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SINGULAR LIMIT OF SOME QUASILINEAR WAVE EQUATIONS WITH DAMPING TERMS

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Abstract. We consider a relation between a mixed problem for a class of quasilinear wave equations with small parameter ϵ and a reduced problem of a parabolic type. By constructing the stable set the global existence of solutions can be discussed. It is shown that the solution u_ϵ of the mixed problem converges, uniformly on any finite time interval, to the solution u of the parabolic equation in an appropriate Hilbert space as $\epsilon \to 0$. Several ϵ weighted energy estimates will be obtained in order to evaluate the difference norm of $u_\epsilon - u$.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We now consider the mixed problem

$$\epsilon^2 u_{tt} - \Big(\alpha + 2\beta \int_\Omega |\nabla u(t,y)|^2 dy\Big) \triangle u + \delta u_t = \mu u^3, \quad t > 0, \quad x \in \Omega, \tag{1.1}$$

$$u(0,x) = u_{0\epsilon}(x), \quad u_t(0,x) = u_{1\epsilon}(x), \quad x \in \Omega, \tag{1.2}$$

$$u(t,x)|_{\partial\Omega} = 0, \quad t \geq 0. \tag{1.3}$$

Here ϵ is a positive parameter with $0 < \epsilon \leq 1$. $\alpha > 0$, $\beta \geq 0$, $\delta > 0$ and μ are given constants. $u = u(t,x)$ is an unknown real-valued function on $[0,\infty) \times \Omega$. $\triangle$ is the Laplace operator in $\mathbb{R}^3$.

If ϵ is a fixed positive number, then the existence and uniqueness for the equations of this kind is investigated by several authors. In [8] Hosoya and Yamada proved the existence and uniqueness of a global solution for the semilinear wave equations under restoring term $\mu|u|^\gamma u$ ($\gamma > 0$, $\mu < 0$). In [11] Ikehata and Okazawa constructed the unstable set and discussed the blowing-up property of the mixed problem (1.1), (1.2), (1.3) with $\delta = 0$ and $\mu > 0$. In [10] Ikehata constructed the stable set and a global solution for the mixed problem (1.1), (1.2), (1.3) with $\mu > 0$. For the related results, see [7–11] and the references therein.

Our main interest is to study the behavior of the solution u_ϵ for the mixed problem (1.1), (1.2), (1.3) as $\epsilon \to 0$. For $\mu = 0$, in [5] Esham and Weinacht discussed the local singular limit of the mixed problem (1.1), (1.2), (1.3) with an inhomogeneous term in one space dimension; that is, the convergence in a local time. In [6] Esham and Weinacht proved the singular limit of the quasilinear hyperbolic equations in the

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same manner. They prescribed initial data $u_{1\epsilon}$ which was not uniform with respect to ϵ and considered the mixed problem (1.1), (1.2), (1.3) in a classical sense which required "boundary initial layer correctors" to obtain convergence uniform in space and time. In the case when $\beta \equiv 0$, there are many works. For the related results, see [2, 5, 6, 15] and the references therein. On the other hand, we shall impose the uniformity of the initial data $u_{1\epsilon}$ with respect to ϵ. Furthermore our problem can be treated within the framework of the global strong solutions in the usual Sobolev spaces by virtue of the results of Ikehata ([10]). We derive ϵ weighted energy estimates of equation (1.1) and evaluate the difference of the solutions in an appropriate Hilbert space.

We briefly survey the singular limit problem for equation (1.1). Formally, letting $\epsilon \to 0$, we can consider the mixed problem (1.1), (1.2), (1.3) as the following mixed problem:

$$\delta u_t - \Big(\alpha + 2\beta \int_\Omega |\nabla u(t,y)|^2 dy\Big)\triangle u = \mu u^3, \quad t > 0, \quad x \in \Omega, \tag{1.4}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.5}$$

$$u(t,x)|_{\partial\Omega} = 0, \quad t \geq 0. \tag{1.6}$$

This suggests that the solution u_ϵ of the mixed problem (1.1), (1.2), (1.3) converges to the solution u of the mixed problem (1.4), (1.5), (1.6) as $\epsilon \to 0$. We shall call this convergence a singular limit. In [14] Laptev discussed the existence of a weak solution of the mixed problem (1.4), (1.5), (1.6) with $\mu = 0$ in $\mathbb{R}^n$. On the other hand, the equation (1.4) can be regarded as the parabolic type with subdifferential operators. It is well known that the mixed problem (1.4), (1.5), (1.6) is solvable from the subdifferential operators point of view. In [16] $\widehat{\text{O}}$tani studied the nonlinear parabolic equations of the subdifferential type containing the problem (1.4), (1.5), (1.6). In [12] Ishii constructed the stable set and a global solution of the problem containing our problem.

Now we define a potential $J(u)$ associated with equation (1.1) and equation (1.4) by

$$J(u) = \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{\beta}{2}\|\nabla u\|_2^4 - \frac{\mu}{4}\|u\|_4^4, \qquad \text{for} \quad u \in H_0^1(\Omega).$$

Also we put

$$d = \inf\{\sup_{\lambda \geq 0} J(\lambda u); u \in H_0^1(\Omega), u \neq 0\}.$$

Then it follows that

$$0 < \frac{1}{4}\alpha^2(\mu C(\Omega,4)^4 - 2\beta)^{-1} \leq d < +\infty, \tag{1.7}$$

if we assume that β and μ of equation (1.1) satisfy $\mu C(\Omega,4)^4 > 2\beta$ with $\mu > 0$. For the proof, see Ikehata and Okazawa ([11]). Here the constant $C(\Omega,4)$ appearing in (1.7) is a Sobolev-Poincaré one; that is, it is introduced by the following Sobolev-Poincaré inequality: let p be any real number with $2 \leq p \leq 6$; then we have for $u \in H_0^1(\Omega)$

$$\|u\|_p \leq C(\Omega,p)\|\nabla u\|_2, \tag{1.8}$$

where $\|\cdot\|_p$ stands for the usual $L^p(\Omega)$ norm.

Finally we introduce a stable set $\mathcal{W}^*$ defined by

$$\mathcal{W}^* = \{u \in H_0^1(\Omega); J(u) < d, \ \alpha\|\nabla u\|_2^2 + 2\beta\|\nabla u\|_2^4 > \mu\|u\|_4^4\}.$$

Then we can obtain the singular limit of (1.1), (1.2), (1.3).

Theorem. *Let* $\alpha > 0$, $\beta \geq 0$, $\delta > 0$ *and* $\mu > 0$ *such that* $\mu C(\Omega, 4)^4 > 2\beta$. *Assume that* $u_{0\epsilon} \in \mathcal{W}^* \cap H^2(\Omega)$, $u_{1\epsilon} \in H_0^1(\Omega)$ *and* $u_0 \in \mathcal{W}^* \cap H^2(\Omega)$ *such that*

$$\sup_{0<\epsilon\leq 1} \{\|u_{1\epsilon}\|_2^2 + 2J(u_{0\epsilon})\} < 2d, \qquad \sup_{0<\epsilon\leq 1} (\|\triangle u_{0\epsilon}\|_2^2 + \|\nabla u_{1\epsilon}\|_2^2) < \eta_0, \qquad \|\triangle u_0\|_2^2 < \eta_1$$

for some $\eta_0 > 0$ *(depending on* $\sup_{0<\epsilon\leq 1}\|\nabla u_{0\epsilon}\|_2$ *and* $\sup_{0<\epsilon\leq 1}\|u_{1\epsilon}\|_2$*),* $\eta_1 > 0$*, independent of* ϵ*, and*

$$u_{0\epsilon} \to u_0 \quad \text{strongly in } H_0^1(\Omega) \cap H^2(\Omega)$$

as $\epsilon \to 0$. *Then, for any finite time interval* $[0, T]$ *the solution* u_ϵ *of the mixed problem* (1.1), (1.2), (1.3) *converges to the solution* u *of the mixed problem* (1.4), (1.5), (1.6) *strongly in* $C([0,T]; H_0^1(\Omega))$ *as* $\epsilon \to 0$. *Furthermore,* u_ϵ' *converges to* u' *strongly in* $L^2((0,T)\times\Omega)$ *as* $\epsilon \to 0$.

Our plan in this paper is as follows. In Section 2 we summarize the results of Ikehata ([10]): the global existence of solutions for the mixed problem (1.1), (1.2), (1.3) using the well-known Sattinger's stable set. To make the context self-contained we prove the existence and uniqueness for the mixed problem (1.4), (1.5), (1.6). In Section 3 we briefly survey the local existence of solutions for the abstract parabolic equations. In Section 4 we prove the unique local existence of the solution for the mixed problem (1.4), (1.5), (1.6). In Section 5 we give the global solvability of the mixed problem (1.4), (1.5), (1.6) using the stable set. In Section 6 we show some energy estimates playing an important role in the proof of our main theorem and prove Theorem.

We conclude this section by stating several notations. Let $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$. We often suppress the space variable x when no confusion arises. Also we often abbreviate $\frac{d}{dt}u_\epsilon(t)$, $\frac{d^2}{dt^2}u_\epsilon(t)$ and $\frac{d}{dt}u(t)$ to $u_\epsilon'(t)$, $u_\epsilon''(t)$ and $u_t(t)$ (or $u'(t)$), respectively.

2. Global existence and ϵ weighted energy estimates of the hyperbolic equation. In this section we briefly describe the results of Ikehata ([10]) concerning the global existence and uniqueness of the solution for (1.1), (1.2), (1.3).

First we formulate our problem precisely. In [9, 10] it was shown that the mixed problem (1.1), (1.2), (1.3) has a unique local solution.

Proposition 2.1. *Assume that* $\alpha > 0$, $\beta \geq 0$, $\delta \in \mathbb{R}$ *and* $\mu \in \mathbb{R}$. *Let* ϵ *be any number with* $0 < \epsilon \leq 1$. *Then, for any* $u_{0\epsilon} \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_{1\epsilon} \in H_0^1(\Omega)$ *there exists a number* $T_\epsilon > 0$ *such that the problem* (1.1), (1.2), (1.3) *has a unique solution* $u_\epsilon(t,x)$ *which satisfies*

$$u_\epsilon \in C([0,T_\epsilon); H_0^1(\Omega) \cap H^2(\Omega))$$

with $\epsilon u_\epsilon' \in C([0,T_\epsilon); H_0^1(\Omega))$ *and* $\epsilon^2 u_\epsilon'' \in C([0,T_\epsilon); L^2(\Omega))$. *Furthermore, if* $T_\epsilon < +\infty$*, then*

$$\lim_{t \nearrow T_\epsilon} \left[\epsilon^2 \left\|\frac{d}{dt}\nabla u_\epsilon(t)\right\|_2^2 + \|\triangle u_\epsilon(t)\|_2^2\right] = +\infty.$$

On the other hand, we have the following energy identity:

Lemma 2.2. *Assume that* $\delta > 0$ *and* $\mu > 0$. *Let* $u_\epsilon(t,x)$ *be as in Proposition* 2.1. *Then it holds that*

$$\begin{aligned}&\frac{\epsilon^2}{2}\Big\|\frac{d}{dt}u_\epsilon(t)\Big\|_2^2 + J(u_\epsilon(t)) + \delta\int_0^t \Big\|\frac{d}{dt}u_\epsilon(s)\Big\|_2^2\,ds\\ &= \frac{\epsilon^2}{2}\|u_{1\epsilon}\|_2^2 + \frac{\alpha}{2}\|\nabla u_{0\epsilon}\|_2^2 + \frac{\beta}{2}\|\nabla u_{0\epsilon}\|_2^4 - \frac{\mu}{4}\|u_{0\epsilon}\|_4^4 \quad \text{on} \quad [0,T] \qquad (2.1)\end{aligned}$$

for any T *with* $0 < T < T_\epsilon$.

Now by Lemma 2.2 it can be easily shown that $u_\epsilon(t,x)$ is bounded in $H_0^1(\Omega)$. That is to say, we have the following:

Lemma 2.3. *Let* $u_\epsilon(t,x)$ *be as in Lemma* 2.2 *with initial data* $u_{0\epsilon} \in \mathcal{W}^* \cap H^2(\Omega)$ *and* $u_{1\epsilon} \in H_0^1(\Omega)$ *satisfying*

$$\frac{1}{2}\|u_{1\epsilon}\|_2^2 + \frac{\alpha}{2}\|\nabla u_{0\epsilon}\|_2^2 + \frac{\beta}{2}\|\nabla u_{0\epsilon}\|_2^4 - \frac{\mu}{4}\|u_{0\epsilon}\|_4^4 < d. \qquad (2.2)$$

Then $u_\epsilon(t) \in \mathcal{W}^*$ *on* $[0,T_\epsilon)$ *and we have*

$$\alpha\|\nabla u_\epsilon(t)\|_2^2 < 4d \quad \text{on} \quad [0,T_\epsilon), \qquad (2.3)$$

$$\epsilon^2\Big\|\frac{d}{dt}u_\epsilon(t)\Big\|_2^2 < 2d \quad \text{on} \quad [0,T_\epsilon). \qquad (2.4)$$

Proof. The first part of the assertion is due to the discussion following Sattinger ([18]) and Tsutsumi ([20]). We omit the proof.

By Lemma 2.2 we have

$$\begin{aligned}&2\epsilon^2\Big\|\frac{d}{dt}u_\epsilon(t)\Big\|_2^2 + \alpha\|\nabla u_\epsilon(t)\|_2^2 + \{\alpha\|\nabla u_\epsilon(t)\|_2^2 + 2\beta\|\nabla u_\epsilon(t)\|_2^4 - \mu\|u_\epsilon(t)\|_4^4\}\\ &\le 4\Big(\frac{\epsilon^2}{2}\|u_{1\epsilon}\|_2^2 + \frac{\alpha}{2}\|\nabla u_{0\epsilon}\|_2^2 + \frac{\beta}{2}\|\nabla u_{0\epsilon}\|_2^4 - \frac{\mu}{4}\|u_{0\epsilon}\|_4^4\Big) < 4d. \qquad (2.5)\end{aligned}$$

Since $u_\epsilon(t) \in \mathcal{W}^*$ on $[0,T_\epsilon)$, we have

$$\alpha\|\nabla u_\epsilon(t)\|_2^2 + 2\beta\|\nabla u_\epsilon(t)\|_2^4 - \mu\|u_\epsilon(t)\|_4^4 > 0. \qquad (2.6)$$

From (2.5) and (2.6) it follows that

$$2\epsilon^2\Big\|\frac{d}{dt}u_\epsilon(t)\Big\|_2^2 + \alpha\|\nabla u_\epsilon(t)\|_2^2 < 4d.$$

Therefore the proof of the second part is completed. □

In Section 4 we need some ϵ weighted energy estimates. So it is important to see the ϵ dependence on various energy estimates. Therefore we shall derive higher order energy estimates for u_ϵ.

Let

$$Z_\epsilon(t) = \epsilon^2\|\nabla u'_\epsilon(t)\|_2^2 + M(\|\nabla u_\epsilon(t)\|_2^2)\|\triangle u_\epsilon(t)\|_2^2,$$

where $M(s) \equiv \alpha + 2\beta s$. Then we have the following.

Lemma 2.4. *Let $u_\epsilon(t)$ be as in Lemma 2.3. Then for some positive constants C_1 and C_2 we have the following estimates:*

$$\begin{aligned} & Z_\epsilon(t) + \frac{3\delta}{2}\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds \\ \le & Z_\epsilon(0) + \frac{C_1}{2\delta}\int_0^t \|\triangle u_\epsilon(s)\|_2^6\,ds + C_2\int_0^t \|\triangle u_\epsilon(s)\|_2^2 \|\nabla u'_\epsilon(s)\|_2\,ds, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & \epsilon^2\delta(\nabla u'_\epsilon(t), \nabla u_\epsilon(t)) + \frac{\delta^2}{2}\|\nabla u_\epsilon(t)\|_2^2 + \alpha\delta\int_0^t \|\triangle u_\epsilon(s)\|_2^2\,ds \\ & \le H_\epsilon + \delta\mu\int_0^t (u_\epsilon(s)^3, -\triangle u_\epsilon(s))\,ds + \epsilon^2\delta\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds, \end{aligned} \tag{2.8}$$

where H_ϵ is defined by

$$H_\epsilon = \epsilon^2\delta(\nabla u_{1\epsilon}, \nabla u_{0\epsilon}) + \frac{\delta^2}{2}\|\nabla u_{0\epsilon}\|_2^2.$$

For details of the discussion, see [10, Lemma 4.3]. We shall omit the proof. Next we set

$$W_\epsilon(t) \equiv Z_\epsilon(t) + \epsilon^2\delta(\nabla u'_\epsilon(t), \nabla u_\epsilon(t)) + \frac{\delta^2}{2}\|\nabla u_\epsilon(t)\|_2^2.$$

Then we have the following.

Lemma 2.5. *Let $u_\epsilon(t)$ be as in Lemma 2.3. Then we have the following estimates:*

$$\begin{aligned} & W_\epsilon(t) + \frac{\delta}{4}\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds \\ & \le W_\epsilon(0) + \int_0^t \big\{\frac{C_1}{2\delta}\|\triangle u_\epsilon(s)\|_2^4 + \big(\frac{4C_2 + \delta^2\mu C_3}{\delta}\big)\|\triangle u_\epsilon(s)\|_2^2 - \alpha\delta\big\}\|\triangle u_\epsilon(s)\|_2^2\,ds \end{aligned} \tag{2.9}$$

for some positive constant C_3 and

$$C_0\{\epsilon^2\|\nabla u'_\epsilon(t)\|_2^2 + \|\triangle u_\epsilon(t)\|_2^2\} \le W_\epsilon(t) \tag{2.10}$$

for some positive constant C_0.

Proof. First the second term in the right-hand side of (2.8) is estimated as follows:

$$|(u_\epsilon(s)^3, -\triangle u_\epsilon(s))| \le \|u_\epsilon(s)\|_6^3 \|\triangle u_\epsilon(s)\|_2.$$

Sobolev's inequality gives $\|u_\epsilon(s)\|_6 \le C\|u_\epsilon(s)\|_{H^2}$ and the regularity theory of elliptic equations assures $\|u_\epsilon(s)\|_{H^2} \le C\|\triangle u_\epsilon(s)\|_2$. Therefore, we get

$$|(u_\epsilon(s)^3, -\triangle u_\epsilon(s))| \le C\|\triangle u_\epsilon(s)\|_2^4. \tag{2.11}$$

Noting (2.11) and the definition of $W_\epsilon(t)$, we add (2.7) to (2.8) to obtain

$$\begin{aligned} & W_\epsilon(t) + \frac{3\delta}{2}\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds + \alpha\delta\int_0^t \|\triangle u_\epsilon(s)\|_2^2\,ds \\ & \le W_\epsilon(0) + \frac{C_1}{2\delta}\int_0^t \|\triangle u_\epsilon(s)\|_2^6\,ds + C_2\int_0^t \|\triangle u_\epsilon(t)\|_2^2\|\nabla u'_\epsilon(s)\|_2\,ds \\ & \quad + \delta\mu\int_0^t (u_\epsilon(s)^3, -\triangle u_\epsilon(s))\,ds + \epsilon^2\delta\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds \\ & \le W_\epsilon(0) + \frac{C_1}{2\delta}\int_0^t \|\triangle u_\epsilon(s)\|_2^6\,ds + \frac{4C_2}{\delta}\int_0^t \|\triangle u_\epsilon(s)\|_2^4\,ds \\ & \quad + \frac{\delta}{4}\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds + \delta\mu C_3\int_0^t \|\triangle u_\epsilon(s)\|_2^4\,ds + \delta\int_0^t \|\nabla u'_\epsilon(s)\|_2^2\,ds. \end{aligned} \tag{2.12}$$

Hence the desired estimate (2.9) is equivalent to (2.12).

We note that $M(\|\nabla u_\epsilon(t)\|_2^2) \ge \alpha$. Then it is easy to see that

$$Z_\epsilon(t) \ge \epsilon^2\|\nabla u'_\epsilon(t)\|_2^2 + \alpha\|\triangle u_\epsilon(t)\|_2^2. \tag{2.13}$$

We observe here that

$$\begin{aligned} \epsilon^2\delta(\nabla u'_\epsilon(t), \nabla u_\epsilon(t)) & \ge -\epsilon^2\delta\|\nabla u'_\epsilon(t)\|_2\|\nabla u_\epsilon(t)\|_2 \ge -\frac{\epsilon^4}{2}\|\nabla u'_\epsilon(t)\|_2^2 - \frac{\delta^2}{2}\|\nabla u_\epsilon(t)\|_2^2 \\ & \ge -\frac{\epsilon^2}{2}\|\nabla u'_\epsilon(t)\|_2^2 - \frac{\delta^2}{2}\|\nabla u_\epsilon(t)\|_2^2. \end{aligned} \tag{2.14}$$

Hence from (2.13), (2.14) we deduce that

$$W_\epsilon(t) \ge \frac{\epsilon^2}{2}\|\nabla u'_\epsilon(t)\|_2^2 + \alpha\|\triangle u_\epsilon(t)\|_2^2.$$

If we choose C_0 so that $C_0 = \min\{\frac{1}{2}, \alpha\}$, then (2.10) can be obtained. This ends the proof of Lemma 2.5. □

Lemma 2.6. *Let $u_\epsilon(t)$ be as in Lemma 2.3 with initial data $u_{0\epsilon}$ and $u_{1\epsilon}$ satisfying*

$$\sup_{0<\epsilon\le 1}\Big\{\frac{C_1}{2\delta}\|\triangle u_{0\epsilon}\|_2^4 + \Big(\frac{4C_2+\delta^2\mu C_3}{\delta}\Big)\|\triangle u_{0\epsilon}\|_2^2\Big\} \le \frac{\alpha\delta}{4}, \tag{2.15}$$

and

$$\sup_{0<\epsilon\le 1}\Big\{\frac{C_1 W_\epsilon(0)^2}{2\delta C_0^2} + \Big(\frac{4C_2+\delta^2\mu C_3}{\delta C_0}\Big)W_\epsilon(0)\Big\} < \frac{\alpha\delta}{2}. \tag{2.16}$$

Then we have

$$\sup_{0<\epsilon\le 1}\Big\{\frac{C_1}{2\delta}\|\triangle u_\epsilon(t)\|_2^4 + \Big(\frac{4C_2+\delta^2\mu C_3}{\delta}\Big)\|\triangle u_\epsilon(t)\|_2^2\Big\} < \frac{\alpha\delta}{2} \quad \text{on } [0, T_\epsilon). \tag{2.17}$$

Proof. Suppose that there exists a number $t^* \in (0, T_\epsilon)$ such that

$$\frac{C_1}{2\delta}\|\triangle u_\epsilon(t)\|_2^4 + \big(\frac{4C_2+\delta^2\mu C_3}{\delta}\big)\|\triangle u_\epsilon(t)\|_2^2 < \frac{\alpha\delta}{2}, \quad t \in [0, t^*), \tag{2.18}$$

and

$$\frac{C_1}{2\delta}\|\triangle u_\epsilon(t^*)\|_2^4 + \big(\frac{4C_2+\delta^2\mu C_3}{\delta}\big)\|\triangle u_\epsilon(t^*)\|_2^2 = \frac{\alpha\delta}{2}. \tag{2.19}$$

Then it follows from (2.9) with $t = t^*$ and (2.18) that

$$W_\epsilon(t^*) + \frac{\delta}{4}\int_0^{t^*} \|\nabla u_\epsilon'(s)\|_2^2\, ds \le W_\epsilon(0). \tag{2.20}$$

Furthermore, using (2.10) we have

$$\|\triangle u_\epsilon(t^*)\|_2^2 \le \frac{W_\epsilon(t^*)}{C_0} \le \frac{W_\epsilon(0)}{C_0}. \tag{2.21}$$

Therefore, combining (2.20) with (2.21) we get

$$\begin{aligned}&\frac{C_1}{2\delta}\|\triangle u_\epsilon(t^*)\|_2^4 + \big(\frac{4C_2+\delta^2\mu C_3}{\delta}\big)\|\triangle u_\epsilon(t^*)\|_2^2\\ &\qquad \le \frac{C_1 W_\epsilon(0)^2}{2\delta C_0^2} + \big(\frac{4C_2+\delta^2\mu C_3}{\delta C_0}\big)W_\epsilon(0) < \frac{\alpha\delta}{2},\end{aligned}$$

which contradicts to (2.19). This completes the proof of Lemma 2.6. □

(2.17) does not necessarily imply

$$u_\epsilon \in C([0,T]; H_0^1(\Omega)\cap H^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$$

since we cannot expect that $u_\epsilon'(t)$ is uniformly bounded in $H_0^1(\Omega)$ for any ϵ with $0 < \epsilon \le 1$. However we have the following ϵ weighted estimates.

Lemma 2.7. *Let $u_\epsilon(t)$ be as in Lemma* 2.3 *with initial data $u_{0\epsilon}$ and $u_{1\epsilon}$ satisfying* (2.15), (2.16). *Then we have the following estimates:*

$$\sup_{0<\epsilon\le 1}\int_0^t \|\nabla u_\epsilon'(s)\|_2^2\, ds \le C \qquad \text{on} \ \ [0,T] \tag{2.22}$$

and

$$\sup_{0<\epsilon\le 1} \epsilon^2\|\nabla u_\epsilon'(t)\|_2^2 \le C \qquad \text{on} \ \ [0,T] \tag{2.23}$$

for any T with $0 < T < T_\epsilon$.

Proof. This lemma is an immediate consequence of Lemma 2.5 and Lemma 2.6. This completes the proof of Lemma 2.7. □

Since (2.17), (2.22) and (2.23) imply the estimates needed for the proof of the global existence of solutions to the problem (1.1), (1.2), (1.3), we have the following result concerning the global existence of solutions for the problem (1.1), (1.2), (1.3).

Theorem 2.8. *Let* $\alpha > 0$, $\beta \geq 0$, $\delta > 0$ *and* $\mu > 0$. *Suppose that* $\mu C(\Omega, 4)^4 > 2\beta$ *and the initial data* $\{u_{0\epsilon}, u_{1\epsilon}\}$ *satisfies* $u_{0\epsilon} \in \mathcal{W}^* \cap H^2(\Omega)$, $u_{1\epsilon} \in H_0^1(\Omega)$ *and* (2.2) *in Lemma* 2.3. *Then for any* ϵ *with* $0 < \epsilon \leq 1$ *there exists a number* η_0, *independent of* ϵ, *such that if* $u_{0\epsilon}$ *and* $u_{1\epsilon}$ *satisfy* $\sup_{0<\epsilon\leq 1}(\|\triangle u_{0\epsilon}\|_2^2 + \|\nabla u_{1\epsilon}\|_2^2) < \eta_0$, *then the problem* (1.1), (1.2), (1.3) *has a unique solution* $u_\epsilon(t,x)$ *such that*

$$u_\epsilon \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \quad \epsilon u_\epsilon' \in C([0,T]; H_0^1(\Omega)), \quad \epsilon^2 u_\epsilon'' \in C([0,T]; L^2(\Omega))$$

for any T *with* $T > 0$.

Proof. By using Proposition 2.1, and Lemmas 2.5, 2.6 and 2.7 we can prove Theorem 2.8. So we shall omit the details. □

3. Local existence for the abstract equations of parabolic type. In this section we describe the general theory concerning the unique local solvability of the parabolic equations.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let A be a linear operator in H with domain $D(A)$ dense in H. For simplicity, suppose $D(A)$ to be separable. With the graph norm of A, $D(A)$ is a real Hilbert space and its injection in H is continuous. We assume that this injection is compact. Suppose A is self-adjoint and positive. Let $C_p > 0$, a constant, be such that

$$(Au, u) \geq C_p^2 \|u\|^2 \qquad \text{for} \quad u \in D(A). \tag{3.1}$$

With the graph norm of $A^{1/2}$ in $D(A^{1/2})$, $D(A)$ is dense in $D(A^{1/2})$. Furthermore, we assume that the injection $D(A) \subset D(A^{1/2})$ is compact.

We are concerned with the abstract initial value problem of the form

$$u'(t) + M(\|A^{1/2}u(t)\|^2)Au(t) = f(u(t)), \quad t > 0, \tag{3.2}$$

$$u(0) = u_0, \tag{3.3}$$

where f is a nonlinear operator from $D(A)$ into H, and $M(s)$ is a C^1-class function satisfying

$$M(s) \geq m_0 > 0 \tag{3.4}$$

with a constant $m_0 > 0$.

We first state the assumptions to be made in the theorems.

(F1) Let f be a nonlinear operator from $D(A)$ into $D(A)$. We assume that there exists a constant L_k for any $k > 0$ such that

$$\|Af(u)\| \leq l(\|Au\|)\|Au\|,$$

$$\|f(u) - f(v)\| \leq L_k \|A^{1/2}u - A^{1/2}v\|$$

for any u, $v \in D(A)$ with $\|u\| + \|Au\| \leq k$, $\|v\| + \|Av\| \leq k$, where $l(\cdot)$ is a nonnegative, nondecreasing and continuous function defined on $[0, \infty)$.

(F2) f is a local Lipschitz operator from $D(A)$ into $D(A)$: for any $M > 0$ there is $L_M > 0$ such that

$$\|f(u) - f(v)\| + \|A(f(u)) - A(f(v))\| \leq L_M(\|u - v\| + \|Au - Av\|)$$

for any u, v with $\|u\| + \|Au\|$, $\|v\| + \|Av\| \leq M$.

If $u_0 \in H$, then we cannot expect the differentiability of $u(t)$ at $t = 0$. But, if $u_0 \in D(A)$, then (F2) assures the differentiability of $u(t)$ at $t = 0$ (see Ôtani, [17]). Hence we can define the notion of the solutions to the problem (3.2), (3.3).

Definition 3.1. A function $u : [0, T] \to H$ is called a solution of (3.2), (3.3) on $[0, T]$ if
(i) $u \in C([0, T]; D(A)) \bigcap C^1([0, T]; H)$,
(ii) u satisfies (3.2) on $[0, T]$,
(iii) $u(0) = u_0$.

Then we can state the uniqueness and local existence for the problem (3.2), (3.3).

Theorem 3.2. *Suppose that* (F1) *and* (F2) *hold. Let* $u_0 \in D(A)$*. Then there exists a unique solution* $u(t)$ *of the problem* (3.2), (3.3) *on some interval* $[0, T_m)$ *such that*

$$u \in C([0, T_m); D(A)) \cap C^1([0, T_m); H),$$

where we may assume that either $T_m = +\infty$ *or* $T_m < +\infty$, $\lim_{t \nearrow T_m} (\|Au(t)\| + \|u(t)\|) = +\infty$.

In order to prove Theorem 3.2 we shall briefly survey the works of E.E. Levi concerning the construction of evolution operator of the parabolic equation with time dependent coefficients. We refer to Tanabe ([19]) and Asano ([1]) for details.

We consider the abstract Cauchy problem in H for the linear equation

$$u'(t) + A(t)u(t) = f(t), \quad t > 0, \tag{3.5}$$

$$u(0) = u_0. \tag{3.6}$$

We impose some assumptions as follows.

(A1) $-A(t)$ is a generator of an analytic semigroup for each t with $0 \leq t \leq T$.

(A2) $A(t)$ is a closed linear densely defined operator on H for each t with $0 \leq t \leq T$. The resolvent set $\rho(A(t))$ of $A(t)$ contains the set $\{\lambda \in \mathbb{C}; Re\lambda \leq 0\}$ and $(1+|\lambda|)(A(t)-\lambda)^{-1}$ is uniformly bounded in $0 \leq t \leq T$, $Re\lambda \leq 0$.

(A2) implies that there exist positive constants C and $\theta \in (0, \pi/2)$ such that $\rho(A(t))$ contains a closed sector $\Sigma = \{\lambda; |\arg\lambda| \geq \theta\} \cup \{0\}$ and it holds that

$$\|(A(t) - \lambda)^{-1}\| \leq C/(1 + |\lambda|)$$

for $0 \leq t \leq T$, $\lambda \in \Sigma$.

(A3) The domain $D(A(t)) \equiv D$ of $A(t)$ is independent of t, hence $A(t)A(0)^{-1}$ is a bounded operator, and a Hölder continuous function of t with respect to the operator $B(H)$ norm from H into H. That is to say, there exist positive constants L and α with $0 < \alpha \leq 1$ such that

$$\|A(t)A(0)^{-1} - A(s)A(0)^{-1}\| \leq L|t - s|^\alpha$$

for any s, t with $0 \leq s \leq T$, $0 \leq t \leq T$.

Then we can construct an evolution operator of (3.5).

Proposition 3.3. *Suppose that* (A1)–(A3) *hold. Then there exists an evolution operator* $U(t,s)$ *of* (3.5) *such that* $U(t,s)D \subset D$ *for* $0 \le s < t \le T$, $(\partial/\partial t)U(t,s)$ *exists as an element of* $B(H)$. *Furthermore the following can be obtained.*

$$\|U(t,s)\| \le C_0, \tag{3.7}$$

$$\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s), \tag{3.8}$$

$$\frac{\partial}{\partial s}U(t,s) = U(t,s)A(s), \tag{3.9}$$

$$\Big\|\frac{\partial}{\partial t}U(t,s)\Big\| = \|A(t)U(t,s)\| \le C_1|t-s|^{-1}, \tag{3.10}$$

$$\|A(t)U(t,s)A(s)^{-1}\| \le C_2, \tag{3.11}$$

$$\|A(t)^{\alpha}U(t,s)\| \le C_3|t-s|^{-\alpha} \tag{3.12}$$

for any α *with* $0 < \alpha < 1$.

By virtue of Proposition 3.3 we can define the solution of the problem (3.5), (3.6) as follows.

Definition 3.4. A function $u : [0,T] \to H$ is called a solution of the problem (3.5), (3.6) if

(i) $u \in C([0,T];H) \cap C^1([0,T];H)$ with $u(t) \in D$ for each t with $0 \le t \le T$, $A(t)u(t) \in C([0,T];H)$,

(ii) u satisfies (3.5),

(iii) $u(0) = u_0$.

Theorem 3.5. *Let* $u_0 \in D$. *Assume that* $f(t)$ *is a Hölder-continuous function. Then the problem* (3.5), (3.6) *has a unique solution* $u(t)$ *which satisfies*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds.$$

4. Proof of Theorem 3.2. In this section we shall prove Theorem 3.2 by the standard method of contraction mapping. As an application of Theorem 3.2 we can prove the unique local existence for the problem (1.4), (1.5), (1.6). H, A, and the constants C_i $(i = 0, \dots, 3)$ are denoted by the same as in Section 3.

We now consider the initial value problem (3.2), (3.3) in H on $[0, T_0]$. Let

$$\mathcal{A}(u(t)) = M(\|A^{1/2}u(t)\|^2)A. \tag{4.1}$$

Then the problem (3.2), (3.2) in H can be written in the quasilinear evolution equation

$$\frac{d}{dt}u(t) + \mathcal{A}(u(t))u(t) = f(u(t)) \quad \text{in } H \text{ on } [0,T_0], \tag{4.2}$$

$$u(0) = u_0 \in D(A). \tag{4.3}$$

Let k be an arbitrary constant satisfying

$$k \geq 2m_0^{-1} C_2 M(\|A^{1/2}u_0\|^2)\|Au_0\|. \tag{4.4}$$

We set

$$M_0 = \max\{M(r); 0 \leq r \leq k^2\}, \qquad M_1 = \max\{|M'(r)|; 0 \leq r \leq k^2\}.$$

Moreover let T_0 be a constant satisfying

$$m_0^{-1} C_2 l(k) M_0 T_0 < \frac{1}{2}, \tag{4.5}$$

where $l(\cdot)$ is a nonnegative, nondecreasing and continuous function on $[0, \infty)$ appearing in (F1).

Then we define the set $\mathcal{K}$ by

$$\mathcal{K} = \big\{v(\cdot) : [0, T_0] \to H; \quad v(0) = u_0, \quad \|Av(t)\| \leq k,$$

$$\|A^{1/2}v(t) - A^{1/2}v(s)\| \leq L|t-s|^{1/2}\big\},$$

where

$$L \equiv L(k) = [2M(C_p k^2)k^2 + 2l(k)k^2]^{1/2}.$$

For each $v(\cdot) \in \mathcal{K}$ we shall consider the linearized problem

$$\frac{d}{dt}u(t) + \mathcal{A}(v(t))u(t) = f(v(t)) \quad \text{in} \quad H \text{ on } [0, T_0], \tag{4.6}$$

$$u(0) = u_0. \tag{4.7}$$

Namely, the problem (4.6), (4.7) is nothing but (4.2), (4.3) with $\mathcal{A}(u(t))$ and $f(u(t))$ replaced by $\mathcal{A}(v(t))$ and $f(v(t))$, respectively.

We first consider the integral equation

$$u(t) = U^v(t,0)u_0 + \int_0^t U^v(t,s) f(v(s))\, ds \tag{4.8}$$

as the integral version of the problem (4.6), (4.7), where $\{U^v(t,s); 0 \leq s \leq t \leq T_0\}$ is an evolution operator associated with the family $\{\mathcal{A}(v(t)); 0 \leq t \leq T_0\}$ of linear operators (see Section 3). Therefore, to get a mild solution $u(t)$ we have only to construct an evolution operator $\{U^v(t,s)\}$ associated with the family $\{\mathcal{A}(v(t))\}$.

Lemma 4.1. *The operator $\mathcal{A}(v(t))$ satisfies* (A1)–(A3) *for each $v \in \mathcal{K}$. Furthermore we have*

$$|M(\|A^{\frac{1}{2}}v(t)\|^2) - M(\|A^{\frac{1}{2}}v(s)\|^2)| \leq 2kC_p^{\frac{1}{2}}M_1\|A^{\frac{1}{2}}v(t) - A^{\frac{1}{2}}v(s)\| \leq 2kC_p^{\frac{1}{2}}M_1L|t-s|^{\frac{1}{2}}. \tag{4.9}$$

Proof. From the definition of $\mathcal{A}(v(t))$ (see (4.1)) and (3.4) it follows that $\mathcal{A}(v(t))$ is a strongly elliptic operator uniformly on $[0, T_0]$, since we have for any $u \in D$

$$\|\mathcal{A}(v(t))u\| = M(\|A^{1/2}v(t)\|^2)\|Au\| \geq m_0\|Au\|.$$

Hence, by virtue of the well-established general theory of analytic semigroup, $-\mathcal{A}(v(t))$ generates an analytic semigroup for each t with $0 \leq t \leq T_0$, from which it follows that the assumption (A1) is satisfied.

To see (A2), it is sufficient to show that

$$\|(\mathcal{A}(v(t)) - \lambda)^{-1}\| \leq \frac{C}{1+|\lambda|} \tag{4.10}$$

for $v(\cdot) \in \mathcal{K}$ and any λ with $\operatorname{Re}\lambda \leq 0$. Let $\lambda = \mu + i\sigma$ with $\mu \leq 0$, $\sigma \in \mathbb{R}$. Let $u \in D$. By (3.1) and (3.4) we have

$$\begin{aligned} |(\mathcal{A}(v(t))u - \lambda u, u)| &= |M(\|A^{1/2}v(t)\|^2)\|A^{1/2}u\|^2 - (\lambda u, u)| \\ &\geq [(m_0C_p - \mu)^2\|u\|^4 + \sigma^2\|u\|^4]^{1/2} \\ &= [(m_0C_p - \mu)^2 + \sigma^2]^{1/2}\|u\|^2. \end{aligned} \tag{4.11}$$

We apply the Schwarz inequality to the left-hand side of (4.11) to obtain

$$\begin{aligned} \|\mathcal{A}(v(t))u - \lambda u\| &\geq [(m_0C_p - \mu)^2 + \sigma^2]^{1/2}\|u\| \geq \frac{1}{\sqrt{2}}(m_0C_p + |\lambda|)\|u\| \\ &\geq \frac{\min\{m_0C_p, 1\}}{\sqrt{2}}(1+|\lambda|)\|u\|. \end{aligned}$$

Hence we obtain (4.10).

Finally the coefficient $M(\|A^{1/2}v(t)\|^2)$ of $\mathcal{A}(v(t))$ satisfies the following estimate:

$$\begin{aligned} &|M(\|A^{1/2}v(t)\|^2) - M(\|A^{1/2}v(s)\|^2)| \leq M_1|\|A^{1/2}v(t)\|^2 - \|A^{1/2}v(s)\|^2| \\ &\leq M_1(\|A^{1/2}v(t)\| + \|A^{1/2}v(s)\|)|\|A^{1/2}v(t)\| - \|A^{1/2}v(s)\|| \\ &\leq 2kC_p^{1/2}M_1\|A^{1/2}v(t) - A^{1/2}v(s)\| \leq 2kC_p^{1/2}M_1L|t-s|^{1/2}, \end{aligned}$$

which implies (A3). This completes the proof of Lemma 4.1. $\square$

By virtue of Lemma 4.1 we can construct an evolution operator $\{U^v(t,s)\}$ associated with the family $\{\mathcal{A}(v(t))\}$ of generators such that $u(t)$ given by (4.8) is a mild solution of the problem (4.6), (4.7). Furthermore, in view of the construction of the evolution operator (see [19, Theorem 3.1 in Chapter 3]), it is easy to see that the evolution operator $\{U^v(t,s)\}$ is uniformly bounded in $\mathcal{K}$; that is, there exist positive constants C_0, C_2 and C_3, independent of $v(\cdot) \in \mathcal{K}$, such that

$$\|U^v(t,s)\| \le C_0, \tag{4.12}$$

$$\|\mathcal{A}(v(t))U^v(t,s)\mathcal{A}(v(s))^{-1}\| \le C_2, \tag{4.13}$$

$$\|\mathcal{A}(v(t))^\alpha U^v(t,s)\| \le C_3|t-s|^{-\alpha} \tag{4.14}$$

for any $v \in \mathcal{K}$ and α with $0 < \alpha < 1$. On the other hand, since the assumption (F1) holds and $v(\cdot) \in \mathcal{K}$, it follows that $f(v(\cdot)) \in C([0,T_0]; D)$ and $f(v(t))$ is a Hölder-continuous function. Thus it follows from Theorem 3.5 and (F2) that the mild solution $u(t)$ is a strong solution of the problem (4.6), (4.7) satisfying

$$u \in C([0,T_0]; D) \cap C^1([0,T_0]; H), \qquad u(0) = u_0.$$

Then we can define the mapping $\Phi : \mathcal{K} \to H$ by $\Phi v = u$ for $v \in \mathcal{K}$. For such a mapping Φ we shall show that Φ maps $\mathcal{K}$ into $\mathcal{K}$. In fact, we have the following.

Lemma 4.2. *Φ maps $\mathcal{K}$ into $\mathcal{K}$.*

Proof. Let $u(t) = (\Phi v)(t)$. From (4.8), (F1), (F2), (A3), (4.4), (4.5), and Proposition 3.3 we have

$$\begin{aligned} m_0\|Au(t)\| &\le \|\mathcal{A}(v(t))u(t)\| \\ &\le \|\mathcal{A}(v(t))U^v(t,0)u_0\| + \int_0^t \|\mathcal{A}(v(t))U^v(t,s)f(v(s))\|\,ds \\ &\le \|\mathcal{A}(v(t))U^v(t,0)(\mathcal{A}(u_0)^{-1})(\mathcal{A}(u_0)u_0)\| \\ &\quad + \int_0^t \|(\mathcal{A}(v(t))U^v(t,s)\mathcal{A}(v(s))^{-1})(\mathcal{A}(v(s))f(v(s)))\|\,ds \\ &\le C_2M(\|A^{1/2}u_0\|^2)\|Au_0\| + C_2M(k^2)T_0 \sup_{t\in[0,T_0]} \|Af(v(t))\| \\ &\le C_2M(\|A^{1/2}u_0\|^2)\|Au_0\| + C_2M_0l(k)T_0k \le m_0k. \end{aligned}$$

Hence we obtain

$$\|Au(t)\| \le k. \tag{4.15}$$

We note here from equation (4.2) and (F1) that

$$\begin{aligned} \|u'(t)\| &\le M(\|A^{1/2}v(t)\|^2)\|Au(t)\| + \|f(v(t))\| \\ &\le M(C_pk^2)\|Au(t)\| + l(k)k \le M(C_pk^2)k + l(k)k, \end{aligned}$$

since $v(\cdot) \in \mathcal{K}$. Hence we have

$$\|u(t)-u(s)\| \le \int_s^t \|u'(r)\|\,dr \le [M(C_pk^2)k + l(k)k]|t-s|. \tag{4.16}$$

Also noting that

$$\|A^{1/2}u(t) - A^{1/2}u(s)\|^2 = (u(t) - u(s), Au(t) - Au(s)) \\ \le (\|Au(t)\| + \|Au(s)\|)\|u(t) - u(s)\| \le 2k\|u(t) - u(s)\|. \tag{4.17}$$

Then we have from (4.16) and (4.17)

$$\|A^{1/2}u(t) - A^{1/2}u(s)\| \le L|t - s|^{1/2}. \tag{4.18}$$

Therefore we conclude by (4.15) and (4.18) that $u(t) \in \mathcal{K}$. This completes the proof of Lemma 4.2. □

In the following we always assume that

$$C(T_0) \equiv 2m_0^{-1/2}k^2C_3M_1LT_0^{1/2} + m_0^{-1/2}C_3L_kT_0^{1/2} < 1.$$

Lemma 4.3. *Let $C(T_0) < 1$. Then Φ is a strict contraction with respect to the metric of $\mathcal{K}$ defined by*

$$d(v, w) \equiv \sup_{t\in[0,T_0]} \|A^{1/2}v(t) - A^{1/2}w(t)\|. \tag{4.19}$$

Proof. We observe from (4.2) that

$$(\Phi v)'(t) + \mathcal{A}(v(t))(\Phi v)(t) = f(v(t)),$$
$$(\Phi w)'(t) + \mathcal{A}(w(t))(\Phi w)(t) = f(w(t)),$$

from which it follows that $(\Phi v)(t) - (\Phi w)(t)$ satisfies

$$(\Phi v - \Phi w)'(t) + \mathcal{A}(w(t))(\Phi v - \Phi w)(t) \\ = [\mathcal{A}(w(t)) - \mathcal{A}(v(t))](\Phi v)(t) + f(v(t)) - f(w(t)). \tag{4.20}$$

Noting that $(\Phi v)(0) = (\Phi w)(0) = u_0$, we reformulate (4.20) as an integral equation to obtain

$$(\Phi v)(t) - (\Phi w)(t) \\ = \int_0^t U^w(t, s)\{[\mathcal{A}(w(s)) - \mathcal{A}(v(s))](\Phi v)(s) + f(v(s)) - f(w(s))\}\, ds. \tag{4.21}$$

Using Lemma 4.1 we apply $A^{1/2}$ to both sides of (4.21) to obtain

$$\|A^{1/2}(\Phi v)(t) - A^{1/2}(\Phi w)(t)\| \\ \le \int_0^t \|A^{1/2}U^w(t, s)[M(\|A^{1/2}v(s)\|^2) - M(\|A^{1/2}w(s)\|^2)]A(\Phi v)(s)\|\, ds \\ + \int_0^t \|A^{1/2}U^w(t, s)(f(v(s)) - f(w(s)))\|\, ds \\ \le 2kM_1L \int_0^t \|A^{1/2}U^w(t, s)A(\Phi v)(s)\|\|A^{1/2}v(s) - A^{1/2}w(s)\|\, ds \\ + \int_0^t \|A^{1/2}U^w(t, s)(f(v(s)) - f(w(s)))\|\, ds, \quad t \in [0, T_0].$$

By (4.13) and (4.14) we obtain

$$\begin{aligned}
&\|A^{1/2}(\Phi v)(t) - A^{1/2}(\Phi w)(t)\| \\
&\le 2m_0^{-1/2} k C_3 M_1 L \int_0^t |t-s|^{-1/2} \|A^{1/2}v(s) - A^{1/2}w(s)\| \|A(\Phi v)(s)\| \, ds \\
&\quad + m_0^{-1/2} C_3 L_k \int_0^t |t-s|^{-1/2} \|A^{1/2}v(s) - A^{1/2}w(s)\| \, ds \\
&\le m_0^{-1/2}(2k^2 C_3 M_1 L T_0^{1/2} + C_3 L_k T_0^{1/2}) \, d(v,w).
\end{aligned}$$

We thus have

$$d(\Phi v, \Phi w) \le C(T_0)\, d(v,w).$$

This ends the proof of Lemma 4.3. □

Proof of Theorem 3.2 completed. Since $\mathcal{K}$ is not expected to be complete with respect to the metric $d(v,w)$, we shall show by iteration that there is $u \in \mathcal{K}$ such that u is a fixed point of $\Phi : \mathcal{K} \to \mathcal{K}$.

First we inductively define $\{u_n(\cdot)\}$ in $\mathcal{K}$ as follows. Let u_0 be initial data of the problem (4.2), (4.3). Then we define

$$u_0(t) \equiv u_0, \quad u_n(t) \equiv (\Phi u_{n-1})(t) \quad t \in [0, T_0], \quad (n \in \mathbb{N}).$$

It follows from Lemma 4.3 that there exists $u \in C([0,T_0]; D(A^{1/2}))$ such that

$$u_n \to u \quad \text{strongly in} \quad C([0,T_0]; D(A^{1/2})) \quad (n \to \infty). \tag{4.22}$$

Since $\{u_n\} \subset \mathcal{K}$, we have

$$u_n(0) = u_0, \tag{4.23}$$

$$\|Au_n(t)\| \le k, \tag{4.24}$$

$$\|A^{1/2}u_n(t) - A^{1/2}u_n(s)\| \le L|t-s|^{1/2}. \tag{4.25}$$

Letting $n \to \infty$, we have by (4.22), (4.23) and (4.25)

$$u(0) = u_0, \tag{4.26}$$

$$\|A^{1/2}u(t) - A^{1/2}u(s)\| \le L|t-s|^{1/2}. \tag{4.27}$$

By virtue of (4.22) and (4.24) there exists a subsequence $\{u_{n_j}(t)\}$ of $\{u_n(t)\}$ such that

$$Au_{n_j}(t) \to Au(t) \quad \text{weakly in} \quad H, \tag{4.28}$$

as $j \to \infty$ uniformly on $[0, T_0]$. From (4.24) and (4.28) it follows that

$$\|Au(t)\| \le k. \tag{4.29}$$

Hence from (4.26), (4.27), and (4.29) we conclude that $u \in \mathcal{K}$.

Next we shall show that u is a desired unique fixed point of $\Phi : \mathcal{K} \to \mathcal{K}$. In fact, we have

$$\begin{aligned} d(u, \Phi u) &\leq d(u, u_n) + d(u_n, \Phi u) = d(u, u_n) + d(\Phi u_{n-1}, \Phi u) \\ &\leq d(u, u_n) + C(T_0) d(u_{n-1}, u). \end{aligned}$$

Letting $n \to \infty$, we obtain $d(u, \Phi u) = 0$ which implies $\Phi u = u$.

Finally we see by the standard argument that u is the strong solution of the problem (3.2), (3.3). This completes the proof of Theorem 3.2. □

We conclude this section by stating the existence and uniqueness for the problem (1.4), (1.5), (1.6).

Let $A = -\triangle$, $D = H_0^1(\Omega) \cap H^2(\Omega)$ and $f(u) = \mu u^3$. Then a simple calculation yields that $f(u)$ satisfies (F1) and (F2). So we shall omit it. Therefore we can apply Theorem 3.2 to the problem (1.4), (1.5), (1.6).

Theorem 4.4. *Assume that $\alpha > 0$, $\beta \geq 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. Then, for any $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists a number $T_m > 0$ depending on $\|\triangle u_0\|_2$ and $\|\nabla u_1\|_2$ such that the problem* (1.4), (1.5), (1.6) *has a unique solution $u(t, x)$ satisfying*

$$u \in C([0, T_m); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, T_m); L^2(\Omega)).$$

Furthermore, we may assume that if $T_m < +\infty$, then

$$\lim_{t \nearrow T_m} (\|\triangle u(t)\|_2^2 + \|\nabla u(t)\|_2^2) = +\infty.$$

5. Global existence for the parabolic equation. In this section we discuss the global existence and uniqueness for the mixed problem (1.4), (1.5), (1.6).

First we describe an energy identity for (1.4) to obtain the global existence for (1.4), (1.5), (1.6).

Lemma 5.1. *Let $u(t)$ be a solution of the problem* (1.4), (1.5), (1.6) *with initial data $u_0 \in H_0^1(\Omega)$. Then it holds that*

$$\begin{aligned} &\delta \int_0^t \Big\| \frac{d}{dt} u(s) \Big\|_2^2 \, ds + \frac{\alpha}{2} \|\nabla u(t)\|_2^2 + \frac{\beta}{2} \|\nabla u(t)\|_2^4 - \frac{\mu}{4} \|u(t)\|_4^4 \\ &\quad = \frac{\alpha}{2} \|\nabla u_0\|_2^2 + \frac{\beta}{2} \|\nabla u_0\|_2^4 - \frac{\mu}{4} \|u_0\|_4^4. \end{aligned}$$

Lemma 5.1 is precisely the identity of Lemma 2.2 when $\epsilon = 0$ and can be derived in the same way.

By virtue of Theorem 4.4 we have the following.

Lemma 5.2. *Let $\alpha > 0$, $\beta \geq 0$, $\delta > 0$ and $\mu > 0$. Suppose that $\mu C(\Omega, 4)^4 > 2\beta$. Assume that $u_0 \in \mathcal{W}^* \cap H^2(\Omega)$. Then the problem (1.4), (1.5), (1.6) has a unique solution $u(t,x)$ such that*

$$u(t) \in \mathcal{W}^*, \quad t \in [0, T_m). \tag{5.1}$$

Furthermore we have

$$\alpha \|\nabla u(t)\|_2^2 < 4d, \quad t \in [0, T_m). \tag{5.2}$$

The proof is the same argument as in Tsutsumi ([20]). We shall omit the proof.

As before we set $M(s) \equiv \alpha + 2\beta s$. Then we have the following estimates. The proof of the idea comes from Ikehata ([10]).

Lemma 5.3. *Let $u(t)$ be as in Lemma 5.2. Then for some positive constant C_1 we have the following estimates:*

$$M(\|\nabla u(t)\|_2^2)\|\triangle u(t)\|_2^2 \leq M(\|\nabla u_0\|_2^2)\|\triangle u_0\|_2^2 + \frac{C_1}{\delta}\int_0^t (\|\triangle u(s)\|_2^6 + \|\triangle u(s)\|_2^4)\, ds, \tag{5.3}$$

and

$$\frac{\delta}{2}\|\nabla u(t)\|_2^2 + \alpha \int_0^t \|\triangle u(s)\|_2^2\, ds \leq \frac{\delta}{2}\|\nabla u_0\|_2^2 + \mu \int_0^t (u(s)^3, -\triangle u(s))\, ds \tag{5.4}$$

for $t \in [0, T_m)$.

Proof. We shall use the idea of Yosida approximations. Let $A = -\triangle$ and $A_\lambda = AJ_\lambda$. J_λ denotes the resolvent of A by $J_\lambda = (I + \lambda A)^{-1}$ $(\lambda > 0)$. Furthermore we define a nonlinear operator $f : H_0^1(\Omega) \to L^2(\Omega)$ as

$$f(u)(x) \equiv u(x)^3$$

for $u \in H_0^1(\Omega)$. Note that this mapping is well defined because of (1.8). Then the equation (1.4) can be written in $L^2(\Omega)$ as

$$\delta u'(t) + M(\|\nabla u(t)\|_2^2)Au(t) = \mu f(u(t)). \tag{5.5}$$

We apply J_λ to both sides of (5.5) to obtain

$$\delta J_\lambda u'(t) + M(\|\nabla u(t)\|_2^2)A_\lambda u(t) = \mu J_\lambda f(u(t)). \tag{5.6}$$

Since we have $\|A_\lambda u'(t)\|_2 \leq \dfrac{1}{\lambda}\|u'(t)\|_2$ (see Brezis, [3]), the $L^2(\Omega)$ inner product of (5.6) with $2A_\lambda u'(t)$ is well defined. Multiplying both sides of (5.6) by $2A_\lambda u'(t)$, we get

$$\begin{aligned} &2\delta\|\nabla J_\lambda u'(t)\|_2^2 + \frac{d}{dt}[M(\|\nabla u(t)\|_2^2)\|A_\lambda u(t)\|_2^2] \\ &\quad = 2\mu(J_\lambda f(u(t)), A_\lambda u'(t)) + 4\beta(\triangle u(t), u'(t))\|A_\lambda u(t)\|_2^2. \end{aligned} \tag{5.7}$$

We integrate (5.7) with respect to t and use the property of Yosida approximations to obtain

$$M(\|\nabla u(t)\|_2^2)\|A_\lambda u(t)\|_2^2 + 2\delta\int_0^t \|\nabla J_\lambda u'(t)\|_2^2\,ds \tag{5.8}$$
$$\le M(\|\nabla u_0\|_2^2)\|\triangle u_0\|_2^2 + 2\mu\int_0^t (A_\lambda f(u(s)), J_\lambda u'(s))\,ds$$
$$+ 4\beta\int_0^t (\triangle u(s), u'(s))\|\triangle u(s)\|_2^2\,ds, \quad t\in[0,T_m).$$

Here we shall estimate the second term in the right-hand side of (5.8). It follows from Sobolev's inequality, the regularity theory of elliptic equations and the property of Yosida approximations that

$$\begin{aligned}|(A_\lambda f(u(s)), J_\lambda u'(s))| &\le 6|(u(s)|\nabla u(s)|^2, u'(s))| + 3|(u(s)^2\triangle u(s), u'(s))|\\ &\le 6\|u(s)|\nabla u(s)|^2\|_2\|u'(s)\|_2 + 3\|u(s)^2\triangle u(s)\|_2\|u'(s)\|_2\\ &\le 6\|u(s)\|_\infty\|\nabla u(s)\|_4^2\|u'(s)\|_2 + 3\|u(s)\|_\infty^2\|\triangle u(s)\|_2\|u'(s)\|_2\\ &\le C\|u(s)\|_{H^2}^3\|u'(s)\|_2 \le C\|\triangle u(s)\|_2^3\|u'(s)\|_2.\end{aligned} \tag{5.9}$$

By the same argument as in (5.9) we have

$$\|f(u(s))\|_2 = \|u(s)\|_6^3 \le C\|\triangle u(s)\|_2^3. \tag{5.10}$$

Hence by (5.5) and (5.10) we get

$$\delta\|u'(t)\|_2 \le M(\|\nabla u(t)\|_2^2)\|\triangle u(t)\|_2 + \|f(u(t))\| \le C(\|\triangle u(t)\|_2 + \|\triangle u(t)\|_2^3). \tag{5.11}$$

Therefore, from (5.8), (5.9) and (5.11) we obtain

$$M(\|\nabla u(t)\|_2^2)\|A_\lambda u(t)\|_2^2 \le M(\|\nabla u_0\|_2^2)\|\triangle u_0\|_2^2 + C_1\int_0^t \|\triangle u(s)\|_2^3\|u'(s)\|_2\,ds$$
$$\le M(\|\nabla u_0\|_2^2)\|\triangle u_0\|_2^2 + \frac{C_1}{\delta}\int_0^t (\|\triangle u(s)\|_2^6 + \|\triangle u(s)\|_2^4)\,ds, \quad t\in[0,T_m).$$

Letting $\lambda\to 0$ we get the desired estimate (5.3).

Now we multiply $-\triangle u$ to both sides of (5.5) to obtain

$$\frac{\delta}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2 + \alpha\|\triangle u(t)\|_2^2 + 2\beta\|\nabla u(t)\|_2^2\|\triangle u(t)\|_2^2 = \mu(f(u(t)), -\triangle u(t)). \tag{5.12}$$

We integrate (5.12) over $[0,T_m)$ to obtain

$$\frac{\delta}{2}\|\nabla u(t)\|_2^2 + \alpha\int_0^t \|\triangle u(s)\|_2^2\,ds \le \frac{\delta}{2}\|\nabla u_0\|_2^2 + \mu\int_0^t (u(s)^3, -\triangle u(s))\,ds, \quad t\in[0,T_m).$$

We thus obtain (5.4). This ends the proof of Lemma 5.3. □

Now we set $W(t)$ as

$$W(t) = M(\|\nabla u(t)\|_2^2)\|\triangle u(t)\|_2^2 + \frac{\delta}{2}\|\nabla u(t)\|_2^2.$$

Then we have the following.

Lemma 5.4. *Let $u(t)$ be as in Lemma 5.2. Then we have the following estimates:*

$$W(t) \le W(0) + \int_0^t \big\{ \frac{C_1}{\delta} \|\triangle u(s)\|_2^4 + \big(\frac{C_1}{\delta} + C_2\big) \|\triangle u(s)\|_2^2 - \alpha \big\} \|\triangle u(s)\|_2^2 \, ds, \quad (5.13)$$

and

$$C_3 \Big\{ \|\triangle u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \Big\} \le W(t) \le C_4 \Big\{ \|\triangle u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \Big\} \quad (5.14)$$

for some positive constants C_i $(i = 1, 2, 3, 4)$.

Proof. Since $H^2(\Omega) \subset L^6(\Omega)$, it follows from the regularity theory of elliptic equations that

$$\mu |(u(s)^3, -\triangle u(s))| \le \mu \|u(s)\|_6^3 \|\triangle u(s)\|_2 \le C_2 \|\triangle u(t)\|_2^4. \quad (5.15)$$

Then (5.13) is an immediate consequence of Lemma 5.3 and (5.15). (5.14) is obvious. This ends the proof of Lemma 5.4. □

Lemma 5.5. *Let $u(t)$ be as in Lemma 5.2 with initial data u_0 satisfying*

$$\frac{C_1}{\delta} \|\triangle u_0\|_2^4 + \big(\frac{C_1}{\delta} + C_2\big) \|\triangle u_0\|_2^2 < \frac{\alpha}{4}, \quad \frac{C_1}{\delta} \cdot \frac{W(0)^2}{C_3^2} + \big(\frac{C_1}{\delta} + C_2\big) \cdot \frac{W(0)}{C_3} < \frac{\alpha}{2}.$$

Then we have

$$\frac{C_1}{\delta} \|\triangle u(t)\|_2^4 + \big(\frac{C_1}{\delta} + C_2\big) \|\triangle u(t)\|_2^2 < \frac{\alpha}{2}$$

for any $t \in [0, T_m)$.

The proof of Lemma 5.5 is the same as in Lemma 2.6. So we shall omit the details.

Finally we can obtain the global existence of the solution to the problem (1.4), (1.5), (1.6).

Theorem 5.6. *Suppose that α, β, δ, and μ are as in Lemma 5.2. Let $u_0 \in \mathcal{W}^* \bigcap H^2(\Omega)$. Then there exists a number $\eta_1 > 0$ (depending on α and δ) such that if u_0 satisfies $\|\triangle u_0\|_2^2 < \eta_1$, then the problem (1.4), (1.5), (1.6) has a unique solution $u(t, x)$ such that*

$$u \in C([0, +\infty); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, +\infty); L^2(\Omega)).$$

Proof. This is an immediate consequence of Theorem 4.4, Lemma 5.4, and Lemma 5.5. □

6. Proof of Theorem. In this section we derive several energy estimates of (1.1) and (1.4) needed for the proof of our main theorem and prove Theorem. In the course of calculations below, various constants are simply denoted by C and change from line to line.

First we have the following estimate for (1.4).

Lemma 6.1. *Let $u(t,x)$ be a solution of the problem* (1.4), (1.5), (1.6) *with initial data u_0 which satisfy the same assumption as in Theorem* 5.6. *Then, for any $T>0$ we have the following estimates:*

$$\|\nabla u(t)\|_2^2 \le C, \quad t \in [0,T], \tag{6.1}$$

$$\|\triangle u(t)\|_2^2 \le C, \quad t \in [0,T], \tag{6.2}$$

$$\Big\|\frac{d}{dt}u(t)\Big\|_2^2 \le C, \quad t \in [0,T]. \tag{6.3}$$

Proof. This is an immediate consequence of Lemma 5.2 and Theorem 5.6. □

We next derive some estimates for (1.1).

Lemma 6.2. *Let $u_\epsilon(t,x)$ be a solution of the problem* (1.1), (1.2), (1.3) *with the initial data $u_{0\epsilon}$ and $u_{1\epsilon}$ which satisfy the same assumption as in Theorem* 2.8. *Then, for any $T>0$ we have the following estimates:*

$$\sup_{0<\epsilon\le 1}\Big\{\|\nabla u_\epsilon\|_{C([0,T];L^2(\Omega))} + \|\triangle u_\epsilon\|_{C([0,T];L^2(\Omega))}\Big\} \le C, \tag{6.4}$$

$$\sup_{0<\epsilon\le 1}\|u'_\epsilon\|_{L^2((0,T)\times\Omega)} \le C, \tag{6.5}$$

$$\sup_{0<\epsilon\le 1}\|\nabla u'_\epsilon\|_{L^2((0,T)\times\Omega)} \le C, \tag{6.6}$$

$$\sup_{0<\epsilon\le 1}\epsilon\|\nabla u'_\epsilon\|_{C([0,T];L^2(\Omega))} \le C. \tag{6.7}$$

Proof. This is an immediate consequence of Lemmas 2.2, 2.5, 2.6, 2.7 and Theorem 2.8. □

The following estimates are essential to the proof of main theorem.

Lemma 6.3. *Let $u_{0\epsilon}$ and $u_{1\epsilon}$ be as in Theorem* 2.8. *Then we have the following estimates:*

$$\sup_{0<\epsilon\le 1}\Big\|\frac{d}{dt}u_\epsilon\Big\|_{C([0,T];L^2(\Omega))} \le C, \tag{6.8}$$

$$\sup_{0<\epsilon\le 1}\epsilon^2\Big\|\frac{d^2}{dt^2}u_\epsilon\Big\|_{C([0,T];L^2(\Omega))} \le C, \tag{6.9}$$

$$\sup_{0<\epsilon\le 1}\epsilon\Big\|\frac{d^2}{dt^2}u_\epsilon\Big\|_{L^2((0,T)\times\Omega)} \le C, \tag{6.10}$$

for any T with $T>0$.

Proof. Let $M(s) \equiv \alpha + 2\beta s$. In the sequel we suppress ϵ of u_ϵ. In view of equation (1.1) and using the imbedding $H^2(\Omega) \subset L^6(\Omega)$ we have

$$\begin{aligned}\delta\|u_t(t)\|_2 &\le \epsilon^2\|u_{tt}(t)\|_2 + M(\|\nabla u(t)\|_2^2)\|\triangle u(t)\|_2 + \mu\|u(t)^3\|_2 \\ &\le \epsilon^2\|u_{tt}(t)\|_2 + M(\|\nabla u(t)\|_2^2)\|\triangle u(t)\|_2 + C\|\triangle u(t)\|_2^3 \le \epsilon^2\|u_{tt}(t)\|_2 + C,\end{aligned}$$

from which it is sufficient to prove (6.9).

Now we differentiate equation (1.1) with respect to t to obtain

$$\epsilon^2 u_{ttt} - M(\|\nabla u\|_2^2)\triangle u_t + \delta u_{tt} = \frac{d}{dt}\Big[M(\|\nabla u\|_2^2)\Big]\triangle u + \mu(u^3)_t. \tag{6.11}$$

If necessary, we can use the method of difference quotients.

We multiply $2u_{tt}$ to both sides of (6.11) and integrate over Ω to obtain

$$\begin{aligned}
&\epsilon^2 \frac{d}{dt}\|u_{tt}(t)\|_2^2 + M(\|\nabla u(t)\|_2^2)\frac{d}{dt}\|\nabla u_t(t)\|_2^2 + 2\delta\|u_{tt}(t)\|_2^2 \\
&= 2\frac{d}{dt}\Big[M(\|\nabla u(t)\|_2^2)\Big](\triangle u(t), u_{tt}(t)) + 6\mu\int_\Omega u^2 u_t u_{tt}\,dx \\
&= 8\beta(\nabla u_t(t), \nabla u(t))(\triangle u(t), u_{tt}(t)) + 6\mu\int_\Omega u^2 u_t u_{tt}\,dx.
\end{aligned} \tag{6.12}$$

Hence we get

$$\begin{aligned}
&\epsilon^2 \frac{d}{dt}\|u_{tt}(t)\|_2^2 + \frac{d}{dt}\Big[M(\|\nabla u(t)\|_2^2)\|\nabla u_t(t)\|_2^2\Big] + 2\delta\|u_{tt}(t)\|_2^2 \\
&\le 8\beta(\nabla u_t(t), \nabla u(t))(\triangle u(t), u_{tt}(t)) \\
&+ 4\beta(\nabla u_t(t), \nabla u(t))\|\nabla u_t(t)\|_2^2 + 6\mu\int_\Omega u^2 u_t u_{tt}\,dx.
\end{aligned} \tag{6.13}$$

Here we use (6.4), the imbedding $H^2(\Omega) \subset L^\infty(\Omega)$, and the Schwartz inequality to obtain

$$\begin{aligned}
&\epsilon^2 \frac{d}{dt}\|u_{tt}(t)\|_2^2 + \frac{d}{dt}\Big[M(\|\nabla u(t)\|_2^2)\|\nabla u_t(t)\|_2^2\Big] + 2\delta\|u_{tt}(t)\|_2^2 \\
&\le C\|\nabla u_t(t)\|_2\|u_{tt}(t)\|_2 + C\|\nabla u_t(t)\|_2^3 + C\|u(t)\|_\infty^2\|u_t(t)\|_2\|u_{tt}(t)\|_2 \\
&\le \frac{C}{\epsilon^2}\|\nabla u_t(t)\|_2^2 + \frac{\epsilon^2}{2}\|u_{tt}(t)\|_2^2 + C\|\nabla u_t(t)\|_2^3 + C\|u_t(t)\|_2\|u_{tt}(t)\|_2 \\
&\le \frac{C}{\epsilon^2}\|\nabla u_t(t)\|_2^2 + \frac{\epsilon^2}{2}\|u_{tt}(t)\|_2^2 + C\|\nabla u_t(t)\|_2^3 + \frac{C}{\epsilon^2}\|u_t(t)\|_2^2 + \frac{\epsilon^2}{2}\|u_{tt}(t)\|_2^2 \\
&\le \frac{C}{\epsilon^2}\|\nabla u_t(t)\|_2^2 + \frac{C}{\epsilon^2}\|u_t(t)\|_2^2 + C\|\nabla u_t(t)\|_2^3 + \epsilon^2\|u_{tt}(t)\|_2^2.
\end{aligned} \tag{6.14}$$

From our hypothesis that $u_{0\epsilon}$ and $u_{1\epsilon}$ are sufficiently small it is easy to see that

$$\sup_{0<\epsilon\le 1} \epsilon^2\Big\|\frac{d^2}{dt^2}u_\epsilon(0)\Big\|_2 \le \sup_{0<\epsilon\le 1}\big[M(\|\nabla u_{0\epsilon}\|_2^2)\|\triangle u_{0\epsilon}\|_2 + \delta\|u_{1\epsilon}\|_2 + \mu\|\triangle u_{0\epsilon}\|_2^3\big] \le C, \tag{6.15}$$

and

$$M(\|\nabla u_{0\epsilon}\|_2^2)\|\nabla u_{1\epsilon}\|_2^2 \le C. \tag{6.16}$$

Hence, multiplying ϵ^2 to both sides of (6.14), using (6.5), (6.15), (6.16), and integrating with respect to t, we obtain

$$\begin{aligned}
&\epsilon^4\|u_{tt}(t)\|_2^2 + \epsilon^2 M(\|\nabla u(t)\|_2^2)\|\nabla u_t(t)\|_2^2 + 2\epsilon^2\delta\int_0^t \|u_{tt}(s)\|_2^2\,ds \qquad (6.17)\\
&\le C + C\|\nabla u_t\|^2_{L^2(0,T;L^2(\Omega))} + C\|u_t\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad + C\epsilon^2\int_0^t \|\nabla u_t(s)\|_2^3\,ds + \epsilon^4\int_0^t \|u_{tt}(s)\|_2^2\,ds\\
&\le C + C\|\nabla u_t(t)\|^2_{L^2(0,T;L^2(\Omega))} + C\epsilon^2\int_0^t \|\nabla u_t(s)\|_2^3\,ds\\
&\quad + \epsilon^4\int_0^t \|u_{tt}(s)\|_2^2\,ds, \quad t\in[0,T].
\end{aligned}$$

Here making use of (6.6), (6.7), we estimate the third term in the right-hand side of (6.17) by

$$\begin{aligned}
C\epsilon^2\int_0^t \|\nabla u_t(s)\|_2^3\,ds &\le C\epsilon^2\|\nabla u_t\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla u_t\|^2_{L^2(0,T;L^2(\Omega))} \qquad (6.18)\\
&\le C\epsilon^4\|\nabla u_t\|^2_{C([0,T];L^2(\Omega))} + C\|\nabla u_t\|^4_{L^2(0,T;L^2(\Omega))} \le C.
\end{aligned}$$

Thus combining (6.17) with (6.18) we have by (6.6)

$$\epsilon^4\|u_{tt}(t)\|_2^2 + 2\epsilon^2\delta\int_0^t \|u_{tt}(s)\|_2^2\,ds \le C + \epsilon^4\int_0^t \|u_{tt}(s)\|_2^2\,ds, \quad t\in[0,T]. \qquad (6.19)$$

Therefore, applying Gronwall's inequality to (6.19), we deduce that

$$\sup_{0<\epsilon\le 1}\epsilon^2\|u_{tt}\|_{C([0,T];L^2(\Omega))} \le C, \qquad \sup_{0<\epsilon\le 1}\epsilon\|u_{tt}\|_{L^2((0,T)\times\Omega)} \le C.$$

This ends the proof of Lemma 6.3. □

Proof of Theorem completed. We are now in a position to prove our main theorem. At first we subtract equation (1.4) from equation (1.1). Then we have

$$\begin{aligned}
\epsilon^2\frac{d^2u_\epsilon}{dt^2} - M(\|\nabla u_\epsilon\|_2^2)(\triangle u_\epsilon - \triangle u) + \delta\big(\frac{du_\epsilon}{dt} - \frac{du}{dt}\big)&\\
= 2\beta(\|\nabla u_\epsilon\|_2^2 - \|\nabla u\|_2^2)\triangle u + \mu(u_\epsilon^3 - u^3),& \qquad (6.20)
\end{aligned}$$

where $M(s)\equiv\alpha+2\beta s$. We take the $L^2(\Omega)$ inner product of (6.20) with $2(u_\epsilon' - u')$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[M(\|\nabla u_\epsilon\|_2^2)\|\nabla u_\epsilon - \nabla u\|_2^2\Big] + 2\delta\|u_\epsilon' - u'\|_2^2 \qquad (6.21)\\
&\le \big|\frac{d}{dt}M(\|\nabla u_\epsilon\|_2^2)\big|\|\nabla u_\epsilon - \nabla u\|_2^2 + 4\beta(\|\nabla u_\epsilon\|_2 + \|\nabla u\|_2)\|\nabla u_\epsilon\\
&\quad - \nabla u\|_2\|\triangle u\|_2\|u_\epsilon' - u'\|_2 + 2\mu\|u_\epsilon^3 - u^3\|_2\|u_\epsilon' - u'\|_2 + 2\epsilon^2\|u_\epsilon''\|_2\|u_\epsilon' - u'\|_2.
\end{aligned}$$

Here we note that

$$\begin{aligned}\Big|\frac{d}{dt}M(\|\nabla u_\epsilon\|_2^2)\Big| &= 4\beta|(\nabla u_\epsilon, \nabla u_\epsilon')| = 4\beta|(\triangle u_\epsilon, u_\epsilon')| \\ &\le 4\beta\|\triangle u_\epsilon\|_2\|u_\epsilon'\|_2 \le C, \end{aligned} \tag{6.22}$$

$$\begin{aligned}\|u_\epsilon^3 - u^3\|_2 &\le (\|u_\epsilon\|_\infty^2 + \|u_\epsilon\|_\infty\|u\|_\infty + \|u\|_\infty^2)\|u_\epsilon - u\|_2 \\ &\le C(\|\triangle u_\epsilon\|_2^2 + \|\triangle u\|_2^2)\|u_\epsilon - u\|_2 \le C\|u_\epsilon - u\|_2 \end{aligned} \tag{6.23}$$

because of (6.4) and (6.8). Then from Poincaré's inequality, (6.21), (6.22) and (6.23) it follows that

$$\begin{aligned}&\frac{d}{dt}\Big[M(\|\nabla u_\epsilon\|_2^2)\|\nabla u_\epsilon - \nabla u\|_2^2\Big] + 2\delta\|u_\epsilon' - u'\|_2^2 \\ &\le C\|\nabla u_\epsilon - \nabla u\|_2^2 + C\|\nabla u_\epsilon - \nabla u\|_2\|u_\epsilon' - u'\|_2 + \frac{2\epsilon^4}{\delta}\|u_\epsilon''\|_2^2 + \frac{\delta}{2}\|u_\epsilon' - u'\|_2^2 \\ &\le C\|\nabla u_\epsilon - \nabla u\|_2^2 + \delta\|u_\epsilon' - u'\|_2^2 + \frac{2\epsilon^4}{\delta}\|u_\epsilon''\|_2^2. \end{aligned} \tag{6.24}$$

Hence using (6.10) and $M(\|\nabla u_\epsilon\|_2^2) \ge \alpha$, we integrate over $[0,T]$ to get

$$\begin{aligned}&\alpha\|\nabla u_\epsilon - \nabla u\|_2^2 + \delta\int_0^t \|u_\epsilon'(s) - u'(s)\|_2^2\,ds \\ &\le C\epsilon^2 + M(\|\nabla u_{0\epsilon}\|_2^2)\|\nabla u_{0\epsilon} - \nabla u_0\|_2^2 + C\int_0^t \|\nabla u_\epsilon(s) - \nabla u(s)\|_2^2\,ds, \quad t \in [0,T]. \end{aligned}$$

Therefore we deduce from Gronwall's inequality that

$$\|\nabla u_\epsilon(t) - \nabla u(t)\|_2^2 \le C\epsilon^2 + C\|\nabla u_{0\epsilon} - \nabla u_0\|_2^2, \tag{6.25}$$

$$\int_0^t \|u_\epsilon'(s) - u'(s)\|_2^2\,ds \le C\epsilon^2 + C\|\nabla u_{0\epsilon} - \nabla u_0\|_2^2 \tag{6.26}$$

on $[0,T]$. We thus conclude from (6.25) and (6.26) that

$$u_\epsilon \to u \quad \text{strongly in } C([0,T];H_0^1(\Omega)), \quad u_\epsilon' - u' \quad \text{strongly in } L^2((0,T)\times\Omega)$$

as $\epsilon \to 0$. This completes the proof of Theorem. $\square$

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