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THE FAST DIFFUSION EQUATION WITH LOGARITHMIC NONLINEARITY AND THE EVOLUTION OF CONFORMAL METRICS IN THE PLANE

Juan L. Vazquez, Juan R. Esteban
Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

Ana Rodriguez
Departamento de Matemática Aplicada, E.T.S. Arquitectura
Universidad Politécnica de Madrid, 28040 Madrid, Spain

Abstract. We consider the nonlinear equation

$$u_t = \Delta \log u$$

posed in two space dimensions. For the Cauchy problem with radially symmetric data, we investigate the existence of solutions, both global and local in time, as well as the question of uniqueness/multiplicity. The most striking result is as follows: for every radial $u(x,0) \in L^1(\mathbf{R}^2)$ there exists a unique *maximal* solution $u \in C^\infty(\mathbf{R}^2 \times (0,T))$ of the Cauchy problem, characterized by the additional property

$$\int_{\mathbf{R}^2} u(x,t)\,dx = \int_{\mathbf{R}^2} u(x,0)\,dx - 4\pi\, t\,, \tag{*}$$

and, accordingly, the existence time is $T = \int u(x,0)\,dx/4\pi$. We then interpret the solutions as the conformal factor of a metric in $\mathbf{R}^2$ evolving by Ricci flow; formula (*) is a version of Gauss-Bonnet's Theorem. The solution here described is not unique if one weakens the equality (*) into an inequality $\le$. We thus obtain infinitely many nonmaximal solutions of the Cauchy problem having different behaviors (more precisely *fluxes*) at $r = +\infty$. One of these options, namely the solution corresponding to formula (*) with last term $-8\pi t$, describes the evolution of a complete compact surface under Ricci flow. For data $u(x,0)$ with infinite integral solutions are unique.

Introduction. This paper is devoted to studying the remarkable properties of the nonlinear diffusion equation

$$u_t = \nabla \cdot (\frac{\nabla u}{u}) = \Delta \log u \tag{0.1}$$

posed in two space dimensions. For definiteness we consider the Cauchy problem for equation (0.1) with initial data

$$u(x,0) = u_0(x)\,, \qquad x \in \mathbf{R}^2\,. \tag{0.2}$$

Let us call this Cauchy problem (CP). We assume throughout that the initial data are nonnegative, locally integrable and radially symmetric, $u_0 = u_0(r)$ with $r = |x|$, and look for a positive solution $u(x,t)$ defined in a strip $Q_T = \mathbf{R}^2 \times (0,T)$ for some $T > 0$ which will be also radially symmetric in x. Some times we can take $T = +\infty$, *global*

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continuation in time, while in other cases the maximal time T is finite because there is a phenomenon of *complete extinction* at that time. In that case the solution is positive and C^∞ smooth for all $x \in \mathbf{R}^2$ and $0 < t < T$ and vanishes identically at $t = T$, *simultaneous extinction*.

Equation (0.1) is a limit case ($m = 0$) of the *filtration equation*

$$u_t = \nabla \cdot (u^{m-1} \nabla u) = \frac{1}{m}\, \Delta u^m\,, \qquad m > 0\,. \tag{0.3}$$

This is the *linear heat equation* for $m = 1$, the *porous medium equation* for $m > 1$ and is usually called the *fast-diffusion equation* for $m < 1$. However, as we will show, the properties of (0.3) and (0.1) are strikingly different. Thus, it is well-known that the Cauchy problem is well-posed for equation (0.3) with data $u_0 \in L^1(\mathbf{R}^2)$ when we consider weak solutions $u \in C([0, +\infty) : L^1(\mathbf{R}^2)) \cap C^\infty(Q)$, $Q = \mathbf{R}^2 \times (0, +\infty)$. In fact, the maps $S_t : u_0 \mapsto u(\cdot, t)$ generate a semigroup of contractions in $L^1(\mathbf{R}^2)$ and the law of conservation of mass holds: $\int_{\mathbf{R}^2} u(x,t)\, dx = \int_{\mathbf{R}^2} u_0(x)\, dx$ for all $t > 0$.

The picture is completely different for $m = 0$. To begin with, the fact that $m = 0$ is a borderline case for the well-posedness of the Cauchy problem for (0.3) was demonstrated in the work [15] where it was proved that *no solution* exists when $m < 0$ and $u_0 \in L^1(\mathbf{R}^2)$, and this is explained as a phenomenon of extinction in zero time in the limit of approximation by solutions u_ε with data $u_{0,\varepsilon}(x) = u_0(x) + \varepsilon$, a technique that will be reused in Section 3 below. An idea of the novelties of this case was also shown by means of the explicit example

$$u(x,t) = \frac{8\,(T-t)}{(1+r^2)^2}\,, \tag{0.4}$$

a family of classical solutions of (0.1) vanishing in finite time with a constant mass-decay rate

$$\frac{d}{dt} M(t) = -8\pi\,, \qquad 0 < t < T\,.$$

As usual, the mass is defined by the formula

$$M(t) = \int_{\mathbf{R}^2} u(x,t)\, dx = 2\pi \int_0^{+\infty} r\, u(r,t)\, dr\,. \tag{0.5}$$

The latter part applies of course only when u_0 is radially symmetric. Mass considerations will play a strong role in what follows, but the number 8π will turn out to be secondary, the critical mass decay rate being 4π. Both numbers will appear below as manifestation of the *Gauss-Bonnet* formula of differential geometry.

Indeed, our work on this subject has been strongly motivated by an application in differential geometry proposed by Hamilton ([6]) and developed by L.F. Wu in [16, 17]. It is as follows: In $\mathbf{R}^2$ any metric can be expressed as

$$ds^2 = e^v\,(dx_1^2 + dx_2^2)\,,$$

with $x = (x_1, x_2)$. The so-called *Ricci flow* prescribes the evolution of the metric according to the rule

$$\frac{\partial}{\partial t} ds^2 = -R\, ds^2 = -2K\, ds^2\,,$$

where K is the Gauss curvature. Taking into account that

$$K = -\frac{1}{2}\,\frac{\Delta v}{e^v}\,,$$

one gets equation (0.1) for the conformal factor $u = e^v$. Observe that v satisfies the equivalent nondivergent equation

$$v_t = e^{-v}\Delta v\,,$$

and no sign restriction is now required. Let us remark that the total area of the surface at time t is given by

$$A(t) = \int u\,dx_1 dx_2\,.$$

It thus coincides with our total mass function $M(t)$. More details are given in Section 7. Let us point out here that Wu's work is concerned with solutions with infinite mass (area).

Here we make a complete study of the Cauchy problem (CP) in (0.1)–(0.2) both with integrable and nonintegrable data under the assumption of radial symmetry which amounts to considering revolution surfaces. Though it is a strong condition, let us point out that we do not need any additional regularity assumption on the data. This is a consequence of typical regularizing effects of diffusion problems which are by now quite well understood. Actually, our approach is based on the nonlinear diffusion theory with the geometry as motivation. The justification of the symmetry assumption lies in the existence of the change of variables

$$v(s,t) = r^2\,u(r,t)\,, \qquad s = \log r\,, \tag{0.6}$$

that allows one to translate the problem to one space dimension. As we shall see in Section 7, v is the angular factor of the metric in geodesic polar coordinates. The point of introducing $v(s,t)$ is that it satisfies the one-dimensional analogue to (0.1):

$$v_t = (\log v)_{ss} \qquad \text{for } s \in \mathbf{R}\,,\ 0 < t < T\,, \tag{0.7}$$

with initial data

$$v_0(s) = e^{2s}\,u_0(e^s)\,, \qquad s \in \mathbf{R}\,.$$

Now, this latter problem has arisen in several applications and has been the object of a certain interest; *cf. e.g.* [1], [5], [18] where further references can be found. A rather complete study of the existence and multiplicity of solutions of the (CP) has been performed by the authors in a series of papers, [4], [11, 12, 13], and this will provide the cornerstone for the present analysis. A complete study of the existence of solutions for (CP) without the radial symmetry assumption needs new techniques and is left for a separate work, [10]. But let us point out that the basic features of the Cauchy problem are demonstrated under the present restrictions. It can also be useful to note that we are able to uniquely characterize our radial solutions by means of additional flux data, and such uniqueness holds in the framework of general solutions, not necessarily symmetric. This is a summary of our main results:

1) Take $u_0(r)$ integrable in $\mathbf{R}^2$ with initial mass $M > 0$. Then there exists a unique *maximal* classical solution of (0.1)–(0.2) with mass decay rate 4π, hence defined up to an extinction time given by

$$T(u_0) = \frac{M}{4\pi}\,,$$

with

$$\frac{dM}{dt} = -4\,\pi\,, \qquad \text{for all} \;\; 0 < t < T;$$

see Theorems 3.1 and 3.2 below. Therefore, not only conservation of mass fails, but also a simultaneous extinction of the solution at finite time T exists. If we allow for the extended definition $u(\cdot,t) \equiv 0$ for $t \ge T(u_0)$, then we still generate a semigroup of contractions in $L^1(\mathbf{R}^2)$. This is interpreted in geometry as the existence for any given conformal metric of finite area of an evolution with maximal area that shrinks to a point in a maximal time T and satisfies at all times $0 < t < T$ the Gauss-Bonnet formula

$$\int K dA = 2\pi.$$

These maximal surfaces are complete with infinite diameter.
2) Take again $u_0(r)$ integrable with mass M. For every function $\Phi_\infty(t)$ in $L^\infty_{\rm loc}[0,+\infty)$ such that $\Phi_\infty(t) \ge 2$, there exists a unique classical solution of (CP) with mass

$$M(t) = M - 2\pi \int_0^t \Phi_\infty(\tau)\,d\tau\,,$$

and defined in a maximal time interval $(0,T)$ where T is given by the equality

$$M = 2\pi \int_0^T \Phi_\infty(\tau)\,d\tau\,; \tag{0.8}$$

see Theorem 4.1 below. We call Φ_∞ the *flux* function at $+\infty$ since

$$\lim_{r\to+\infty} \frac{r\,u_r(r,t)}{u(r,t)} = -\Phi_\infty(t)\,. \tag{0.9}$$

Now we obtain under the same geometrical assumptions a unique evolution with arbitrarily prescribed area function $M(t)$ such that $M'(t) \le -4\pi$. When $M'(t) < -4\pi$ it represents an incomplete surface with a singularity at $r = +\infty$ unless $M'(t) = -8\pi$. In the latter case we get a complete compact surface evolving under Ricci flow and the Gauss-Bonnet formula holds with Euler characteristic 2, just like in example (0.4).
3) Let now u_0 be locally integrable with infinite mass; *i.e.,* $u_0 \notin L^1(\mathbf{R}^2)$. Then there exists a unique global solution of (CP) having infinite mass for all times $0 < t < +\infty$; see Theorem 6.1. This means that we have a unique evolution formed by complete surfaces with infinite area.

The study that follows produces a number of complementary results, like the nonexistence of solutions with a point (Dirac) mass as initial data, Section 5. Moreover, we are naturally led by our technique to consider a more general problem, namely the existence of solutions defined for $r \ne 0$ with a possible singularity at $r = 0$. We show that these new solutions can be explained as solutions of the inhomogeneous equation

$$u_t = \Delta \log u - 2\pi\,\Phi_0(t)\,\delta_0(x)$$

with $\Phi_0(t) \ge -2$, so that the singularity at 0 is either a *source*, if $\Phi_0 > 0$, or a *sink*, if $\Phi_0 < 0$. The *flux* at $x = 0$, $\Phi_0(t)$ is given by

$$\Phi_0(t) = \lim_{r\to 0} \frac{r\, u_r(r,t)}{u(r,t)}\,. \tag{0.10}$$

We obtain for the new problem existence and multiplicity results generalizing **1)–3)** above. The previous solutions without singularities correspond to the choice $\Phi_0 = 0$ as expected.

These results are completed with an exhaustive description of the families of self-similar solutions of (0.1), which is collected as two Appendices. Though based on a simple phase plane analysis it can serve as a guide to the reader into the wealth of more or less explicit solutions, and they provide concrete examples of all the different behaviors described in **1)–3)** above. In order to show the variety of possible large-time behaviors for infinite-area evolutions (cf. Wu's work) let us just mention here that corresponding to every initial data of the form $u_0(x) = c|x|^\gamma$ with $c > 0$ and $\gamma > -2$ there exists a self-similar evolution of the form $u(x,t) = t^{-\alpha}U(|x|t^{-\beta})$ with time exponent $\alpha = -\gamma/(\gamma+2)$ covering the whole range $(-1,\infty)$ and $\beta = 1/(\gamma+2) > 0$ (see Theorem B.3). Spatial decay $\gamma = -2$ (which is the limit case for infinite area) corresponds to a different self-similarity with exponential time decay; cf. Theorem B.5. On the other hand, the analysis leads in a natural way to considerations that fall out of our present scope, like the existence of solutions with nonintegrable singularities which may or may not disappear in time, or solutions defined in moving or stationary annuli with singular boundary conditions.

Among the remarkable properties of equation (0.1) we can mention its *invariance under the conformal group*, a property that singles it out among the family of equations (0.3). Thus, given a conformal transformation $\phi : \Omega' \to \Omega$ where Ω' and Ω are subdomains of $\mathbf{R}^2$, preservation of the metric element ds^2 leads to the formula

$$\overline{u}(x,t) = u(\phi(x),t)|\phi'(x)|^2, \tag{0.11}$$

which transforms any solution $u(x,t)$ of (0.1) defined for $x \in \Omega$ and $0 < t < T$ into another solution $\overline{u}$ of (0.1) defined for $x \in \Omega'$, $0 < t < T$. Here $\phi'(x)$ denotes the complex derivative and the transformation preserves local length and area. In this way a number of non-radially symmetric solutions can be obtained from our results. Let us just point out that the map $\phi(z) = e^z$ produces the change of variables (0.6), which applies even without radial symmetry, while the choice $\phi(z) = 1/z$ produces the *inversion formula*

$$\overline{u}(x,t) = \frac{1}{|x|^4}\, u(\frac{x}{|x|^2}, t) \tag{0.12}$$

which in the radial case reads $\overline{u}(r,t) = r^{-4}\, u(r^{-1},t)$. If u is a solution of (0.1) in $Q_T^* = (\mathbf{R}^2\backslash\{0\}) \times (0,T)$ so is $\overline{u}$. Observe that (0.12) is an isometric transformation which changes the behavior of u at $r = 0$ (respectively $r = +\infty$) into the behavior of $\overline{u}$ at $r = +\infty$ (respectively $r = 0$). Thus, we have

$$\Phi_0(\overline{u}) = -4 + \Phi_\infty(u)\,, \qquad \Phi_\infty(\overline{u}) = 4 + \Phi_0(u)\,.$$

Notice also that formula (0.4) gives a self-inverse solution in separated variables. More in general, if there are points at which ϕ' vanishes they become points at which $\overline{u}$ has a zero, and moreover the equation ceases to hold because of the appearance of a delta singularity in $\Delta \log \overline{u}$; see Theorem 1.3 below.

As for related works let us mention that a very complete formal study of the filtration equation (0.3) for the whole range of parameters $m \in \mathbf{R}$ and dimensions $N \geq 1$ has been recently performed by King ([7]). Though the study does not contain formal proofs nor does it discuss in detail existence and uniqueness/multiplicity questions, it does contain a wealth of asymptotic results and techniques. In particular, for equation (0.3) it uses transformation (0.6) and performs a preliminary analysis of the self-similar solutions. The transformation into a one-dimensional problem appears also in the recent work of Rosenau ([9]) who constructs and discusses a number of explicit solutions. He uses Bäcklund transforms to produce new solutions, a powerful method that we also used in [13] for uniqueness purposes.

Let us end this introduction by recalling the result of [15] according to which, in dimensions $N \geq 3$ and exponent $m \leq 0$, the Cauchy problem (0.3), (0.2) does not admit *any* solution with finite mass. Precise growth conditions on the data which allow for existence of solutions of such a problem are found by Daskalopoulos-del Pino [3].

1. Reduction to a one-dimensional problem. Let $u = u(r,t)$ be a radial solution of

$$u_t = \Delta \log u = \frac{1}{r}\left(\frac{r\,u_r}{u}\right)_r \qquad \text{for } r > 0 \text{ and } t > 0\,. \tag{1.1}$$

We introduce the change of variables (0.6) and find that $v(s,t)$ satisfies the one-dimensional analogue to (1.1):

$$v_t = (\log v)_{ss} \qquad \text{for } s \in \mathbf{R}\,,\ t > 0\,. \tag{1.2}$$

The flux functions of both equations are related by

$$\frac{r\,u_r}{u} = -2 + \frac{v_s}{v}\,,$$

which gives for the endpoint limits the identities

$$\lim_{r\to 0} \frac{r\,u_r}{u} = -2 + \lim_{s\to -\infty} \frac{v_s}{v}\,, \qquad \text{and} \qquad \lim_{r\to +\infty} \frac{r\,u_r}{u} = -2 + \lim_{s\to +\infty} \frac{v_s}{v}\,,$$

whenever these limits exist. This can be written as

$$\Phi_0(t) = f(t) - 2\,, \qquad \Phi_\infty(t) = g(t) + 2\,, \tag{1.3}$$

where Φ_0 and Φ_∞ are the fluxes of u defined in (0.10) and (0.9) and the corresponding fluxes for equation (1.2) are defined by

$$f(t) = \lim_{s\to -\infty} \frac{v_s(s,t)}{v(s,t)}\,, \qquad g(t) = -\lim_{s\to +\infty} \frac{v_s(s,t)}{v(s,t)}\,. \tag{1.4}$$

Observe that the inversion formula (0.12) translates in one dimension into the symmetry $\overline{v}(s,t) = v(-s,t)$.

We define the mass of the solution u at time t by formula (0.5). If $M(t)$ is finite for $0 \leq t < T$ we say that u is a finite-mass solution. Finite mass and conservation of the total mass hold simultaneously for u and v, as the following computation shows:

$$M(t) = 2\pi \int_0^{+\infty} r\,u(r,t)\,dr = 2\pi \int_{-\infty}^{+\infty} v(s,t)\,ds = 2\pi\, M_1(t)\,,$$

where $M_1(t)$ denotes the mass of v at time t. By the same token it is clear that the condition $u(\cdot,t) \in L^1_{\rm loc}(\mathbf{R}^2)$ becomes a condition of local integrability in $\mathbf{R}$ plus integrability at $s=-\infty$ for v.

With these observations we are able to translate the existence and uniqueness theory developed in the papers [4], [11, 12, 13]. We recall the main facts for the reader's convenience.

Theorem 1.1. *For every radial initial function $u_0 \in L^1(\mathbf{R}^2)$ and every pair of flux functions $\Phi_0(t), \Phi_\infty(t) \in L^\infty_{\rm loc}[0,+\infty)$ with $\Phi_0 \ge -2$, $\Phi_\infty \ge 2$, there exists a time $T>0$ and a classical solution of* (1.1) *in $Q^*_T = (\mathbf{R}^2\backslash\{0\}) \times (0,T)$ such that the initial data*

$$\lim_{t\to 0} u(x,t) = u_0(x)$$

is taken in $L^1(\mathbf{R}^2)$ and the flux relations

$$\lim_{r\to 0} \frac{r\,u_r}{u} = \Phi_0(t)\,, \qquad \lim_{r\to+\infty} \frac{r\,u_r}{u} = -\Phi_\infty(t)\,,$$

hold in $L^1_{\rm loc}(0,T)$. We have

$$T = \sup\big\{t>0 : \int_{\mathbf{R}^2} u_0(x)\,dx > 2\pi \int_0^t (\Phi_0(t)+\Phi_\infty(t))\,dt\,\big\}\,.$$

and u vanishes identically at $t=T$; i.e., $\lim_{t\nearrow T} u(x,t) \equiv 0$.

We next make sure that finite-mass solutions indeed have fluxes.

Theorem 1.2. *For every finite mass solution of* (1.1) *in Q^*_T, the mass is nonincreasing in time and the fluxes $\Phi_0(t)$, $\Phi_\infty(t)$ always exist in $L^1_{\rm loc}(0,T)$, and moreover, $\Phi_0 \ge -2$, $\Phi_\infty \ge 2$.*

These results come from Theorems 1, 2 and 5 in [11], where further details can be found and easily translated. In particular, let us mention that the natural comparison result applies, *cf.* [11, Theorem 3], which implies that the maximal solution with given initial data is obtained for the minimal flux pair $f(t)=g(t)=0$. However, the translated solution is *not* a global solution of (1.1) in $Q=\mathbf{R}^2\times(0,+\infty)$ because, by virtue of (1.3), it corresponds to a nonzero flux at $r=0$, $\Phi_0=-2$, thus implying that $u(r,t)$ exhibits a singularity near $r=0$ of the order of r^{-2} (in first approximation). Global (in space) solutions of (1.1), *i.e.,* solutions which are regular at $r=0$ and have $\Phi_0=0$, correspond to solutions of (1.2) with outgoing flux at $-\infty$, $f(t)=2$. Of course, this equivalence needs proof, which will be given below in Theorem 4.1.

Let us now make sure that solutions with the "wrong" flux at zero correspond to solving the equation with a special forcing term in the form of a Dirac mass located at $r=0$ for $0<t<T$.

Theorem 1.3. *Let $u=u(r,t)$ be a positive, radially symmetric function satisfying* (1.1) *in Q^*_T and assume that the limit*

$$\lim_{r\to 0} \frac{r\,u_r(r,t)}{u(r,t)} = \Phi_0(t)$$

exists in $L^1_{\rm loc}(0,T)$. Then $u(r,t)$ is a distributional solution of

$$u_t = \Delta\,\log u - 2\pi\,\Phi_0(t)\,\delta_0(x) \qquad\qquad in\ \mathbf{R}^2\times(0,T)\,. \tag{1.5}$$

Proof. Let $\zeta \in C_0^\infty(\mathbf{R}^2 \times (0,T))$ and let $\varepsilon > 0$. Integrating by parts and using the equation we have

$$\int_0^T \int_{|x|>\varepsilon} (\log u\, \Delta\zeta + u\, \zeta_t)\, dx\, dt$$
$$= \int_0^T \int_{|x|=\varepsilon} \log u(\varepsilon,t) \frac{\partial \zeta}{\partial r}\, d\sigma\, dt - \int_0^T \int_{|x|=\varepsilon} \zeta \left. \frac{u_r}{u}\right|_{r=\varepsilon} d\sigma\, dt.$$

The first integral of the second member goes to 0 as $\varepsilon \to 0$ while the last one goes to

$$-2\pi \int_0^T \zeta(0,t)\Phi_0(t)\, dt\, .$$

It follows from this that solutions in Q_T^* with the right flux $\Phi_0 = 0$ solve the exact equation (1.1) in Q_T in the sense of distributions. It is now a piece of more or less standard regularity theory for quasilinear parabolic equations to conclude that u is in fact bounded, positive and smooth at $r = 0$ for all times $0 < t < T$, hence a classical solution in Q_T. However, we will give a direct proof of this fact in Theorems 3.1 and 3.2.

Let us remark at this point that the application of formula (0.11) with $\phi(z) = z^m$, with m an integer (we use complex notation for the space variable) produces the transformation of a standard solution $u(z,t)$ defined in Q_T into a solution

$$\overline{u}(z,t) = m^2|z|^{2m-2}u(z^m,t)$$

defined in Q_T^* which satisfies (1.5) with $\Phi_0 = 2(m-1)$.

2. Some explicit solutions. Before proceeding with the main bulk of the theory we illustrate the relation between the one- and two-dimensional problems with a number of explicit solutions which are easily constructed in dimension one.

Example 1. The family of solutions of (1.2) in *separated variables* is

$$v_{\mu,T}(s,t) = \frac{2\,\mu^2(T-t)}{\cosh^2(\mu s)}\, ,$$

with parameters μ and $T > 0$; *cf.* [10, page 993]. Its most remarkable property is extinction in finite time $v_{\mu,T}(s,T) = 0$. The solution corresponding to (1.1) is then

$$u_{\mu,T}(r,t) = \frac{8\,\mu^2(T-t)}{r^{2(1+\mu)} + r^{2(1-\mu)} + 2\,r^2}\, . \tag{2.1}$$

Some of its properties are:

i) The total mass is finite and changes according to

$$M(t) = \int_{\mathbf{R}^2} u_{\mu,T}(x,0)\, dx - 8\pi\mu\, t\, .$$

ii) Extinction occurs at the time

$$T = \frac{1}{8\pi\mu} \int_{\mathbf{R}^2} u_{\mu,T}(x,0)\, dx = \frac{1}{4\,\mu} \int_0^{+\infty} r\, u_{\mu,T}(r,0)\, dr\, .$$

iii) The fluxes at zero and infinity are constant and given by

$$\Phi_0 = -2\,(1-\mu) > -2\,, \qquad \Phi_\infty = 2\,(1+\mu) > 2\,.$$

iv) Behavior at $r = 0$:

$$\begin{array}{rll} \text{smooth:} & u_{1,T}(r,t) = \dfrac{8\,(T-t)}{(1+r^2)^2} & \text{if } \mu = 1\,, \\ \text{infinite singularity:} & u_{\mu,T}(r,t) = O(r^{-2(1-\mu)}) & \text{if } 0 < \mu < 1\,, \\ \text{vanishing singularity:} & u_{\mu,T}(0,t) = 0 & \text{if } \mu > 1\,. \end{array}$$

Thus, only the function $u_{1,T}(r,t)$, namely (0.4), is a global solution of $u_t = \Delta \log u$ in Q_T. The remaining functions are solutions of (1.5) with a source at $r = 0$ if $0 < \mu < 1$ or a sink if $\mu > 1$; *cf.* Theorem 1.3.

v) Any solution in (2.1) is invariant by the inversion formula (0.12).

Example 2. Let us mention another family of solutions having infinity mass at any positive time. It originates from the family of global *traveling wave* solutions for (1.2) given by

$$v_{\beta,k,C}(s,t) = \frac{C}{\beta}\,\frac{1}{1+k\,e^{-C(s-\beta t)}}\,,$$

for $\beta, k, C > 0$; *cf.* [18]. From them we can write the solutions of (1.1),

$$u_{\beta,k,C}(r,t) = \frac{C}{\beta}\,\frac{1}{r^2 + k\,r^{2-C}\,e^{\beta C t}}\,.$$

They have fluxes $\Phi_\infty(t) = 2$ and $\Phi_0(t) = C - 2 > -2$, and present three different behaviors at $r = 0$,

$$\begin{array}{rll} \text{smooth:} & u_{\beta,k,2}(0,t) = \dfrac{2}{\beta k}\,e^{-2\beta t} & \text{if } C = 2\,, \\ \text{singularity:} & u_{\beta,k,C}(r,t) = O(r^{-(2-C)}) & \text{if } 0 < C < 2\,, \\ \text{vanishing:} & u_{\beta,k,C}(0,t) = 0 & \text{if } C > 2\,. \end{array}$$

The same comments made above apply concerning the existence/nonexistence of a singularity at $r = 0$.

Example 3. The fundamental solution for (1.2) reads

$$v(s,t;M_1) = \frac{2\,t}{(\frac{2\pi t}{M_1})^2 + s^2}\,;$$

cf. [10, page 991], and corresponds to a Dirac mass as initial data $v(s,0,M_1) = M_1\,\delta_0(s)$. Translating into two dimensions gives

$$u(x,t;M) = \frac{2\,t}{r^2\,(\,(\frac{4\pi^2\,t}{M})^2 + (\log r)^2\,)}\,, \tag{2.2}$$

(where $r=|x|$) as a solution of (1.1) in $Q^*=(\mathbf{R}^2\backslash\{0\})\times(0,+\infty)$. We next list some of its properties.

i) Mass conservation: $M(t)=2\pi\, M_1$ is constant for any $t\ge 0$.

ii) Behavior at $r=0$: We have for the solution a singularity of the form

$$u(r,t;M)\sim \frac{2\,t}{r^2\,(\log r)^2}\,, \qquad \text{as } r\to 0\,,$$

while for the flux we have $\Phi_0(t)=-2$ for every $t>0$.

iii) Behavior at infinity. We have

$$u\sim \frac{2\,t}{r^2\,(\log r)^2}\,, \qquad \text{as } r\to +\infty\,,$$

and constant outgoing flux $\Phi_\infty(t)=2$. Observe that the sum of the two fluxes must be zero since the total mass is conserved.

iv) Initial data: u takes as $t=0$ singular initial data at $r=0$ and $r=1$ (a uniform mass on $|x|=1$), being zero otherwise.

v) Positivity and space-time decay:

$$0<u(r,t;M)\le \frac{2\,t}{r^2\,(\log r)^2}\,.$$

vi) Integrability: Though $u(\cdot,t;M)\in L^1(\mathbf{R^2})$, it holds that $u(\cdot,t;M)\notin L^{1+\varepsilon}_{\rm loc}(\mathbf{R^2})$ for any $\varepsilon>0$. This is a consequence of the type of singularity at $r=0$.

vii) u is a solution of the modified equation (1.5) with forcing term; *i.e.,*

$$u_t=\Delta\,\log u+4\pi\,\delta_0(x) \qquad\qquad \text{in } \mathcal{D}'(\mathbf{R}^2\times(0,+\infty))\,.$$

viii) $u(r,t;M)$ is invariant by the inversion formula (0.12).

Example 4. Letting $M\to+\infty$ in (2.2), we find the solution

$$u_\infty(r,t)=\frac{2\,t}{r^2\,(\log r)^2}\,, \tag{2.3}$$

which has flux $\Phi_\infty(t)=2$ at infinity and $\Phi_0(t)=-2$ at $r=0$. It is integrable both at $r=0$ and $r=+\infty$ but it has a singularity of the form

$$u_\infty\sim\frac{2\,t}{(\log r)^2}$$

on the circumference $r=1$, which comes from the singularity of the corresponding one-dimensional solution, $v_\infty=2\,t/s^2$ at $s=0$. Observe that (2.3) is self-invertible by (0.12).

3. Maximal solutions with finite mass. We envisage here a different method of construction of the solutions to the Cauchy problem (CP) in the whole plane $x\in\mathbf{R}^2$ which has been developed in [15]. It consists in adding to the initial data a quantity $\varepsilon>0$ so that the problem is no more degenerate, and then solving the problem

$$\begin{cases} u_t=\Delta\log u & \text{in } Q=\mathbf{R}^2\times(0,+\infty)\,,\\ u(x,0)=u_0(x)+\varepsilon & x\in\mathbf{R}^2. \end{cases} \tag{P$_\varepsilon$}$$

Suppose the initial data is bounded. As it is remarked in [15, Section I], problem (P_ε) has a unique and classical solution in Q for which the maximum principle and L^1-comparison apply, so that

$$\varepsilon < u_\varepsilon(x,t) < ||u_0||_\infty + \varepsilon\,, \qquad ||u_\varepsilon(\cdot,t) - \varepsilon||_1 \le ||u_0||_1\,.$$

Moreover, the family $\{u_\varepsilon\}$ is decreasing as $\varepsilon \searrow 0$; *i.e.*, $u_{\varepsilon_1} > u_{\varepsilon_2}$ if $\varepsilon_1 > \varepsilon_2$; hence we may pass to the limit to get

$$u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t) \tag{3.1}$$

as a candidate solution to the original problem. However, as explained in [15], there is a catch in this argument, namely the possibility that the solution vanishes identically for $t > 0$, thus refusing to satisfy the initial data because of the occurrence of an initial layer. This is precisely what happens for $N \ge 3$ ([15]). We will show that this does not happen in our case. We have

Theorem 3.1. *For every radially symmetric $u_0 \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ there is a maximal classical solution of the Cauchy problem* (CP). *It is defined for a time*

$$T = \frac{1}{4\pi}\int_{\mathbf{R}^2} u_0(x)\,dx\,, \tag{3.2}$$

and it has constant flux $\Phi_\infty = 2$ and, accordingly, mass

$$M(t) = \int_{\mathbf{R}^2} u_0(x)\,dx - 4\pi\,t\,, \qquad 0 < t < T.$$

It can be obtained as the limit of the approximation process (P_ε). It also coincides with the solution obtained from dimension one with corresponding initial data and with flux data $f = 2$, $g = 0$; cf. (1.4). *Finally, we have for two solutions u_1 and u_2*

$$\int (u_1 - u_2)_+(x,t)\,dx \le \int (u_1 - u_2)_+(x,0)\,dx\,. \tag{3.3}$$

Proof. Suppose that $u_0 \in L^1 \cap L^\infty$ and let u_ε and u be as in (3.1).

Step 1. Let us show that $u(r,t) > 0$ for $r > 0$ and $0 < t < T$. To eliminate the possibility of u being identically zero, we compare the u_ε with the solution $\overline{u}$ of the problem in a strip $Q^*_{T'}$ obtained from Theorem 1.1 with u_0 as initial data and constant flux conditions $\overline{\Phi}_0 > 0$ and $\overline{\Phi}_\infty = 2$. This $\overline{u}$ is a positive, smooth solution of (1.1) for $r > 0$ and $0 < t < T'$, and it takes on the flux data in $L^\infty_{\rm loc}[0,T')$. It has a sink at zero, $\overline{u}(0,t) = 0$, and it decays to zero at infinity. Now, for any $\delta > 0$, a classical comparison in the region $\{(x,t) : |x| > \delta, 0 < t < T'\}$ gives $u_\varepsilon(r,t) \ge \overline{u}(r,t)$ for $0 < r < +\infty$, $0 < t < T'$, independently of ε. Hence, the same comparison holds in the limit as $\varepsilon \to 0$ and we arrive at the conclusion that $u(x,t)$ given by (3.1) is positive for $0 < r < +\infty$ and $0 < t < T'$. From Theorem 1.1 we also know that

$$T' = \frac{1}{2\pi\,(2 + \overline{\Phi}_0)}\int_{\mathbf{R}^2} u_0(x)\,dx\,,$$

which can be taken arbitrarily close to T as defined in (3.2).

Step 2. Let us show that $u(0,t) > 0$ for all $0 < t < T$. We first fix any $0 < \tau_0 < T$ and a compactly supported and non-negative $\tilde{u}_0(x) = \varphi(|x - x_0|)$ such that $0 \notin \text{supp } \tilde{u}_0$, $\tilde{u}_0(x) \le u(x, \tau_0)$ for all $x \in \mathbf{R}^2$. Next we use Theorem 1.1 with this initial radial data now centered at x_0 and constant fluxes $\Phi_0 > 0$ and $\Phi_\infty = 2$. Let us call $\tilde{u}$ this solution of (1.1) in $(\mathbf{R}^2 \backslash \{x_0\}) \times (0, \tilde{T})$, where $\tilde{T} < T$. Comparison of u with $\tilde{u}(x, t - \tau_0)$ in $\{(x,t) : |x - x_0| > \delta,\ \tau_0 < t < \tilde{T}\}$ gives $u(x,t) \ge \tilde{u}(x, t - \tau_0)$ in this region, and in particular

$$u(0,\tau) > \tilde{u}(0, \tau - \tau_0) > 0 \qquad \text{for all } \tau \text{ with } \tau_0 < \tau < \tilde{T}\,.$$

To end this step, we use the a priori estimate

$$u_{\varepsilon,t} \le \frac{u_\varepsilon}{t}\,,$$

which comes from [15, Lemma 3]. It means that $u_\varepsilon(0,t)/t$ is a nonincreasing function of t, therefore

$$u_\varepsilon(0,t) \ge \frac{t}{\tau}\, u_\varepsilon(0,\tau) \ge \frac{t}{\tau}\, \tilde{u}(0, \tau - \tau_0) > 0 \qquad \text{for all } 0 < t < \tau\,.$$

We conclude by letting $\varepsilon \to 0$ and observing that τ_0 can be taken arbitrarily close to T.

Step 3. The fact that $\Phi_0(t)$ and $\Phi_\infty(t)$ exist in $L^1_{\text{loc}}(0,T)$ and moreover $\Phi_\infty(t) \ge 2$ follows by [15, Lemma 8]. Actually, since $u(0,t) > 0$ and $u(\cdot,t) \in L^1(\mathbf{R}^2)$, then necessarily $\Phi_0(t) = 0$. As a consequence of Theorem 1.3 and regularity theory of quasilinear parabolic PDE's $u(r,t)$ is a smooth solution of (CP) in Q_T.

Step 4. We next prove that $\Phi_\infty(t) = 2$. We compare with the solution $\tilde{u}$ which Theorem 1.1 assigns to u_0 as initial data and constant fluxes $\tilde{\Phi}_0 > 0, \tilde{\Phi}_\infty > 2$. This $\tilde{u}$ has a sink at $r = 0$ and behaves like

$$\frac{\log \tilde{u}}{\log r} \to -\tilde{\Phi}_\infty \qquad \text{as } r \to +\infty\,.$$

We thus obtain

$$u_\varepsilon(r,t) \ge \tilde{u}(r,t) \qquad \text{for all } r > 0 \text{ and } 0 < t < T\,,$$

and letting $\varepsilon \to 0$ and taking logarithms,

$$-\lim_{r\to+\infty} \frac{\log u(r,t)}{\log r} \le -\lim_{r\to+\infty} \frac{\log \tilde{u}(r,t)}{\log r} = \tilde{\Phi}_\infty\,.$$

Since we already know that $\Phi_\infty(t) \ge 2$ exists and $\tilde{\Phi}_\infty > 2$ is arbitrary, this last limit implies $\Phi_\infty(t) = 2$.

Step 5. The solution u translates in one dimension into a solution $v(s,t)$ of (1.2) with corresponding initial data and with fluxes $f = 2$, $g = 0$. Then we know that this solution is unique by [12, Theorem 1], and that its finite extinction time corresponds exactly to T as given in (3.2).

Step 6. Standard L^1-comparison arguments show that u is maximal among all the solutions of (CP). Formula (3.3) has been proved in [11] in one dimension.

Theorem 3.2. *Theorem* 3.1 *holds true for every* $u_0 \in L^1(\mathbf{R}^2)$.

Proof. If u_0 is not bounded we define approximations $u_{0,n}(x) = \min\{u_0(x), n\}$, and consider the maximal solution u_n given by Theorem 3.1. We translate into two dimensions the one-dimensional smoothing effect to get

$$u_n(r,t) \le C\,\frac{M^2}{r^2\,t}\,; \tag{3.4}$$

cf. [4, Proposition 2.6.iii]. We thus conclude that the increasing limit $u(r,t)$ exists and is a classical solution of (1.1) except possibly at $r = 0$, and it satisfies the estimate (3.4).

We still have to check that $u(0,t)$ is bounded above for every $0 < t < T$. For this we will need the solutions constructed in Appendix B. From the flux data in one dimension we know that $\log v \sim s^{-2}$; *cf.* [11, Theorem 3]; hence

$$\frac{\log u}{\log r} \to 0 \qquad \text{as } r \to 0\,, \tag{3.5}$$

for any $t > 0$, which is not sufficient. We complete the argument as follows. Fix any $\tau > 0$ and use the limit (3.5) to obtain $\delta > 0$ such that

$$u(r,\tau) < \frac{1}{r} \qquad \text{for } 0 < r < \delta\,.$$

With this estimate in hand we compare in $\{(x,t) : |x| < \delta\,,\ \tau < t < T\}$ our $u(r,t)$ with the self-similar solution $u_\gamma(r, t-\tau)$ constructed in Theorem B.3 (no circularity in the argument!) for $-2 < \gamma < -1$, $\kappa = 0$ and initial data $u_\gamma(r,0) = C\,r^\gamma$. We choose the constant $C > 1$ so that

$$u_\gamma(\delta, T) = \min_{\tau \le t \le T} u_\gamma(\delta, T) \ge \frac{M^2}{\delta^2\,\tau} \ge \frac{M^2}{\delta^2\,t} \ge u(\delta, t)\,,$$

for all $\tau \le t \le T$. We finally obtain

$$u(0,t) \le u_\gamma(0, t-\tau) < +\infty \qquad \text{for all } \tau < t \le T\,.$$

Finally, we get a more precise estimate of the behavior of maximal solutions.

Theorem 3.3. *Let* u_0 *and* $u(r,t)$ *be as above. Then* u *behaves as* $r \to +\infty$ *like*

$$\frac{2\,t}{r^2\,(\log r)^2} \le u(r,t) \le C_\tau\,\frac{T-t}{r^2}\,, \tag{3.6}$$

where the last inequality holds for $0 < \tau \le t \le T$. *Moreover, if* u_0 *is compactly supported we have exact equality to first approximation order in the left-hand limit above.*

Proof. The second inequality is nothing but (3.4) combined with [11, page 162]. The first inequality comes from the one-dimensional problem as in [12, Lemma 2]. See also [4, Theorem 4] for the case of compactly supported u_0.

4. General theory of finite mass solutions. We summarize here the classification of finite mass and radially symmetric solutions of the Cauchy problem (CP). By means of the one-dimensional analogy we proved in Theorem 1.1 a basic existence and uniqueness theorem in which we can freely assign a flux function at infinity under the sole restriction of $\Phi_\infty(t) \ge 2$. We can also choose the flux at the origin $\Phi_0(t)$ as long as $\Phi_0 \ge -2$. We still need to prove that global (in space) solutions correspond to $\Phi_0 = 0$. We have just proved this fact for maximal solutions, where $\Phi_\infty = 2$. The general result is

Theorem 4.1. *Every finite-mass solution of* (1.1) *in* Q_T^* *having zero flux at* $r = 0$ *is actually positive and smooth in* $\mathbf{R}^2 \times (0,T)$ *.*

Proof. Comparison with the maximal solution shows that it is bounded at $r = 0$ for $t > 0$.

To show that $u(0,t) > 0$ for $0 < t < T$, we consider a sequence of piecewise-constant fluxes $\Phi_{\infty,n}(t) \ge \Phi_\infty(t)$ and such that $\Phi_{\infty,n} \to \Phi_\infty$ in $L^1_{\rm loc}(0,T)$. Let $u_n(x,t)$ be the solution with fluxes $\Phi_0 = 0$ and $\Phi_{\infty,n}$ and with the same initial data as $u(x,t)$. By arguing as in Step 2 in Theorem 3.1 we know that $u_n(0,t) > 0$ for $0 < t < T_n$. Since $u \ge u_n$ and $T_n \to T$ by the convergence of $\Phi_{\infty,n}$, we conclude that $u(0,t) > 0$ for $0 < t < T$.

By using Theorems 4.1 and 1.1 we can summarize:

Theorem 4.2. *For every radially symmetric* $u_0 \in L^1(\mathbf{R}^2)$ *and every flux data* $\Phi_\infty(t) \in L^\infty_{\rm loc}[0,+\infty)$ *such that* $\Phi_\infty \ge 2$ *, there is a unique classical solution of the mixed problem*

$$\begin{cases} u_t = \Delta \log u \\ u(x,0) = u_0(x) \\ \displaystyle\lim_{r\to+\infty} \frac{r\,u_r(r,t)}{u(r,t)} = -\Phi_\infty(t)\,. \end{cases}$$

In addition, there exists $0 < T < +\infty$ *such that*

$$\begin{aligned} u(\cdot,t) &> 0 \qquad \textit{for } t < T\,, \\ u(\cdot,t) &\equiv 0 \qquad \textit{for } t = T\,, \end{aligned}$$

where T *is given by*

$$T = \sup\Big\{\, t > 0 : \int_{\mathbf{R}^2} u_0(x)\,dx > 2\pi \int_0^t \Phi_\infty(t)\,dt \,\Big\}\,.$$

The existence of $\Phi_\infty(t) \ge 2$ above implies the behavior

$$\lim_{r\to+\infty} \frac{\log u(r,t)}{\log r} = -\Phi_\infty(t) \qquad \text{in} \;\; L^\infty_{\rm loc}(0,T)\,.$$

Using standard integration by parts, this gives a characterization of the maximal solution.

Theorem 4.3. *The maximal solution can be characterized as the unique solution of the Cauchy problem satisfying*

$$\lim_{r\to+\infty} \frac{\log u(r,t)}{\log r} = -2 \qquad \textit{in} \;\; L^\infty_{\rm loc}(0,T)\,.$$

5. Nonexistence of fundamental solutions. To end this section we show that no classical solution of (1.1) can exist corresponding to initial data a Dirac mass. We begin our investigation by showing what happens when we try to obtain the fundamental solution as limit of solutions with integrable initial data. Let $M > 0$ be any initial mass, let us take any admissible mass function $M(t)$ (*i.e.,* such that $M(0) = M$ and $M'(t) \le -4\pi$) and let T be the time at which $M(t)$ vanishes. Let us then take a radially symmetric integrable function $\phi(x)$, which may be smooth, with mass 1, and let us consider the solutions $u_k(x,t)$ to the approximate problems

$$\begin{cases} u_t = \Delta \log u \\ u(x,0) = M\phi_k(x) \\ \int_{\mathbf{R}^2} u(x,t)\,dx = M(t) \end{cases} \qquad (\mathrm{P}_k)$$

with $\phi_k(x) = k^2\phi(kx)$. Observe that the maximal mass choice $M(t) = (M - 4\pi\, t)_+$ with $T = M/4\pi$ corresponds to maximal solutions of the Cauchy problem. Now, since the time-independent rescaling

$$\overline{u}(x,t) = k^2\, u(kx,t)\,, \qquad k > 0\,,$$

transforms solutions of (1.1) into solutions of (1.1) keeping the mass intact, due to the invariance of the initial data and the uniqueness of the problem we have

$$u_k(x,t) = k^2\, u_1(kx,t)\,. \tag{5.1}$$

All these solutions exist for the same time interval $(0,T)$. It is clear from formula (5.1) that for every $0 \le t < T$ we have

$$u_k(x,t) \to M(t)\,\delta_0(x)\,.$$

It immediately follows that

Proposition 5.1. *In the limit of any sequence of approximate problems* (P_k) *as above we have*

$$\lim_{k\to\infty} u_k(x,t) = M(t)\,\delta_0(x)\,. \tag{5.2}$$

Though the limits (5.2) are very strange candidates for a fundamental solution they do represent in a simple configuration a deep phenomenon, and they deserve the name of *limit solutions*. On the one hand, there is vanishing diffusion when $u \to +\infty$, so that a part of the mass $M(t)$ stays put at the origin. On the other hand, the part that goes away does reach $r = +\infty$, due to the opposite effect of infinite diffusion coefficient for $u = 0$. The maximal limit solution is thus $(M - 4\pi\, t)_+\delta(x)$.

Before we prove the nonexistence result we need to recall the concept of *concentration comparison* from [14] and apply it to our case. For a radially symmetric and nonnegative locally integrable function f we define the concentration function as

$$F(r) = \int_{|x|<r} f(x)\,dx\,.$$

We say that f_1 is more concentrated than f_2, $f_1 \succ f_2$, if the relation $F_1(r) \ge F_2(r)$ holds for every $r > 0$ for the respective concentrations. The basic concentration result is

Lemma 5.2 (Concentration Lemma)**.** *Let u and $\overline{u}$ be solutions of* (1.1) *with concentrations U and $\overline{U}$. Assume that the existence times and masses satisfy $T(u) \geq T(\overline{u})$ and $M_u(t) \leq M_{\overline{u}}(t)$ for $0 < t < T(\overline{u})$. Assume moreover that u is more concentrated than $\overline{u}$ at $t = 0$. Then the same property is true for every $t > 0$.*

For the proof we refer to [14]. With this lemma and Proposition 5.1 we prove

Theorem 5.3. *There exists no classical solution of* (1.1) *in a strip $\mathbf{R}^2 \times (0, T)$ taking initial data $M\delta(x)$ in the sense that*

$$\lim_{t \to 0} \int_{\mathbf{R}^2} u(x,t)\, \phi(x)\, dx = M\, \phi(0)\,.$$

Proof. Let $u(x,t)$ be such a solution with mass function $M(t)$. We will make a comparison of u with the family of solutions $u_k(x,t)$ with continuous mass $\overline{M}(t) \leq M(t)$, defined as above, by means of the concentration lemma. Thus, given $k > 0$ there exists $\varepsilon > 0$ such that the solution $u(x, \varepsilon)$ is more concentrated around the origin than $(M - \varepsilon)\, \phi_k(x)$. As a consequence of the lemma we have

$$u(x, t + \varepsilon) \succ u_k(x, t; M - \varepsilon)\,.$$

Let now $k \to +\infty$ and $\varepsilon \to 0$ to get

$$u(x,t) \succ \overline{M}(t)\, \delta(x)\,.$$

But since

$$\int_{\mathbf{R}^2} u(x,t)\, dx \leq M(t)$$

and we can choose $\overline{M} \to M(t)$, we get the conclusion that $u(x,t) = M(t)\, \delta(x)$, a contradiction.

Remark. *The limit $m \to 0$ in the fast diffusion equation* (0.3)*.* There is another natural way of obtaining the fundamental solution for equation (1.1), by taking the fundamental solution

$$B_m(x,t;M) = \Big(\big(\frac{4\pi\, t}{M}\big)^{(1-m)/m} + \frac{1-m}{4m}\frac{r^2}{t}\Big)^{-1/(1-m)},$$

for the fast-diffusion equation (0.3) with $1 > m > 0$ and passing to the limit $m \searrow 0$. It is then easily checked that as $m \to 0$ we get again the limit

$$B_m \to (M - 4\pi\, t)_+\, \delta_0\,,$$

so the maximal limit solution (5.1) is still recovered. Observe that $B_m(\cdot, t)$ tends to 0 uniformly in $\mathbf{R}^2$ for $t > T$, while for $t = T$ we have $B_m(x,T) \to 0$ if $x \neq 0$ and $B_m(0,T) \to 1$. Curiously enough, $B_m(x,t)^m \to 1$ for $t < T$.

6. Solutions in $L^1_{\text{loc}}(\mathbf{R}^2)$. We now consider the case of initial data $u_0 \in L^1_{\text{loc}}(\mathbf{R}^2)$ and radially symmetric. This implies that the initial function v_0 for the one-dimensional problem is integrable at $-\infty$; *i.e.*, $v_0 \in L^1(-\infty, 0)$.

Theorem 6.1. *Let $u_0 \in L^1_{\rm loc}(\mathbf{R}^2)$, non-negative and radially symmetric. Assume moreover that $u_0 \notin L^1(\mathbf{R}^2)$. Then there exists a unique $u = u(r,t)$ positive, radially symmetric, and a smooth solution of*

$$\begin{cases} u_t = \Delta \log u & \text{in } \mathbf{R}^2 \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbf{R}^2\,, \end{cases}$$

which exists and has infinite mass for all times $t > 0$. It also satisfies

$$\lim\inf_{r\to+\infty} \frac{\log u(r,t)}{\log r} \geq -2 \qquad \text{locally uniformly in } t \in (0,T)\,. \tag{6.1}$$

Proof. Translating the problem into one dimension we have to find a solution v of (1.2) with flux data $f(t) = 2$ at $s = -\infty$ and infinite mass at $s = +\infty$; *i.e.,*

$$\int_0^{+\infty} v(s,t)\, ds = +\infty \quad \text{for all } t > 0\,.$$

The existence of this solution is a consequence of Theorem 4 of [13]. Theorem 5 of the same paper shows that such a solution is unique under the extra condition

$$-\log v = o(s) \qquad \text{as } s \to +\infty\,;$$

i.e., (6.1), a condition that is obtained in the construction of v as the limit of solutions v_n of approximate problems with finite initial mass

$$\int_{-\infty}^{+\infty} v_{0n}(s)\, ds = n\,,$$

same condition at $s = -\infty$, and maximality at $s = +\infty$.

We show next how to improve this theorem into a clean uniqueness result by eliminating the growth restriction. This is not immediate (for us). Let $\overline{v}$ be the solution just mentioned and let v be any other solution. We can now repeat literally the comparison argument of [13, Theorem 5] (based on the standard technique of multiplication by suitable functions, subtraction and integration by parts) to show that $v \leq \overline{v}$, in other words [13] shows that $\overline{v}$ is the *maximal solution.* Let us now integrate in s to obtain

$$V(s,t) = \int_{-\infty}^{s} v(y,t)\, dy + 2\,t\,, \qquad \overline{V}(s,t) = \int_{-\infty}^{s} \overline{v}(y,t)\, dy + 2\,t\,.$$

The desired conclusion $\overline{v} = v$ is equivalent to $\overline{V} = V$. Now, the inequality $\overline{v} \geq v$ implies that $\overline{V} \geq V$. Hence the equality $\overline{V} = V$ will follow if we establish the following

Claim. $\overline{V} \leq V$ *in* $\mathbf{R} \times (0,T)$.

Proof. Both functions satisfy the equation

$$V_t = \frac{V_{ss}}{V_s}$$

with boundary condition at $s=-\infty$

$$V(-\infty,t)=\overline{V}(-\infty,t)=2\,t\,,$$

and same initial conditions

$$V(s,0)=\overline{V}(s,t)=\int_{-\infty}^{s} v_0(y)\,dy\,.$$

We compare now V and $\overline{V}$ in a rectangle $\mathcal{R}=(-\infty,R)\times(0,T)$ for some large $R>0$. In order to arrive at the conclusion $\overline{V}\le V$ in $\mathcal{R}$ we only have to check the ordering on the lateral boundary $s=R$, $0<t<T$. Since v has infinite mass at $+\infty$, given t_1 and a constant $C=2n$ there exists $R=R(t_1,n)\gg 1$ such that

$$\int_{-\infty}^{R} v(y,t_1)\,dy\ge C\,.$$

It is clear that we can take $R\to+\infty$ as $n\to\infty$. Let w be the maximal solution starting at $t=t_1$ with initial data $v(\cdot,t_1)$. Then for every $t>t_1$

$$\int_{-\infty}^{R} v(y,t)\,dy\le\int_{-\infty}^{R} w(y,t)\,dy\,.$$

According to standard integration by parts for maximal solutions,

$$\int_{-\infty}^{R} w(y,t)\,dy\le\int_{-\infty}^{2R} w(y,t_1)\,dy+(t-t_1)\frac{o(R)}{R}-2(t-t_1).$$

Taking into account that $w(\cdot,t_1)=v(\cdot,t_1)$ we get

$$\int_{-\infty}^{2R} v(y,t_1)\,dy\ge\int_{-\infty}^{R} v(y,t)\,dy-\varepsilon(t-t_1)\ge 2n-\varepsilon(t-t_1).$$

Let us now return to the maximal solution $\overline{v}$. It is obtained as a limit of solutions $\overline{v}_n$ with finite mass, so that

$$\int_{-\infty}^{+\infty}\overline{v}_n(y,t)\,dy\le n\,.$$

It means that we can compare V and $\overline{V}_n$ and conclude that $\overline{V}_n(s,t)\le V(s,t)$ for $0<t<T$ and $-\infty<s<R_n$. Since $R_n\to+\infty$ as $n\to\infty$ the claim is proved.

Corollary. *The solution constructed in Theorem* 6.1 *is maximal among all the* (*positive, radially symmetric and smooth*) *solutions defined in a strip* $\mathbf{R}^2\times(0,T)$.

7. Geometrical discussion. Evolution of conformal metrics. We next discuss our results in terms of the problem of evolution of a metric by Ricci flow mentioned in the Introduction; *cf.* [6], [16], [17]. We have a metric ds with conformal factor u,

$$ds^2=u\,(dx_1^2+dx_2^2)=u\,(dr^2+r^2d\theta^2)\,,$$

which is

$$ds^2 = d\rho^2 + \mathcal{G}(\rho)\, d\theta^2 \,,$$

in geodesic polar coordinates, with

$$d\rho = \sqrt{u}\, dr \qquad \text{and} \qquad \mathcal{G}(\rho) = r^2\, u = v\,.$$

The area element is

$$dA = u\, dx_1 dx_2 = r\, u\, dr\, d\theta = r\,\sqrt{u}\, d\rho\, d\theta\,, \tag{7.1}$$

and the total area $A(t)$ of the surface coincides with our total mass function $M(t)$.

As for the Gauss curvature K of the surface we have

$$K = -\frac{1}{2}\,\frac{\Delta \log u}{u} = -\frac{1}{r\,\sqrt{u}}\,\frac{\partial^2}{\partial \rho^2}(r\,\sqrt{u})\,; \tag{7.2}$$

cf. [2, pages 237, 288]. We also have the following expression for the geodesic curvature of the lines $\rho = \text{const.}$,

$$k_g\big|_{\rho=\text{const.}} = \frac{1}{2\,r\,\sqrt{u}}\,(2 + \frac{r\,u_r}{u})\,,$$

and, by the Gauss-Bonnet theorem

$$\int_{|x|=r} k_g\, ds + \int_{|x|<r} K\, dA = 2\pi\,; \tag{7.3}$$

cf. [2, page 274]. The distance from $r = 0$ to $r = +\infty$ is

$$d(t) = \int_0^{+\infty} ds = \int_0^{+\infty} \sqrt{u}\, dr\,. \tag{7.4}$$

1. Let us state the results in Sections 3 and 4 for a metric of finite area A_0 with conformal factor u_0 which evolves by Ricci flow.

Theorems 3.1 and 3.2 imply the existence and uniqueness of a flow of maximal area. By the lower estimate (3.6) we have distance $d(t) = +\infty$ for every $0 < t < T$. It easily follows that the metric is *complete*. The surfaces shrink to a point in a time $T = A(0)/4\pi$. From (7.3), we also have that

$$\int K\, dA = 2\pi \tag{7.5}$$

(Gauss-Bonnet's formula with Euler characteristic 1), which agrees with our mass formula since by the equation

$$\frac{dA}{dt} = 2\int K\, dA\,.$$

There is no self-similar global model, but we have the local model (2.3):

$$\frac{2t}{r^2\,(\log r)^2}$$

for $r>1$ (or $0<r<1$), which has negative curvature $K=-1/(2t)$ (a *pseudosphere*).

According to Theorem 4.2, given any admissible area function; *i.e.,* a function $A(t)$, $0<t<T$, such that $A(0)=A_0$ and $A'(t)<-4\pi$, there exists a unique evolution $u(\cdot,t)$ of the conformal factor with the prescribed finite area at any t. Since $d(t)<+\infty$ in this case, the metric is not complete. In all cases we have

$$\int K dA = -\frac{1}{2}A'(t) \geq 2\pi\,,$$

a kind of Gauss-Bonnet inequality. Now $r=+\infty$ is at finite distance so the surfaces admit (topological) one-point compactification.

The particular case $A'(t)=-8\pi$ deserves special attention, since by (0.12) the inversion produces a smooth function in a neighborhood of $r=0$, which is a way of stating that u is smooth at $r=+\infty$. Therefore, we obtain a family of compact surfaces, topological spheres. The example (0.4) is precisely the family of spheres with radius $(T-t)$ under stereographical projection. Indeed, it is remarkable that the Gauss-Bonnet theorem gives

$$\int K dA = 2\pi\,\chi(S) = 4\pi\,.$$

In the remaining cases where $\Phi_\infty \neq 4$ we get surfaces with a singular point at $r=+\infty$.

Finally, Theorem A.1 in Appendix A contains some examples of self-similar, finite area evolutions with finite $d(t)$, and with strictly positive curvature; *cf.* Remark A.3 below.

2. Let us now translate the results of Section 6 for an initial conformal factor u_0 with infinite area. Theorem 6.1 states the existence of a *unique* flow of complete surfaces with infinite area that exists for all times $t>0$.

Theorem B.3 in Appendix B contains examples of self-similar evolutions of complete, infinite area surfaces. They have variable $K(r,t)$ which does not change sign but $K(r,t)\to 0$ as $r\to+\infty$.

Theorem B.5 contains an example of complete, infinite area evolution with variable $K(r,t)>0$ which tends to 0 as $r\to+\infty$. These latter solutions are called *cigar-like* solitons in [16].

The remaining self-similar solutions in Appendix A and B provide one with some examples of metrics with singularities at $r=0$, or defined only in disks or annuli.

Appendix A. A study of self-similarity. Finite-time solutions. Due to its scale invariance, equation (1.1) admits self-similar solutions of the general form

$$u(x,t)=\phi(t)\,U(\xi)\,,\quad \xi=r/\psi(t)\,.$$

A closer analysis of the possible scaling factors ϕ and ψ leads to the conclusion that there are only these possibilities:

I. Standard (forward) self-similarity, of the form $u(x,t)=t^{-\alpha}\,U(\xi)$, $\xi=r\,t^{-\beta}$, for real exponents α and β satisfying $\alpha-2\beta=-1$.

II. Backward self-similarity, of the form $u(x,t)=(T-t)^{\alpha}\,U(\xi)$, $\xi=r\,(T-t)^{\beta}$, and now the similarity exponents are related by $\alpha-2\beta=1$.

III. Exponential self-similarity, $u(x,t)=e^{-\alpha t}\,U(\xi)$, $\xi=r\,e^{-\beta t}$, with $\alpha-2\beta=0$.

IV. Special self-similarity due to the absence of one of the scaling factors. It takes three forms. Firstly, a genuine new case $u(x,t) = U(r)$, where $\Delta \log U = 0$. Hence all stationary log-harmonic functions are solutions of (1.1). Then

$$u(x,t) = \pm\, t\, U(r)\,, \qquad \text{with} \quad \Delta \log U = \pm U$$

(choosing matching signs). Finally, $u(x,t) = U(r/\sqrt{(\pm\, t)})$. The latter two fall anyway into the classes **I** or **II**.

We have for the profile U the equation

$$-\alpha\, U - \beta\, \xi U_\xi = \frac{1}{\xi}\left(\frac{\xi\, U_\xi}{U}\right)_\xi, \qquad \xi > 0\,. \tag{A.1}$$

The new variables $F = \log(\xi^2 U)$, $\eta = \log \xi$, transform the O.D.E. (A.1) into

$$F'' + (\alpha - 2\beta + \beta F')\, e^F = 0\,, \tag{A.2}$$

or equivalently, into the autonomous system

$$\begin{cases} F' = G\,, \\ G' = -(\alpha - 2\beta + \beta G)\, e^F\,. \end{cases} \tag{A.3}$$

The fluxes can be calculated from

$$\frac{r\, u_r}{u} = \frac{\xi\, U_\xi}{U} = G(\eta) - 2\,,$$

and in particular

$$\Phi_0 = \lim_{\xi\to 0} \frac{\xi\, U_\xi}{U} = \lim_{\eta\to-\infty} G(\eta) - 2 \qquad \Phi_\infty = -\lim_{\xi\to+\infty} \frac{\xi\, U_\xi}{U} = 2 - \lim_{\eta\to+\infty} G(\eta)\,,$$

(the involved limits will be shown to exist). Notice also that Φ_0, Φ_∞ are constant for any self-similar function u of all types. We recall that global solutions of (1.1) will correspond to orbits of (A.3) which exist for the whole range $-\infty < \eta < +\infty$ and satisfy the flux condition $\Phi_0 = 0$. The flux at infinity measures the rate of decay of the solution, $u(r,t)$ being of the order of $r^{-\Phi_\infty}$ as $r \to +\infty$.

Finally, we remark that by translation, $\eta \to \eta + c$, we obtain new solutions of the system (A.3) corresponding to the same orbit in the (F,G)-plane. In terms of equation (A.1) it is equivalent to applying the rescaling

$$\tilde{U}(\xi) = \frac{1}{k^2}\, U(\frac{\xi}{k})\,,$$

with $k = e^{-c}$. We will say that they are the same solution up to rescaling.

We begin our study with self-similar solutions of type **II**, existing for times $-\infty < t < T$, which will produce a number of solutions with finite mass vanishing in finite time. Without loss of generality we may put $T = 0$ and then $-\infty < t < 0$. Since $\alpha - 2\beta = 1$ in this case, the system (A.3) reduces to

$$\begin{cases} F' = G\,, \\ G' = -(1 + \beta G)\, e^F\,. \end{cases} \tag{A.4}$$

Our main result is the following

Theorem A.1. *For every $\beta \in \mathbf{R}$ there exists a unique solution* (*up to rescaling*) *of equation* (A.1) *with $\Phi_0 = 0$* (*thus giving rise to a global solution of* (1.1) *defined for $t < 0$*). *The flux at infinity is implicitly determined by the formula*

$$\beta\,\Phi_\infty + \log \frac{1 + (2 - \Phi_\infty)\,\beta}{1 + 2\,\beta} = 0\,, \quad \alpha = 1 + 2\beta$$

and covers the range $2 < \Phi_\infty < +\infty$ with $\Phi_\infty = 4$ for $\beta = 0$.

The proof follows from a rather standard phase-plane analysis, which at the same time produces solutions of (1.1) having a singularity at zero, with $\Phi_0 \neq 0$, under the sole restriction $\Phi_0 > -2$. We will divide the study into the cases $\beta = 0$, $\beta > 0$ and $\beta < 0$.

Case $\beta = 0$: This is very simple. It is clear that we will obtain separated variables. Actually, we will get the family (2.1). The phase-plane analysis will allow us to fix the ideas for the more involved cases $\beta \neq 0$. We have

$$\frac{1}{2}\,G^2 + e^F = C > 0 \tag{A.5}$$

as a first integral of (A.4), whence on any orbit $|G|$ is bounded, and F is bounded from above. Since G is strictly decreasing, the limits

$$G(-\infty) = \lim_{\eta \to -\infty} G(\eta)\,, \qquad G(+\infty) = \lim_{\eta \to +\infty} G(\eta)\,,$$

necessarily exist and are finite. Then we find in (A.5) that corresponding limits exist for $F(\eta)$, their only admissible end-values (as $\eta \to \pm\infty$) being

$$\lim_{\eta \to -\infty} F(\eta) = \lim_{\eta \to +\infty} F(\eta) = -\infty\,.$$

We get $G(-\infty)^2 = G(+\infty)^2 = 2C$ for the constant in (A.5), and since G is strictly decreasing $G(-\infty) = -G(+\infty) > 0$. In terms of Φ_0, and Φ_∞, this is $\Phi_\infty - \Phi_0 = 4$, $\Phi_0 > -2$, $\Phi_\infty > 2$. The first order O.D.E. in (A.5) can be integrated explicitly to obtain

$$-\sqrt{2C}\,\eta = \log \frac{\sqrt{2C} + G}{\sqrt{2C} - G}$$

(where we have set $G = 0$ for $\eta = 0$, and thus $e^{F(0)} = C$), and

$$F(\eta) = \log \frac{C}{\cosh^2(\sqrt{C/2}\,\eta)}\,.$$

Theorem A.2. *For $\beta = 0$ the self-similar solutions of type* **II** *coincide with the family of solutions in separated variables* (2.1). *We have a solution* (*up to a scale factor*) *for every constant flux $\Phi_0 > -2$, with $\Phi_\infty = \Phi_0 + 4$. There is only one global solution, the one with $\Phi_0 = 0$, $\Phi_\infty = 4$, given by* (0.4).

Comparing our notation with (2.1) we have $\mu = \sqrt{C/2} = 1 + \Phi_0/2$.

Case $\beta > 0$: The phase plane is more involved since there are two regions, $\{G > -1/\beta\}$ and $\{G < -1/\beta\}$ separated by the horizontal line $\{G = -1/\beta\}$. This line corresponds to a particular solution of the system

$$F(\eta) = F(0) - \eta/\beta\,, \qquad G(\eta) = -1/\beta\,,$$

which gives a *stationary* solution of (1.1) of the form $u(r,t) = C\,r^{-\alpha/\beta} = C\,r^{-(2+1/\beta)}$, with fluxes related by $\Phi_\infty + \Phi_0 = 0\,, \Phi_0 = -\alpha/\beta$. We remark that the total mass of u is infinite, since the integral $\int_{\mathbf{R}^2} u(x,t)dx$ diverges at $r = 0$.

The half-plane $\{G > -1/\beta\}$ gives *finite mass* solutions of equation (1.1). As in the previous case we have a first integral of (A.4) which now reads

$$\frac{1}{\beta}\big(G - \frac{1}{\beta}\log(1+\beta G)\big) + e^F = C\,. \tag{A.6}$$

Let us consider the first term in (A.6) as function of G:

$$C_\beta(G) = \frac{1}{\beta}\big(G - \frac{1}{\beta}\log(1+\beta G)\big)\,,$$

defined for $G > -1/\beta$. It decreases from $+\infty$ to 0 in the interval $-1/\beta < G < 0$ and increases from 0 to $+\infty$ when $0 < G < +\infty$. On an orbit we have $0 \le C_\beta(G) < C$ from (A.6), so that $|G|$ is bounded and F is bounded from above. As in the case $\beta = 0$, the limits $G(-\infty) > 0 > G(+\infty)$ exist and are finite, $F(-\infty) = F(+\infty) = -\infty$, and $C_\beta(G(-\infty)) = C_\beta(G(+\infty)) = C > 0$ so that for every value of the constant $C > 0$ we get two values for the endpoints, one for $G(-\infty) \in (0,+\infty)$ and one for $G(+\infty) \in (-1/\beta, 0)$. Taking into account that $G(-\infty) = 2+\Phi_0$ and $G(+\infty) = 2-\Phi_\infty$ we arrive at the final conclusion.

Theorem A.3. *Let $\beta > 0$. For any $\Phi_0 > -2$ there exists a unique* (*up to scaling*) *solution of* (A.1) *with flux at infinity $\Phi_\infty \in (2, 2+1/\beta)$ determined by*

$$\Phi_0 + \Phi_\infty = \frac{1}{\beta}\log\frac{\alpha + \beta\,\Phi_0}{\alpha - \beta\,\Phi_\infty}\,.$$

The corresponding solution $u(x,t)$ of (1.1) *has finite total mass. The only global solution, corresponding to $\Phi_0 = 0$, is positive and smooth in Q_T.*

Proof. We still need to show that the global solution is continuous at $r = 0$. This will follow from a more detailed analysis of the behavior of the solutions as $\xi \to 0$. Since $\xi\,U_\xi/U$ strictly decreases from Φ_0 down to $-\Phi_\infty$, we obtain that $\xi^{-\Phi_0}U$ is strictly decreasing. We also have

$$\lim_{\xi\to 0}\frac{\log U}{\log \xi} = \Phi_0 = 0\,.$$

We conclude that

$$\begin{aligned} &U \text{ is strictly decreasing}, \quad U \ge C > 0\,, \text{ for } 0 < \xi < 1\,,\\ &U = o(1/\xi^p) \text{ as } \xi \to 0\,, \text{ for any } p > 0\,. \end{aligned}$$

Let us also prove that U is bounded from above, which implies its continuity. Since $G \to 2$ as $\eta \to -\infty$, we have $F(\eta)/\eta \to 2$. Now given $\varepsilon > 0$, we can write

$$a\,e^{(2+\varepsilon)\eta} < -G' < A\,e^{(2-\varepsilon)\eta} \tag{A.7}$$

for η sufficiently close to $-\infty$, where $a = 1+(2-\varepsilon)\beta$, $A = 1+(2+\varepsilon)\beta$. By integrating from $-\infty$ to η we obtain

$$\frac{a}{2+\varepsilon}e^{(2+\varepsilon)\eta} < 2 - G < \frac{A}{2-\varepsilon}e^{(2-\varepsilon)\eta}\,,$$

and a new integration, this time from η' to η, gives

$$\frac{a}{(2+\varepsilon)^2}(e^{(2+\varepsilon)\eta}-e^{(2+\varepsilon)\eta'})<(F(\eta')-2\eta')-(F(\eta)-2\eta)<\frac{A}{(2-\varepsilon)^2}(e^{(2-\varepsilon)\eta}-e^{(2-\varepsilon)\eta'})\,.$$

Since $F(\eta)-2\eta$ is decreasing and concave, the limit $L=\lim_{\eta\to-\infty}(F(\eta)-2\eta)\le+\infty$ exists. By letting $\eta'\to-\infty$, in the estimate above we obtain that L is finite and

$$L-\frac{A}{(2-\varepsilon)^2}e^{(2-\varepsilon)\eta}<F(\eta)-2\eta<L-\frac{a}{(2+\varepsilon)^2}e^{(2+\varepsilon)\eta}\,.$$

This shows that U is bounded for ξ close to 0; actually $U(0)=e^L$.

Remark A.1. We can use that L is finite to improve (A.7) and write

$$a\,e^{L-\varepsilon}\,e^{2\eta}<-G'<A\,e^L\,e^{2\eta}\,.$$

We then integrate twice as above to obtain

$$\exp\{-\frac{A\,U(0)}{4}\xi^2\}<\frac{U(\xi)}{U(0)}<\exp\{-\frac{a\,e^{L-\varepsilon}}{4}\xi^2\}\,,$$

and let $\varepsilon\to0$ to conclude that

$$\Delta U(0)=-\alpha\,U(0)^2\,.$$

Remark A.2. When $\Phi_0\neq0$, computations as above prove that $\xi^{-\Phi_0}\,U(\xi)$ takes a finite and positive limit as $\xi\to0$. Similarly for $\xi^{\Phi_\infty}\,U(\xi)$ as $\xi\to+\infty$. This amounts to

$$u(r,t)\sim(-t)^{\alpha+\beta\Phi_0}\,r^{\Phi_0}\qquad\text{as}\quad r\to0\,,$$
$$u(r,t)\sim(-t)^{\alpha-\beta\Phi_\infty}\,r^{-\Phi_\infty}\qquad\text{as}\quad r\to+\infty\,,$$

for the solution of (1.1).

Remark A.3. The Gauss curvature of the solution of type **II** is

$$K(r,t)=\frac{1}{2\,(-t)}\,(1+\beta\,G)\,.$$

The global, finite-mass solutions in Theorem A.1 have positive $K(r,t)$ decreasing from $K(0,t)=\alpha/(2\,(-t))$ down to

$$\lim_{r\to+\infty}K(r,t)=\frac{\alpha-\beta\Phi_\infty}{2\,(-t)}>0\,.$$

We next study the region $\{G<-1/\beta\}$. We find in this case a new and strange family of solutions defined in the exterior of a moving disk.

Theorem A.4. *Let $\beta > 0$. For every $\Phi_\infty > 2 + 1/\beta$ and every $\xi_0 > 0$ there exists a unique solution of* (A.1) *in an interval* $0 < \xi_0 < \xi < +\infty$. *Consequently, u is defined in a domain of the form $\{(x,t) : t < 0\,, |x| > \xi_0\,(-t)^{-\beta}\}$ with a singularity on the moving circumference $\{x : |x| = \xi_0\,(-t)^{-\beta}\}$ whose radius tends to $+\infty$ as $t \to 0$.*

Proof. In this region we know that $G' > 0$ and $F' < 0$. Moreover, $G < -1/\beta$. We obtain a first integral of the form

$$\frac{1}{\beta}\big(G - \frac{1}{\beta}\log|1 + \beta G|\big) + e^F = C\,. \tag{A.8}$$

It follows that $G(+\infty)$ is actually less than $-1/\beta$ and $F(+\infty) = -\infty$. Moreover, we obtain that

$$\frac{d^2G}{dF^2} = \big(1 + \frac{e^F}{G^2}\big)\frac{dG}{dF} < 0\,, \tag{A.9}$$

hence $G(F)$ is decreasing and concave. Therefore, $G \to -\infty$ in the other end. Going back to (A.8) we see that necessarily $F \to +\infty$.

Let us prove that η tends to a finite value as $F \to +\infty$, $G \to -\infty$. From (A.9) we derive the inequality

$$-\frac{d^2G}{dF^2} > -\frac{dG}{dF} > 0\,,$$

which we integrate to obtain

$$-G(F_1) > A(F) + B(F)\,e^{F_1} \qquad \text{for all } \; F < F_1\,,$$

where we have set

$$A(F) = -G(F) + \frac{dG(F)}{dF}\,, \qquad B(F) = -e^{-F}\frac{dG(F)}{dF} = \frac{1 + \beta\,G(F)}{G(F)} > 0\,.$$

Using the formula

$$\frac{d\eta}{dF} = \frac{1}{G(F)}\,,$$

we get

$$\eta(F_1) - \eta(F) = \int_F^{F_1}\frac{dF}{G(F)} > -\frac{1}{A(F)}\log\big(\frac{e^{F_1}}{A(F) + B(F)\,e^{F_1}}\,\frac{A(F) + B(F)\,e^F}{e^F}\big)$$

for all $F < F_1$. Letting $F_1 \to +\infty$ we conclude that the limit

$$\eta_0 = \lim_{F_1 \to +\infty}\eta(F_1)$$

(which exists since η decreases as F_1 increases) is actually finite: $\eta_0 > -\infty$. Moreover,

$$0 < \eta - \eta_0 \le -\frac{1}{-G}\,\frac{1}{1 - \dfrac{1 + \beta G}{G}\,\dfrac{e^F}{-G}}\,\log\left(\frac{1 + \beta G}{G}\,\frac{e^F}{-G}\right)\,. \tag{A.10}$$

Remark A.4. Using the fact that

$$\frac{e^F}{-G} \to \frac{1}{\beta} \qquad \text{as } \eta \searrow \eta_0\,,$$

we obtain from (A.10) the development

$$F(\eta) \le -\log(\beta(\eta-\eta_0)) \qquad \text{as } \eta \searrow \eta_0\,.$$

This amounts to

$$U(\xi) \le \frac{1}{\beta\,\xi^2\,\log\dfrac{\xi}{\xi_0}} \qquad \text{as } \xi \searrow \xi_0 = e^{\eta_0}$$

for the profile $U(\xi)$, and

$$u(r,t) \le \frac{-t}{\beta\,r^2\,\log\dfrac{r\,(-t)^{\beta}}{r_0}} \qquad \text{as } r \searrow r_0\,(-t)^{-\beta}\,,$$

for the solution of (1.1).

Case $\beta < 0$: The inversion formula (0.12) reduces this case to the case $\beta > 0$. In our self-similar setting we put

$$f(\eta) = F(-\eta)\,, \quad g(\eta) = -G(-\eta)\,, \quad \tilde\beta = -\beta\,,$$

to transform system (A.3) for F, G and $\beta < 0$ into the same system for f, g and $\tilde\beta > 0$. Moreover the limits as $\eta \to -\infty$ change into limits for $\eta \to +\infty$ and vice versa. In terms of the profile, it corresponds to changing $U(\xi)$ into $\frac{1}{\xi^4}\,U(\frac{1}{\xi})$, again a solution of (A.1).

Theorem A.5. *Let $\beta < 0$. There is a stationary solution of the form*

$$u(r,t) = C\,r^{-(2-1/\beta)},$$

which is singular at $r = 0$. In the region $\{G < -1/\beta\}$, for every $\Phi_0 \in (-2, -2-1/\beta)$ there exists a unique solution up to scaling which has finite mass and flux at infinity $\Phi_\infty > 2$. The global solution corresponds to flux $\Phi_0 = 0$. Finally, the region $\{G > -1/\beta\}$ contains solutions defined in the interior of a shrinking disk with singularity on the border $\{x : |x| = \xi_0\,(-t)^{-\beta}\}$, which focuses towards 0 as $t \nearrow 0$.

Appendix B. Self-similar solutions existing for infinite forward time. We perform in this section the study of the self-similar solutions of the types **I** and **III**, which are defined for times $0 < t < +\infty$. All of them have infinite mass.

B.1. Self-similarity of type I. We use the form

$$u(x,t) = t^{-\alpha}\,U(\xi)\,, \quad \xi = r\,t^{-\beta}\,, \qquad \text{with } \alpha - 2\,\beta = -1\,.$$

The profile U satisfies (A.1), and the autonomous system (A.3) takes now the form

$$\begin{cases} F' = G\,, \\ G' = (1-\beta G)\,e^F\,. \end{cases} \tag{B.1}$$

Let us perform the phase-plane analysis.

Case $\beta = 0$**.** We have a first integral of the form

$$e^F = \frac{1}{2}\, G^2 + C\,. \tag{B.2}$$

For $C = 0$ we obtain the solution $e^F = 2/\eta^2$ with a singularity at $\eta = 0$ (or any other η by translation). It gives again the explicit solution (2.3) with singularities at $r = 0$ and $r = 1$. This last singularity is not integrable.

If $C < 0$, say $C = -2\mu^2$ for a positive μ, then (B.2) can be integrated explicitly and gives the profile

$$U(\xi) = \frac{8\mu^2}{\xi^{2(1+\mu)} + \xi^{2(1-\mu)} - 2\,\xi^2} \qquad \text{for } \xi \neq 1\,,$$

which corresponds to the solution

$$u(r,t) = \frac{8\mu^2\, t}{r^{2(1+\mu)} + r^{2(1-\mu)} - 2\, r^2} \qquad r \neq 1\,,\ t > 0\,,$$

with a nonintegrable singularity at $r = 1$, but which is integrable both at $r = 0$ and at $r = +\infty$, and moreover it has fluxes $\Phi_0 = 2(\mu - 1) > -2$ and $\Phi_\infty = 2(\mu + 1) > 2$. Notice that the case $\mu = 1$ above is simply

$$u(r,t) = \frac{8\mu^2\, t}{(1 - r^2)^2}\,.$$

Finally, if $C = 2\mu^2 > 0$, (B.2) can also be integrated. The result can be written (up to translations in η) as

$$e^F = \frac{2\mu^2}{\cos^2(\mu(\eta - \pi/2))}\,,$$

which corresponds to the profile

$$U(\xi) = \frac{2\mu^2}{\xi^2\, \cos^2(\mu \log(e^{-\pi/2}\xi))} \qquad \text{for } e^{(1-1/\mu)\pi/2} < \xi < e^{(1+1/\mu)\pi/2}\,.$$

Case $\beta > 0$**.** We first have the particular solution $G = 1/\beta\,, F = \eta/\beta$, which corresponds to the stationary solution

$$U(\xi) = C\,\xi^{1/\beta - 2}, \qquad u(r,t) = C\, r^{1/\beta - 2}\,,$$

which is either singular or vanishes at $r = 0$, unless it is constant when $\beta = 1/2$.

Besides this solution, we have

$$e^F = C_\beta(G) + C\,, \tag{B.3}$$

as first integral of (B.1), where C_β is the following function:

$$C_\beta(G) = -\frac{1}{\beta}\big(G + \frac{1}{\beta} \log|1 - \beta\, G|\big)\,.$$

The region $\{G > 1/\beta\}$ contains orbits that increase in F from $-\infty$ to $+\infty$ and decrease in G. As $F \to +\infty$ we have $G \searrow 1/\beta$, hence $\eta \to +\infty$, and since by (B.3) G is bounded from above, we also have a finite value for G, as $F \to -\infty$, hence $\eta \to -\infty$. The value of $G(-\infty) = 2 + \Phi_0$ is the unique solution of $C = C_\beta(G(-\infty))$ in the interval $1/\beta < G < +\infty$.

In the region $\{G < 1/\beta\}$ we first see that on any orbit G increases with η, and also that $C_\beta(G)$ is nonnegative. The behavior of the orbits depends on the sign of the constant C in (B.3).

Thus, if $C \le 0$ we get from (B.3) that $C_\beta(G) > |C| \ge 0$; actually, either $1/\beta > G > G_1 \ge 0$ or $G < G_2 \le 0$, where G_1 and G_2 are the two solutions of $C_\beta(G) = |C|$.

In the first case, $G \nearrow 1/\beta$ as $F \to +\infty$, and then $\eta \to +\infty$. As $F \to -\infty$, we have $G \searrow G_1$, and then $\eta \to -\infty$. In particular, we find fluxes $\Phi_0 = G_1 - 2 \ge -2$, and $\Phi_\infty = 2 - 1/\beta < 2$, so the corresponding solution $u(r,t)$ of (1.1) has infinite total mass, though it is locally integrable.

In the second case, as $F \to -\infty$ we find that $G \nearrow G_2 \le 0$, thus $\eta \to +\infty$. We get a flux at $+\infty$, $\Phi_\infty = 2 - G_2 \ge 2$. On the opposite end of this orbit we have $G \to -\infty$ as $F \to +\infty$. The fact that $\eta \searrow \eta_0 > -\infty$ in this orbit follows from

$$\frac{d^2G}{dF^2} = \Big(1 - \frac{e^F}{G^2}\Big)\frac{dG}{dF} \qquad \text{and} \qquad \frac{e^F}{G} \to -\frac{1}{\beta}$$

with arguments as in Theorem A.4.

When $C > 0$ in (B.3), we see that F is bounded from below. As in the previous case, as $F \to +\infty$, we have $G \nearrow 1/\beta$ and $\eta \to +\infty$, and again $\Phi_\infty = 2 - 1/\beta < 2$. Then (B.3) and

$$\frac{dG}{dF} = \frac{1 - \beta G}{G}\, e^F < 0 \qquad \text{for } G < 0,$$

imply that as $G \to -\infty$, $F \to +\infty$, and this occurs for a finite value of η, $\eta \searrow \eta_0 > -\infty$, as in the previous case.

Theorem B.1. *For every $\beta > 0$ and every $\Phi_0 \ge -2$ there is a unique solution (up to rescaling) of the system (B.1) defined for $\eta \in \mathbf{R}$. It has infinite mass and flux at infinity $\Phi_\infty = 2 - 1/\beta < 2$. When $\Phi_0 = 0$ the quantity $\xi^{2-1/\beta} U$ is increasing (decreasing) if $\beta > 1/2$ (respectively $\beta < 1/2$).*

Case $\beta < 0$**.** As explained in Appendix A we can reduce this case to the preceding one by making the changes $\eta \to -\eta$, $F \to F$, $G \to -G$, the phase-plane is reflected on the F axis and the sense of the arrows is reversed. We then get the following result:

Theorem B.2. *For every $\beta < 0$ and every $\Phi_\infty \ge 2$ there is a unique solution (up to rescaling) of the system (B.1) defined for $\eta \in \mathbf{R}$. It has infinite mass and flux at zero $\Phi_0 = -2 + 1/\beta < -2$.*

It is interesting to translate the above results in terms of the Cauchy problem for equation (1.1).

Theorem B.3. *For every $\gamma > -2$ there is a unique self-similar solution of (1.1) of type* **I** *taking initial data*

$$u_0(r) = c\, r^\gamma, \qquad c > 0, \tag{B.4}$$

which is defined and smooth for all $x \in \mathbf{R}$ and $t > 0$. Moreover, for every $\kappa \le 2$, there is one such solution in Q^* which behaves at $r = 0$ as follows:

$$u(r,t) \sim r^{-\kappa} \tag{B.5}$$

for every $t > 0$. *All these solutions behave as* $r \to +\infty$ *like* $u(r,t) \sim r^{\gamma}$ *for* $0 < t < +\infty$.

Remark B.1. Since in this case the initial data (B.4) are locally integrable the existence of a unique classical solution comes from Section 6. The self-similarity comes from the fact that both the equation and the data are invariant under a suitable rescaling. This is a powerful type of argument which has been widely used recently in porous media equations.

For the proof, we first choose $\beta > 0$ so that $\gamma = -2 + 1/\beta$ and then select the self-similar profile with end-values $\Phi_0 = -\kappa$ and $\Phi_\infty = -\gamma$ in Theorem B.1. In a similar way, Theorem B.2 can be rephrased as

Theorem B.4. *For every* $\gamma \le -2$ *and every* $\kappa > 2$ *there is a unique self-similar solution of* (1.1) *in* Q^* *taking on initial data* (B.4) *and having a singularity at* 0 *of the form* (B.5) (*which is nonintegrable*).

Remark B.3. The Gauss curvature of the solutions of type **I** is

$$K(r,t) = -\frac{1}{2\,t}\,(1-\beta\,G)\,.$$

The global, infinite-mass solutions in Theorem B.3 have positive (respectively, negative) variable curvature $K(r,t)$ if $\beta > 1/2$ (respectively $\beta < 1/2$). Moreover, $\lim_{r\to+\infty} K(r,t) = 0$.

B.2. Exponential self-similarity. These are solutions that exist for times ranging from $-\infty$ to $+\infty$. They have the form

$$u(x,t) = e^{-\alpha t}\,U(\xi)\,, \quad \xi = r\,e^{-\beta t}\,, \qquad \text{with } \alpha = 2\beta\,. \tag{B.6}$$

Notice that this form of self-similarity corresponds to traveling wave solutions for the one-dimensional problem (0.7). The system (A.3) takes now the form

$$\begin{cases} F' = G\,, \\ G' = -\beta\,G\,e^F\,, \end{cases}$$

which integrates to give

$$F' + \beta\,e^F = C\,. \tag{B.7}$$

For $\beta = 0$ we obtain again all the stationary log-harmonic solutions of (1.1).

For $\beta > 0$ and $C > 0$ we integrate (B.7) in the region $\{C > \beta e^F\}$ and find a solution defined for all $0 < \xi < +\infty$,

$$e^F = \frac{C}{\beta}\,\frac{1}{1 + k\,\xi^{-C}}$$

with parameter $k > 0$. It corresponds to the profile

$$U(\xi) = \frac{C}{\beta}\,\frac{1}{\xi^2 + k\,\xi^{2-C}}\,,$$

and a solution with zero flux at $r = 0$ is obtained for $C = 2$.

The remaining solutions of (B.7) are *not* defined for all $0 < \xi < +\infty$. They are

$$e^F = \frac{C}{\beta}\,\frac{1}{1 - k\,\xi^{-C}} \qquad \text{if } C \ne 0\,,$$

and

$$e^F = \frac{1}{\beta}\,\frac{1}{k + \log\xi} \qquad \text{if } C = 0\,,$$

again with parameter $k > 0$.

The case $\beta < 0$ transforms into the case of positive β's thanks to the inversion formula (0.12).

Theorem B.5. *For every* $\beta > 0$ *there is a unique global solution of the form* (B.6) (*up to rescaling*). *It is given by*

$$U(\xi) = \frac{2}{\beta}\,\frac{1}{k+\xi^2}\,, \qquad u(r,t) = \frac{2}{\beta}\,\frac{1}{r^2 + k\,e^{2\beta t}}\,. \tag{B.8}$$

It has infinite mass and has flux at $+\infty$ *exactly* 2. *There exists also for every* $C > 0$ *one such solution in* Q^* *with flux at zero* $\Phi_0 = C - 2 > -2$ (*hence an integrable infinite singularity if* $C < 2$, *a sink if* $C > 2$). *They have* $\Phi_\infty = 2$.

Remark B.4. The Gauss curvature of the solutions in Theorem B.5 is

$$K(r,t) = \frac{\beta}{2\,t}\,G > 0\,.$$

As $r \to +\infty$ we have $K(r,t) \to 0$.

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