ and $\tilde{v}(x) = \sup\{f(x,\xi) \mid \xi \in \widetilde{M}\}$.

Since M is a C^2 submanifold of S^{N-1} and since $e_N \in M$ and $T_{e_N}M = \operatorname{span}\{e_1, \dots, e_d\}$, there are a $\delta > 0$, a neighborhood U of the origin in $\mathbf{R}^d$ and C^2 functions $\xi_{d+1}, \dots, \xi_N$ on U such that

$$\begin{cases} M \cap B(e_N,\delta) = \big\{\big(u, \xi_{d+1}(u), \dots, \xi_N(u)\big) : u \in U\big\}, \\ \xi_{d+1}(0) = \cdots = \xi_{N-1}(0) = 0,\ \xi_N(0) = 1, \\ D\xi_k(0) = 0 \ \text{ for all } \ k = d+1, \dots, N. \end{cases} \tag{6.9}$$

Choosing $\delta > 0$ small enough, we may assume that $\overline{M} \cap B(e_N,\delta) = M \cap B(e_N,\delta)$. By (6.8), there is a $\gamma > 0$ such that for every $x \in B(e_N,\gamma)$,

$$\begin{aligned} v(x) &= \max\{f(x,\xi) : \xi \in M \cap B(e_N,\delta)\} \\ &= \max\big\{f\big(x,u,\xi_{d+1}(u), \dots, \xi_N(u)\big) : u \in U\big\}. \end{aligned}$$

By virtue of (6.4), we may assume that

$$f\big(x,u,\xi_{d+1}(u), \dots, \xi_N(u)\big) = g\big(x,u,\xi_{d+1}(u), \dots, \xi_N(u)\big)$$

for all $x \in B(e_N,\gamma)$ and $u \in U$.

We are going to use the implicit function theorem in order to deduce the C^2 regularity of v near e_N. Define $h \in C^2\big(B(e_N,\gamma) \times U\big)$ by

$$h(x,u) = f\big(x,u,\xi_{d+1}(u), \dots, \xi_N(u)\big).$$

We note that $D_uh(e_N,0) = 0$ since the function $h(e_N,u)$ attains a maximum at $u = 0$. Using (6.5), (6.6) and (6.9), by simple computation we obtain $D_u^2h(e_N,0) = 4D^2\xi_N(0)$. Differentiating the relation

$$|u|^2 + \sum_{k>d} \xi_k(u)^2 = 1$$

twice, we get $D^2\xi_N(0) = -I$. Hence, we have $D_u^2 h(e_N, 0) = -4I$. Now, by the implicit function theorem, there are an $\varepsilon \in (0, \gamma]$ and a unique function $\varphi \in C^1\big(B(e_N, \varepsilon), U\big)$ such that $\varphi(e_N) = 0$ and $D_u h\big(x, \varphi(x)\big) = 0$ for all $x \in B(e_N, \varepsilon)$. It is clear that $v(x) = h\big(x, \varphi(x)\big)$ for all $x \in B\big(e_N, \varepsilon\big)$. Differentiating this relation and using that $D_u h\big(x, \varphi(x)\big) = 0$, we have $Dv(x) = D_x h\big(x, \varphi(x)\big)$ for all $x \in B(e_N, \varepsilon)$. This shows that $v \in C^2\big(B(e_N, \varepsilon)\big)$. Also, we have $Dv(e_N) = 4e_N$ by (6.5).

We continue to calculate the second derivatives of v at $e_N \in M$. We have

$$\begin{aligned} v_{x_i x_j}(e_N) &= h_{x_i x_j}(e_N, 0) + \sum_{k=1}^{d} h_{x_i u_k}(e_N, 0)\varphi_{k, x_j}(e_N), \\ h_{x_i x_j}(e_N, 0) &= f_{x_i x_j}(e_N, e_N), \end{aligned}$$

and

$$h_{x_i u_k}(e_N, 0) = f_{x_i \xi_k}(e_N, 0) + \sum_{l > d} f_{x_i \xi_l}(e_N, e_N)\xi_{l, u_k}(0)$$

for all $i, j = 1, \ldots, N$ and $k = 1, \ldots, d$. Therefore, using (6.6), (6.7) and (6.9), we see that $v_{x_i x_j}(e_N) = 0$ if $d < i < N$ and $j = 1, \ldots, N$. This means that $D^2 v(e_N) \in \mathbf{S}^N(e_N) = \mathbf{S}^N(Dv(e_N))$. We now conclude that v is a C^2 function on a neighborhood of M and that for any $z \in M$, $Dv(z) = 4z$ and $D^2 v(z) \in \mathbf{S}^N(Dv(z))$.

Step II. Our intention now is to replace v by a $C^{1,1}$ function on $\mathbf{R}^N$. To this end, we use the regularization techniques of inf-convolutions. We define $w : \mathbf{R}^N \to \mathbf{R}$ by

$$w(x) = \inf_{y \in \mathbf{R}^N} \big(v(y) + \frac{1}{2}|x - y|^2\big) = \min_{y \in \mathbf{R}^N} \big(v(y) + \frac{1}{2}|x - y|^2\big).$$

It is easily checked (and also well known) that $w \in C^{1,1}(\mathbf{R}^N)$. Moreover, since v is strictly convex on $\mathbf{R}^N$, so is w. By the strict convexity of the function: $y \mapsto v(y) + (1/2)|x - y|^2$, for each $x \in \mathbf{R}^N$ there is a unique $z(x) \in \mathbf{R}^N$ such that

$$w(x) = v(z(x)) + \tfrac{1}{2}|x - z(x)|^2.$$

It is well known that the function $z : \mathbf{R}^N \to \mathbf{R}^N$ is Lipschitz continuous and surjective. Furthermore, z is a homeomorphism of $\mathbf{R}^N$. Also, we have

$$\begin{aligned} &Dw(x) \in D^- v\big(z(x)\big), \quad D^- v\big(z(x)\big) + z(x) - x \ni 0, \\ &w(x) = v\big(z(x)\big) + \frac{1}{2}|x - z(x)|^2 \end{aligned}$$

for all $x \in \mathbf{R}^N$, where $D^- v(\xi)$ denotes the subdifferential of v at ξ. The value $z(x)$ is also characterized as the unique solution y of $D^- v(y) + y - x = 0$.

Now, let V be a neighborhood of M such that $v \in C^2(V)$. Set $W = z^{-1}(V)$. Clearly, this set W is a neighborhood of $z^{-1}(M)$. We will prove that $w \in C^2(W)$. Setting $k(x, y) = Dv(y) + y - x$ and differentiating this relation, we get

$$D_y k(x, y) = D^2 v(y) + I \geq I \quad \text{for all} \quad y \in V.$$

Noting that $y = z(x)$ is a solution of $k(x,y) = 0$, by the implicit function theorem we deduce that $z \in C^1(W)$. Hence, from the relation $Dw(x) = Dv\big(z(x)\big)$ it follows that $w \in C^2(W)$.

Let $p \in M$. Then $Dv(p) = p$. Hence, $y = p$ is a solution of $Dv(y) + y - 2p = 0$. That is, $z(2p) = p$. Therefore, $Dw(2p) = Dv\big(z(2p)\big) = Dv(p) = p$. Now, the strict convexity of w implies that $x = 2p$ is the unique solution of $Dw(x) = p$. Thus for every $p \in M$ we have $Dw(x) = p$ if and only if $x = 2p$. The relation $w(x) = v\big(z(x)\big) + \frac{1}{2}|x - z(x)|^2$ yields that $w(2p) = \frac{1}{2}$ for all $p \in M$. Observe also that since $z(2p) = p$ for $p \in M$, the set W is a neighborhood of $2M$.

Next, we want to prove that

$$D^2 w(2p) \in \mathbf{S}^N\big(Dw(2p)\big) \quad \text{for all} \quad p \in M.$$

Fix $p \in M$. Differentiating both of the relations

$$Dw(x) = Dv\big(z(x)\big) \quad \text{and} \quad Dv\big(z(x)\big) + z(x) = x,$$

which is valid in W, we get

$$D^2 w(2p) = \big(I + D^2 v(p)\big)^{-1} D^2 v(p).$$

We then observe that if $A, \pi \in \mathbf{S}^N$ satisfy $\pi A \pi = A$ and if $I + A$ is nonsingular, then $\pi (I + A)^{-1} A \pi = (I + A)^{-1} A$. Indeed, we then have

$$\pi(I + A)^{-1} A \pi = \pi (I + A)^{-1} A = \pi A (I + A)^{-1} = A(I + A)^{-1} = (I + A)^{-1} A.$$

Since $D^2 v(p) \in \mathbf{S}^N(p)$, we thus conclude that $D^2 w(2p) \in \mathbf{S}^N(p) = \mathbf{S}^N\big(Dw(2p)\big)$.

Step III. Define

$$K = \{x \in \mathbf{R}^N : w(x) \le \tfrac{1}{2}\}.$$

Then it is clear that K is a compact, strictly convex set. Since $v \ge w$ by the definition of w, it follows that $B(0,1) \subset K$. In particular, we have $0 \in \operatorname{Int} K$. Since $w(x) - w(y) \ge \langle Dw(y), x - y\rangle$ for all $x, y \in \mathbf{R}^N$ by the convexity of w, and $w(0) < 0$, we see that if $w(y) = 1/2$, then $\langle Dw(y), y\rangle > 1/2$. Obviously, we have $\partial K = \{x \in \mathbf{R}^N : w(x) = 1/2\}$.

Finally, we define $\psi : \mathbf{R}^N \to \mathbf{R}$ by

$$\psi(x) = \min\{t \ge 0 : x \in tK\}.$$

This is the so-called Minkowski functional of K. It is well known and easily seen that ψ is convex and positively homogeneous of degree 1 on $\mathbf{R}^N$ and that $\psi(x) > 0$ if $x \ne 0$. Since w is in $C^{1,1}(\mathbf{R}^N)$ and $\langle x, Dw(x)\rangle > 0$ for all $x \in \partial K$, the set ∂K can be represented locally as the graph $x_N = g(x_1, \ldots, x_{N-1})$ of a $C^{1,1}$ function g after an orthogonal change of variables. This guarantees that $\psi \in C^{1,1}(\mathbf{R}^N \setminus \{0\})$. Similarly, since w is a C^2 function in a neighborhood of M, we see that ψ is a C^2 function in a neighborhood of $\mathbf{R}_+ M$.

By the definition of ψ, if $x \neq 0$ then $y = x/\psi(x) \in \partial K$. Hence, $w(x/\psi(x)) = 1/2$ for all $x \neq 0$. Now, differentiating this produces

$$D\psi(x) = \frac{1}{\langle y, Dw(y)\rangle} Dw(y), \quad \text{with } y = \frac{x}{\psi(x)}, \quad \text{for } x \neq 0.$$

In particular, for every $p \in M$, $D\psi(2p) = (1/2)p$ since $\psi(2p) = 1$ and moreover, $D\psi(tp) = (1/2)p$ for all $t > 0$. The strict convexity of K or w now implies that $D\psi(x) \in \mathbf{R}_+ M$ if and only if $x \in \mathbf{R}_+ M$.

Simple but tedious calculations yield

$$\begin{aligned} D^2\psi(2p) = \tfrac{1}{2} D^2 w(2p) - \tfrac{1}{2}\big[(D^2 w(2p)\cdot p)\otimes p + p \otimes (D^2 w(2p)\cdot p)\big] \\ + \tfrac{1}{2}\langle D^2 w(2p)\cdot p, p\rangle p \otimes p. \end{aligned}$$

Noting that if $\pi, A \in \mathbf{S}^N$ satisfy $\pi A \pi = A$ and if $\pi p = p$ then

$$\pi\big((Ap)\otimes p\big)\pi = (Ap)\otimes p, \quad \pi\big(p \otimes (Ap)\big)\pi = p \otimes (Ap),$$

we see that $D^2\psi(2p) \in \mathbf{S}^N(p)$ for all $p \in M$ and hence, $D^2\psi(x) \in \mathbf{S}^N(D\psi(x))$ for all $x \in \mathbf{R}_+ M$. The function ψ has all the required properties. $\square$

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