Advances in Differential Equations Volume 1, Number 3, May 1996, pp. 453 – 498

MINIMIZATION PROBLEMS FOR NONCOERCIVE FUNCTIONALS SUBJECT TO CONSTRAINTS II*

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Abstract. The paper establishes several minimization theorems for noncoercive functionals defined on a Hilbert (or reflexive Banach) space which are subject to constraints. Applications to critical point theory and variational inequalities are given. The results are also applied to obtain the existence of solutions of several nonlinear boundary and unilateral problems.

1. Introduction—Abstract results. This paper constitutes a continuation of our earlier work, [13], where we have studied a class of noncoercive functionals defined on a reflexive Banach space which are subject to constraint manifolds. This class of functionals was characterized by a property (Property (P)) which together with a compatibility condition on the constraint manifold allowed us to use regularization procedures to prove the existence of minimizers. Several consequences, such as critical-point theorems and existence results for solutions of boundary-value problems of semilinear and nonlinear partial differential equations were also given.

In this paper we specialize mostly to the Hilbert space case and provide more general properties which allow for extensions and considerable refinements of our earlier results. The results obtained here will, in turn, be used to establish several existence results for variational inequalities and provide a unified manner of treating classes of boundary value problems which may be called problems of Landesman-Lazer type.

In order to save writing we shall adopt the notation and conventions of our earlier work, [13]. Thus that paper is requisite for the reading of the present one.

The paper is organized as follows. We first present some notation and present a discussion of the basic defining property (Property (P)) of our class of functionals and discuss its relationship with similar properties used earlier ([13]). We then discuss some basic abstract results for the existence of minima of functionals subject to constraints. These results are again in the spirit of [13] but allow for greater generality. While the results are presented in a Hilbert space setting, it should be noted that results of that type will be valid in reflexive Banach spaces as well, provided one uses a setting similar to the present one motivated by the discussion in [13]. We shall not strive for this greater generality here. We next employ the abstract results to discuss a class of boundary-value problems for semilinear elliptic equations (Landesman-Lazer-type problems), which have been discussed extensively in the literature since the publication

Received for publication May 1995.
*Research supported by a grant from NSF.
AMS Subject Classifications: 35J15, 35J85, 49J40, 58E05.

of [12]. In fact, our method of treating these problems has its roots in a paper by Hess ([8]) which gave a different proof, using variational methods, of the results in [12]. Our results extend and unify much of what is known about such problems. The next sections formulate and establish similar results from a more general problem area, namely, variational inequalities, and we obtain results which are extensions of the work in [15], [7] and [4]. We conclude the paper with further applications to critical-point theory for variational inequalities and present some applications which are in the spirit of recent work in [2].

1.1. Some notation and assumptions—Property (P).

• Let V be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ (we shall also use $\langle\cdot,\cdot\rangle$ for the pairing between V and V^*). We assume that A and φ are as in Section 2.1, [13]; i.e., $A : V \to V^*$, with $\varphi(u) = \langle A(u), u\rangle$ weakly lower semicontinuous (in this case, $\langle\cdot,\cdot\rangle$ is the pairing between V and V^*; in most later instances, it will be clear from the context what $\langle\cdot,\cdot\rangle$ denotes). Further, $j : V \to \mathbb{R}\cup\{\infty\}$ is a weakly lower semicontinuous functional such that $j(0) = 0$. Also ψ, S, Y are as in Section 2.1 of [13]; i.e., Y is a Banach space, $\psi : V \to Y$ a completely continuous mapping, and $S = \{u \in V : \psi(u) = \gamma\} \neq \emptyset$ (for some $\gamma \in Y$). We denote by $F : V \to \mathbb{R}\cup\{\infty\}$, the mapping $F = \varphi + j$ and suppose that there exist $c \in \mathbb{R}$, $\beta > 0$ such that

$$F(u) \geq c\|u\|^\beta \tag{1.1}$$

for all u with $\|u\|$ sufficiently large.

• As earlier in [13], we consider the minimization problem

$$u \in S : F(u) = \min_{v\in S} F(v). \tag{1.2}$$

• In the work to follow we shall employ the following extension of property (P) of [13].

Definition 1.1. We say that F has property (P) on S if the following condition is satisfied:

If $\{v_n\} \subset S$ is any sequence in S satisfying

$$\|v_n\| \to \infty, \tag{1.3}$$

$$w_n = v_n/\|v_n\| \rightharpoonup 0, \tag{1.4}$$

$$\forall\lambda > 0, \quad \limsup \frac{F(v_n)}{\|v_n\|^\lambda} \leq 0, \tag{1.5}$$

then there exists $v_0 \in S$ such that

$$\limsup F(v_n) > F(v_0).$$

Remark 1.1. The only difference between the definition of property (P) used here and that in [13] is that in [13] condition (1.5) is for φ instead of F, and $\lambda = 2$ (cf. condition (2.5), [13]).

Note that if j is convex then

$$\liminf \frac{j(v_n)}{\|v_n\|^2} \geq 0,$$

which implies that

$$\limsup \frac{\varphi(v_n)}{\|v_n\|^2} \leq \limsup \frac{\varphi(v_n)}{\|v_n\|^2} + \liminf \frac{j(v_n)}{\|v_n\|^2} \leq \limsup \frac{F(v_n)}{\|v_n\|^2}.$$

Hence, by choosing $\lambda = 2$ in (1.5), we see that (1.5) implies the third condition in (2.4), [13]. Thus the definition of property (P) presented above contains that in [13]. In particular, the special cases of property (P) in [13] (Propositions 2.1, 2.2, 2.3) also hold for the new definition (with j convex).

1.2. Existence results.

Theorem 1.1. *Let F satisfy property (P) on S. Suppose the following compatibility condition is satisfied*:

If $w \in V$ is such that there exists a sequence $\{u_n\} \subset V$ satisfying:

$$\|u_n\| \to \infty, \;\; w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w, \tag{1.6}$$

$$\limsup \frac{F(u_n)}{\|u_n\|^\lambda} \leq 0, \; \forall \lambda > 0, \tag{1.7}$$

$$\lim \frac{\psi(u_n)}{\|u_n\|^\lambda} = 0, \; \forall \lambda > 0, \tag{1.8}$$

then we have

$$\begin{cases} u - w \in S, \; \forall u \in S \\ F(u-w) \leq F(u), \; \forall u \in S. \end{cases} \tag{1.9}$$

Under these assumptions (1.2) *has a solution.*

Proof. Choose $\alpha > \beta$. For $\epsilon > 0$, we consider the regularized functionals

$$F_\epsilon(u) = F(u) + \epsilon \|u\|^\alpha, \; u \in V, \tag{1.10}$$

and the regularized minimization problems

$$u_\epsilon \in S : F_\epsilon(u_\epsilon) = \min_{v \in S} F_\epsilon(v). \tag{1.11}$$

Since F and $\|\cdot\|^\alpha$ are weakly lower semicontinuous, F_ϵ is also weakly lower semicontinuous on V. Moreover for each fixed $\epsilon > 0$, F_ϵ is coercive on V (and hence on S). In fact, for $\|u\|$ sufficiently large, one has

$$F_\epsilon(u) = F(u) + \epsilon\|u\|^\alpha \geq \epsilon\|u\|^\alpha + c\|u\|^\beta.$$

Since $\alpha > \beta, \lim_{\|u\|\to\infty} F_\epsilon(u) = \infty$. Since S is weakly closed, these properties of F_ϵ imply that (1.11) has a solution $u_\epsilon \in S$; i.e.,

$$F(u_\epsilon) + \epsilon\|u_\epsilon\|^\alpha \le F(v) + \epsilon\|v\|^\alpha, \ \forall v \in S. \tag{1.12}$$

As in [13], we will show that $\{u_\epsilon\}$ is bounded, which will permit us to choose a subsequence $\{u_{\epsilon_n}\} \subset \{u_\epsilon\}$ such that $u_{\epsilon_n} \rightharpoonup u_0$ in V. Since S is weakly closed, $u_0 \in S$. Moreover, once $\{u_{\epsilon_n}\}$ is shown to be bounded,

$$\epsilon_n\|u_{\epsilon_n}\|^\alpha \to 0, \ \epsilon_n\|v\|^\alpha \to 0 \ (n \to \infty), \quad \text{for all } v \in S.$$

Hence, for $v \in S$, by the weak lower semicontinuity of F,

$$\begin{aligned} F(u_0) &\le \liminf F(u_{\epsilon_n}) \le \liminf[F(u_{\epsilon_n}) + \epsilon_n\|u_{\epsilon_n}\|^\alpha] \\ &\le \liminf[F(v) + \epsilon_n\|v\|^\alpha] = F(v). \end{aligned}$$

Thus $u_0 \in S$ is a solution of (1.2).

We now prove that $\{u_\epsilon\}$ is bounded. Suppose for the sake of contradiction that there exists a subsequence

$$\{u_n = u_{\epsilon_n}\} \subset \{u_\epsilon\}, \ \ \epsilon_n \to 0 \text{ as } n \to \infty$$

such that $\|u_n\| \to \infty$ $(n \to \infty)$. For $\lambda > 0$, we have for a fixed $v \in V$, $F(v) < \infty$,

$$\begin{aligned} \limsup\Big[\frac{F(u_n)}{\|u_n\|^\lambda}\Big] &\le \limsup\Big[\frac{F_{\epsilon_n}(u_n)}{\|u_n\|^\lambda}\Big] \le \limsup\Big[\frac{F_{\epsilon_n}(v)}{\|u_n\|^\lambda}\Big] \\ &= \limsup\Big[\frac{F(v) + \epsilon_n\|v\|^\alpha}{\|u_n\|^\lambda}\Big] = 0. \end{aligned} \tag{1.13}$$

Now, passing to a subsequence, if necessary, we can assume without loss of generality that

$$w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w \quad (n \to \infty).$$

We see that $\{u_n\}, \{w_n\}, w$ satisfy (1.6), (1.7). Now since $u_n \in S$, we obtain that $\psi(u_n) = \gamma$, $\forall n$. Hence

$$\lim \frac{\psi(u_n)}{\|u_n\|^\lambda} = 0.$$

Thus (1.8) is true. By hypothesis, we must have (1.9).

Letting $u = u_n - w \in S$ in (1.12) (with $\epsilon = \epsilon_n$), one has

$$F(u_n) + \epsilon_n\|u_n\|^\alpha \le F(u_n - w) + \epsilon_n\|u_n - w\|^\alpha \le F(u_n) + \epsilon_n\|u_n - w\|^\alpha$$

(by (1.9)). Hence $\|u_n\|^\alpha \le \|u_n - w\|^\alpha$, and then $\|u_n\| \le \|u_n - w\|$, $\forall n$. This inequality implies that

$$\langle u_n, w\rangle \le \tfrac{1}{2}\|w\|^2, \quad \text{and} \quad \langle w_n, w\rangle \le \tfrac{1}{2}\frac{\|w\|^2}{\|u_n\|}.$$

Thus $\|w\|^2 = \lim_{n\to\infty}\langle w_n, w\rangle = 0$; i.e., $w = 0$.

We have checked (1.3), (1.4) in the definition of property (P) (with $\{v_n\}$ replaced by $\{u_n\}$). Condition (1.5) follows from (1.13). Hence, since F satisfies property (P), we have $u_0 \in S$ such that

$$\limsup F(u_n) > F(u_0).$$

However, (1.12) (with $\epsilon = \epsilon_n$) implies that

$$\limsup F(u_n) \le \limsup[F(u_n) + \epsilon_n\|u_n\|^\alpha] \le \limsup[F(u_0) + \epsilon_n\|u_0\|^\alpha] = F(u_0).$$

This contradiction proves that $\{u_\epsilon\}$ is bounded, which completes our proof. □

Remark 1.2. (i) In applications, we usually replace (1.7) by the weaker conditions:

$$\liminf \frac{\langle Au_n, u_n\rangle}{\|u_n\|^\lambda} + \limsup \frac{j(u_n)}{\|u_n\|^\lambda} \le 0, \tag{1.14}$$

or

$$\limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|^\lambda} + \liminf \frac{j(u_n)}{\|u_n\|^\lambda} \le 0. \tag{1.15}$$

(ii) Suppose now that j is convex. Letting j_∞ be the asymptotic derivative

$$j_\infty(u) = \lim_{t\to\infty} \frac{1}{t} j(tu),$$

then with $\{u_n\}$, $\{w_n\}, w$ as in Theorem 1.1, we already know ([13]) that

$$\liminf \frac{j(u_n)}{\|u_n\|} \ge j_\infty(w),$$

and

$$\liminf \frac{j(u_n)}{\|u_n\|^2} \ge 0.$$

Hence from (1.15) with $\lambda = 1$ and $\lambda = 2$, we get

$$\limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|} + j_\infty(w) \le 0, \tag{1.16}$$

and

$$\limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} \le 0. \tag{1.17}$$

Conditions (1.16) and (1.17) appear in (2.13), Theorem 2.4, [13], and in (2.4), Definition of property (P), [13]. The above arguments show that they are more restrictive than condition (1.5). Hence Theorem 2.5 in [13] is a corollary of Theorem 1.1 in the case j is convex.

We now derive a consequence of Theorem 1.1 in the case that $j(u)$ and $\langle Au, u\rangle$ do not tend to $-\infty$ very fast as $\|u\| \to \infty$, in the sense that

$$\liminf_{\|u\|\to\infty} \frac{j(u)}{\|u\|^2} \ge 0 \tag{1.18}$$

and

$$\limsup_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} \ge 0. \tag{1.19}$$

Under these assumptions, we have:

Corollary 1.1. *Assume that* F *satisfies property* (P) *as stated in Section* 2.2, [13], *and that* (1.18) *and* (1.19) *hold. Suppose furthermore that the following compatibility condition is satisfied*:

If $w, \{u_n\}, \{w_n\}$ *satisfy* (1.6), (1.8), *and*

$$\liminf \frac{j(u_n)}{\|u_n\|} \leq 0, \tag{1.20}$$

$$\limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} = 0, \tag{1.21}$$

then (1.9) *holds. Then* (1.2) *has a solution.*

Proof. We first prove that if F has property (P) as stated in Section 2.2, [13], and if (1.18), (1.19) are satisfied, then F has property (P) defined above. In fact, assume that $\{v_n\} \subset S$ satisfies (1.3), (1.4), (1.5).

Letting $\lambda = 2$ in (1.5), we get

$$\begin{aligned} 0 &\geq \limsup \Big[\frac{\langle Av_n, v_n\rangle}{\|v_n\|^2} + \frac{j(v_n)}{\|v_n\|^2}\Big] \geq \limsup \frac{\langle Av_n, v_n\rangle}{\|v_n\|^2} + \liminf \frac{j(v_n)}{\|v_n\|^2} \\ &\geq \limsup \frac{\langle Av_n, v_n\rangle}{\|v_n\|^2} \quad \text{(by (1.18))}. \end{aligned} \tag{1.22}$$

Thus (2.4) of [13] holds. Since F has property (P) in the sense of [13], (2.5) of [13] also holds, which implies that F has property (P) in the sense of Section 1.1.

We now can apply Theorem 1.1, by proving that the compatibility condition [(1.6)–(1.8) $\Rightarrow$ (1.9)] is satisfied. Assume $w, \{u_n\}, \{w_n\}$ satisfy (1.6), (1.7), (1.8). From (1.7) with $\lambda = 1$, we obtain

$$\begin{aligned} 0 &\geq \limsup \Big[\frac{\langle Au_n, u_n\rangle}{\|u_n\|} + \frac{j(u_n)}{\|u_n\|}\Big] \geq \limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|} + \liminf \frac{j(u_n)}{\|u_n\|} \\ &\geq \liminf \frac{j(u_n)}{\|u_n\|} \quad \text{(by (1.19))}. \end{aligned}$$

Hence we have (1.20). On the other hand, by letting $\lambda = 2$ in (1.7), we obtain (1.22). Moreover, we have from (1.19) that

$$\limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} \geq 0.$$

This, together with (1.22), gives us (1.21). Therefore, (1.6), (1.8), (1.20), and (1.21) are satisfied. By hypothesis, we must have (1.9). We have hence verified the compatibility conditions in Theorem 1.1. By this theorem, (1.2) has a solution. □

1.3. Existence results of Landesman-Lazer-type for semilinear elliptic equations. In this section, we apply Theorem 1.1 and Corollary 1.1 above to prove existence results for noncoercive semilinear elliptic equations, which generalize the classical Landesman-Lazer theorem at the first eigenvalue.

Let $N, m \geq 1$, and let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let V be a closed subspace of $H^m(\Omega)$ such that $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. We consider a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ with some growth conditions to be specified later. Denote by G the primitive of g with respect to the second variable

$$G(x, u) = \int_0^u g(x, \xi)\, d\xi, \quad u \in \mathbb{R}.$$

Then G is also a Carathéodory function.

We consider on V the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ induced by the usual norm and inner product of $H^m(\Omega)$.

Let A be the following elliptic operator on V: $A : V \to V^*$,

$$\langle Au, v \rangle = \int_\Omega \sum_{|i|,|j| \leq m} a_{ij} D^i u D^j v = a(u, v), \quad \forall u, v \in V, \tag{1.23}$$

where $a_{ij} \in L^\infty(\Omega), a_{ij} = a_{ji}$, $\forall i, j$, and a_{ij} satisfy the usual ellipticity condition

$$\sum_{|i|=|j|=m} a_{ij}(x) \xi_1^{i_1 + j_1} \cdots \xi_N^{i_N + j_N} \geq C |\xi|^{2m}, \tag{1.24}$$

for almost every $x \in \Omega$, all $\xi \in \mathbb{R}^N$. (Here $i = (i_1, \ldots, i_N)$, $j = (j_1, \ldots, j_N)$). We assume that A is nonnegative on V; i.e.,

$$\langle Au, u \rangle = a(u, u) \geq 0, \quad \forall u \in V,$$

and that $\ker A = \{u \in V : \langle Au, u \rangle = 0\}$ is a finite-dimensional subspace of V.

We consider the following minimization problem:

$$u_0 \in V : \langle Au_0, u_0 \rangle + j(u_0) = \min_{u \in V} [\langle Au, u \rangle + j(u)], \tag{1.25}$$

where

$$j(u) = \int_\Omega G(x, u(x))\, dx, \; u \in V. \tag{1.26}$$

This minimization problem corresponds to the semilinear elliptic equation (of order m) on Ω,

$$\sum_{|i|,|j| \leq m} (-1)^{|i|} D^i (a_{ij} D^j u) + g(\cdot, u) = 0,$$

with various boundary conditions, depending on the choice of the subspace V.

We now consider some assumptions on g, G, and A.

• First, we assume that G satisfies the following growth condition:

$$|G(x, u)| \leq a(x) + b(x)|u|^s, \forall u \in \mathbb{R}, \quad \text{a.e. } \; x \in \Omega, \tag{1.27}$$

where $a \in L^1(\Omega), b \in L^r(\Omega), a, b \geq 0$, and $1 \leq s < 2$, and there exists $p \geq 1$ such that

$$\begin{cases} p < \frac{2N}{N-2m} & \text{if} \quad N - 2m > 0 \\ p < \infty & \text{if} \quad N - 2m \leq 0, \end{cases} \tag{1.28}$$

and that

$$1 \leq s \leq p \quad \text{and} \quad r \geq \big(\frac{p}{s}\big)' = \frac{p}{p-s}. \tag{1.29}$$

From (1.28), we know that the embedding

$$H^m(\Omega) \hookrightarrow L^p(\Omega) \tag{1.30}$$

is compact. Hence for $u \in V$, we have $|u| \in L^p(\Omega)$ and then $|u|^s \in L^{p/s}(\Omega)$. By Hölder's inequality, $b|u|^s \in L^1(\Omega)$. Hence (1.27) implies that $G(\cdot, u) \in L^1(\Omega)$ and then the integral in (1.26) is finite. j is therefore well defined.

(1.27) also implies (via the Vainberg-Krasnosels'kii theorem) that the mapping

$$u \mapsto G(\cdot, u) \tag{1.31}$$

is continuous from $L^p(\Omega)$ to $L^1(\Omega)$. Now, by the compactness of the imbedding (1.30) we see that (1.31) is a completely continuous mapping from V to $L^2(\Omega)$. Hence j defined by (1.26) is completely continuous from V to $\mathbb{R}$.

We also assume that there exist $t_0 > 0$, $a_1 \in L^1(\Omega)$, $b_1 \in L^r(\Omega)$, $a_1, b_1 \geq 0$ almost everywhere on Ω, such that

$$\frac{1}{t} G(x, tu) \geq -a_1(x) - b_1(x)|u|^s, \tag{1.32}$$

for all $t \geq t_0$, all $u \in \mathbb{R}$, and almost every $x \in \Omega$. We denote by

$$G^+(x) = \liminf_{u \to \infty} \frac{G(x,u)}{u}, \quad G^-(x) = \limsup_{u \to -\infty} \frac{G(x,u)}{u}, \ x \in \Omega. \tag{1.33}$$

Then G^+, G^- are measurable functions on Ω.

For a function w defined almost everywhere on Ω, we denote by

$$\Omega_w^\pm = \{x \in \Omega : w(x) > 0 \text{ (respectively} < 0)\}, \tag{1.34}$$

• We also need the following assumption for the principal part of the operator A: There exist $C, D > 0$ such that

$$\int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i u D^j u \geq C|u|_m^2 - D\|u\|_{H^{m-1}(\Omega)}^2, \tag{1.35}$$

for all $u \in V$, where

$$|u|_m = \big(\int_\Omega \sum_{|i|=m} |D^i u|^2\big)^{1/2}.$$

Remark 1.3. (i) We note that both (1.27) and (1.32) are satisfied if the following condition holds:

$$|G(x,u)| \leq a(x) + b(x)|u|, \quad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{1.36}$$

with $a \in L^1(\Omega)$, $b \in L^r(\Omega)$, $a, b \geq 0$, $r \geq p' = \frac{p}{p-1}$. In fact, we chose in this case $s = 1$. Then (1.29), (1.27) are satisfied. Moreover, for $t \geq 1, u \in \mathbb{R}$, $x \in \Omega$,

$$\frac{1}{t}|G(x,tu)| \leq \frac{a(x)}{t} + \frac{b(x)|tu|}{t} \leq a(x) + b(x)|u|.$$

Hence (1.32) is also satisfied with $a = a_1, b = b_1$.

(ii) Some examples where (1.35) is satisfied are the following:

(a) $m = 1$; i.e., A is a second-order elliptic operator. In this case, we have from the ellipticity condition (1.24) that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq C|\xi|^2, \quad \forall \xi \in \mathbb{R}^N.$$

Therefore, for $u \in H^1(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij} D^i u D^j u \geq C \int_\Omega \sum_{i=1}^{N} |D^i u|^2 = C \int_\Omega |\nabla u|^2.$$

Hence, we have (1.35) with $D = 0$.

(b) $V = H_0^m(\Omega)$. In fact, since the elliptic form B given by

$$\langle Bu, v\rangle = \int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i u D^j v,$$

is strongly, uniformly elliptic on Ω (by (1.24)), Gårding's inequality ([1]) gives us

$$\int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i u D^j u \geq C\|u\|^2_{H^m(\Omega)} - D\|u\|^2_{L^2(\Omega)}$$
$$\geq C|u|_m^2 - D\|u\|^2_{H^{m-1}(\Omega)}, \quad \forall u \in H_0^m(\Omega).$$

(c) $m = 2$, and A is the operator in plate theory:

$$\langle Au, v\rangle = a(u,v) = \int_\Omega [\,\partial_{11}u\,\partial_{11}v + \partial_{22}u\,\partial_{22}v +$$
$$+ \nu(\partial_{11}u\,\partial_{22}v + \partial_{22}u\,\partial_{11}v) + 2(1-\nu)\partial_{12}u\,\partial_{12}v],$$

for all $u, v \in H^2(\Omega)$ ($0 < \nu < 1/2$ is the Poisson ratio). For $u \in H^2(\Omega)$, we have

$$a(u,u) = \int_\Omega \sum_{|i|=|j|=2} a_{ij} D^i u D^j u$$
$$\geq (1-\nu) \int_\Omega [(\partial_{11}u)^2 + (\partial_{22}u)^2 + (\partial_{12}u)^2] \geq \tfrac{1}{2}(1-\nu)|u|_2^2.$$

We have (1.35) with $C = \frac{1}{2}(1-\nu)$, $D = 0$.

With these settings, we have the following result.

Theorem 1.2. *Suppose the following condition is satisfied: If $w \in \ker A$ and*

$$\int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w \le 0, \tag{1.37}$$

then

$$\int_\Omega G(x, u(x) - w(x)) dx \le \int_\Omega G(x, u(x))\, dx, \quad \forall u \in V. \tag{1.38}$$

Then (1.25) *has a solution.*

Proof. We have $\langle Au, u\rangle \ge 0$, $\forall u \in V$, and by (1.27),

$$\begin{aligned} j(u) &\ge -\int_\Omega a(x)\, dx - \int_\Omega b(x)|u(x)|^s\, dx \ge -\|a\|_{L^1(\Omega)} - \|b\|_{L^{(p/s)'}(\Omega)} \||u|^s\|_{L^{p/s}(\Omega)} \\ &= -\|a\|_{L^1(\Omega)} - \|b\|_{L^{(p/s)'}(\Omega)} \|u\|^s_{L^p(\Omega)} \ge -\|a\|_{L^1(\Omega)} - C_0 \|b\|_{L^r(\Omega)} \|u\|^s \end{aligned} \tag{1.39}$$

(since $L^r(\Omega) \subset L^{(p/s)'}(\Omega)$, we have by Hölder's inequality that

$$\|b\|_{L(p/s)'(\Omega)} \le C_1 \|b\|_{L^r(\Omega)},$$

and by the embedding (1.30),

$$\|u\|_{L^p(\Omega)} \le C_2 \|u\|, \ \forall u \in H^m(\Omega).)$$

Hence $F(u) = \langle Au, u\rangle + j(u)$ $(u \in V)$ is bounded from below in the sense of (1.1).

Now, we check that the assumptions of Corollary 1.1 are satisfied. First, we show that F has property (P) in the sense of Section 2.2, [13]. Suppose $\{v_n\}$ is a sequence in $V \subset H^m(\Omega)$ such that $\|v_n\| \to \infty$,

$$w_n = v_n / \|v_n\| \rightharpoonup 0 \ \ (n \to \infty),$$

and

$$\limsup \frac{1}{\|v_n\|^2} \langle Av_n, v_n\rangle \le 0.$$

By the linearity of A,

$$\limsup \langle Aw_n, w_n\rangle = \limsup \frac{\langle Av_n, v_n\rangle}{\|v_n\|^2} \le 0.$$

Since A is nonnegative, we have

$$\lim \langle Aw_n, w_n\rangle = 0. \tag{1.40}$$

On the other hand,

$$\langle Aw_n, w_n\rangle = \int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i w_n D^j w_n + \int_\Omega \sum_{|i|<m \text{ or } |j|<m} a_{ij} D^i w_n D^j w_n. \tag{1.41}$$

Since $w_n \rightharpoonup 0$ in V, we have $w_n \rightharpoonup 0$ in $H^m(\Omega)$, and hence $w_n \to 0$ in $H^{m-1}(\Omega)$. Then for $|i| < m$, we have $D^i w_n \to 0$ in $L^2(\Omega)$, and then

$$a_{ij} D^i w_n \to 0 \text{ in } L^2(\Omega). \tag{1.42}$$

Moreover, since $D^j : H^m(\Omega) \to L^2(\Omega), u \mapsto D^j u$ is linear, bounded hence weakly continuous, for all $|j| \le m$, we have $D^j w_n \rightharpoonup 0$ in $L^2(\Omega)$. This and (1.42) show that

$$\int_\Omega a_{ij} D^i w_n D^j w_n \to 0 \quad (n \to \infty),$$

whenever $|i| < m$ or $|j| < m$. Together with (1.40) and (1.41), this implies that

$$\lim \int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i w_n D^j w_n = 0. \tag{1.43}$$

Now, from (1.35), we have

$$\int_\Omega \sum_{|i|=|j|=m} a_{ij} D^i w_n D^j w_n \ge C|w_n|_m^2 - D\|w_n\|^2_{H^{m-1}(\Omega)}. \tag{1.44}$$

Since $\|w_n\|_{H^{m-1}(\Omega)} \to 0$, we have from (1.43) and (1.44) that $\lim |w_n|_m^2 = 0$. Therefore

$$\|w_n\|^2 = \|w_n\|^2_{H^{m-1}(\Omega)} + |w_n|_m^2 \to 0 \quad \text{as } n \to \infty.$$

This contradicts the fact that $\|w_n\| = 1, \forall n$, proving that F has property (P).

We now notice that (1.19) is obviously satisfied since $\langle Au, u\rangle \ge 0, \forall u$. To verify (1.18), we observe that from (1.39), we have

$$\frac{j(u)}{\|u\|^2} \ge -\frac{1}{\|u\|^2}\|a\|_{L^1(\Omega)} - C_0\|b\|_{L^r(\Omega)}\frac{1}{\|u\|^{2-s}}.$$

Since $s < 2$, the right-hand side of this inequality tends to 0 as $\|u\| \to \infty$, proving that

$$\liminf_{\|u\|\to\infty} \frac{j(u)}{\|u\|^2} \ge 0.$$

We have (1.18). Now, we check the compatibility condition in Corollary 1.1. Let $w, \{u_n\}, \{w_n\}$ satisfy (1.6). (1.8), (1.20) and (1.21). Since the mapping $u \mapsto a(u,u)$, $u \in V$, is weakly lower semicontinuous, we have from (1.21) that

$$0 \le a(w,w) \le \liminf a(w_n, w_n) = \liminf \frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} \le \limsup \frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} = 0.$$

Hence $a(w,w) = 0$; i.e., $w \in \ker A$. Now, since $w_n \to w$ in $L^p(\Omega)$, by passing to a subsequence, if necessary, we can assume that there exists $\overline{w} \in L^p(\Omega)$ such that

$w_n \to w$ almost everywhere in Ω, and $|w_n| \le \overline{w}$ almost everywhere in $\Omega, \forall n$. Since $G(x,0) = 0$, $x \in \Omega$, we have

$$\begin{aligned}\frac{j(u_n)}{\|u_n\|} &= \frac{1}{\|u_n\|}\big[\int_{\Omega_w^+} G(x,u_n(x)) + \int_{\Omega_w^-} G(x,u_n(x))\big]\\ &= \int_{\Omega_w^+} \frac{G(x,u_n(x))}{\|u_n\|} + \int_{\Omega_w^-} \frac{G(x,u_n(x))}{\|u_n\|}\\ &= \int_{\Omega_w^+} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} + \int_{\Omega_w^-} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|}.\end{aligned}$$

Since $\|u_n\| \to \infty$, we have, by (1.32),

$$\frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \ge a_1(x) - b_1(x)|w_n(x)|^2 \ge -a_1(x) - b_1(x)[\overline{w}(x)]^2, \quad \text{a.e.} \quad x \in \Omega, \tag{1.45}$$

for all n sufficiently large. Moreover

$$a_1 + b_1\overline{w}^s \in L^1(\Omega). \tag{1.46}$$

Now, for $x \in \Omega_w^+$, we have $w_n(x) > 0$, $\forall n$ sufficiently large, and hence $\|u_n\|w_n(x) \to \infty$ $(n \to \infty)$. Therefore

$$\begin{aligned}\liminf \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} &= \liminf \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|w_n(x)} \cdot w_n(x) \\ &\ge \liminf_{u\to\infty} \frac{G(x,u)}{u} \cdot \lim w_n(x) = G^+(x)w(x).\end{aligned} \tag{1.47}$$

By (1.45), (1.46), we can apply Fatou's lemma and get

$$\liminf \int_{\Omega_w^+} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \ge \int_{\Omega_w^+} \liminf \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \ge \int_{\Omega_w^+} G^+(x)w(x).$$

Similarly, for $x \in \Omega_w^-$, we have $w_n(x) < 0$ for all n large, and then

$$\liminf \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \ge \limsup_{u\to-\infty} \frac{G(x,u)}{u} \cdot \lim w_n(x) = G^-(x)w(x).$$

Again by Fatou's lemma,

$$\liminf \int_{\Omega_w^-} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \ge \int_{\Omega_w^-} G^-(x)w(x).$$

It follows from (1.20) that

$$\begin{aligned}0 \ge \liminf \frac{j(u_n)}{\|u_n\|} &= \liminf \big[\big(\int_{\Omega_w^+} + \int_{\Omega_w^-}\big)\frac{G(x,\|u_n\|w_n(x))}{\|u_n\|}\big] \\ &\ge \liminf \int_{\Omega_w^+} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} + \liminf \int_{\Omega_w^-} \frac{G(x,\|u_n\|w_n(x))}{\|u_n\|} \\ &\ge \int_{\Omega_w^+} G^+w + \int_{\Omega_w^-} G^-w.\end{aligned} \tag{1.48}$$

Hence, we have (1.37). By hypothesis, we must have (1.38); i.e.,

$$j(u-w) \le j(u), \quad \forall\, u \in V.$$

Since

$$\langle Au, w\rangle = \langle Aw, u\rangle = 0, \ \forall u \in V,$$

we have

$$F(u-w) = \langle A(u-w), u-w\rangle + j(u-w) \le \langle Au, u\rangle + j(u) \ = \ F(u)$$

(since $\langle A(u-w), u-w\rangle = \langle Au, u\rangle$, $j(u-w) \le j(u)$). Hence, we have (1.9), and the proof is complete by Corollary 1.1. □

An immediate consequence of Theorem 1.2 is the following:

Corollary 1.2. *Assume G satisfies* (1.27)–(1.29), (1.32) *and furthermore*

$$\int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w > 0, \quad \forall w \in \ker A\backslash\{0\}. \tag{1.49}$$

Then (1.25) *has a solution.*

Proof. Suppose (1.49) is satisfied. Then $w \in \ker A$ satisfies (1.37) only in the case $w = 0$, which obviously implies (1.38). □

Now, we consider the particular case where $V = H_0^m(\Omega)$, and

$$g(x,\xi) = g_0(x,\xi) - h(x), \ x \in \Omega, \ \xi \in \mathbb{R}.$$

Here $h \in L^2(\Omega)$, and

$$|g_0(x,\xi)| \le M(x) \ \text{ for a.e. } \ x \in \Omega, \text{ all } \xi \in \mathbb{R}, \tag{1.50}$$

with $M \in L^q(\Omega), q > \frac{2N}{N+2m}$ (and $q > 1$). Assume furthermore that the limits

$$g_0(x,\pm\infty) = \lim_{\xi\to\pm\infty} g_0(x,\xi) \tag{1.51}$$

exist for almost every $x \in \Omega$. From (1.50) we have

$$|g_0(x,\pm\infty)| \le M(x), \ x \in \Omega,$$

and hence $g(\cdot,\pm\infty) \in L^q(\Omega)$. Under these settings we have the following corollary.

Corollary 1.3. *Assume* (1.49), (1.51) *are satisfied, and that*

$$\int_{\Omega_w^+} g(\cdot,\infty)w + \int_{\Omega_w^-} g(\cdot,-\infty)w > \int_\Omega hw, \tag{1.52}$$

for all $w \in \ker A\backslash\{0\}$. *Then* (1.25) *has a solution.*

Proof. We have, for $u \in \mathbb{R}$,

$$G(x,u) = \int_0^u [g_0(x,\xi) - h(x)]\,d\xi = \int_0^u g_0(x,\xi)\,d\xi - h(x)u.$$

Hence, by (1.50),

$$|G(x,u)| \le \int_0^{|u|} M(x)\,d\xi + |h(x)||u| = (M(x) + |h(x)|)|u|,\ x \in \Omega,\ u \in \mathbb{R}. \tag{1.53}$$

Since $\frac{2N}{N+2m} < 2$, we can assume without loss of generality that $\frac{2N}{N+2m} < q < 2$. Now, let $a \equiv 0$, $b = M + |h|$, $s = 1$, $r = q$, and $p = q' = \frac{q}{q-1}$. Since

$$\frac{2N}{N+2m} < q = p' = 1 + \frac{1}{p-1},$$

we have $\frac{1}{p-1} > \frac{N-2m}{N+2m}$, and hence $p < \frac{2N}{N-2m}$ in the case $N > 2m$.

Since $|h| \in L^2(\Omega) \subset L^q(\Omega)$, we have that $b = M + |h| \in L^q(\Omega)$. We have (1.36), which in turn, implies (1.27) and (1.35) (Remark 1.3 (i)).

On the other hand, by Remark 1.3 (ii), we see that (1.35) is satisfied by Gårding's inequality. For $x \in \Omega$, we have

$$\frac{G(x,u)}{u} = \frac{1}{u}\int_0^u g_0(x,\xi)\,d\xi - h(x).$$

Hence, by L'Hôpital's rule,

$$\begin{aligned} G^\pm(x) = \lim_{u\to\pm\infty} \frac{G(x,u)}{u} &= \lim_{u\to\pm\infty} \frac{\int_0^u g_0(x,\xi)\,d\xi}{u} - h(x) \\ &= \lim_{u\to\pm\infty} g_0(x,u) - h(x) = g_0(x,\pm\infty) - h(x). \end{aligned}$$

For $w \in \ker A$, we have

$$\begin{aligned} \int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w &= \int_{\Omega_w^+} [g(\cdot,\infty) - h]w + \int_{\Omega_w^-} [g(\cdot,-\infty) - h]w \\ &= \int_{\Omega_w^+} g(\cdot,\infty)w + \int_{\Omega_w^-} g(\cdot,-\infty)w - \int_\Omega hw. \end{aligned}$$

Therefore, (1.52) and (1.49) are the same in this case. Corollary 1.3 follows from Corollary 1.2. $\square$

Remark 1.4. In the case where $g_0(x,\xi) = g_0(\xi)$ is a bounded function which depends only on ξ, and the limits

$$g_{\pm\infty} = \lim_{\xi\to\pm\infty} g_0(\xi)$$

are constants, we see that assumption (1.52) becomes

$$g_\infty \int_{\Omega_w^+} w + g_{-\infty} \int_{\Omega_w^-} w > \int_\Omega hw, \quad \forall w \in \ker A\backslash\{0\}.$$

This is the usual Landesman-Lazer condition (cf. [12], [8], [16]).

In Corollary 1.3, the dependence of g_0 on $x \in \Omega$ is permitted, and we do not have to assume that $g_{-\infty} \le g_0 \le g_\infty$, and that g_0 is bounded on $\mathbb{R}$. We have similar results in the cases where $g_{\pm\infty}$ have values $\pm\infty$.

For further results and references concerning problems of the above type we also refer to [6], [10], and [14].

2. Existence results of Landesman-Lazer type for noncoercive, semilinear variational inequalities.

2.1. Notations—General results. Let $(V, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a Hilbert space as in Sections 1.1 and 1.2, $K \subset V$ be a closed, convex set, and $a : V \times V \to \mathbb{R}$ be a continuous symmetric, nonnegative bilinear form on V. We assume furthermore that:

• $q : V \to V^*$ is the Gâteaux derivative of a continuous functional Q from V to $\mathbb{R}$; i.e.

$$\langle q(u), v\rangle = \lim_{t\to 0} \frac{Q(u+tv) - Q(u)}{t}, \; u, v \in V,$$

• $h \in V^*$, and

• $\varphi : V \to \mathbb{R} \cup \{\infty\}$ is a proper, convex, lower semicontinuous functional, such that $D(\varphi) \cap K \neq \emptyset$ ($D(\varphi) = \{u \in V : \varphi(u) < \infty\}$ is the effective domain of φ). We consider the following variational inequality on K:

$$\begin{cases} a(u, v-u) + \langle q(u), v-u\rangle + \varphi(v) - \varphi(u) \ge \langle h, v-u\rangle, & \forall v \in K \\ u \in K. \end{cases} \tag{2.1}$$

For a noncoercive bilinear form a, this problem does not generally have solutions for all $h \in V^*$. In this section, we will apply the results above to find conditions on h, q, and φ such that (2.1) is solvable.

We need some more assumptions on Q. We assume that

$$Q \;\text{ is weakly lower semicontinuous on }\; V \tag{2.2}$$

(in applications, Q is usually completely continuous), and

$$\liminf_{\|u\|\to\infty} \frac{Q(u)}{\|u\|^2} \ge 0. \tag{2.3}$$

(This means that for large $\|u\|$, Q does not go to $-\infty$ faster than quadratic functions.) Consider the following functional on V, $F : V \to \mathbb{R} \cup \{\infty\}$,

$$F(u) = \tfrac{1}{2}a(u,u) + Q(u) + \varphi(u) + I_K(u) - \langle h, u\rangle \tag{2.4}$$
$$= \Phi(u) + j(u), \tag{2.5}$$

with

$$\Phi(u) = \tfrac{1}{2}a(u,u), \quad j(u) = Q(u) + \varphi(u) + I_K(u) - \langle h, u\rangle,\ u \in V.$$

We shall need the following lemma, which is a standard result.

Lemma 2.1. *F is weakly lower semicontinuous on V, and every solution of the minimization problem*

$$u \in V : F(u) = \min_{v\in V} F(v), \tag{2.6}$$

is a solution of the variational inequality (2.1).

The following theorem is a consequence of Theorem 1.1, applied to the functional F given by (2.4) (and $S = V$; i.e., $\psi \equiv 0$). We shall call the set rcK defined by the following formula the *recession cone* of K

$$rcK = \bigcap\nolimits_{t>0} tK.$$

Theorem 2.1. *Assume F given by* (2.4) *has property* (P)*, and that the following compatibility condition is satisfied: If $w \in \ker a \cap rcK$ is such that there exists a sequence $\{u_n\} \subset K$ satisfying*

$$\|u_n\| \to \infty, \quad \frac{u_n}{\|u_n\|} = w_n \rightharpoonup w, \tag{2.7}$$

and

$$\varphi_\infty(w) + \limsup \frac{Q(u_n)}{\|u_n\|} \le \langle h, w\rangle, \tag{2.8}$$

then $-w \in rcK$, and

$$Q(v - w) + \varphi(v - w) \le Q(v) + \varphi(v) - \langle h, w\rangle, \tag{2.9}$$

for all $v \in K$. Then (2.1) *has a solution.*

Proof. By Lemma 2.1, we only need to prove that the minimization problem (2.6) has a solution. From Lemma 2.1, F is weakly lower semicontinuous on V. Moreover, there exists $C \in \mathbb{R}$ such that $F(u) \ge C\|u\|^2$, for all $u, \|u\|$ sufficiently large. In fact, we have

$$\tfrac{1}{2}a(u,u) + I_K(u) \ge 0,$$

and

$$|\langle h, u\rangle| \le \|h\|_*\|u\|, \quad u \in V.$$

Moreover, since φ is convex, $D(\varphi) \neq \emptyset$, by the separation theorem of convex sets ([5]), we have $a, b \in \mathbb{R}$ such that

$$\varphi(u) \geq a + b\|u\|,\ \forall u \in V. \tag{2.10}$$

Now, by (2.3), we have

$$-1 < \liminf_{\|u\|\to\infty} \frac{Q(u)}{\|u\|^2},$$

and hence

$$\frac{Q(u)}{\|u\|^2} > -1, \quad \text{or} \quad Q(u) > -\|u\|^2$$

for all $u, \|u\|$ sufficiently large. We therefore have

$$F(u) \geq -\|u\|^2 - (\|h\|_* - b)\|u\| + a \geq C\|u\|^2,$$

for all $u, \|u\|$ large, where C is some negative number.

Now we check that the compatibility condition in Theorem 1.1 is satisfied with F given above, and $\psi \equiv 0$.

Suppose now that $u, \{u_n\}, \{w_n\}$ satisfy (1.6) and (1.7). From (1.7) with $\lambda = 2$, we have

$$\begin{aligned} 0 \geq \limsup \frac{F(u_n)}{\|u_n\|^2} &= \limsup\big[\tfrac{1}{2}\frac{a(u_n,u_n)}{\|u_n\|^2} + \frac{Q(u_n)}{\|u_n\|^2} + \frac{\varphi(u_n)}{\|u_n\|^2} + \frac{I_K(u_n)}{\|u_n\|^2} - \frac{\langle h, u_n\rangle}{\|u_n\|^2}\big] \\ &\geq \limsup\big[\tfrac{1}{2}a(w_n,w_n) + \frac{I_K(u_n)}{\|u_n\|^2}\big] + \liminf \frac{Q(u_n)}{\|u_n\|^2} + \liminf \frac{\varphi(u_n)}{\|u_n\|^2} - \lim \frac{\langle h, u_n\rangle}{\|u_n\|^2}. \end{aligned} \tag{2.11}$$

Since φ is convex, we have from (2.10) that

$$\liminf \frac{\varphi(u_n)}{\|u_n\|^2} \geq \liminf\big[\frac{a}{\|u_n\|^2} + \frac{b}{\|u_n\|}\big] = 0.$$

Also

$$\lim \frac{\langle h, u_n\rangle}{\|u_n\|^2} = \lim \frac{\langle h, w_n\rangle}{\|u_n\|} = 0,$$

since h is linear and $\{u_n\}$ is bounded. Together with (2.3), we have from (2.11) that

$$\limsup\big[\tfrac{1}{2}a(w_n,w_n) + \frac{I_K(u_n)}{\|u_n\|^2}\big] \leq 0.$$

But $a \geq 0, I_K \geq 0$ on V; we must have

$$\lim a(w_n,w_n) = \lim \frac{I_K(u_n)}{\|u_n\|^2} = 0.$$

Since $I_K(u_n) = \infty$ if $u_n \notin K$, this implies that $I_K(u_n) = 0$; i.e., $u_n \in K$ for all n sufficiently large. Hence $\{u_n\} \subset K$ $(n \geq n_0)$. Moreover,

$$0 \leq a(w,w) \leq \liminf a(w_n, w_n) = 0.$$

Hence, $a(w,w) = 0$; i.e., $w \in \ker a$.

Now, let $u \in K$. Since $\|u_n\| \to \infty$, we have $0 < \|u_n\|^{-1} < 1$ for n sufficiently large, and since $u_n, u \in K$, it follows from the convexity of K that

$$u\big(1 - \frac{1}{\|u_n\|}\big) + w_n = \big(1 - \frac{1}{\|u_n\|}\big)u + \frac{1}{\|u_n\|}u_n \in K,$$

for all n large. Letting $n \to \infty$, since K is weakly closed, one gets $u + w \in K$. Since this holds for all $u \in K$, we have $w \in rcK$. We have proved that $w \in rcK \cap \ker a$. Now, by letting $\lambda = 1$ in (1.7), and noticing that $u_n \in K$, we get

$$\begin{aligned} 0 &\geq \limsup \frac{F(u_n)}{\|u_n\|} = \limsup \big[\frac{1}{2}\frac{a(u_n,u_n)}{\|u\|} + \frac{Q(u_n)}{\|u_n\|} + \frac{\varphi(u_n)}{\|u_n\|} - \frac{\langle h, u_n\rangle}{\|u_n\|}\big] \\ &\geq \limsup \big[\frac{Q(u_n)}{\|u_n\|} + \frac{\varphi(u_n)}{\|u_n\|} - \frac{\langle h, u_n\rangle}{\|u_n\|}\big] \ \ (\text{since } \frac{a(u_n,u_n)}{\|u_n\|} \geq 0,\ \forall n) \\ &\geq \limsup \frac{Q(u_n)}{\|u_n\|} + \liminf \frac{\varphi(u_n)}{\|u_n\|} - \lim \langle h, w_n\rangle. \end{aligned} \tag{2.12}$$

Since $w_n \rightharpoonup w$, we have $\lim\langle h, w_n\rangle = \langle h, w\rangle$. Since $\|u_n\| \to \infty$, we have from the convexity of φ that

$$\varphi_\infty(w) \leq \liminf \frac{\varphi(u_n)}{\|u_n\|}.$$

Substituting these into (2.12), we obtain

$$0 \geq \limsup \frac{Q(u_n)}{\|u_n\|} + \varphi_\infty(w) - \langle h, w\rangle.$$

Hence (2.7), (2.8) are satisfied. By hypothesis, we have $-w \in rcK$ and (2.9). Let $u \in K$. Then $u - w \in K$ and $I_K(u-w) = 0$. Moreover, $a(u-w, u-w) = a(u,u)$. Therefore,

$$\begin{aligned} F(u-w) &= \tfrac{1}{2}a(u-w,u-w) + Q(u-w) + \varphi(u-w) - \langle h, u-w\rangle \\ &\leq \tfrac{1}{2}a(u,u) + Q(u) + \varphi(u) - \langle h, w\rangle - \langle h, u-w\rangle \\ &= \tfrac{1}{2}a(u,u) + Q(u) + \varphi(u) - \langle h, u\rangle = F(u). \end{aligned}$$

If $u \notin K$ then we also have $F(u-w) \leq \infty = F(u)$. This shows that the conditions of Theorem 1.1 are satisfied. Hence using Lemma 2.1, we obtain that (2.6) and thus (2.1) are solvable. □

Now we consider some particular cases in which F given by (2.4) has property (P). We have the following proposition.

Proposition 2.1. *Let F be given by (2.4). Then F has property (P) whenever one of the following conditions is satisfied*:

(a) *F has property (P) as stated in Section 2.2, [13].*

(b) *If $\{v_n\} \subset K$ satisfies*

$$\|v_n\| \to \infty, \quad w_n = v_n/\|v_n\| \rightharpoonup 0, \tag{2.13}$$

and

$$\lim a(w_n, w_n) = 0 \tag{2.14}$$

then

$$\limsup \frac{F(v_n)}{\|v_n\|} > 0. \tag{2.15}$$

(c) *If $\{v_n\} \subset K$, $v_n/\|v_n\| \rightharpoonup w$, and*

$$\sup F(v_n) < \infty, \tag{2.16}$$

then $v_n/\|v_n\| \to w$.

(d) *There exist $P_0, P_1 : V \to \mathbb{R}^+$ such that*

$$\begin{cases} \bullet\ P_0(K) \text{ is bounded} \\ \bullet\ \exists s > 0 : P_0(\lambda x) \leq \lambda^s P_0(x),\ \forall x \in K,\ \lambda \in [0,1] \\ \bullet\ 0 \in K \quad \text{if } \ P_0 \not\equiv 0 \\ \bullet\ \forall \{x_n\} \subset V \text{ with } x_n \rightharpoonup 0, \quad \text{we have } \ P_1(x_n) \to 0 \end{cases} \tag{2.17}$$

and $\exists\, C, \alpha > 0$ such that

$$a(u,u) + P_0(v) + P_1(u) \geq C\|u\|^\alpha, \quad \forall v \in V. \tag{2.18}$$

(e) *We have*

$$\begin{cases} \bullet\ Q(0) = \varphi(0) = 0,\ Q \ \text{ is completely continuous,} \\ \bullet\ \liminf\limits_{\|u\| \to \infty} \dfrac{Q(u)}{\|u\|} \geq 0, \end{cases} \tag{2.19}$$

and there exist P_0, P_1 satisfying (2.17) and $C, \alpha > 0$ such that

$$a(u,u) + P_0(v) + P_1(v) + j^+(v) \geq C\|v\|^\alpha, \quad \forall v \in V. \tag{2.20}$$

(f) *The bilinear form a is coercive off its kernel; i.e.,*

- $\ker\, a$ *is a finite-dimensional subspace of V*
- $\exists\, C > 0$ *such that* $a(v,v) \geq C\|v\|^2,\ \forall v \in (\ker a)^\perp.$

Proof. (a) Since $\langle Au, u\rangle = a(u,u) \geq 0$, $\forall u \in V$, one has

$$\limsup_{\|u\|\to\infty} \frac{a(u,u)}{\|u\|} \geq 0.$$

We have (1.19). Now, as in the proof of Theorem 2.1, we have by the convexity of φ and the linearity of h that

$$\liminf_{\|u\|\to\infty} \frac{\varphi(u)}{\|u\|^2} \geq 0 = \lim_{\|u\|\to\infty} \frac{\langle h, u\rangle}{\|u\|^2}.$$

Together with (2.3), this gives

$$\liminf_{\|u\|\to\infty} \frac{j(u)}{\|u\|^2} \geq \liminf_{\|u\|\to\infty} \frac{Q(u)}{\|u\|^2} + \liminf_{\|u\|\to\infty} \frac{\varphi(u)}{\|u\|^2} + \liminf_{\|u\|\to\infty} \frac{I_K(u)}{\|u\|^2} - \lim_{\|u\|\to\infty} \frac{\langle h, u\rangle}{\|u\|^2} \geq 0;$$

i.e., we have (1.18).

Now, by the proof of Corollary 1.1, we know that if F has property (P) as stated in Section 2.2, [13], and if (1.18), (1.19) hold, then F has property (P) as stated in Section 1.1.

(b) Assume $\{v_n\}$ satisfies (2.13), and

$$\limsup \frac{a(v_n, v_n)}{\|v_n\|^2} \leq 0.$$

Since $a \geq 0$, we have $\lim a(w_n, w_n) = \lim \frac{a(v_n,v_n)}{\|v_n\|^2} = 0$; i.e., (2.14) is satisfied. By our assumptions, we have (2.15) which, in turn, implies that

$$\limsup F(v_n) = \limsup \frac{F(v_n)}{\|v_n\|} \cdot \lim \|v_n\| = \infty.$$

Hence (2.5) of [13] is clearly satisfied, proving that F has property (P) as stated in [13]. By (a) F has property (P).

(c) The proof of (c) is similar to that of Proposition 2.2, [13], and we omit it.

(d) Suppose $\{v_n\}$ satisfies (2.13), (2.14). If $0 \in K$ then since $v_n \in K$ and $\|v_n\| \geq 1$ for all n large, we have $w_n \in K$. We have

$$P_0(w_n) \leq \|v_n\|^{-s} P_0(v_n) \leq \|v_n\|^{-s}(\sup P_0(K)),$$

for n large and then, by (2.17), $\lim P_0(w_n) = 0$. From (2.18) we have

$$a(w_n, w_n) + P_0(w_n) + P_1(w_n) \geq C\|w_n\|^\alpha = C.$$

From (2.14) and (2.17), we see that the left-hand side of this inequality tends to 0 as $n \to \infty$. This contradiction proves that F has property (P).

In the case where $0 \notin K$, we have $P_0 \equiv 0$ in (2.18) and the proof is still valid.

(e) Assume now that (2.19), (2.20) hold and that $\{v_n\}$ satisfies (2.13), (2.14). We prove that (2.15) is also satisfied. Suppose otherwise that

$$\limsup \frac{1}{\|v_n\|}[a(v_n, v_n) + j(v_n)] \leq 0.$$

Hence

$$\limsup \frac{j(v_n)}{\|v_n\|} \leq 0.$$

Since $v_n \in K$, we have

$$\frac{j(v_n)}{\|v_n\|} = \frac{Q(v_n)}{\|v_n\|} + \frac{\varphi(v_n)}{\|v_n\|} - \langle h, w_n \rangle.$$

By (2.13), we have $\langle h, w_n \rangle \to 0$ $(n \to \infty)$. Therefore,

$$\begin{aligned} 0 &\geq \limsup\Big[\frac{Q(v_n)}{\|v_n\|} + \frac{\varphi(v_n)}{\|v_n\|} - \langle h, w_n \rangle\Big] = \limsup\Big[\frac{Q(v_n)}{\|v_n\|} + \frac{\varphi(v_n)}{\|v_n\|}\Big] \\ &\geq \liminf \frac{Q(v_n)}{\|v_n\|} + \limsup \frac{\varphi(v_n)}{\|v_n\|} \geq \limsup \frac{\varphi(v_n)}{\|v_n\|} \text{ (by (2.19))} \\ &\geq \liminf \frac{\varphi(v_n)}{\|v_n\|} \geq \liminf \varphi\Big(\frac{v_n}{\|v_n\|}\Big) = \liminf \varphi(w_n). \end{aligned} \tag{2.21}$$

On the other hand, we have from (2.20) that

$$a(w_n, w_n) + P_0(w_n) + P_1(w_n) + j^+(w_n) \geq C.$$

As above, we have from (2.13), (2.14), and (2.17) that

$$\lim a(w_n, w_n) = \lim P_0(w_n) = \lim P_1(w_n) = 0.$$

It follows that

$$\liminf j^+(w_n) \geq C,$$

and then

$$j^+(w_n) > \frac{C}{2} > 0 \text{ for all } n \text{ large}.$$

Hence

$$j(w_n) = j^+(w_n) > \frac{C}{2};$$

i.e.,

$$Q(w_n) + \varphi(w_n) > \langle h, w_n \rangle + \frac{C}{2} \quad \text{for all } n \text{ large}.$$

Since Q is completely continuous, we have $\lim Q(w_n) = Q(0) = 0$, and then

$$\liminf \varphi(w_n) = \liminf [Q(w_n) + \varphi(w_n)] = \lim[\langle h, w_n\rangle + C/2] = C/2 > 0.$$

This contradicts (2.21), proving that (2.15) is satisfied. Our conclusion now follows from (b).

(f) Let P be the orthogonal projection of V onto $\ker a$. Since $\dim(\ker a) < \infty$, P is a compact linear mapping from V to V.

For $v \in V$, let $v = v_1 + v_2$, $v_1 \in \ker a$, $v_2 \in (\ker a)^\perp$. One has $v_1 = Pv$, and

$$a(v, v) = a(v_1 + v_2, v_1 + v_2) = a(v_2, v_2) \geq C\|v_2\|^2,$$

and therefore

$$\begin{aligned} a(v, v) + \|Pv\|^2 &\geq C\|v_2\|^2 + \|v_1\|^2 \geq \min(C, 1)(\|v_1\|^2 + \|v_2\|^2) \\ &= \min(C, 1)\|v\|^2, \quad \forall v \in V. \end{aligned}$$

Let $P_0 \equiv 0$, $P_1(v) = \|Pv\|^2$. Then (2.17), (2.18) are satisfied. Our conclusion now follows from (e).

2.2. Landesman-Lazer-type theorems for semilinear variational inequalities containing elliptic operators. In this section, we apply the above abstract results to establish existence theorems for variational inequalities containing elliptic operators as considered in Section 1.3 for equations.

Let Ω, V, A, g and G be defined as in Section 1.3, with the assumptions (1.23)–(1.29), (1.32), (1.35). We also use the notation $G^\pm, \Omega_w^\pm$ as in (1.33) and (1.34).

Here, we assume that K is a closed, convex subset of V, and $\varphi : V \to \mathbb{R} \cup \{\infty\}$ is a convex, proper, lower-semicontinuous functional such that $D(\varphi) \cap K \neq \emptyset$. Also, let $h \in V^*(= V)$, and $q : V \to V^*$, $Q : V \to \mathbb{R}$, be given by

$$\langle q(u), v\rangle = \int_\Omega g(x, u(x))v(x)\, dx, \tag{2.22}$$

$$Q(u) = \int_\Omega G(x, u(x))\, dx,\ u, v \in V. \tag{2.23}$$

As remarked in Section 1.3, we have that q, Q are well defined in V, $Q \in C^1(V, \mathbb{R})$ and $Q' = q$. Moreover, Q is completely continuous on V, and $Q(0) = 0$.

With these settings, we consider the variational inequality (2.1). Applying Theorem 2.1, we have the following existence result in this case:

Theorem 2.2. *Suppose the following compatibility condition is satisfied: If $w \in \ker a \cap rcK$ is such that*

$$\int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w + \varphi_\infty(w) \leq \langle h, w\rangle, \tag{2.24}$$

then $-w \in rcK$, *and*

$$\int_\Omega G(x, u(x) - w(x))\, dx + \varphi(u - w) \leq \int_\Omega G(x, u(x)) + \varphi(u) - \langle h, w \rangle, \tag{2.25}$$

for all $u \in K$. *Then* (2.1) *has a solution.*

Proof. We verify that with the above assumptions on a, φ, K, h and Q, all the conditions of Theorem 2.1 are satisfied.

We first observe that Q has the properties (2.2) and (2.3). In fact, since Q is completely continuous, (2.2) is satisfied. Moreover, in the proof of Theorem 1.2, we have shown that for G satisfying (1.28), (1.29),

$$\liminf_{\|u\| \to \infty} \frac{1}{\|u\|^2} \int_\Omega G(x, u(x))\, dx \geq 0 \tag{2.26}$$

(cf. (1.26) and (1.18), which together with (2.23), implies (2.3)).

Now consider F defined by (2.4) with Q given by (2.23). Using arguments as in the proof of Theorem 1.2 and Proposition 3.1, we will check that F has property (P).

In fact, in the proof of Theorem 1.2, we have proved that there does not exist any sequence $\{v_n\} \subset V (\subset H^m(\Omega))$ such that $\|v_n\| \to \infty$, $w_n = v_n/\|v_n\| \rightharpoonup 0$, and

$$\limsup \frac{a(v_n, v_n)}{\|v_n\|^2} \leq 0;$$

i.e., there does not exist any sequence $\{v_n\}$ satisfying (2.13) and (2.14). Hence F has property (P) by Proposition 2.1 (b). Now, we check the compatibility [(2.7)–(2.8) $\Rightarrow$ (2.9)] in Theorem 2.1.

Suppose $\{u_n\} \subset K$ satisfies (2.7) and (2.8). From (1.48) in the proof of Theorem 1.2, we have already observed that

$$\begin{aligned} &\limsup \frac{Q(u_n)}{\|u_n\|} = \limsup \frac{1}{\|u_n\|} \int_\Omega G(x, u_n(x))\, dx \\ &\geq \liminf \frac{1}{\|u_n\|} \int_\Omega G(x, u_n(x))\, dx \geq \int_{\Omega_w^+} G^+(x) w(x)\, dx + \int_{\Omega_w^-} G^-(x) w(x)\, dx. \end{aligned}$$

Hence (2.8) implies that

$$\varphi_\infty(w) + \int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w \leq \langle h, w \rangle;$$

i.e., (2.24) is satisfied.

By hypothesis, we have (2.25), which is exactly (2.9). Our conclusion now follows from Theorem 2.1. □

From this theorem, we immediately have the following consequence:

Corollary 2.1. *Under the above assumptions, if the condition*

$$\int_{\Omega_w^+} G^+ w + \int_{\Omega_w^-} G^- w + \varphi_\infty(w) > \langle h, w\rangle, \quad \forall w \in (\ker a \cap rcK)\backslash\{0\} \tag{2.27}$$

is satisfied, then (2.1) *has a solution.*

Proof. (2.27) implies that $w = 0$ is the only point in $\ker a \cap rcK$ that satisfies (2.24), hence (2.25) is obviously satisfied. □

Applying these results in the following particular case, we obtain a Landesman-Lazer-type theorem for variational inequalities. We assume that g satisfies the growth condition (1.49) and the limit condition (1.50); i.e.,

$$|g(x,\xi)| \le M(x), \quad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}, \tag{2.28}$$

with $M \in L^q(\Omega)$, $q > 2N(N+2m)^{-1}$, and the limits

$$g(x, \pm\infty) = \lim_{\xi\to\pm\infty} g(x,\xi) \tag{2.29}$$

exist for almost every $x \in \Omega$.

Arguing as in the proof of Corollary 1.3, we see that the primitive G of g (with respect to ξ) satisfies (1.36) and then (1.27), (1.32). Moreover, we have

$$G^{\pm}(x) = g(x, \pm\infty), \quad x \in \Omega.$$

Therefore, in this case, Corollary 2.1 becomes the following:

Corollary 2.2. *Under the assumptions* (2.28), (2.29), *if we have*

$$\int_{\Omega_w^+} g(\cdot,\infty) w + \int_{\Omega_w^-} g(\cdot,-\infty) w + \varphi_\infty(w) > \langle h, w\rangle, \tag{2.30}$$

for all $w \in (\ker A \cap rcK)\backslash\{0\}$*, then* (2.1) *has a solution.*

We now restrict ourselves to an even more special case of second-order elliptic operators, i.e., where $m = 1$, and $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and A is a second-order symmetric elliptic operator of the form

$$\langle Au, v\rangle = a(u,v) - \int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i u D_j v + (b-\lambda) uv\Big], \quad u, v \in V. \tag{2.31}$$

Let g satisfy (2.28) (with $m = 1$); i.e., $|g(x,\xi)| \le M(x)$ (almost every $x \in \Omega$, $\forall \xi \in \mathbb{R}$) with $M \in L^q(\Omega)$, $q > 2N(N+2)^{-1}$, and the limits $g(x,\xi) \to k(x)$, for almost every $x \in \Omega$ (as $\xi \to \infty$) exist. Suppose now that Γ is a closed subset of $\partial\Omega$, and consider $V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$, ($V = H_0^1(\Omega)$ if $\Gamma = \partial\Omega$, and $V = H^1(\Omega)$ if $\Gamma = \emptyset$).

Let K be a closed, convex subset of V. With $\varphi \equiv 0$, the variational inequality (2.1) becomes in this particular case the following problem:

$$a(u, v-u) + \int_\Omega g(x, u(x))[v(x) - u(x)]\, dx \geq \langle h, v-u \rangle, \ \forall v \in K, \quad u \in K, \tag{2.32}$$

with $h \in V^*$ (h can be taken as

$$\langle h, u \rangle = \int_\Omega hu, \ u \in V,$$

with $h \in L^s(\Omega), s > 2N/(N+2)$ if $N > 3$). Let $\lambda = \lambda_0$ be the first eigenvalue of the operator A_0,

$$\langle A_0 u, v \rangle = \int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i u D_j v + buv \Big],$$

with Dirichlet conditions on Γ and Neumann condition on $\partial\Omega \backslash \Gamma$, i.e., the first eigenvalue of the variational equality

$$a(u, v) = 0, \ \forall v \in V, \quad u \in V, \ u \neq 0. \tag{2.33}$$

It is known (Lemma 2.2, [15]) that λ_0 is simple, and we can choose the eigenvector e corresponding to λ_0 such that

$$e(x) > 0, \quad x \in \Omega. \tag{2.34}$$

By (2.33), we have $\ker a = \operatorname{span}\{e\} = \mathbb{R}e$.

In this particular case, we have the following consequence of Corollary 2.2.

Corollary 2.3. *With the above settings, we have*:

(a) *If* $\ker a \cap rcK = \{0\}$, *then* (2.1) *has a solution.*

(b) *If* $\ker a \cap rcK = \mathbb{R}^+ e (= \{te : t \geq 0\})$, *and if*

$$\int_\Omega ke > \langle h, e \rangle, \tag{2.35}$$

then (2.32) *has a solution.* □

Proof. (a) follows directly from Corollary 2.2.

(b) If $w \in (\ker a \cap rcK) \backslash \{0\}$ then by assumption, $w = te, \ t > 0$. Since $e > 0$ on Ω, we have $w > 0$ on Ω; i.e., $\Omega_w^+ = \Omega$, $\Omega_w^- = \emptyset$. We have $g(\cdot, \infty) = k$, and (2.30) becomes in this case

$$\int_\Omega k(te) > \langle h, te \rangle.$$

Since $t > 0$ this is equivalent to (2.35). Our conclusion now follows from Corollary 2.2.

Remark 2.1. (i) In [3], we considered existence theorems for variational inequalities of the form (2.1), where g was assumed to have a bounded support (i.e., g vanishes outside some ball in V). Theorems 2.1 and 2.2 and their consequences give similar existence results in the case where g does not necessarily have bounded support (but has instead a variational structure).

(ii) The same result as Corollary 2.3 (b) was derived by Szulkin (Theorem 3.1, [15]), under some restriction on the convex set K (condition 11 (ii)) to guarantee the coerciveness of the problem. Here, by replacing the coerciveness of the corresponding functional by property (P) and a compatibility condition, we can drop this assumption and therefore improve Szulkin's result. Also, Corollary 2.3 (a) complements Corollary 2.3 (b) in the case where $e \notin rcK$.

In the last part of this section, we consider another application to the equilibrium problem of a plate resting on an elastic foundation with unilateral boundary condition (cf. Example 5, [3]). The problem is formulated as the following boundary value problem:

$$\Delta^2 u + F(u) = h \text{ on } \Omega, \tag{2.36}$$

with the boundary condition:

$$\begin{cases} F_3 = 0 \text{ on } \Omega, \\ \begin{cases} M_\tau \le 0, \ \partial_\nu u \ge 0, \\ M_\tau \cdot \partial_\nu u = 0 \text{ on } \partial\Omega. \end{cases} \end{cases} \tag{2.37}$$

Here Ω is a bounded domain in $\mathbb{R}^2$, occupied by the plate in its resting position, h denotes the vertical force acting on the plate, $F(u)$ represents the restoring force from the elastic foundation, F_3 the normal shear force, M_τ the twisting moment on the boundary, and $\partial_\nu u$ the outward normal derivative on $\partial\Omega$. We assume that F can be written as $F = g + g_0$, where $g_0 = g_0(\xi)$ is an increasing function for $\xi \in \mathbb{R}$, and $g = g(\xi)$ satisfies the assumptions (2.28) and (2.29) above.

The problem (2.36)–(2.37) can be formulated as the following variational inequality:

$$a(u, v-u) + \int_\Omega g(u)(v-u) + \varphi(v) - \varphi(u) \ge \langle h, v-u\rangle, \ \forall v \in K, \quad u \in K, \tag{2.38}$$

where

- $V = H^2(\Omega)$, $K = \{u \in V : \partial_\nu u \ge 0 \text{ on } \partial\Omega\}$,
- $a : V \times V \to \mathbb{R}$ is the usual bilinear form in the theory of plates (cf. Remark 1.3),
- $\varphi(v) = \int_\Omega \psi(v(x))\, dx, \ v \in V$,

with

$$\psi(s) = \int_0^s g_0(t)\, dt, \quad s \in \mathbb{R}.$$

By Remark 1.3 (c), we know that

$$a(u,u) + \|u\|^2_{H^1(\Omega)} \ge (1-\nu)\|u\|^2_{H^2(\Omega)}, \quad \forall u \in V,$$

with $0 < \nu < 1/2$. Since the mapping $P_1 : H^2(\Omega) \to \mathbb{R}$, $P_1(u) = \|u\|^2_{H^1(\Omega)}$ is completely continuous, we have (2.17), (2.18) with $\alpha = 2$, $P_0 \equiv 0$. Now, let $Q(u) = \int_\Omega G(u(x))\,dx,\ u \in H^2(\Omega)$, and

$$G(t) = \int_0^t g(s)\,ds,\ t \in \mathbb{R}.$$

By Proposition 2.1 (d), we have that F defined by (2.4) satisfies property (P) in this case. Moreover, we know that $\ker a$ consists of restrictions of polynomials of degree 1 on Ω:

$$\ker a = \{p : \Omega \to \mathbb{R} : \exists\, q_0, q_1, q_2 \in \mathbb{R} : p(x) = q_0 + q_1 x_1 + q_2 x_2,\ \forall x = (x_1, x_2) \in \Omega\}.$$

It is clear that K is a closed convex cone in $H^2(\Omega)$, and by using an argument in [3], Theorem 3, we have

$$\ker a \cap rcK = \ker a \cap K = \mathbb{R}, \tag{2.39}$$

and moreover

$$\varphi_\infty(w) = \begin{cases} w|\Omega|\psi^+ & \text{if } w > 0 \\ w|\Omega|\psi^- & \text{if } w < 0, \end{cases} \tag{2.40}$$

with $\psi^\pm = \lim_{t\to\pm\infty} \frac{\psi(t)}{t}$ (the limits exist since ψ is convex in $\mathbb{R}$).

Now, using (2.39), (2.40) and applying Corollary 2.2, we have the following Landesman-Lazer-type theorem for (2.38).

Corollary 2.4. *If*

$$g(-\infty) + \psi^- < \frac{\langle h, 1\rangle}{|\Omega|} < g(\infty) + \psi^+,$$

then (2.38) *has a solution.*

2.3. Existence theorem for noncoercive variational inequalities containing homogeneous functionals. We derive in this section some corollaries of Theorem 2.2 in cases where the functional Q is homogeneous on V.

Consider the variational inequality (2.1) and the associated functionals F, Φ, j defined by (2.4), and the minimization problem (2.6) corresponding to (2.1). We assume that the assumptions about q, Q, h, φ in Section 2.1 are satisfied.

In this section, we assume furthermore that Q is positive homogeneous of degree $\beta > 1$; that is,

$$Q(tu) = t^\beta Q(u), \quad \forall t \geq 0,\ u \in V. \tag{2.41}$$

(This can also be stated as $\langle Q'(u), u\rangle = \beta Q(u)$, $\forall u \in V$.) As above, we assume that Q satisfies (2.2), (2.3), and moreover Q is bounded (i.e., Q maps bounded sets of V into bounded sets of $\mathbb{R}$). (We note that these conditions hold if Q is completely continuous.)

Now, we remark that if $1 < \beta < 2$ then (2.3) is satisfied. In fact, suppose otherwise that there exists a sequence $\{u_n\} \subset V, \|u_n\| \to \infty$ $(n \to \infty)$ such that

$$\lim_{n\to\infty} \frac{Q(u_n)}{\|u_n\|^2} < 0. \tag{2.42}$$

Let $w_n = u_n/\|u_n\|$. Passing to a subsequence, if necessary, we can assume that $w_n \rightharpoonup w$. Since the sequence $\{Q(w_n)\}$ is bounded, we have by the homogeneity of Q ((2.41)):

$$\frac{Q(u_n)}{\|u_n\|^2} = \frac{Q(\|u_n\|w_n)}{\|u_n\|^2} = \frac{\|u_n\|^\beta Q(w_n)}{\|u_n\|^2} = \frac{Q(w_n)}{\|u_n\|^{2-\beta}} \to 0 \text{ as } n \to \infty,$$

contradicting (2.42). Hence we have (2.3).

In the case $\beta \geq 2$, we observe that (2.3) is satisfied if and only if

$$Q(u) \geq 0, \quad \forall u \in V. \tag{2.43}$$

In fact, if $Q(u) < 0$ for some $u \in V$, then by letting $u_n = nu$, $n = 1, 2, \dots$, one has

$$\frac{Q(u_n)}{\|u_n\|^2} = n^{\beta-2}\frac{Q(u)}{\|u\|^2} \to \begin{cases} Q(u)/\|u\|^2 < 0 & \text{if } \beta = 2 \\ -\infty & \text{if } \beta > 2. \end{cases}$$

Conversely, (2.43) clearly implies (3.3).

With these settings, we can now prove the following consequence of Theorem 2.1:

Theorem 2.3. *Suppose F has property (P) and that the following compatibility condition is satisfied*:

$$Q(w) \geq 0, \quad \forall w \in \ker a \cap rcK, \tag{2.44}$$

and if $w \in \ker a \cap rcK$ satisfies

$$Q(w) = 0, \quad \varphi_\infty(w) \leq \langle h, w\rangle \ (\text{in case } \beta \geq 2), \tag{2.45}$$

then $-w \in rcK$, and

$$Q(u - w) + \varphi(u - w) \leq Q(u) + \varphi(u) - \langle h, w\rangle, \ \forall v \in K. \tag{2.46}$$

Then (2.1) *has a solution.*

Proof. We check that the assumptions in Theorem 2.1 are satisfied. For this purpose, let $w, \{w_n\}, \{u_n\}$ satisfy (2.7) and (2.8).

Since $w \in \ker a \cap rcK$, we have from (2.44) that $Q(w) \geq 0$. Now, we prove that one cannot have $Q(w) > 0$. In fact, if this is the case then by noting that

$$\liminf Q(w_n) \geq Q(w) > \tfrac{1}{2}Q(w),$$

we have

$$Q(w_n) \geq \frac{Q(w)}{2} > 0,$$

for all n sufficiently large. This implies that

$$\frac{Q(u_n)}{\|u_n\|} = \frac{\|u_n\|^\beta Q(w_n)}{\|u_n\|} = \|u_n\|^{\beta-1}Q(w_n) \geq \|u_n\|^{\beta-1}\frac{Q(w)}{2},$$

for all n sufficiently large. Since $\beta > 1$, and $\|u_n\| \to \infty$, it follows that $\lim \frac{Q(u_n)}{\|u_n\|} = \infty$. On the other hand, we know that $\varphi_\infty(v) > -\infty$, $\forall v \in V$ ([3], Section 3). Hence

$$\varphi_\infty(w) + \lim \frac{Q(u_n)}{\|u_n\|} = \infty.$$

This contradicts (2.8), proving that $Q(w) > 0$ cannot happen. Hence $Q(w) = 0$.

In the case $\beta \geq 2$, we have $Q \geq 0$ on V (cf. (2.3) and (2.43)). In this case, we have $\limsup \frac{Q(u_n)}{\|u_n\|} \geq 0$, and it follows from (2.8) that $\varphi_\infty(w) \leq \langle h, w\rangle$.

We have (2.45). By hypotheses, we have (2.46) which is the same as (2.9). Therefore, the compatibility condition in Theorem 2.1 is satisfied. Our conclusion thus follows from that theorem. □

An immediate consequence of Theorem 2.3 is the following:

Corollary 2.5. *Assume F has property (P) and that*

$$Q(w) > 0, \quad \forall w \in (\ker a \cap rcK)\backslash\{0\}. \tag{2.47}$$

Then (2.1) *has a solution.*

Proof. If (2.47) holds then (2.44)–(2.45) implies that $w = 0$, for which (2.46) is obviously satisfied. □

Now we consider the case where $\varphi(0) = 0,\ 0 \in K$, and find conditions for the existence of nontrivial solutions of (2.1). We have the following consequence of Corollary 2.5.

Corollary 2.6. *Let F satisfy property (P).*

(a) *Assume* (2.47) *is satisfied and that*

$$\exists\, z \in V : F(z) < 0, \tag{2.48}$$

and

$$\langle h, w\rangle \leq 0 \leq \varphi(w),\ \forall w \in \ker a \cap rcK. \tag{2.49}$$

Then (2.1) *has a solution $u \notin \ker a \cap rcK$.*

(b) *Assume* (2.48) *is satisfied and that*

$$Q(w) > 0,\ \forall w \in \ker a\backslash\{0\}, \tag{2.50}$$

and

$$\langle h, w\rangle \leq 0 \leq \varphi(w),\ \forall w \in \ker a. \tag{2.51}$$

Then (2.1) *has a solution $u \notin \ker a$.*

Proof. (a) Let u be a solution of (2.6) and then (2.1), whose existence is guaranteed by Corollary 2.5 (under the assumption (2.47)).

Since $Q(0) = \langle h, 0\rangle = \varphi(0) = I_K(0) = 0$, we have that $F(0) = 0$, which together with (2.48), implies that 0 is not a solution of (2.6); i.e., $u \neq 0$.

Assume now that $u \in (\ker a \cap rcK)\backslash\{0\}$. Letting $v = 0$ in (2.1), we get

$$-a(u,u) - \langle q(u), u\rangle - \varphi(u) \geq -\langle h, u\rangle,$$

or

$$\langle h, u\rangle \geq \langle Q'(u), u\rangle + \varphi(u) = \beta Q(u) + \varphi(u).$$

By (2.48) and (2.49), one has $Q(u) > 0$ and then

$$\beta Q(u) + \varphi(u) \geq \beta Q(u) > 0.$$

But $\langle h, u\rangle \geq 0$. This contradiction proves that $u \notin (\ker a \cap rcK)\backslash\{0\}$ and then $u \notin \ker a \cap rcK$.

(b) is proved in the same way. We just observe that (2.50) implies (2.47), and again we can apply Corollary 2.5.

Remark 2.2. (i) Consider the particular case where $K = V$, $h \equiv 0$, and φ is positive homogeneous of degree $\alpha \geq 1$. If $\alpha > 1$ then from the convexity of φ, we have $\varphi(x) \geq 0$, $\forall x \in V$. Hence (2.51) is obviously satisfied. In this case, Corollary 2.6 therefore reduces to Theorem 2.1, [7].

However, as seen above in Corollaries 2.5, 2.6 and Theorem 2.3, we do not have to assume that φ is α-homogeneous, and $\alpha < \beta < 2$, as in the quoted theorem. By Proposition 2.1 (f), if a is coercive off the kernel (semicoercive, as called in [7]) then F has Property (P). In Theorem 2.3, we do not assume that $Q(w) > 0$ for $w \in \ker a\backslash\{0\}$. In Theorem 2.3 and Corollaries 2.5, 2.6, we consider closed convex sets K which are not necessarily cones (whose indicator functions are homogeneous).

We observe that the theorems of Szulkin ([15]) and Goeleven, Nguyen, and Willem ([7]), being special cases of Corollaries 2.2 and 2.6, can be treated in a unified way by our analysis.

(ii) Suppose we have the following condition:

$$\exists\, z \in rcK \cap D(\varphi_\infty) : \varphi_\infty(z) - \langle h, z\rangle < 0. \tag{2.52}$$

Then (2.48) is satisfied. In fact, by Proposition 1.2 of [3], we know that $tz \in D(\varphi_\infty)$, $\varphi_\infty(tz) = t\varphi_\infty(z)$, $\forall z \in D(\varphi_\infty)$, $t \geq 0$, and $\varphi(u) \leq \varphi_\infty(u)$, $\forall u \in V$. Let $w = tz$, $t > 0$. Since $0 \in K$, we have $w \in rcK \subset K$; i.e., $I_K(w) = 0$. Therefore

$$\begin{aligned} F(w) &= a(w,w) + Q(w) + \varphi(w) - \langle h, w\rangle \\ &\leq a(tz,tz) + Q(tz) + \varphi_\infty(tz) - \langle h, tz\rangle \\ &= t^2 a(z,z) + t^\beta Q(z) + t\varphi_\infty(z) - t\langle h, z\rangle \\ &= t[ta(z,z) + t^{\beta-1}Q(z) + \varphi_\infty(z) - \langle h, z\rangle]. \end{aligned}$$

Since $\beta > 1$, we have $ta(z,z) + t^{\beta-1}Q(z) \to 0$ $(t \to 0^+)$. By (2.52), we see that $F(w) < 0$ for $t > 0$ sufficiently small, proving (2.48).

If φ is positive homogeneous of degree $\alpha > 1$ then by using arguments as above, we see that (2.48) is satisfied if

$$\exists\, z \in rcK \cap D(\varphi) : \ \langle h, z\rangle > 0. \tag{2.53}$$

(iii) Now we consider the case where $h \equiv 0$.

• Suppose φ is α-homogeneous with $\alpha > \beta$ (this holds in particular if $\varphi \equiv 0$), and $\beta < 2$. If

$$\exists\, z \in \ rcK \cap D(\varphi) : Q(z) < 0, \tag{2.54}$$

then (2.48) is satisfied.

In fact, using the above arguments, we have $w = tz \in K$ $(t > 0)$, and

$$\begin{aligned} F(w) &= a(w,w) + Q(w) + \varphi(w) + I_K(w) = t^2 a(z,z) + t^\beta Q(z) + t^\alpha \varphi(z) \\ &= t^\beta [t^{2-\beta} a(z,z) + t^{\alpha-\beta}\varphi(z) + Q(z)]. \end{aligned}$$

Again we have $t^{2-\beta}a(z,z)$, $t^{\alpha-\beta}\varphi(z) \to 0$ $(t \to 0^+)$ and hence $F(w) < 0$ for $t > 0$ sufficiently small; i.e., (2.48) holds.

Note that we can omit the assumption "$\alpha > \beta$" if (2.55) is replaced by:

$$\exists\, z \in rcK : Q(z) < 0, \ \varphi(z) \le 0. \tag{2.55}$$

The proof for this is the same as above.

• Now, we let $\Psi = \varphi + I_K$. Hence Ψ is a convex, lower-semicontinuous functional from V to $\mathbb{R} \cup \{\infty\}$ and $\Psi(0) = 0$, and

$$F(u) = \tfrac{1}{2} a(u,u) + Q(u) + \Psi(u), \ u \in V.$$

We suppose that Ψ_0 is the derivative of order α of Ψ at 0 in the following sense:

$\Psi_0 : V \to \mathbb{R} \cup \{\infty\}$ is a convex, lower-semicontinuous functional such that:

(a) If $v_n \rightharpoonup v$, $\sigma_n \to 0^+$, then

$$\Psi_0(u) \le \liminf \frac{\Psi(\sigma_n u_n)}{\sigma_n^\alpha},$$

and:

(b) For each $u \in V$, each sequence $\{\sigma_n\}$, $\sigma_n \to 0^+$, we can find a sequence $\{v_n\} \subset V$ such that

$$v_n \to v \ \text{ in } \ V, \ \text{ and } \ \frac{\Psi(\sigma_n v_n)}{\sigma_n^\alpha} \to \Psi_0(v).$$

We prove that if

$$\exists\, z \in D(\Psi_0) : Q(z) < 0 \ \text{ (in the case } \ \alpha > \beta), \tag{2.56}$$

or

$$\exists\, z \in D(\Psi_0) : \ Q(z) + \Psi_0(z) < 0 \ \text{ (in the case } \ \alpha = \beta), \tag{2.57}$$

then (2.48) is satisfied.

In fact, choose $t_n = n^{-1} \to 0$ $(n \to \infty)$, and $z_n \in V$ $(n = 1, 2, \dots)$ such that $z_n \to z$ and

$$\frac{\Psi(t_n z_n)}{t_n^\alpha} \to \Psi_0(z). \tag{2.58}$$

We have

$$\begin{aligned} F(t_n z_n) &= t_n^2 a(z_n, z_n) + t_n^\beta Q(z_n) + \Psi(t_n z_n) \\ &= t_n^\beta \big[t_n^{2-\beta} a(z_n, z_n) + Q(z_n) + \frac{\Psi(t_n z)n)}{t_n^\alpha}\, t_n^{\alpha-\beta} \big]. \end{aligned}$$

If $\alpha \geq \beta$ then

$$\lim \frac{\Psi(t_n z_n)}{t_n^\alpha}\, t_n^{\alpha-\beta} = \lim \frac{\Psi(t_n z_n)}{t_n^\alpha} \cdot \lim t_n^{\alpha-\beta} = \begin{cases} \Psi_0(z) \cdot 0 = 0 & \text{if } \alpha < \beta \\ \Psi_0(z) \cdot 1 = \Psi_0(z) & \text{if } \alpha = \beta. \end{cases}$$

Hence

$$\begin{aligned} \lim \big[t_n^{2-\beta} a(z_n, z_n) + Q(z_n) + \frac{\Psi(t_n z_n)}{t_n^\alpha} + t_n^{\alpha-\beta} \big] &= \begin{cases} Q(z) & \text{if } \alpha > \beta \\ Q(z) + \Psi_0(z) & \text{if } \alpha = \beta \end{cases} \\ &< 0. \end{aligned}$$

Then $F(t_n z_n) < 0$ for n sufficiently large, and (2.48) is satisfied. □

We note that if $K = V$ and φ is α-homogeneous then we can check that $\Psi_0 = \Psi = \varphi$, and therefore (2.56) becomes (2.54).

We conclude this section with some applications of Theorem 2.3 and Corollaries 2.5 and 2.6.

Example 2.1. Consider the following variational inequality:

$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla (u - u) + \int_\Omega g(x, u(x))[v(x) - u(x)]\, dx \\ \displaystyle + \int_\Omega \psi(x, v(x))\, dx - \int_\Omega \psi(x, u(x))\, dx \geq \int_\Omega h(v - u),\ \forall v \in K, \quad u \in K. \end{cases} \tag{2.59}$$

Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is an open bounded domain with smooth boundary, Ω_0 is a subdomain of Ω and its measure $|\Omega_0| > 0$, $V = H^1(\Omega)$, and

- $K = \{u \in H^1(\Omega) : u \geq \zeta$ almost everywhere on $\Omega_0\}$, ζ is a given function in $L^\infty(\Omega_0)$, $\zeta \leq 0$ almost everywhere on Ω_0 (so that $0 \in K$). We also assume that $h \in L^2(\Omega)$, and
- $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $\psi \geq 0$ and $\psi(x, \xi)$ is convex with respect to ξ for almost every $x \in \Omega$. Moreover, assume that $\psi(x, 0) = 0$, $x \in \Omega$.
- $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $g(x, \xi)$ is positive homogeneous with respect to ξ if degree $\beta - 1$, $\beta > 1$; i.e.,

$$g(x, t\xi) = t^{\beta-1} g(x, \xi), \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R},\ t \geq 0.$$

Let $G(x,u) = \int_0^u g(x,\xi)\,d\xi$, $u \in \mathbb{R}$, $x \in \Omega$. Then G is a Carathéodory function, and G is homogeneous of degree β with respect to u. We assume that G satisfies the following growth condition:

There exists p such that

$$1 < \beta \leq p < 2^* = \begin{cases} 2N(N-2)^{-1} & \text{if } N > 2 \\ \infty & \text{if } N = 1, 2, \end{cases} \tag{2.60}$$

and

$$G(\cdot, \pm 1) \in L^{p/(p-\beta)}(\Omega). \tag{2.61}$$

• Let $s = \beta$, $b = |G(\cdot, 1)| + |G(\cdot, -1)| \in L^{p/(p-\beta)}(\Omega)$. By the β-homogeneity of G, we have

$$|G(x,u)| \leq b|u|^\beta, \quad \text{a.e. } x \in \Omega, \ \forall u \in \mathbb{R},$$

showing that G satisfies (1.27), (1.28), (1.29). It follows that the functional $Q : V(= H^1(\Omega)) \to \mathbb{R}$,

$$Q(u) = \int_\Omega G(x, u(x))\,dx, \ u \in V,$$

is well-defined and completely continuous on V, and $Q \in C^1(V, \mathbb{R})$ with

$$\langle Q'(u), v\rangle = \langle q(u), v\rangle = \int_\Omega g(\cdot, u)v, \ \forall u, v \in V.$$

By the homogeneity of G with respect to u, we immediately have that Q is positive homogeneous of degree β on V. In the case $\beta \geq 2$, we also assume that $G(\cdot, \pm 1) \geq 0$ so that $Q \geq 0$, and then (2.2), (2.3) are satisfied.

• We note that a particular case of g and G is that

$$g(x,\xi) = r(x)|\xi|^{\beta-2}\xi, \quad G(x,\xi) = \frac{1}{\beta} r(x)|\xi|^\beta, \ x \in \Omega, \ \xi \in \mathbb{R},$$

where $r \in L^\infty(\Omega)$.

• Let $\varphi : V \to [0, \infty]$ be given by

$$\varphi(u) = \int_\Omega \psi(x, u(x))\,dx, \ u \in V.$$

Since $\psi \geq 0$, φ is well defined. By the convexity of ψ with respect to u, we have that φ is convex on V, and moreover $\varphi(0) = 0$. By using Fatou's lemma, we can verify that φ is lower semicontinuous on V.

• By a direct verification, we see that K is a closed convex set in V.

With these settings, we now have that (2.59) is of the form (2.1) with

$$a : V \times V \to \mathbb{R}, \quad a(u,v) = \int_\Omega \nabla u \nabla v, \ u, v \in V,$$

and Q, φ, K, h defined above. Applying Corollaries 2.5 and 2.6 to this particular case, we have the following result:

Corollary 2.7. (a) *If*

$$\int_\Omega G(x,1)\,dx > 0, \tag{2.62}$$

then (2.59) *has a solution.*

(b) *If* (2.62) *is satisfied, and*

$$\int_\Omega h \le 0, \tag{2.63}$$

and

$$\exists\, z \in K : \frac{1}{2}\int_\Omega |\nabla z|^2 + Q(z) + \varphi(z) < \langle h, z\rangle, \tag{2.64}$$

then (2.59) *has a solution* $u \notin [0,\infty)$.

(c) *Assume that* (2.64) *is satisfied, and*

$$\int_\Omega G(x,\pm 1)dx > 0, \tag{2.65}$$

and

$$\int_\Omega h = 0. \tag{2.66}$$

Then (2.66) *has a nonconstant solution.*

Proof. (a) Let $P_1(u) = \int_\Omega u^2, u \in V$. Then P_1 is completely continuous from V to $\mathbb{R}$, and

$$a(u,u) + P_1(u) = \|u\|^2_{H^1(\Omega)} = \|u\|^2, \quad \forall u \in V.$$

This shows that (2.17), (2.18) are satisfied (with $P_0 = 0$), and therefore F given by (2.4) has property (P) by Proposition 2.1. Moreover, it is clear that $\ker a = \mathbb{R}$. Now we check that $rcK = \{u \in H^1(\Omega) : u \ge 0$ almost everywhere on $\Omega_0\}$. In fact, if $u \ge 0$ almost everywhere on Ω_0, then for all $t \ge 0$, all $w \in K$, we have

$$(w + tu)(x) = w(x) + tu(x) \ge w(x) \ge \zeta(x),$$

for almost every $x \in \Omega_0$; i.e., $w + tu \in K$, showing that $u \in rcK$. Conversely, if $u \in rcK$, then $nu \in rcK \subset K$, $\forall n$; i.e., $nu(x) \ge \zeta(x)$ for almost every $x \in \Omega_0$. Hence $u(x) \ge \frac{1}{n}\zeta(x)$ for almost every $x \in \Omega_0$, $\forall n$. Letting $n \to \infty$, we have $u \ge 0$ almost everywhere on Ω_0.

We have therefore $\ker a \cap rcK = \{u \in \mathbb{R} : u \ge 0 \text{ on } \Omega_0\} = [0,\infty)$ (since $|\Omega_0| > 0$).

Now suppose (2.62) is satisfied. Then for $u \in (\ker a \cap rcK)\backslash\{0\} = (0,\infty)$, one has

$$Q(u) = \int_\Omega G(x,u)\,dx = u^\beta \int_\Omega G(x,1)\,dx > 0.$$

Hence (2.47) holds, and our conclusion follows from Corollary 2.5.

(b) Since $\varphi \ge 0$ on V, we see that (2.63) implies (2.49). Together with (2.64), this shows that all the conditions in Corollary 2.6 (a) are satisfied, proving (b).

(c) From (2.66), we have that

$$\langle h, w\rangle = w\int_\Omega h = 0,\ \forall w \in \mathbb{R} = \ker a.$$

We thus have (2.51). Let $w \in \mathbb{R}\backslash\{0\}$. It follows from (2.65) that

$$Q(w) = \int_\Omega G(x,w)\,dx = \int_\Omega G(x,|w|\text{sign}w)\,dx = \begin{cases} w^\beta \displaystyle\int_\Omega G(x,1)\,dx & \text{if } w>0 \\ |w|^\beta \displaystyle\int_\Omega G(x,-1)\,dx & \text{if } w<0 \end{cases}$$
$$> 0.$$

Hence (2.50) is satisfied, and our conclusion follows From Corollary 2.6 (b). □

From Remark 2.2 (b), (c), we have that (2.64) is satisfied if ψ is α-homogeneous with respect to ξ ($\alpha > 1$), and either

- $\int_\Omega hz > 0$ for some $z \in H^1(\Omega),\ z \geq 0$ almost everywhere on Ω_0 and $\psi(\cdot, z) \in L^1(\Omega)$,

or

- $h \equiv 0$, $\beta < \min\{2, \alpha\}$, and

$$\int_\Omega G(x, z(x))\,dx < 0$$

for some $z \geq 0$ almost everywhere on Ω_0, and $\psi(\cdot, z) \in L^1(\Omega)$.

Now we consider a semilinear variational inequality containing the plate operator.

Example 2.2. Consider the following variational inequality:

$$\begin{cases} a(u, v-u) + \beta\displaystyle\int_\Omega [k|\nabla u|^{\beta-2}\nabla u\nabla(v-u) + r|u|^{\beta-2}u(v-u)] \\ \geq \displaystyle\int_\Omega h(v-u),\ \forall v \in K, \quad u \in K. \end{cases} \tag{2.67}$$

Here Ω is a bounded domain in $\mathbb{R}^2$, $V = H^2(\Omega)$, and $a : V \times V \to \mathbb{R}$ is the bilinear form in the plate theory (cf. Remark 1.3 (c)). We also assume the following:

- $h \in L^2(\Omega)$, $k, r \in L^\infty(\Omega)$ and $\beta > 1$, and
- $K = \{u \in H^2(\Omega) : \partial_\nu u \geq \psi \text{ on } \partial\Omega\}$, where $\psi \in C(\partial\Omega)$ is a given function.

K is a closed convex set in V, due to the continuity of the trace operator $H^2(\Omega) \to H^1(\partial\Omega)$. The minimization problem associated with (2.67) is in this case

$$u \in H^2(\Omega) : F(u) = \min_{v\in H^2(\Omega)} F(v), \tag{2.68}$$

with $F(v) = \frac{1}{2}a(v,v) + Q(u) + I_K(v) - \langle h, v\rangle,\ v \in H^2(\Omega)$, and

$$Q(u) = \int_\Omega [k|\nabla v|^\beta + r|v|^\beta], \quad v \in H^2(\Omega). \tag{2.69}$$

We assume that $k, r \geq 0$ (so that $Q \geq 0$) in the case $\beta \geq 2$.

Now, we have the following corollary of Theorem 2.3:

Corollary 2.8. *If either*

$$r = 0 \quad \text{and} \quad \int_\Omega h = 0, \qquad \text{or} \tag{2.70}$$

$$\int_\Omega r > 0, \tag{2.71}$$

then (2.67) *has a solution.*

Proof. As observed in Section 2.2, a satisfies the conditions in Proposition 2.1 (d). Hence F has property (P). Moreover

$$\ker a = \{w : \Omega \to \mathbb{R} : \exists q_0, q_1, q_2 \in \mathbb{R} : w(x) = q_0 + q_1 x_1 + q_2 x_2,\ x = (x_1, x_2) \in \Omega\}.$$

From (2.69), we see that Q is homogeneous of degree β on V. By the compactness of the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ and the continuity of the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ $(1 < q < \infty)$, we see that the mappings $u \mapsto |\nabla u|, |u|$ are completely continuous from $H^2(\Omega)$ to $L^\beta(\Omega)$. Hence Q given by (2.69) is completely continuous on V. Moreover, $Q \in C^1(V, \mathbb{R})$. We see that (2.2) and (2.3) are satisfied. Now we check that

$$rcK = \{w \in H^2(\Omega) :\ \partial_\nu w \geq 0 \ \text{ a.e. on } \ \partial\Omega\}. \tag{2.72}$$

In fact, if $\partial_\nu w \geq 0$ on $\partial\Omega$, then for $u \in K$, i.e., $\partial_\nu u \geq \psi$ on Ω, we have

$$\partial_\nu(u + tw) = \partial_\nu u + t\partial_\nu w \geq \partial_\nu u \geq \psi$$

on Ω; i.e., $u + tw \in K$, proving $w \in rcK$. Conversely, if $w \in rcK$ then fixing $u \in K$, we have

$$\partial_\nu u + t\partial_\nu w = \partial_\nu(u + tw) \geq \psi, \text{ for all } t > 0,$$

for almost every $x \in \partial\Omega$. Dividing this inequality by $t = t_n = n \in \mathbb{N}$, and letting $n \to \infty$, we obtain $\partial_\nu w \geq 0$ almost everywhere on Ω. Now, as in (2.39), we have from (2.72) (and the arguments used in [3]) that

$$\ker a \cap rcK = \mathbb{R}. \tag{2.73}$$

We now apply Theorem 2.3 and Corollary 2.5. From (2.70), (2.71), we have $\int_\Omega r \geq 0$, and then for $w \in \ker a \cap rcK = \mathbb{R}$, $|\nabla w| = 0$ on Ω, and

$$Q(w) = \int_\Omega r|w|^\beta = |w|^\beta \int_\Omega r \geq 0 \ (\forall w \in \mathbb{R}). \tag{2.74}$$

We have (2.44). Now, if $r = 0$ then by (2.74), $Q(w) = 0$, $\forall w \in \mathbb{R}$. For $w \in \mathbb{R}$, we have $-w \in \mathbb{R} \subset rcK$, and since $\varphi \equiv 0$ in our problem, (2.70) implies that

$$Q(v - w) = \int_\Omega k|\nabla(v - w)|^\beta = \int_\Omega k|\nabla v|^\beta = Q(v) = Q(v) - \int_\Omega hw, \quad \forall v \in K.$$

Hence, (2.46) is satisfied and our conclusion follows from Theorem 2.3.

If (2.71) holds, then from (2.74), we see that $Q(w) = 0$ only at $w = 0$. We have (2.47) and our conclusion follows from Corollary 2.5. □

By Remark 2.2 (b), (c), and Corollary 2.6, we see that if $\int_\Omega r = 0$ and either

- $\int_\Omega h = 0$ and $\exists\, z \in H^2(\Omega) : \partial_\nu z \geq 0$ on $\partial\Omega$ and $\int_\Omega hz < 0$, or
- $h \equiv 0$ on Ω and $\exists\, z \in H^2(\Omega) : \partial_\nu z \geq 0$ on $\partial\Omega$ and $\int_\Omega [k|\nabla z|^\beta + r|z|^\beta] < 0$, then

(2.67) has a nonconstant solution.

2.4. Further results on critical points. Let a, q, Q, h, φ, K be as in Section 2.1, and S, ψ be as in Section 1.1. Let F, Φ, j be defined by (2.4). We assume that $S \cap K \cap D(\varphi) \neq \emptyset$ and consider now the minimization of F on S.

$$u \in S : F(u) = \min_{v \in S} F(v), \tag{2.75}$$

or, equivalently,

$$u \in S \cap K : F(u) = \min_{v \in S \cap K} F(v). \tag{2.76}$$

Using the proof of Theorem 2.1, taking account now that $S \neq V$, we have the following result:

Theorem 2.4. *Let F given by* (2.4) *satisfy property* (P) *on S, and let the following compatibility condition hold*:

If $w \in \ker a \cap rcK$ is such that there exist sequences $\{u_n\}, \{w_n\}$ satisfying (2.7), (2.8), *and* (1.8), *then we have $-w \in rcK$,*

$$u - w \in S,\ \forall u \in S, \tag{2.77}$$

and (2.9). *Then* (2.75) *has a solution.*

We now consider particular situations where Theorem 2.4 and a Lagrange multiplier result for variational inequalities are used to obtain the existence of nontrivial critical points of variational inequalities. We assume that $h = 0$, Q and q are (nonnegative) homogeneous of degree 2, and $\psi \in C^1$ is homogeneous of degree $\alpha \neq 2$. We also assume that K is a cone in V, and

$$Q(v) \geq 0,\ \forall v \in K. \tag{2.78}$$

Let F have property (P) on S. Note that this holds if a, Q, φ, K satisfy one of the conditions (a)–(f) in Proposition 2.1. Under these conditions, we have:

Corollary 2.9. (a) *Assume the following compatibility condition*:

For all $w \in K \cap \ker \psi \cap \ker a \cap \ker \varphi \cap \ker Q$, we have

$$-w \in K \cap \ker \varphi \tag{2.79}$$

and

$$\psi(v - w) = \psi(v),\ \forall v \in V, \quad Q(v - w) \leq Q(v),\ \forall v \in K. \tag{2.80}$$

Then (2.75) *has a solution.*

(b) *Assume*:

(i) $\psi(u) < 0$ *for some* $u \in K \cap D(\varphi)$, (ii) $\psi(u) \geq 0$ *for all* $u \in K \cap \ker a \cap \ker \varphi \cap \ker Q$, *and if* $w \in K \cap \ker a \cap \ker \varphi \cap \ker Q$ *is such that* $\psi(u) = 0$ *then* (2.79) *and* (2.80) *are satisfied.*

Under these conditions, there exists a nontrivial solution $u \notin \ker \psi$ *of the following variational inequality*:

$$a(u, v-u) + \langle q(u) + \psi'(u), v-u \rangle + \varphi(v) - \varphi(u) \geq 0, \ \forall v \in K, \quad u \in K. \tag{2.81}$$

Proof. (a) We check the conditions in Theorem 2.4. Since K is a cone $K = rcK$. Let $w \in \ker a \cap K$ such that (2.7), (2.8), and (1.8) hold. We show that in fact $w \in \ker \psi \cap \ker \varphi \cap \ker Q$. Choosing in (1.8) $\lambda = \alpha$, and using the α-homogeneity of ψ, we get

$$\lim \psi(w_n) = \lim \frac{\psi(u_n)}{\|u\|^\alpha} = 0.$$

From the weak continuity of ψ and (2.7), we have $\psi(w) = \lim \psi(w_n) = 0$; i.e., $w \in \ker \psi$. Now, the homogeneity of φ implies

$$\varphi_\infty(w) = \lim_{t\to\infty} \frac{\varphi(tw)}{t} = \lim_{t\to\infty} t\varphi(w) = \begin{cases} 0 & \text{if } \ \varphi(w) = 0 \\ \infty & \text{if } \ \varphi(w) > 0 \end{cases} \tag{2.82}$$

(since $t \mapsto \varphi(tw)/t$ is increasing for $t > 0$, $\varphi_\infty(w) > -\infty$ thus $\varphi(w) \geq 0$). As in the proof of Theorem 2.3, we note that

$$\limsup \frac{Q(u_n)}{\|u_n\|} = \begin{cases} \infty & \text{if } \ Q(w) > 0 \\ \geq 0 & \text{if } \ Q(w) = 0. \end{cases} \tag{2.83}$$

In fact, if $Q(w) > 0$ then by the weak lower semicontinuity of Q,

$$\liminf Q\big(\frac{u_n}{\|u_n\|}\big) \geq Q(w) > 0.$$

Hence

$$\liminf \frac{Q(w_n)}{\|u_n\|} = \liminf \|u_n\| \frac{Q(u_n)}{\|u_n\|^2} = \liminf \|u_n\| Q\big(\frac{(u_n)}{\|u_n\|}\big) = \infty \cdot Q(w) = \infty.$$

However, since $u_n \in K$, we have $u_n/\|u_n\| \in K$ and $Q(u_n/\|u_n\|) \geq 0$ by (2.78). Thus (2.83) holds. From (2.82) and (2.83), we see that (2.8) holds; i.e.,

$$\varphi_\infty(w) + \limsup \frac{Q(u_n)}{\|u_n\|} \leq 0$$

only in the case $\varphi(w) = Q(w) = 0$. I.e., we have proved that $w \in K \cap \ker a \cap \ker \psi \cap \ker \varphi \cap \ker Q$. By hypothesis, (2.79) and (2.80) hold. In particular, $u - w \in S$, $\forall u \in S$. Now, let $\xi \in (0,1)$.

Since $-w \in \ker\varphi$, we have from the convexity and homogeneity of φ that

$$\begin{aligned}\varphi(v-w) &= \varphi\big[\xi\frac{v}{\xi} + (1-\xi)\frac{(-w)}{1-\xi}\big] \\ &\leq \xi\varphi\big(\frac{v}{\xi}\big) + (1-\xi)\varphi\big(\frac{-w}{1-\xi}\big) = \frac{\varphi(v)}{\xi} + \frac{\varphi(-w)}{1-\xi} = \frac{\varphi(v)}{\xi}.\end{aligned}$$

Letting $\xi \to 1^-$, one gets $\varphi(v-w) \leq \varphi(v),\ \forall v \in V$. Now, from the second condition of (2.80) follows that

$$\varphi(v-w) + Q(v-w) \leq \varphi(v) + Q(v),\ \forall v \in K.$$

Hence (2.9) is satisfied and our conclusion follows from Theorem 2.4.

(b) For the proof of (b), we use the following result of Kubrusly about a Lagrange multiplier theorem for variational inequalities:

Theorem (Theorem 2, [11])**.** *Let $H, G : V \to \mathbb{R}$ be two (Fréchet) differentiable functionals whose gradients are $A = H',\ B = G'$. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper, convex functional. Let $G_r = \{v \in V : G(v) = r\} \neq \emptyset$ $(r \in \mathbb{R}\backslash\{0\})$, and assume that $u \in G_r$ is a solution of the minimization problem:*

$$(H + \Psi)(u) = \min_{v \in G_r}(H + \Psi)(v). \tag{2.84}$$

Suppose furthermore that u satisfies the following condition:

$$\mathfrak{N}_u^+ \cap D(\Psi) \neq \emptyset, \quad \mathfrak{N}_u^- \cap D(\Psi) \neq \emptyset, \tag{2.85}$$

where

$$\mathfrak{N}_u^+ = \{v \in V : \langle B(u), v-u\rangle > 0\}, \quad \mathfrak{N}_u^- = \{v \in V : \langle B(u), v-u\rangle < 0\}.$$

Then there exists $\lambda \in \mathbb{R}$ such that (u, λ) satisfies the following variational inequality:

$$\langle A(u) - \lambda B(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in V. \tag{2.86}$$

We apply this result, together with (a) to prove (b). Since $\psi(u) < 0$ for some $u \in K \cap D(\varphi)$, and since K is a cone and φ is homogeneous, we have $|\psi(u)|^{-\alpha}u \in K$, $\varphi(|\psi(u)|^{-\alpha}u) < \infty$, and $\psi(|\psi(u)|^{-\alpha}u) = |\psi(u)|^{-1}\psi(u) = -1$. Letting $S = \{u \in V : \psi(u) = -1\}$, we have $S \cap K \cap D(\varphi) \neq \emptyset$, and $\inf_{v\in S} F(v) < \infty$.

Now, let $w \in K \cap \ker a \cap \ker Q \cap \ker\varphi \cap \ker\psi$. (2.79) and (2.80) follow from assumption (ii). Hence, by (a), (2.75) has a solution $u \in S$. Moreover, $u \in K \cap D(\varphi)$ by the above remark. Now, we split the functional F in (2.4) in another way: We put $G = \psi,\ r = -1$, and

$$H(u) = \tfrac{1}{2}a(u,u) + Q(u), \quad \Psi(u) = \varphi(u) + I_K(u), \quad \forall u \in V.$$

Then $G, H \in C^1(V)$, and

$$A = H',\ B = \psi' \quad \text{with} \quad \langle H'(u), v\rangle = a(u,v) + \langle q(u), v\rangle, \quad \forall u, v \in V. \tag{2.87}$$

Moreover, Ψ is a convex, proper functional on V and $D(\Psi) = K \cap D(\varphi)$. We have $F = H + \Psi$ and (2.75) (or (2.76)) becomes (2.84). We check the condition (2.85). Noting that $u \in S \cap K \cap D(\varphi)$, we have by the α-homogeneity of ψ that

$$\langle B(u), u\rangle = \langle \psi'(u), u\rangle = \alpha\psi(u) = -\alpha < 0. \tag{2.88}$$

Putting $v = tu,\ t > 0$, w have $v \in D(\Psi)$ and

$$\langle B(u), v - u\rangle = (t-1)\langle B(u), u\rangle = (1-t)\alpha.$$

This implies that

$$v = tu \in \mathfrak{N}_u^+ \cap D(\Psi) \quad (\text{respectively} \ \ \mathfrak{N}_u^- \cap D(\Psi)),$$

if $0 < t < 1$ (respectively $t > 1$). Hence $\mathfrak{N}_u^\pm \cap D(\Psi) \neq \emptyset$ and (2.85) is verified.

From the quoted result of Kubrusly it follows (using (2.86), (2.87)), that there exists $\mu \in \mathbb{R}$ such that

$$a(u, v-u) + \langle q(u) + \mu\psi'(u), v - u\rangle + (\varphi + I_K)(v) - (\varphi + I_K)(u) \geq 0,\ \forall v \in V,$$

or equivalently,

$$a(u, v-u) + \langle q(u) + \mu\psi'(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K, \quad u \in K. \tag{2.89}$$

Letting $v = \xi u \in K$ ($\xi > 0$) in this inequality, we get

$$\begin{aligned}&(\xi - 1)[a(u,u) + \langle q(u) + \mu\psi'(u), u\rangle] + \xi^2\varphi(u) - \varphi(u)\\ &= (\xi-1)[a(u,u) + \langle q(u) + \mu\psi'(u), u\rangle + (\xi+1)\varphi(u)] \geq 0,\ \forall \xi > 0.\end{aligned}$$

Dividing this inequality by $\xi - 1$ ($\xi \neq 1$), and letting $\xi \to 1^+$ and $\xi \to 1^-$, we obtain

$$a(u,u) + \langle q(u) + \mu\psi'(u), u\rangle + 2\varphi(u) = 0.$$

It follows from (2.88) that

$$-\mu\langle\psi'(u), u\rangle = \mu\alpha = a(u,u) + \langle q(u), u\rangle + 2\varphi(u) = a(u,u) + 2Q(u) + 2\varphi(u) \geq 0$$

(we have $Q(u), \varphi(u) \geq 0$ by the above observations). Hence $\mu \geq 0$. If $\mu = 0$ then $a(u,u) = Q(u) = \varphi(u) = 0$; i.e., $u \in K \cap \ker a \cap \ker Q \cap \ker\varphi$. On the other hand, $\psi(u) = -1$. This contradicts the assumption in (ii) that $\psi \geq 0$ on $K \cap \ker a \cap \ker Q \cap \ker\varphi$. Hence we must have $\mu > 0$. The last step is to rescale u to get a solution on (2.81). Let $u = \mu^{\frac{1}{2-\alpha}} w$ and $v = \mu^{\frac{1}{2-\alpha}} z$. We have $w \in K$, and $v \in K$ if and only if $z \in K$. (2.89) becomes:

$$\begin{aligned}0 &\leq a(\mu^{\frac{1}{2-\alpha}} w, \mu^{\frac{1}{2-\alpha}}(z-w)) + \langle q(\mu^{\frac{1}{2-\alpha}} w) + \mu\psi'(\mu^{\frac{1}{2-\alpha}} w), \mu^{\frac{1}{2-\alpha}}(z-w)\rangle\\ &\quad + \varphi(\mu^{\frac{1}{2-\alpha}} z) - \varphi(\mu^{\frac{1}{2-\alpha}} w)\\ &= \mu^{\frac{2}{2-\alpha}} a(w, z-w) + \mu^{\frac{1}{2-\alpha}}\langle \mu^{\frac{1}{2-\alpha}} q(w) + \mu^{1+\frac{\alpha-1}{2-\alpha}}\psi'(w), z-w\rangle + \mu^{\frac{2}{2-\alpha}}[\varphi(z) - \varphi(w)]\\ &= \mu^{\frac{2}{2-\alpha}}[a(w,z-w) + \langle q(w) + \psi'(w), z-w\rangle + \varphi(z) - \varphi(w)],\ \forall z \in K.\end{aligned}$$

This shows that $w \in K$ is a solution on (2.81). Moreover, $\psi(w) = \mu^{\frac{\alpha}{\alpha-2}}\psi(u) = -\mu^{\frac{\alpha}{\alpha-2}} < 0$, and hence $w \notin \ker\psi$ is a nontrivial solutions of (2.81). $\square$

The following is an immediate consequence of the above corollary.

Corollary 2.10. *If* (i) $\psi(u) < 0$ *for some* $u \in K \cap D(\varphi)$, *and* (ii) $\psi(u) > 0$ *for all* $u \in K \cap \ker a \cap \ker \varphi \cap \ker Q \backslash \{0\}$, *then the variational inequality* (2.81) *has a nontrivial solution* $u \notin \ker \psi$.

To prove this, we just need to note that if these assumptions hold and if w satisfies condition (b) (ii) then necessarily $w = 0$ and (2.79) and (2.80) are obviously satisfied.

Remark 2.3. (i) The above result of Kubrusly is a generalization of the Liusternik theorem for smooth functions. Condition (2.85) is a replacement for the assumption that u is a regular point (i.e., $G'(u) \neq 0$).

(ii) In the case $K = V, \varphi = 0, Q = 0$ then (2.81) becomes the nonlinear equation:

$$a(u, v) + \langle \psi'(u), v \rangle = 0, \ \forall v \in V, \quad u \in V,$$

and moreover, (2.79) and the second condition in (2.80) are immediately satisfied. Corollaries 2.9 and 2.10 therefore reduce to Corollaries 4.2, 4.3 in [13] about existence of nontrivial critical points of the functional $f(u) = \frac{1}{2}a(u, u) + \psi(u)$. Corollaries 2.9 and 2.10 are counterparts of these results for unilateral cases.

(iii) In the above discussion, we concentrate on the minimization problem (2.75) and the variational inequality (2.81) in the particular cases where F is decomposed into several summands as in (2.4). We can also consider these results for a general homogeneous functional F as in Section 1 and Theorem 1.1, Corollary 1.1. However, the particular settings in this section are more direct and convenient for applications.

Now, we consider the case where K is a cone and $\varphi = 0, Q = 0$. We obtain from Corollaries 2.9 and 2.10 the following consequence:

Corollary 2.11.

(a) *If* (i) $\psi(u) < 0$ *for some* $u \in K$, *and* (ii) $\psi(u) \geq 0$ *for all* $u \in K \cap \ker a$, *further if* $w \in K \cap \ker a \cap \ker \psi$ *then* $-w \in K$ *and* $\psi(v - w) = \psi(v)$, $\forall v \in V$, *then there exists a nontrivial solution* u *of the variational inequality*:

$$a(u, v - u) + \langle \psi'(u), v - u \rangle \geq 0, \ \forall v \in K, \quad u \in K. \tag{2.90}$$

(b) *If* (i) *in* (a) *is satisfied, and* $\psi(u) > 0$, $\forall u \in K \cap \ker a \backslash \{0\}$, *then* (2.90) *has a solution* $u \notin \ker \psi$.

Corollary 2.11 improves Corollary 2.1 of [7] by showing the existence of nontrivial solutions of (2.90) with any degree of homogeneity α of ψ, $\alpha \neq 2$, not necessarily $\alpha < 2$ as in that result. Hence Corollary 2.11 is the more natural and general extension of the result in [4] for the unilateral cases.

For instance, consider the problem (P_0) in [7]:

$$\int_0^T \dot{u}(v - u)^{\cdot} \, dt + \int_0^T \nabla_u V(t, u)(v - u) \, dt \geq 0, \ \forall v \in K, \quad u \in K, \tag{2.91}$$

where $K = \{u \in H_T^1 : u(x) \geq 0 \text{ on } [0, T]\}$. Applying Corollary 2.11 (b), we have the existence of nontrivial solutions of (2.91) even in the case V is homogeneous of degree $\alpha > 2$.

Moreover, in the case $Q = 0$, Corollaries 2.9 and 2.10 permit us to consider problems (P) in [7] in the limit case $\beta \neq 2$ and $\alpha = 2$ (in [7], the case $\alpha < \beta < 2$ is considered). For example, consider the variational inequality:

$$\begin{cases} \int_0^T \dot{u}(v-u)^{\cdot}\, dt + \int_0^T \nabla_u V(t,u)(v-u)\, dt + \int_0^T g(t)(|v|^2 - |u|^2)\, dt \geq 0, \\ \forall v \in H_T^1, \quad u \in H_T^1, \end{cases} \tag{2.92}$$

where V is as in (2.91) and g is a nonnegative bounded function on $[0,T]$. Since φ given by

$$\varphi(u) = \int_0^T g(t)|u|^2,\ u \in H_T^1,$$

is convex, continuous, and homogeneous of degree 2 on H_T^1, (2.92) is of the form (2.81) with $q = 0$ and $\psi(u) = \int_0^T V(t,u)\, dt$. Hence, one may apply Corollaries 2.9 and 2.10 to get conditions for the existence of nontrivial solutions of (2.92). Those results may also be applied, if in (2.92), the space H_T^1 is replaced by the cone K given in (2.91).

Next we will, once more, present an example of a unilateral problem that contains the operator in the plate theory. Consider the following variational inequality with a unilateral condition on the boundary:

$$\begin{cases} a(u, v-u) + \alpha \int_\Omega k|\nabla u|^{\alpha-2}\nabla u \nabla(v-u) + \int_{\partial\Omega} r\left[(\partial_\nu v)^+\right]^2 dS \\ - \int_{\partial\Omega} r\left[(\partial_\nu u)^+\right]^2 dS \geq 0,\ \forall v \in K, \quad u \in K. \end{cases} \tag{2.93}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, a is the bilinear form in the theory of plates (cf. Section 6.2, [13]), $\alpha > 0,\ \alpha \neq 2$, and $k \in L^\infty(\Omega)$. Moreover, $r \in L^\infty(\Omega),\ r \geq 0$, and $K = \{v \in H^2(\Omega) : D_\tau v \geq 0 \text{ on } \Omega_0\}$, Ω_0 is a subdomain of Ω, τ is a given (unit) vector in $\mathbb{R}^2$ and $D_\tau v = \nabla v \cdot \tau$ is the directional derivative of v along the direction τ. K is a closed, convex cone in $H^2(\Omega)$. We define

$$\psi(v) = \int_\Omega k|\nabla v|^\alpha, \quad Q(v) = 0, \quad \varphi(v) = \int_{\partial\Omega} r\left[(\partial_\nu v)^+\right]^2 dS,$$

for $v \in V = H^2(\Omega)$. By the compactness of the embedding $H^1(\Omega) \hookrightarrow L^\alpha(\Omega)$ and of the mapping $H^2(\Omega) \hookrightarrow [L^2(\partial\Omega)]^2,\ v \mapsto \nabla v|_{\partial\Omega}$, we see that $\psi \in C^1(V,\mathbb{R})$, ψ is completely continuous on V, and

$$\langle \psi'(u), v\rangle = \alpha \int_\Omega k|\nabla u|^{\alpha-2}\nabla u \nabla v, \quad \forall u, v \in V.$$

Moreover, φ is a convex, continuous mapping from V to $\mathbb{R}^+$ with $D(\varphi) = V$.

With these settings, we see that (2.93) is of the form (2.81) and constant functions are (trivial) solutions of (2.93). On the other hand, a satisfies the condition in Proposition 2.1 (e), hence F has property (P). Applying Corollary 2.9 (b), we get the following result:

Corollary 2.12. *If* $\int_\Omega k > 0$ *and* $\int_\Omega k|\nabla u|^\alpha < 0$ *for some* $u \in K$*, then* (2.93) *has a nontrivial solution.*

Proof. First, we note that condition (i) in Corollary 2.9 (b) holds by hypothesis. We also know that $\ker a$ is the set of all affine polynomials restricted to Ω.

Let $w \in K \cap \ker a$, $w(x) = p_1 x_1 + p_2 x_2 + q$, $(x_1, x_2) \in \Omega$, and $D_\tau w \geq 0$ on Ω_0. We have

$$\psi(w) = (p_1^2 + p_2^2)^{1/2} \int k.$$

Hence $\psi(w) \geq 0$ and $\psi(w) = 0$ only if $p_1 = p_2 = 0$; i.e., $w \in \mathbb{R}$. Consequently, $D_\tau(-w) = D_\tau w = 0$ on Ω_0, and therefore $-w \in K$. Moreover, $\partial_\nu w = 0$ on $\partial\Omega$ and thus $\varphi(w) = 0$. On the other hand,

$$\nabla(v - w) = \nabla v \quad \text{on} \;\; \Omega,$$

hence $\psi(v - w) = \psi(v)$, $\forall v \in V$. Thus we have shown that the conditions (2.79) and (2.80) are satisfied. Our conclusion now follows from Corollary 2.9 (b). $\square$

The above results can also be used to establish the existence of nontrivial solutions of variational inequalities (or equations) containing two nonconvex nonlinearities. In these cases, we can split the nonlinearities, consider one part of the functional, and the other part of the constraint surface. This is illustrated by the following example. Consider the variational inequality:

$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla(v - u) + \int_{\Omega_0} p[u(\zeta_1(x))(v - u)(\zeta_2(x)) + u(\zeta_2(x))(v - u)(\zeta_1(x))]\, dx \\ \displaystyle\qquad + \alpha \int_\Omega k|u|^{\alpha-2}u(v - u) \geq 0, \; \forall v \in K, \quad u \in K. \end{cases} \tag{2.94}$$

Here $1 < \alpha < 2^*$, $\alpha \neq 2$, Ω is a bounded domain in $\mathbb{R}^N$, $K = \{u \in H^1(\Omega) : u \geq 0 \text{ on } \Omega\}$, Ω_0 is a subdomain of Ω, and $k \in L^\infty(\Omega), p \in L^\infty(\Omega_0)$, $p \geq 0$ on Ω_0. Assume furthermore that for $i = 1, 2$, ζ_i is a diffeomorphism from Ω_0 onto $\zeta_i(\Omega_0) \subset \Omega$ such that $J(\zeta_i^{-1})$ is bounded. From this assumption, by changing variables, we see that $u \circ \zeta_i \in L^2(\Omega_0)$ whenever $u \in L^2(\Omega)$, and moreover,

$$\|u \circ \zeta_i\|_{L^2(\Omega_0)} \leq \|u\|_{L^2(\Omega)}, \; u \in L^2(\Omega), \tag{2.95}$$

for some $C > 0$; i.e., the (linear) mapping $u \mapsto u \circ \zeta_i$ is bounded from $L^2(\Omega)$ to $L^2(\Omega_0)$. This shows that the second integral in (2.94) is well defined. Moreover, let $V = H^1(\Omega)$ and

$$Q(u) = \int_{\Omega_0} pu(\zeta_1(x))u(\zeta_2(x))\, dx, \; u \in V.$$

It follows from (2.95) that $Q \in C^1(V, \mathbb{R})$ and

$$\langle Q'(u), v\rangle = \int_{\Omega_0} p[u(\zeta_1(x))v(\zeta_2(x)) + u(\zeta_2(x))v(\zeta_1(x))]\, dx, \; \forall u, v \in V. \tag{2.96}$$

We also note that Q is not convex in general. Some simple examples of Q are the following:

• Ω is symmetric, and we choose $\Omega_0 = \Omega$, $\zeta_1(x) = x$, $\zeta_2(x) = -x$; i.e.,

$$Q(u) = \int_\Omega pu(x)u(-x)\,dx.$$

• $\Omega \subset \mathbb{R}^2$ and Ω_0 is a disc centered at 0 with radius ρ. Then one can choose ζ_1, ζ_2 to be rotations by angles θ_1, θ_2:

$$\zeta_i(x) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} x, \quad x = (x_1, x_2)^T \in \Omega_0.$$

With these settings, we see that (2.94) is of the form (2.81). Again, 0 is always a solution of (2.94). We have the following result for the existence of other solutions:

Corollary 2.13. *If $\int_\Omega k > 0$ and k changes sign on Ω, then* (2.94) *has a nontrivial solution.*

Proof. We note that K is a closed, convex cone in $V = H^1(\Omega)$. Moreover, $Q \geq 0$ on K (since $\zeta_i(\Omega_0) \subset \Omega$ and $p \geq 0$) and Q is homogeneous of degree 2 on V.

Let $\psi(u) = \int_\Omega k|u|^\alpha$, $u \in V$. Since $\alpha < 2^*$, ψ is completely continuous on V. Moreover, ψ is homogeneous of degree α on V, $\psi \in C^1(V, \mathbb{R})$ and $\langle \psi'(u), v\rangle = \alpha \int_\Omega k|u|^{\alpha-2}uv$, $u, v \in V$. ψ is not convex on V since k changes sign on Ω. However, using this assumption, we can choose a function $u \geq 0$ on Ω (i.e., $u \in K$) such that $\int_\Omega k|u|^\alpha = \psi(u) < 0$.

Letting $\varphi = 0$, we see that condition (i) in Corollary 2.10 holds.

As usual, we see that a satisfies the assumptions in Proposition 2.1 and F has Property (P). Furthermore, $\ker a = \mathbb{R}$ and therefore $K \cap \ker a \cap Q \subset \mathbb{R}^+$. If $u \in K \cap \ker a \cap Q \setminus \{0\}$ then u is a positive constant and then

$$\psi(u) = u^\alpha \int_\Omega k > 0.$$

We have condition (ii) of Corollary 2.10. The existence of nontrivial solutions of (2.94) follows from that corollary. $\square$

To conclude, we establish the existence of nontrivial solutions of a Neumann problem containing the p-Laplacian and nonlinearities of different exponents. Consider the following variational inequality:

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u) + s\int_\Omega k|u|^{s-2}u(v-u) + r\int_\Omega g|u|^{r-2}u(v-u) \geq 0, \\ \forall v \in K, \quad u \in K. \end{cases} \tag{2.97}$$

Here Ω is as in the previous example, $1 < r \leq p \leq s < p^*$, and $k, g \in L^\infty(\Omega)$, $k \geq 0$ on Ω, $K = \{u \in H^1(\Omega) : u \geq \zeta \text{ on } \Omega\}$, where Γ is a (relatively) open subset on $\partial\Omega$, $\zeta \in L^\infty(\Gamma)$, $\zeta \leq 0$ on Γ.

(2.97) is the weak formulation of the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + sk|u|^{s-2}u + rg|u|^{r-2}u = 0 \quad \text{in} \ \ \Omega \tag{2.98}$$

with the mixed Neumann and unilateral boundary condition:

$$\begin{cases} \partial_p u = 0 \ \text{ on } \ \partial\Omega\backslash\Gamma \\ \left.\begin{cases} u-\zeta \geq \ 0 \\ \partial_p u \geq \ 0 \\ (u-\zeta)\partial_p u = \ 0 \end{cases}\right\} \ \text{on} \ \ \Gamma, \end{cases} \tag{2.99}$$

$\partial_p u = |\nabla u|^{p-2}\partial_\nu u$. If $\Gamma = \emptyset$, then (2.99) becomes the Neumann boundary condition: $\partial_p u = 0$ on $\partial\Omega$; and if $\Gamma = \partial\Omega$ then we have a unilateral boundary condition on $\partial\Omega$. Note that equations similar to (2.98) (with Dirichlet boundary condition) were studied in [2] and [9].

Let $V = W^{1,p}(\Omega)$, $\varphi(u) = \int_\Omega k|u|^s$, $Q(u) = \int_\Omega g|u|^r$, and

$$\langle q(u), v\rangle = \int_\Omega rg|u|^{r-2}uv, \ u, v \in V.$$

Then φ is convex, continuous on V, $Q \in C^1(V, \mathbb{R})$ and $Q' = q$. By using standard arguments, we can check that (2.97) is equivalent to the following variational inequality:

$$\langle A(u), v-u\rangle + \langle q(u), v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \ \forall v \in K, \quad u \in K. \tag{2.100}$$

Here $\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v$, and $A = \alpha'$ with $\alpha \in C^1(V, \mathbb{R})$ given by

$$\alpha(u) = \frac{1}{p}\int_\Omega |\nabla u|^p. \tag{2.101}$$

The solutions of (2.100) are critical points of the functional $F = \alpha + Q + \varphi$ on K. Note that since α is coercive off its kernel, F has property (P) on V and hence on K.

With some modifications, we can prove results that are extensions of Theorems 1.1, 2.1, 2.2, 2.3, 2.4, Proposition 2.1, and Corollaries 1.1, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.9, 2.10, 2.11 for variational inequalities in Banach spaces. We merely need to replace (1.4) in the definition of property (P) by the assumption:

$$w_n = v_n/\|v_n\| \rightharpoonup w, \quad \text{and} \quad \|v_n\| \leq \|v_n - \lambda w\|, \ \forall n, \ \forall \lambda \geq 1,$$

and we consider, instead of (2.1), the variational inequality:

$$\langle A(u), v-u\rangle + \langle q(u), v-u\rangle + \varphi(v) - \varphi(u) \geq \langle h, v-u\rangle, \tag{2.102}$$

where $A = \alpha'$ with $\alpha : V \to \mathbb{R}$ a continuous bounded and Gâteaux-differentiable functional, Φ in (2.4) is replaced by $\Phi(u) = \alpha(u)$.

We see now that φ (respectively Q) is homogeneous of degree p (respectively r), and then have the following result:

Corollary 2.14. *If* $\int_\Omega g > 0$ *and* g *changes sign on* Ω *then* (2.97) *has a nontrivial solution.*

Proof. We have in this case $rcK = \{u \in V : u \geq 0 \text{ on } \Gamma\}$. Hence

$$rcK \cap \ker \alpha = \begin{cases} \mathbb{R} & \text{if } \Gamma \neq \emptyset \\ \mathbb{R}^+ & \text{if } \Gamma = \emptyset. \end{cases}$$

In all cases, $Q(u) = |u| \int_\Omega g > 0$, $\forall u \in (rcK \cap \ker \alpha)\backslash\{0\}$. Since $\varphi \geq 0 \equiv h$, (2.51) is satisfied. Moreover, since g changes sign on Ω, there exists $u \in C_0^1(\Omega)$ such that $\int_\Omega g|u|^r < 0$. Since $u \in rcK$, (2.54) and then (2.48) are satisfied. Our conclusion follows from the counterpart of Corollary 2.6 for Banach spaces. $\square$

In the case $\zeta = 0$ (or $\gamma = \emptyset$) and $k = 0$ or $s = p$, we can apply the Banach space version of Corollary 2.10 instead of Corollaries 2.5 and 2.6 to see that Corollary 2.14 still holds for $r \in (p, p^*)$.

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