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CONTROLLABILITY OF THE LINEAR SYSTEM OF THERMOELASTIC PLATES

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Abstract. We prove that the linear system of thermoelastic plates, with finite speed of propagation of the elastic components, is controllable in the following sense: If the control time is large enough and we act in the equation of displacement by means of a control supported in a neighborhood of the boundary of the plate, then we may control exactly the displacement and simultaneously the temperature in an approximate way. The method of proof is an adaptation of the techniques developed by the second author for the proof of the exact-approximate controllability for the three-dimensional system of thermolelasticity and combines: (i) a decoupling result based on an idea due to Henry, Lopes and Perissinotto for three-dimensional thermoelasticity, (ii) the variational approach to controllability developed by Fabre, Puel and Zuazua and (iii) some observability inequalities for the system of thermoelastic plates.

1. Introduction and main results. Let Ω be a bounded, open, connected set in $\mathbb{R}^2$ with boundary $\partial\Omega$ of class C^3. Let $T > 0$ and set

$$Q = \Omega \times (0, T), \qquad \Sigma = \partial\Omega \times (0, T).$$

We consider the following system which describes the small vibrations of a thin, homogeneous, isotropic thermoelastic plate in absence of exterior forces and heat sources:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \beta_1 \Delta\theta = 0 & \text{in } Q \\ \beta_2\theta_t - \Delta\theta - \beta_3\Delta w_t = 0 & \text{in } Q \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma \\ \theta = 0 & \text{on } \Sigma \\ w(0) = w^0;\ w_t(0) = w^1;\ \theta(0) = \theta^0 & \text{in } \Omega \end{cases} \tag{1}$$

where $\gamma, \beta_i > 0$. The reader is referred to [11] for a heuristic derivation of the model.

Lagnese ([10]) proved that the system (1) is well posed in $V = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. More precisely, for every initial data $(w^0, w^1, \theta^0) \in V$ there exists a unique solution

$$(w, w_t, \theta) \in C([0, T], V). \tag{2}$$

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This solution is given by $(w(t), w_t(t), \theta) = S(t)(w^0, w^1, \theta^0)$ for $t > 0$ where $S(t) : V \to V$ is the semigroup generated by system (1).

On the other hand, multiplying in (1) by $(w_t, \beta_1\theta/\beta_3)$ we see that the energy

$$E(t) = \frac{1}{2}\left[\int_\Omega |\Delta w|^2 + \int_\Omega |w_t|^2 + \gamma|\nabla w_t|^2 + \frac{\beta_1\beta_2}{\beta_3}\int_\Omega \theta^2\right]$$

satisfies

$$\frac{dE}{dt} = -\frac{\beta_1}{\beta_3}\int_\Omega |\nabla\theta|^2.$$

More precisely,

$$E(t) + \frac{\beta_1}{\beta_3}\int_0^t\int_\Omega |\nabla\theta|^2\,dx\,ds = E(0).$$

We fix a control time $T > 0$ and a control region ω; an open and nonempty subset of Ω. We are allowed to act on the system through the equations of displacement by means of a control function $f(x,t) \in L^1((0,T); H^{-1}(\Omega))$ that represents an exterior force. The support of the control is restricted to the control region ω so that $\operatorname{supp} f(\cdot,t) \subset \omega$ for almost every $t \in (0,T)$. The thermoelastic plate system in presence of the control f reads as follows:

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \beta_1\Delta\theta = f & \text{in } Q \\ \beta_2\theta_t - \Delta\theta - \beta_3\Delta w_t = 0 & \text{in } Q \\ w = \dfrac{\partial w}{\partial\nu} = 0 & \text{on } \Sigma \\ \theta = 0 & \text{on } \Sigma \\ w(0) = w^0;\ w_t(0) = w^1;\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{3}$$

The existence of a unique solution $(w, w_t, \theta) \in C([0,T]; V)$, in terms of the semigroup S, may be proved by the variation-of-constants formula.

The controllability problem we are considering is the following: Find sufficient conditions on (ω, T) (control region and control time) such that for every initial and final data (w^0, w^1, θ^0), $(v^0, v^1, \xi^0) \in V$ and for every $\varepsilon > 0$ there exists a control f such that the solution of (3) satisfies

$$w(T) = v^0,\ w_t(T) = v^1, \quad \|\theta(T) - \xi^0\|_{L^2(\Omega)} \le \varepsilon. \tag{4}$$

In other words, we request the exact controllability of the displacement and the approximate controllability of the temperature. In the sequel, if this property holds we will say that the system (3) is *exact-approximately controllable.* We observe that we can expect (4) to hold only if T is large enough, since, due to the fact that $\gamma \neq 0$, the elastic components of the system (3) propagate with finite speed. In Section 6 we comment briefly the different nature of the control problem when $\gamma = 0$.

Control problems for thermoelastic systems have been studied intensively in the last years, since it is a modelic system in which the conservative effect of the plate equation with the dissipative one of the heat equation are combined . In classic literature (see Lagnese-Lions, [11], or Lions [13], Volume II) some *partial controllability* results are established. That is, the displacement is exactly controlled but no information is provided about the behaviour of the temperature.

In the case of a three-dimensional thermoelastic body the notion of exact-approximate controllability (exact in the displacement and simultaneously approximate in the temperature) has been introduced by the second author in [22]. The results of the present paper are proved by adapting Zuazua's methods to the thermoelastic plate system.

As it is by now well known, to obtain the exact controllability of wave-type equations it is necessary that the support of the control ω satisfies some geometric properties. Namely, the *geometric control conditions* introduced by G. Bardos, G. Lebeau and J. Rauch ([2]). We will restrict ourselves to the simplest case in which ω is a neighborhood of the whole boundary. Other situations will be discussed in Section 6.2.

Thus we shall assume that $\omega \subset \Omega$ is a neighborhood of $\partial\Omega$ (in Ω), i.e., that there exists some neighborhood $\Theta \subset \mathbb{R}^2$ of $\partial\Omega$ such that $\omega = \Omega \cap \Theta$.

We have the following result:

Theorem 1. *Let ω be a neighborhood of $\partial\Omega$ in Ω. Suppose that $T > \sqrt{\gamma}\,\mathrm{diam}(\Omega\backslash\omega)$. Then, system* (3) *is exact-approximately controllable in time T with controls supported in ω.*

One of the main ingredients of the proof of Theorem 1 is the following observability inequality for the adjoint system of thermoelasticity:

$$\begin{cases} \varphi_{tt} - \gamma\Delta\varphi_{tt} + \Delta^2\varphi + \beta_3\Delta\psi_t = 0 & \text{in } Q \\ -\beta_2\psi_t - \Delta\psi + \beta_1\Delta\varphi = 0 & \text{in } Q \\ \varphi = \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma \\ \psi = 0 & \text{on } \Sigma \\ \varphi(T) = \varphi^0;\ \varphi_t(T) = \varphi^1;\ \psi(T) = \psi^0 & \text{in } \Omega. \end{cases} \tag{5}$$

Proposition 1. *Under the assumptions of Theorem* 1, *for every bounded set B of $L^2(\Omega)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_\omega |\nabla\varphi|^2\,dx\,dt$$

for every solution of (5) *with initial data such that*

$$\|\varphi^0\|_{H_0^1(\Omega)} + \|\tilde{C}_0\varphi^1 + \beta_3\Delta\psi^0\|_{H^{-2}(\Omega)} \geq 1, \quad \psi^0 \in B,$$

where $\tilde{C}_0$ is an isomorphism from $\tilde{H}$ onto $H^{-2}(\Omega)$ and $\tilde{H}$ is the dual space of $H_0^2(\Omega)$ with respect to $H_0^1(\Omega)$ (*when the latter is identified with its dual space*).

Remark 1. In the following section we give a precise characterization of the space $\tilde{H}$ and the isomorphism $\tilde{C}_0$.

By suitably adapting the methods developed by Zuazua ([22]), J.L. Lions ([15]) and Fabre, Puel and Zuazua in [5]–[6] we will show that Theorem 1 is a consequence of Proposition 1 and Holmgren's Uniqueness Theorem. As we shall see, the core of the proof of exact-approximate controllability is this proposition in the sense that, roughly, if T and ω are such that the observability inequality of Proposition 1 holds, then Theorem 1 takes place. To prove Proposition 1 we will combine multiplier techniques, Holmgren's Uniqueness Theorem and the following decoupling result, an adaptation of an analogous result for the linear system of three-dimensional thermoelasticity due to Henry, Lopes and Perissinotto ([8]):

Theorem 2. *Let us denote by* $\{S^0(t)\}_{t\geq 0}$ *the semigroup associated to the following decoupled system*:

$$\begin{cases} \tilde{w}_{tt} - \gamma\Delta\tilde{w}_{tt} + \Delta^2\tilde{w} - \beta_3\beta_1\Delta\tilde{w}_t = 0 & \text{in } Q \\ \beta_2\tilde{\theta}_t - \Delta\tilde{\theta} - \beta_3\Delta\tilde{w}_t = 0 & \text{in } Q \\ \tilde{w} = \dfrac{\partial\tilde{w}}{\partial\nu} = 0 & \text{on } \Sigma \\ \tilde{\theta} = 0 & \text{on } \Sigma \\ \tilde{w}(0) = w^0;\ \tilde{w}_t(0) = w^1;\ \tilde{\theta}(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{6}$$

Then, $S(t) - S^0(t) : V \to C([0,T]; V)$ *is continuous and compact.*

Remark 2. Observe that system (6) is obtained from (1) by replacing θ, in the last term of the first equation, by $-\beta_3 w_t$. In view of the boundary conditions this is equivalent to the following equation: $-\Delta\theta - \beta_3\Delta w_t = 0$. This equation is obtained from the second equation of (1) by neglecting the term θ_t. As a consequence, (6) is obtained from (1) by substituting the parabolic equation of (1) by its elliptic version.

Observe that system (6) is uncoupled. It is possible to obtain $\tilde{w}$ independently of $\tilde{\theta}$ by solving the first equation, which is a damped plate equation. This allows us, in particular, to apply J. L. Lions's multiplier techniques (e.g. [12], [11]).

The rest of the paper is organized as follows. In Section 2 we give some preliminary existence results of solutions for the adjoint system (5) and we prove Theorem 2. In Section 3 we give some consequences of Holmgren's Uniqueness Theorem that we will use in the proof of our results. In Section 4 we prove the observability inequality of Proposition 1. In Section 5 we prove the controllability result (Theorem 1). In the last section we briefly analyze the case $\gamma = 0$ and discuss some open problems that arise when ω is not a neighborhood of the whole boundary. At the end of the paper, in an Appendix, we prove in detail some estimates used in the proof of Proposition 1.

2. Preliminaries.

2.1. Existence of solutions for the adjoint system. In order to make precise the notion of solution of the adjoint system (5) we introduce the space $\tilde{H}$ mentioned in Proposition 1. We follow Lagnese ([10]). First of all we introduce the following scalar product in $H_0^1(\Omega)$ given by

$$c_0(v;\hat{v}) = (v,\hat{v}) + \gamma(\nabla v, \nabla\hat{v})$$

where $(: , :)$ denotes both the usual scalar product in $L^2(\Omega)$ and in $(L^2(\Omega))^2$. For $v \in H_0^1(\Omega)$ define $\tilde{v} \in H_0^2(\Omega)$ by

$$\int_\Omega \Delta\tilde{v}\Delta\hat{v}\,dx = c_0(v;\hat{v}), \quad \forall\hat{v} \in H_0^2(\Omega).$$

This is equivalent to solving

$$\begin{cases} \Delta^2\tilde{v} = v - \gamma\Delta v & \text{in } \Omega \\ \tilde{v} \in H_0^2(\Omega). \end{cases}$$

We define $\|v\|_{\tilde{H}} = \|\Delta\tilde{v}\|_{L^2(\Omega)}$ and $\tilde{H}$ as the completion of $H_0^1(\Omega)$ with the norm $\|\cdot\|_{\tilde{H}}$. We have

$$\|v\|_{\tilde{H}} \leq K\|v\|_{H_0^1},\ \forall v \in H_0^1(\Omega),$$

so that $H_0^1 \subset \tilde{H}$ algebraically and topologically.

Let $C_0 = I - \gamma\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$. Then C_0 is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. If $v \in H_0^1(\Omega)$ and $\hat{v} \in H_0^2(\Omega)$, then

$$|\langle C_0 v, \hat{v}\rangle| = |c_0(v,\hat{v})| = |\int_\Omega \Delta\tilde{v}\Delta\hat{v}| \leq \|v\|_{\tilde{H}}\|\hat{v}\|_{H_0^2};$$

hence

$$\|C_0 v\|_{H^{-2}(\Omega)} \leq \|v\|_{\tilde{H}}.$$

Therefore, C_0 can be extended by continuity to a bounded linear operator $\tilde{C}_0$ from $\tilde{H}$ in $H^{-2}(\Omega)$, and $v \in \tilde{H}$ if and only if there exists $\hat{v} \in H_0^2(\Omega)$ such that $\tilde{C}_0 v = \Delta^2\hat{v}$. It follows from this characterization that $\tilde{C}_0$ has closed range in $H^{-2}(\Omega)$ and is an isomorphism from $\tilde{H}$ onto its range. On the other hand, the range of $\tilde{C}_0$ contains $H^{-1}(\Omega)$ and is therefore dense in $H^{-2}(\Omega)$. It follows that $\tilde{C}_0$ is an isomorphism from $\tilde{H}$ onto $H^{-2}(\Omega)$.

Lagnese ([10]) observes that if $v \in H_0^1(\Omega)$,

$$\|v\|_{\tilde{H}} = \sup_{\hat{v}\in H_0^2} \frac{|c_0(v;\hat{v})|}{\|\hat{v}\|_{H_0^2}}. \tag{7}$$

Therefore $\tilde{H}$ is the dual space of $H_0^2(\Omega)$ with respect to $H_0^1(\Omega)$ (when the latter is identified with its dual space).

We can give now the existence and uniqueness theorem for solutions of the adjoint system (see Theorem 2.6 in Lagnese ([10]) for the proof).

Proposition 2. *If $\{\varphi^0, \varphi^1, \psi^0\} \in H_0^1(\Omega) \times \tilde{H} \times L^2(\Omega)$ there exists a unique triplet*

$$\{\varphi, \varphi_t, \psi\} \in C([0,T]; H_0^1(\Omega) \times \tilde{H} \times L^2(\Omega)),$$

a solution of (5) *in the following sense: For every $\hat{\varphi} \in H_0^2(\Omega)$ and $t \in [0,T]$ we have*

$$\begin{aligned} \langle \tilde{C}_0\varphi_t, \hat{\varphi}\rangle - \langle \tilde{C}_0\varphi^1, \hat{\varphi}\rangle - \int_\Omega \Delta(\int_t^T \varphi(s)\,ds)\Delta\hat{\varphi}\,dx \\ + \beta_3\int_\Omega \psi(t)\Delta\hat{\varphi}\,dx - \beta_3\int_\Omega \psi^0\Delta\hat{\varphi}\,dx = 0 \end{aligned} \tag{8}$$

and for every $\hat{\psi} \in H_0^1(\Omega)$ and $t \in [0,T]$:

$$-\left\langle \left(\beta_2\psi(t) + \beta_1\Delta\int_t^T \varphi(s)\,ds\right)_t, \hat{\psi}\right\rangle - \int_\Omega \nabla\psi(t)\nabla\hat{\psi}\,dx = 0. \tag{9}$$

Remark 3. It is not difficult to see that Proposition 2 is a consequence of the first result of existence and uniqueness of finite energy solutions we mentioned for system (1). If we make the change of variables

$$w(t) = -\int_t^T \varphi(s)\,ds + \xi(x); \quad \theta(t) = \psi(t)$$

we observe that if $\xi \in H_0^2(\Omega)$ satisfies $\Delta^2\xi = -\beta_3 - \tilde{C}_0\varphi^1$ then (w,θ) solves a system analogous to (1), but backward in time. More precisely

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \beta_3\Delta\theta = 0 & \text{in } Q \\ -\beta_2\theta_t - \Delta\theta + \beta_1\Delta w_t = 0 & \text{in } Q \\ w = \dfrac{\partial w}{\partial\nu} = 0 & \text{on } \Sigma \\ \theta = 0 & \text{on } \Sigma \\ w(T) = \xi,\ w_t(T) = \varphi^1;\ \theta(T) = \psi^0 & \text{in } \Omega. \end{cases} \tag{10}$$

We deduce the existence and uniqueness of a solution $(w,\theta) \in C([0,T];V)$. Then the solution of (5) we were looking for is $(\varphi,\psi) = (w_t,\theta)$ and satisfies the conditions of Proposition 2.

2.2. Proof of Theorem 2. Let $(w^0,w^1,\theta^0) \in V$. Then, there exists a unique solution $(\tilde{w},\tilde{\theta})$ of the decoupled system (6). The solution $\tilde{w}$ of the plate equation (first equation in (6)), has the following regularity (see [10]):

$$\tilde{w} \in C([0,T];H_0^2(\Omega)) \cap C^1([0,T];H_0^1(\Omega)) \cap C^2([0,T];\tilde{H}).$$

Therefore the solution of the second equation, which is the heat equation, has the following regularity (see e.g. [16], Volume 1, page 257):

$$\tilde{\theta} \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)).$$

Let us analyze separately the plate equation of system (6) that the displacement $\tilde{w}$ satisfies:

$$\begin{cases} \tilde{w}_{tt} - \gamma\Delta\tilde{w}_{tt} + \Delta^2\tilde{w} - \beta_3\beta_1\Delta\tilde{w}_t = 0 & \text{in } Q \\ \tilde{w} = \dfrac{\partial\tilde{w}}{\partial\nu} = 0 & \text{on } \Sigma \\ \tilde{w}(0) = w^0;\ \tilde{w}_t(0) = w^1 & \text{in } \Omega. \end{cases} \tag{11}$$

We define the energy

$$E(t) = \frac{1}{2}\left(\int_\Omega |\tilde{w}_t(t)|^2 + \gamma\int_\Omega |\nabla\tilde{w}_t(t)|^2 + \int_\Omega |\Delta\tilde{w}(t)|^2\right),$$

that satisfies

$$\frac{dE}{dt} = -\beta_1\beta_3\int_\Omega |\nabla\tilde{w}_t(t)|^2.$$

Thus

$$E(T) = E(0) - \beta_1\beta_3\int_0^T\int_\Omega |\nabla\tilde{w}_t|^2\,dx\,dt.$$

On the other hand, the trace of $\tilde{w}$ over Σ has additional regularity properties:

Lemma 1. *Let $\tilde{w}$ be the solution of* (11) *with initial data in $H_0^2(\Omega) \times H_0^1(\Omega)$. Then*

$$\Delta \tilde{w} \in L^2(\Sigma).$$

Moreover, there exists a constant $C > 0$ independent of w and T such that:

$$\int_\Sigma |\Delta \tilde{w}|^2 \leq C(1+T)E(0).$$

Proof. It is enough to prove this result for regular solutions

$$w \in C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H_0^2(\Omega)).$$

The estimates can be extended to finite energy solutions by density arguments.

Since Ω is of class C^3, there exists $h = h(x) \in C^2(\bar{\Omega})$ such that $h = \nu$ in $\partial\Omega$. We multiply (11) by $h_k \frac{\partial \tilde{w}}{\partial x_k}$ and integrate over Q. We have

$$\int_0^T\!\!\int_\Omega \tilde{w}_{tt} h_k \frac{\partial \tilde{w}}{\partial x_k} - \gamma \int_0^T\!\!\int_\Omega \Delta \tilde{w}_{tt} h_k \frac{\partial \tilde{w}}{\partial x_k} + \int_0^T\!\!\int_\Omega \Delta^2 \tilde{w} h_k \frac{\partial \tilde{w}}{\partial x_k} - \beta_1\beta_3 \int_0^T\!\!\int_\Omega \Delta \tilde{w} h_k \frac{\partial \tilde{w}}{\partial x_k} = 0. \tag{12}$$

Let us compute separately the various terms in (12). First we have

$$\int_0^T\!\!\int_\Omega \tilde{w}_{tt} h_k \frac{\partial \tilde{w}}{\partial x_k} = \frac{1}{2} \int_0^T\!\!\int_\Omega \frac{\partial h_k}{\partial x_k} |\tilde{w}_t|^2 + X,$$

where

$$|X| = \left| \int_\Omega \tilde{w}_t h_k \frac{\partial \tilde{w}}{\partial x_k} dx \Big|_0^T \right| \leq CE(0)$$

and C is a constant that depends on $|h_k|_{L^\infty(\Omega)}$.

For the second term we obtain:

$$\begin{aligned} -\gamma \int_0^T\!\!\int_\Omega \Delta \tilde{w}_{tt} h_k \frac{\partial \tilde{w}}{\partial x_k} &= \gamma \int_0^T\!\!\int_\Omega \Delta \tilde{w}_t h_k \frac{\partial \tilde{w}_t}{\partial x_k} - \gamma \int_\Omega \Delta \tilde{w}_t h_k \frac{\partial \tilde{w}}{\partial x_k} \Big|_0^T \\ &= -\frac{\gamma}{2} \int_0^T\!\!\int_\Omega h_k \frac{\partial}{\partial x_k} |\nabla \tilde{w}_t|^2 - \gamma \int_0^T\!\!\int_\Omega \frac{\partial \tilde{w}_t}{\partial x_i} \frac{\partial h_k}{\partial x_i} \frac{\partial \tilde{w}_t}{\partial x_k} + Z \\ &= \frac{\gamma}{2} \int_0^T\!\!\int_\Omega \frac{\partial h_k}{\partial x_k} |\nabla \tilde{w}_t|^2 - \gamma \int_0^T\!\!\int_\Omega \frac{\partial \tilde{w}_t}{\partial x_i} \frac{\partial h_k}{\partial x_i} \frac{\partial \tilde{w}_t}{\partial x_k} + Z, \end{aligned}$$

where

$$|Z| = \left| \int_\Omega \nabla \tilde{w}_t \cdot \nabla (h_k \frac{\partial \tilde{w}}{\partial x_k}) \Big|_0^T \right| \leq CE(0).$$

We have adopted the convention of summation of repeated indexes. Integrating by parts the third term we see that

$$\int_0^T\!\!\int_\Omega \Delta^2 \tilde{w} h_k \frac{\partial \tilde{w}}{\partial x_k} = \int_0^T\!\!\int_\Omega \Delta \tilde{w} \Delta (h_k \frac{\partial \tilde{w}}{\partial x_k}) - \int_\Sigma \frac{\partial (h \cdot \nabla \tilde{w})}{\partial \nu} \Delta \tilde{w}$$

but since $\tilde{w} \in H_0^2(\Omega)$,

$$\frac{\partial \tilde{w}}{\partial \nu \partial x_k} = \nu_k \frac{\partial^2 \tilde{w}}{\partial \nu^2} \quad \text{and} \quad \frac{\partial^2 \tilde{w}}{\partial x_k^2} = \nu_k^2 \frac{\partial^2 \tilde{w}}{\partial \nu^2} \quad \text{on} \quad \partial\Omega.$$

In consequence we obtain:

$$\int_0^T\int_\Omega \Delta^2 \tilde{w} h_k \frac{\partial \tilde{w}}{\partial x_k} = -\frac{1}{2}\int_0^T\int_\Omega \frac{\partial h_k}{\partial x_k}|\Delta \tilde{w}|^2 + \int_0^T\int_\Omega \Delta\tilde{w}\Delta h_k \frac{\partial \tilde{w}}{\partial x_k} + 2\int_0^T\int_\Omega \Delta\tilde{w}[\nabla h_k \cdot \nabla \frac{\partial \tilde{w}}{\partial x_k}] - \frac{1}{2}\int_\Sigma |\Delta\tilde{w}|^2.$$

Finally, for the fourth term we obtain

$$-\beta_1\beta_3\int_0^T\int_\Omega \Delta\tilde{w}_t h_k \frac{\partial \tilde{w}}{\partial x_k} = \beta_1\beta_3\int_0^T\int_\Omega \Delta\tilde{w} h_k \frac{\partial \tilde{w}_t}{\partial x_k} - \beta_1\beta_3 \int_\Omega \Delta\tilde{w} h_k \frac{\partial \tilde{w}}{\partial x_k}\Big|_0^T.$$

Summarizing,

$$\frac{1}{2}\int_\Sigma |\Delta\tilde{w}|^2 \leq C\left[E(0) + \int_0^T E(t)\right] \leq C(1+T)E(0),$$

where C depends on $\|h\|_{W^{2,\infty}(\Omega)}$.

Lemma 2. *Let $\tilde{w} \in C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$ be the solution of* (11). *Then*

$$\tilde{w}_{tt} \in L^2(\Omega \times (0,T)).$$

Moreover, there exists $C > 0$ such that

$$\|\tilde{w}_{tt}\|_{L^2(Q)} \leq CE(0).$$

Proof. Let us write (11) in the following way:

$$(I - \gamma\Delta)\tilde{w}_{tt} = \beta_1\beta_3\Delta\tilde{w}_t - \Delta^2\tilde{w}. \tag{13}$$

We multiply (13) by $\phi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. We obtain

$$\int_0^T\int_\Omega \tilde{w}_{tt}(I-\gamma\Delta)\phi = \beta_1\beta_3\int_0^T\int_\Omega \tilde{w}_t\Delta\phi - \int_0^T\int_\Omega \Delta\tilde{w}\Delta\phi + \int_\Sigma \frac{\partial\phi}{\partial\nu}\Delta\tilde{w}$$

$$\leq C\left[\|\tilde{w}_t\|_{L^2(Q)} + \|\Delta\tilde{w}\|_{L^2(Q)} + \|\Delta\tilde{w}\|_{L^2(\Sigma)}\right]\|\phi\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}.$$

From this we deduce immediately that $\tilde{w}_{tt} \in L^2(Q)$ since $I - \gamma\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is an isomorphism. $\square$

Let us now proceed to the proof of Theorem 2. Let B be a bounded subset of V. We set

$$(w(t), w_t(t), \theta(t)) = [S(t)](w^0, w^1, \theta^0); \quad (\tilde{w}(t), \tilde{w}_t(t), \tilde{\theta}(t)) = [S^0(t)](w^0, w^1, \theta^0)$$

and

$$(v(t), v_t(t), \zeta(t)) = [S(t) - S^0(t)](w^0, w^1, \theta^0)$$

for any $(w^0, w^1, \theta^0) \in B$. We have

$$\begin{cases} v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v + \beta_1 \Delta \zeta = -\beta_1 [\beta_3 \Delta \tilde{w}_t + \Delta \tilde{\theta}] \ \text{ in } Q \\ \beta_2 \zeta_t - \Delta \zeta - \beta_3 \Delta v_t = 0 \ \text{ in } \ Q \\ v = \dfrac{\partial v}{\partial \nu} = 0 \ \text{ on } \Sigma \\ \zeta = 0 \ \text{ on } \Sigma \\ v(0) = v_t(0) = \zeta(0) = 0 \ \text{ in } \Omega. \end{cases} \tag{14}$$

We proceed in two steps:

First step: Let us see that $\beta_3 \Delta \tilde{w}_t + \Delta \tilde{\theta}$ is bounded in $L^1(0, T; H^{-1+\delta}(\Omega))$ for some $\delta > 0$ when (w^0, w^1, θ^0) varies in B.

Let us decompose $\beta_3 \Delta \tilde{w}_t + \Delta \tilde{\theta}$ as $\beta_3 \Delta \tilde{w}_t + \Delta \tilde{\theta} = \Delta z_1 + \Delta z_2$ where z_1 satisfies

$$\begin{aligned} \beta_2 z_{1,t} - \Delta z_1 &= \beta_2 \beta_3 \tilde{w}_{tt} \ \text{ in } Q \\ z_1 &= 0 \quad \text{ on } \Sigma \\ z_1(0) &= 0 \quad \text{ in } \Omega \end{aligned}$$

and z_2 satisfies

$$\begin{aligned} \beta_2 z_{2,t} - \Delta z_2 &= 0 \quad \text{ in } Q \\ z_2 &= 0 \quad \text{ on } \Sigma \\ z_2(0) &= \beta_3 w^1 + \theta^0 \ \text{ in } \Omega. \end{aligned}$$

If $G(t)$ is the semigroup generated by $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions, we can write z_1 as

$$z_1(t) = \beta_3 \int_0^t G_{\beta_2}(t-s) \tilde{w}_{tt}(s)\, ds,$$

where $G_{\beta_2}(t) = G(t/\beta_2)$. Since $G(\cdot)$ is an analytic semigroup, we have (see [18], page 74)

$$|G(t)v|_{L^2(\Omega)} \le |v|_{L^2(\Omega)}; \quad |G(t)v|_{H^2(\Omega)} \le \frac{1}{t} |v|_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

For $1 > \delta > 0$ we can interpolate (see Lions-Magenes, [16]) and we obtain:

$$|G(t)v|_{H^{1+\delta}(\Omega)} \le t^{\frac{-(1+\delta)}{2}} |v|_{L^2(\Omega)}.$$

Therefore

$$\begin{aligned} \int_0^T \|z_1(t)\|_{H^{1+\delta}(\Omega)}\, dt &\le \beta_3 \int_0^T \int_0^t \|G_{\beta_2}(t-s) \tilde{w}_{tt}(s)\, ds\|_{H^{1+\delta}}\, dt \\ &\le C \int_0^T \int_0^t (t-s)^{\frac{-(1+\delta)}{2}} \|\tilde{w}_{tt}(s)\|_{L^2(\Omega)}\, ds\, dt \\ &\le C \int_0^T \|\tilde{w}_{tt}(s)\|_{L^2(\Omega)}\, ds \int_s^T (t-s)^{\frac{-(1+\delta)}{2}}\, dt \\ &= \frac{2C}{(1-\delta)} \int_0^T \|\tilde{w}_{tt}(s)\|_{L^2(\Omega)} (T-s)^{\frac{1-\delta}{2}}\, ds \le C \|\tilde{w}_{tt}\|_{L^2(Q)}. \end{aligned}$$

Therefore Δz_1 is bounded in $L^1(0,T;H^{-1+\delta}(\Omega))$.

In a similar way we see that

$$\int_0^T \|z_2(t)\|_{H^{1+\delta}(\Omega)} \leq CT^{\frac{1-\delta}{2}}\|w^1+\theta^0\|_{L^2(\Omega)}.$$

In consequence $\beta_3\Delta\tilde{w}_t+\Delta\tilde{\theta}$ is bounded in $L^1(0,T;H^{-1+\delta}(\Omega))$.

Second Step: In view of Lemma 2 and since $\tilde{\theta}\in H^1(0,T;H^{-1}(\Omega))$ we deduce that

$$\Delta\tilde{w}_t\in H^1(0,T;H^{-2}(\Omega)); \quad \text{and} \quad \Delta\tilde{\theta}\in H^1(0,T;H^{-3}(\Omega)).$$

Therefore, $\beta_3\Delta\tilde{w}_t+\Delta\tilde{\theta}$ is bounded in $H^1(0,T;H^{-3}(\Omega))$ when the initial data remains in B. We can apply classical compactness results (see Simon, [21], Theorem 5), with $X=H^{-1+\delta}(\Omega)$, $B=H^{-1}(\Omega)$ and $Y=H^{-3}(\Omega)$. So when (w^0,w^1,θ^0) varies in B, there exists a subsequence $\{(w_n^0,w_n^1,\theta_n^0)\}_n\subset B$ and $f\in L^1(0,T;H^{-1}(\Omega))$ such that the corresponding solutions of the decoupled system (6) satisfy

$$\beta_3\Delta\tilde{w}_{n,t}+\Delta\tilde{\theta}_n\to f \text{ strongly in } L^1(0,T;H^{-1}(\Omega)).$$

In consequence, from the elementary properties of system (14), we deduce that the corresponding sequence of solutions (v_n,v_{n_t},ζ_n) of (14) converges strongly in $C([0,T];V)$.

3. Uniqueness results.

Lemma 3. *Let (w,θ) be a solution of the thermoelastic plate system in the set*

$$\mathcal{A}=\bigcup_{0<s<T/2}\{B(0,1+s/\sqrt{\gamma})\times(s,T-s)\}.$$

Suppose that there exist constants c and d such that $(w,\theta)=(c,d)$ in $B(0,1)\times(0,T)$. Then $(w,\theta)=(c,d)$ in $\mathcal{A}$.

Remark 4. By translation invariance (in time and space) and by scaling we deduce that if $(w,\theta)=(c,d)$ in $B(x_0,\rho)\times(t_1,t_2)$ then $(w,\theta)=(c,d)$ in

$$\bigcup_{0<s<(t_2-t_1)/2}\{B(x_0,\rho+s/\sqrt{\gamma})\times(t_1+s,t_2-s)\}.$$

Proof. Without loss of generality we may assume $(c,d)=(0,0)$. We apply to the thermoelastic plate system an argument due to A. Haraux ([7]).

Let us denote by φ_k and ρ_k the eigenfunctions and eigenvalues of the Laplace-Beltrami operator on the sphere S^1 of $\mathbb{R}^2$. Let

$$w_k=\int_{S^1} w\varphi_k\,d\sigma, \quad \theta_k=\int_{S^1}\theta\varphi_k\,d\sigma.$$

(To simplify notation we shall denote $w=w_k$, $\theta=\theta_k$.) Multiplying (1) by φ_k and integrating on S^1 we deduce that $w=w_k$ and $\theta=\theta_k$ satisfies

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}-\gamma\frac{\partial^2}{\partial t^2}\frac{\partial^2 w}{\partial r^2}-\frac{\gamma}{r}\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial r}+\frac{\gamma\rho_k}{r^2}\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial r^4}w+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}\\
&\quad-\frac{1}{r^2}[3\rho_k+1]\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^3}[1+2\rho_k]\frac{\partial w}{\partial r}+\frac{1}{r^4}[\rho_k^2-4\rho_k]w\\
&\quad+\beta_1\frac{\partial^2\theta}{\partial r^2}+\frac{\beta_1}{r}\frac{\partial\theta}{\partial r}-\beta_1\frac{\rho_k}{r^2}\theta=0;\\
&\beta_2\theta_t-\frac{\partial^2\theta}{\partial r^2}-\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\rho_k}{r^2}\theta-\beta_3\frac{\partial}{\partial t}\frac{\partial^2 w}{\partial r^2}-\frac{\beta_3}{r}\frac{\partial}{\partial t}\frac{\partial w}{\partial r}+\frac{\beta_3\rho_k}{r^2}\frac{\partial w}{\partial t}=0.
\end{aligned}\tag{15}$$

Let $w^1 = \partial^2 w/\partial r^2$. We write (15) in the following way:

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial r^2} - w^1 = 0\\
&\frac{\partial^2 w}{\partial t^2} - \gamma\frac{\partial^2 w^1}{\partial t^2} - \frac{\gamma\rho_k}{r^2}\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial r} + \frac{\partial^2 w^1}{\partial r^2} + \frac{2}{r}\frac{\partial w^1}{\partial r} - \frac{1}{r^2}[3\rho_k+1]w^1\\
&\quad + \frac{1}{r^3}[1+2\rho_k]\frac{\partial w}{\partial r} + \frac{1}{r^4}[\rho_k^2 - 4\rho_k]w + \beta_1\frac{\partial^2\theta}{\partial r^2} + \frac{\beta_1}{r}\frac{\partial w}{\partial r}\theta - \beta_1\frac{\rho_k}{r^2}\theta = 0;\\
&\beta_2\theta_t - \frac{\partial^2\theta}{\partial r^2} - \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\rho_k}{r^2}\theta - \beta_3\frac{\partial w^1}{\partial t} - \frac{\beta_3}{r}\frac{\partial}{\partial t}\frac{\partial w}{\partial r} + \frac{\beta_3\rho_k}{r^2}\frac{\partial w}{\partial t} = 0,
\end{aligned}
\tag{16}
$$

where (16) is satisfied in

$$\mathcal{B} = \bigcup_{0<s<T/2}\{(0, 1+s/\sqrt{\gamma}) \times (s, T-s)\}.$$

Following F. John ([9]) we compute the characteristic form of the operator involved in (16). Given $\alpha \in \mathbb{Z}^2$, $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = m$ we denote $\partial^\alpha = \partial^m/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}$. In Schwartz notation, the general form of an m-order linear system of N differential equations in N unknowns, takes the simple form

$$\sum_{|\alpha|\le m} A_\alpha(x)\partial^\alpha u = B(x),$$

where u and B are column vectors with N components and A_α are $N \times N$ square matrices.

Let X be the column vector $X = [w, w^1, \theta]$. Then the principal part of the differential operator involved in system (16) is given by $A_{(2,0)}\partial^{(2,0)}X + A_{(0,2)}\partial^{(0,2)}X$ with

$$A_{(2,0)} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & \beta_1\\ 0 & 0 & -1 \end{bmatrix}, \quad A_{(0,2)} = \begin{bmatrix} 0 & 0 & 0\\ 1 & -\gamma & 0\\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore its characteristic matrix is

$$\Lambda((\xi,\tau)) = \begin{bmatrix} \xi^2 & 0 & 0\\ \tau^2 & \xi^2 - \gamma\tau^2 & \beta_1\xi^2\\ 0 & 0 & -\xi^2 \end{bmatrix}$$

and the principal form of the operator is given by

$$Q(\xi,\tau) = \det(\Lambda(\xi,\tau)) = -\xi^4(\xi^2 - \gamma\tau^2).$$

The line given by

$$\Pi = \{(r,t) \in \mathbb{R}\times\mathbb{R} : \xi r + \tau t = c\}$$

is characteristic with respect to (16) if and only if $\xi^4 = 0$ or

$$\xi^2 - \gamma\tau^2 = 0, \quad \tau = \pm\frac{\xi}{\sqrt{\gamma}}.$$

In consequence, the characteristic lines of the system are

$$t = c, \quad t \pm \sqrt{\gamma} r = c.$$

Since $(w_k, \theta_k) = (0,0)$ for $0 \leq r < 1$ it is enough to consider system (15) in the region

$$\tilde{\mathcal{B}} = \bigcup_{0<s<T/2} \{(1/2, 1 + s/\sqrt{\gamma}) \times (s, T-s)\},$$

where its coefficients are analytic. By Holmgren's Uniqueness Theorem (see F. John, [9]) we deduce that $(w_k, \theta_k) = (0,0)$ in $\tilde{\mathcal{B}}$ (and thus in $\mathcal{B}$) for all $k \in I\!N$. Therefore

$$(w, \theta) = (0,0) \quad \text{in} \ \ \mathcal{A}.$$

Proposition 3. *Suppose that* $T > \sqrt{\gamma}\,\mathrm{diam}(\Omega\backslash\omega)$. *Let* $(w, w_t, \theta) \in C([0,T]; V)$ *be a solution of* (1) *such that* $\nabla w = 0$ *in* $\omega \times (0,T)$. *Then* $w \equiv \theta \equiv 0$ *in* $\Omega \times (0,T)$.

Proof. Let $(u_i, v_i) = (\partial w/\partial x_i, \partial\theta/\partial x_i)$, $i = 1, 2$. Then the (u_i, v_i) satisfy

$$\begin{cases} u_{i,tt} - \gamma\Delta u_{i,tt} + \Delta^2 u_i + \beta_1 \Delta v_i = 0 & \text{in } Q \\ \beta_2 v_{i,t} - \Delta v_i - \beta_3 \Delta u_{i,t} = 0 & \text{in } \ Q. \end{cases} \tag{17}$$

Since $u_i = 0$ in $\omega \times (0,T)$ we deduce from (17) that $\Delta v_i = 0$ in $\omega \times (0,T)$.

Moreover, $(\Delta u_i, \Delta v_i)$ satisfies (17) and $(\Delta u_i, \Delta v_i) = (0,0)$ in $\omega \times (0,T)$ for $i = 1,2$.

In view of Lemma 3 $(\nabla\Delta w, \nabla\Delta\theta) = (0,0)$ in $\tilde{Q}$ where

$$\tilde{Q} = \Omega \times (\sqrt{\gamma}\mathrm{diam}(\Omega\backslash\omega)/2, T - \sqrt{\gamma}\mathrm{diam}(\Omega\backslash\omega)/2).$$

Therefore $\Delta w = f(t)$ in $\tilde{Q}$ but since $\frac{\partial w}{\partial \nu} = 0$ on Σ, $f(t) \equiv 0$ in $\tilde{Q}$. Then w is harmonic in $\tilde{Q}$, $w = 0$ on Σ and Ω is connected. In consequence $w \equiv 0$ in $\tilde{Q}$.

On the other hand, since $\nabla\Delta\theta = 0$ in $\tilde{Q}$ we have $\Delta\theta = g(t)$. However, since $w = 0$ in $\omega \times (0,T)$, by the first equation of (1), we deduce that $\Delta\theta = 0$ in ω and therefore $\Delta\theta = 0$ in $\tilde{Q}$. Since $\theta = 0$ on Σ, this implies that $\theta \equiv 0$ in $\tilde{Q}$. We have obtained

$$(w, \theta) = 0 \text{ in } \Omega \times (\sqrt{\gamma}\ \mathrm{diam}(\Omega\backslash\omega)/2, T - \sqrt{\gamma}\ \mathrm{diam}(\Omega\backslash\omega)/2).$$

By forward uniqueness of solutions of (1) we have $(w, \theta) = 0, t \geq \sqrt{\gamma}\ \mathrm{diam}(\Omega\backslash\omega)/2$. Then $(w, \theta) = 0$ in $\Omega \times (\sqrt{\gamma}\ \mathrm{diam}(\Omega\backslash\omega)/2, T)$. Therefore, it is sufficient to show by backward uniqueness that $(w, \theta) = 0$, for $0 \leq t \leq \sqrt{\gamma}\ \mathrm{diam}(\Omega\backslash\omega)/2$. To do that we observe that, since $\Delta w = 0$ in $\omega \times (0,T)$, then $\Delta w = 0$ on $\partial\Sigma$. Thus, we can think of w as being the displacement of the solution of system (1) in which the boundary conditions $w = \frac{\partial w}{\partial \nu} = 0$, $\theta = 0$ on Σ are replaced by $w = \Delta w = \theta = 0$ on Σ. With these new boundary conditions the backward uniqueness result of the Appendix 7.1 (Lemma 6) can be applied and this completes the proof of Proposition 3.

Proposition 4. *Suppose that* $T > \sqrt{\gamma}\,\mathrm{diam}(\Omega\backslash\omega)$. *Let* (φ, ψ) *be a solution of* (5) *such that* $\nabla\varphi = 0$ *in* $\omega \times (0,T)$. *Then* $\varphi \equiv \psi \equiv 0$ *in* $\Omega \times (0,T)$.

Proof. Reversing the time variable we get

$$\begin{cases} \tilde{\varphi}_{tt} - \gamma\Delta\tilde{\varphi}_{tt} + \Delta^2\tilde{\varphi} - \beta_3\Delta\tilde{\psi}_t = 0 & \text{in } Q \\ \beta_2\tilde{\psi}_t - \Delta\tilde{\psi} + \beta_1\Delta\tilde{\varphi} = 0 & \text{in } Q \\ \tilde{\varphi} = \dfrac{\partial\tilde{\varphi}}{\partial\nu} = 0 & \text{on } \Sigma \\ \tilde{\psi} = 0 & \text{on } \Sigma \end{cases} \tag{18}$$

with $\tilde{\varphi}(t) = \varphi(T-t)$; $\tilde{\psi}(t) = \psi(T-t)$.

Note that $(\tilde{\varphi}, \tilde{\psi}_t)$ are solutions of thermoelastic plate system (1) where some of the constants have changed. Thus, by Proposition 3, we get $\tilde{\varphi} = 0$ and $\tilde{\psi}_t = 0$. Then $\Delta\tilde{\psi} = 0$. Since $\tilde{\psi} = 0$ on Σ and Ω is a connected set, $\tilde{\psi} = 0$.

4. The observability inequality. This section is devoted to the proof of Proposition 1. We will use the notation $\tilde{V} = H_0^1(\Omega) \times \tilde{H} \times L^2(\Omega)$.

First we recall that, as observed in Remark 3, if (φ, ψ) solves (5), then (ϕ, ψ), where

$$\phi(x,t) = -\int_t^T \varphi(x,s)\,ds + \xi(x),$$

with $\xi \in H_0^2(\Omega)$ solving

$$\Delta^2\xi = -\beta_3\Delta\psi^0 - \tilde{C}_0\varphi^1 \quad \text{in } \Omega, \tag{19}$$

satisfies

$$\begin{cases} \phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \beta_3\Delta\psi = 0 & \text{in } Q \\ -\beta_2\psi_t - \Delta\psi + \beta_1\Delta\phi_t = 0 & \text{in } Q \\ \phi = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Sigma \\ \psi = 0 & \text{on } \Sigma \\ \phi(T) = \xi;\ \phi_t(T) = \varphi^0;\ \psi(T) = \psi^0 & \text{in } \Omega. \end{cases} \tag{20}$$

This solution (ϕ, ϕ_t, ψ) of (20) is in the class $C([0,T]; V)$.

We introduce $\rho \in C^3(\bar{\Omega})$ (fixed) satisfying

$$\rho = 1 \ \text{ in } \partial\Omega, \quad 1 \geq \rho > 0 \ \text{ in } \omega, \quad \rho = 0 \ \text{ in } \Omega\backslash\omega \tag{21}$$

$$\nabla\rho/\rho^{3/4} \in L^\infty(\Omega), \quad \Delta\rho/\rho^{1/2} \in L^\infty(\Omega). \tag{22}$$

This function is easy to construct. It is sufficient to take $\rho = \tilde{\rho}^4$ where $\tilde{\rho}$ is any regular function satisfying (21).

Taking into account that $\|\xi\|_{H_0^2(\Omega)}$ and $\|\beta_3\Delta\psi^0 + \tilde{C}_0\varphi^1\|_{H^{-2}(\Omega)}$ are equivalent norms, we see that Proposition 1 is equivalent to the following one:

Proposition 5. *Under the assumptions of Proposition* 1, *for every bounded set B of $L^2(\Omega)$ there exists $\delta = \delta(B) > 0$ such that*

$$\delta \leq \int_0^T \int_\Omega \rho |\nabla\phi_t|^2\,dx\,dt$$

holds for every solution of (20) *with final data such that*

$$\|(\xi,\varphi^0)\|_{H_0^2(\Omega)\times H_0^1(\Omega)} \geq 1, \quad \psi^0 \in B. \tag{23}$$

To prove Proposition 5, first we introduce the decoupled system associated to (20):

$$\begin{cases} \tilde\phi_{tt} - \gamma\Delta\tilde\phi_{tt} + \Delta^2\tilde\phi + \beta_1\beta_3\Delta\tilde\phi_t = 0 & \text{in } Q \\ -\beta_2\tilde\psi_t - \Delta\tilde\psi + \beta_1\Delta\tilde\phi_t = 0 & \text{in } Q \\ \tilde\phi = \dfrac{\partial\tilde\phi}{\partial\nu} = 0 & \text{on } \Sigma \\ \tilde\psi = 0 & \text{on } \Sigma \\ \tilde\phi(T) = \xi;\ \tilde\phi_t(T) = \varphi^0;\ \tilde\psi(T) = \psi^0 & \text{in } \Omega, \end{cases} \tag{24}$$

and consider the subsystem that $\tilde\phi$ satisfies:

$$\begin{cases} \tilde\phi_{tt} - \gamma\Delta\tilde\phi_{tt} + \Delta^2\tilde\phi + \beta_1\beta_3\Delta\tilde\phi_t = 0 & \text{in } Q \\ \tilde\phi = \dfrac{\partial\tilde\phi}{\partial\nu} = 0 & \text{on } \Sigma \\ \tilde\phi(T) = \xi,\ \tilde\phi_t(T) = \varphi^0 & \text{in } \Omega. \end{cases} \tag{25}$$

We have the following observability inequality for system (25).

Proposition 6. *Suppose that $T > \sqrt{\gamma}\,\mathrm{diam}(\Omega\backslash\omega)$. Then, there exists a constant $C > 0$ and a seminorm $X : H_0^2(\Omega)\times H_0^1(\Omega) \to \mathbb{R}^+$ such that*

$$\|\xi\|^2_{H_0^2(\Omega)} + \|\varphi^0\|^2_{H_0^1(\Omega)} \leq C[\int_0^T\int_\Omega \rho|\nabla\tilde\phi_t|^2\,dx\,dt + X^2(\xi,\varphi^0)]$$

holds for every solution of (25)*, $X : H_0^2(\Omega)\times H_0^1(\Omega) \to \mathbb{R}^+$ being continuous and compact.*

The proof of this proposition will be given at the end of this section. Let us now conclude the proof of Proposition 5 by assuming that Proposition 6 holds.

We decompose the solution of (20) as $(\phi,\psi) = (\tilde\phi,\tilde\psi) + (\varrho,\zeta)$ where $(\tilde\phi,\tilde\psi)$ solves (24) and (ϱ,ζ) satisfies

$$\begin{cases} \varrho_{tt} - \gamma\Delta\varrho_{tt} + \Delta^2\varrho = -\beta_3\Delta\psi + \beta_1\beta_3\Delta\tilde\phi_t & \text{in } Q \\ -\beta_2\zeta_t - \Delta\zeta + \beta_1\Delta\varrho = 0 & \text{in } Q \\ \varrho = \dfrac{\partial\varrho}{\partial\nu} = 0 & \text{on } \Sigma \\ \zeta = 0 & \text{on } \Sigma \\ \varrho(T) = \varrho_t(T) = 0;\ \zeta(T) = 0 & \text{in } \Omega. \end{cases} \tag{26}$$

As a consequence of Proposition 6 we have that

$$\|\xi\|^2_{H_0^2(\Omega)} + \|\varphi^0\|^2_{H_0^1(\Omega)} \leq C[\int_0^T\int_\Omega \rho(|\nabla\phi_t|^2 + |\nabla\varrho_t|^2)\,dx\,dt + X^2(\xi,\varphi^0)]. \tag{27}$$

We argue by contradiction. Suppose that Proposition 5 does not hold. Then, there exists a bounded set B of $L^2(\Omega)$ and a sequence of initial data $(\xi_j, \varphi_j^0, \psi_j^0)$ with $\psi_j^0 \in B$ satisfying (23) such that

$$\int_0^T \int_\Omega \rho |\nabla \phi_{j,t}|^2 \, dx \, dt \to 0 \ \text{ as } j \to \infty. \tag{28}$$

In view of (27) and taking into account (28) and that $\|(\xi_j, \varphi_j^0)\|_{H_0^2(\Omega)\times H_0^1(\Omega)} \geq 1$ holds, we deduce that

$$\liminf_{j\to\infty} [\int_0^T \int_\Omega \rho |\nabla \varrho_{j,t}|^2 \, dx \, dt + X^2(\xi_j, \varphi_j^0)] > 0. \tag{29}$$

We introduce the normalized data

$$(\hat{\xi}_j, \hat{\varphi}_j^0, \hat{\psi}_j^0) = (\xi_j, \varphi_j^0, \psi_j^0)/[\|\rho^{1/2} \nabla \varrho_{j,t}\|^2_{L^2(Q)} + X^2(\xi_j, \varphi_j^0)]^{1/2}$$

and the corresponding solutions $(\hat{\phi}_j, \hat{\psi}_j)$ and $(\hat{\varrho}_j, \hat{\zeta}_j)$ of (20) and (26) respectively. We then have

$$\int_0^T \int_\Omega \rho |\nabla \hat{\varrho}_{j,t}|^2 \, dx \, dt + X^2(\hat{\xi}_j, \hat{\varphi}_j^0) = 1, \ \forall j; \quad \int_0^T \int_\Omega \rho |\nabla \hat{\phi}_{j,t}|^2 \, dx \, dt \to 0 \ \text{ as } j \to \infty. \tag{30}$$

In view of (27) we deduce that $(\hat{\xi}_j, \hat{\varphi}_j^0)$ is bounded in $H_0^2(\Omega) \times H_0^1(\Omega)$.

On the other hand, by (29) and taking into account that $\psi_j^0 \in B$ we deduce that $\hat{\psi}_j^0$ remains in a bounded set $\hat{B}$ of $L^2(\Omega)$. By extracting subsequences we deduce that

$$(\hat{\xi}_j, \hat{\varphi}_j^0) \rightharpoonup (\hat{\xi}, \hat{\varphi}^0) \ \text{ weakly in } H_0^2(\Omega) \times H_0^1(\Omega), \quad \hat{\psi}_j^0 \rightharpoonup \hat{\psi}^0 \ \text{ weakly in } L^2(\Omega)$$

and

$$\hat{\phi}_{j,t} \rightharpoonup \hat{\phi}_t \ \text{ weakly in } L^2(0,T; H_0^1(\Omega)), \quad \hat{\varrho}_{j,t} \rightharpoonup \hat{\varrho}_t \ \text{ weakly in } L^2(0,T; H_0^1(\Omega)) \tag{31}$$

as $j \to \infty$, where $(\hat{\phi}, \hat{\psi})$ and $(\hat{\varrho}, \hat{\zeta})$ are respectively the solutions of (20) and (26) corresponding to the limit initial data.

On the other hand, by virtue of Theorem 2 we know that $(\hat{\varrho}_{j,t})$ is relatively compact in $C([0,T]; H_0^1(\Omega))$ and therefore

$$\hat{\varrho}_{j,t} \to \hat{\varrho}_t \ \text{ strongly in } L^2(0,T; H_0^1(\Omega)). \tag{32}$$

As a consequence of (30) and (31) we deduce that

$$\nabla \hat{\varphi} = \nabla \hat{\phi}_t = 0 \quad \text{in } \omega \times (0,T). \tag{33}$$

In view of (33) and applying Proposition 4, we obtain that $(\hat{\varphi}, \hat{\psi}) \equiv 0$ in $\Omega \times (0,T)$ and therefore

$$(\hat{\varphi}^0, \hat{\varphi}^1, \hat{\psi}^0) \equiv 0. \tag{34}$$

In view of (19), this implies that

$$\hat{\xi} \equiv 0 \tag{35}$$

and in particular

$$\hat{\varrho}_t \equiv 0. \tag{36}$$

However, combining (30), (32) and the fact that $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}^+$ is compact, we deduce that

$$\int_0^T\int_\Omega \rho|\nabla \hat{\varrho}_t|^2\,dx\,dt + X^2(\hat{\xi}, \hat{\varphi}^0) = 1$$

and this contradicts (34), (35), (36). □

For the proof of Proposition 6 we make the following change of variables. We define

$$v = e^{\frac{-\beta_1\beta_3}{2\gamma}(t-T)}\tilde{\phi}.$$

Then v satisfies

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + \sqrt{A}v_t + \dfrac{A}{4}v + \dfrac{\gamma A}{4}\Delta v = 0 & \text{in } Q \\ v = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma \\ v(T) = v^0 = \xi,\ v_t(T) = v^1 = \varphi^0 - \dfrac{\sqrt{A}}{2}\xi & \text{in } \Omega \end{cases} \tag{37}$$

where $A = (\frac{\beta_1\beta_3}{\gamma})^2$.

We observe that Proposition 6 is equivalent to the following one:

Proposition 7. *Suppose that* $T > \sqrt{\gamma}\mathrm{diam}(\Omega\backslash\omega)$. *Then there exists a positive constant* $C > 0$ *and a seminorm* $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}^+$ *such that*

$$\|v^0\|^2_{H_0^2(\Omega)} + \|v^1\|^2_{H_0^1(\Omega)} \leq C[\int_0^T\int_\Omega \rho|\nabla v_t|^2\,dx\,dt + X^2(v^0, v^1)]$$

holds for every solution of (37), $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}^+$ *being continuous and compact.*

Proposition 7 states roughly that we can bound the total energy of solutions of the plate equation (37) by means of the energy concentrated in a neighborhood ω of the boundary provided $T > \sqrt{\gamma}\,\mathrm{diam}(\Omega\backslash\omega)$. Similar results for the wave equation were established in Lions ([12], Chapter I). We are going to prove Proposition 7 by similar multiplier techniques. However, system (37) being fourth order, the computations are longer. To make the paper easier to read we give a detailed proof of Proposition 7 in Appendix 7.2.

5. Proof of Theorem 1. First we observe that it is sufficient to prove it when $w^0 \equiv w^1 \equiv \theta^0 \equiv 0$. Indeed, given any initial and final data (w^0, w^1, θ^0), $(v^0, v^1, \xi^0) \in V$ and $\varepsilon > 0$, let $(\tilde{w}, \tilde{\theta})$ be the solution of (1) with initial data (w^0, w^1, θ^0). Let $(\hat{w}, \hat{\theta}) = (w, \theta) - (\tilde{w}, \tilde{\theta})$. It is easy to check that finding $f \in L^1((0,T); H^{-1}(\Omega))$ such that (4) holds is equivalent to finding $f \in L^1((0,T); H^{-1}(\Omega))$ such that the solution $(\hat{w}, \hat{\theta})$ of (3) with this control and zero initial data satisfies

$$\hat{w}(T) = v^0 - \tilde{w}(T), \quad \hat{w}_t(T) = v^1 - \tilde{w}_t(T); \quad \|\hat{\theta}^0 - \xi^0 + \tilde{\theta}(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

Therefore, in the sequel we will assume $w^0 \equiv w^1 \equiv \theta^0 \equiv 0$.

Given any $(\varphi^0, \varphi^1, \psi^0) \in \tilde{V} = H_0^1(\Omega) \times \tilde{H} \times L^2(\Omega)$, $\varepsilon > 0$ and $\rho \in C^\infty(\bar{\Omega})$ fixed satisfying (21)–(22), we introduce the functional $J_\rho : \tilde{V} \to \mathbb{R}$ defined as follows:

$$J_\rho(\varphi^0, \varphi^1, \psi^0) = \frac{1}{2}\int_0^T\int_\Omega \rho|\nabla\varphi|^2 + \varepsilon\beta_2\|\psi^0\|_{L^2} + \langle \tilde{C}_0\varphi^1 + \beta_3\Delta\psi^0, v^0\rangle - \langle \tilde{C}_0\varphi^0, v^1\rangle - \beta_2\int_\Omega \xi^0\psi^0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H_0^2(\Omega)$ and $H^{-2}(\Omega)$, and $\tilde{C}_0$ is the operator defined in the preliminaries.

The functional J_ρ is coercive in $\tilde{V}$. More precisely, if we define in $\tilde{V}$ the norm:

$$|\varphi^0, \varphi^1, \psi^0|_{*\tilde{V}} = |\varphi^0|_{H_0^1(\Omega)} + |\psi^0|_{L^2(\Omega)} + |\tilde{C}_0\varphi^1 + \beta_3\Delta\psi^0|_{H^{-2}(\Omega)},$$

we have

Lemma 4. *Under the assumptions of Theorem* 1,

$$\liminf_{|(\varphi^0,\varphi^1,\psi^0)|_{*\tilde{V}}\to\infty} \frac{J_\rho(\varphi^0, \varphi^1, \psi^0)}{|(\varphi^0, \varphi^1, \psi^0)|_{*\tilde{V}}} \geq \varepsilon\beta_2. \tag{38}$$

Proof. Let us consider a sequence $(\varphi_j^0, \varphi_j^1, \psi_j^0)$ in $\tilde{V}$ such that

$$N_j = |(\varphi_j^0, \varphi_j^1, \psi_j^0)|_{*\tilde{V}} \to \infty \qquad \text{as} \quad j \to \infty.$$

We introduce the normalized data $(\hat{\varphi}_j^0, \hat{\varphi}_j^1, \hat{\psi}_j^0) = (\varphi^0, \varphi^1, \psi^0)/N_j$ and the corresponding solutions of (5): $(\hat{\varphi}_j, \hat{\psi}_j) = (\varphi_j, \psi_j)/N_j$. We have

$$\begin{aligned} I_j/N_j = J_\rho(\varphi_j^0, \varphi_j^1, \psi_j^0)/N_j = \frac{N_j}{2}\int_0^T\int_\Omega \rho|\nabla\hat{\varphi}_j|^2\,dx\,dt + \varepsilon\beta_2|\hat{\psi}_j^0|_{L^2(\Omega)} \\ + \langle \tilde{C}_0\hat{\varphi}_j^1 + \beta_3\Delta\hat{\psi}_j^0, v^0\rangle - \langle \tilde{C}_0\hat{\varphi}_j^0, v^1\rangle - \beta_2\int_\Omega \xi^0\hat{\psi}_j^0. \end{aligned} \tag{39}$$

We distinguish the following two cases:

i)
$$\liminf_{j\to\infty}\int_0^T\int_\Omega \rho|\nabla\hat{\varphi}_j|^2 > 0;$$

ii) There exists a subsequence (still denoted by index j) such that

$$\int_0^T\int_\Omega \rho|\nabla\hat{\varphi}_j|^2 \to 0 \quad \text{as} \quad j \to \infty. \tag{40}$$

In the first case, clearly

$$\liminf_{j\to\infty} I_j/N_j = \infty.$$

Let us consider the second case. Since $(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0)$ is bounded in $\tilde V$ we can extract a subsequence such that

$$(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0) \rightharpoonup (\hat\varphi^0, \hat\varphi^1, \hat\psi^0) \text{ weakly in } \tilde V \text{ as } j \to \infty.$$

We denote by $(\hat\varphi, \hat\psi)$ the corresponding solution to (5). In view of (40) we have that

$$\rho\nabla\hat\varphi \equiv 0 \text{ in } \Omega \times (0,T),$$

and, in particular,

$$\nabla\hat\varphi \equiv 0 \text{ in } \omega \times (0,T).$$

As a consequence of Proposition 4 we have that $(\hat\varphi^0, \hat\varphi^1, \hat\psi^0) \equiv 0$, and therefore,

$$(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0) \rightharpoonup (0,0,0) \text{ in } \tilde V.$$

From (39) we deduce that

$$\liminf_{j\to\infty} I_j/N_j = \liminf_{j\to\infty} \frac{N_j}{2}\int_0^T\!\!\int_\Omega \rho|\nabla\hat\varphi_j|^2 + \varepsilon\beta_2|\hat\psi_j^0|_{L^2(\Omega)}.$$

Clearly we have the result if

$$\liminf_{j\to\infty} \ |\hat\psi_j^0|_{L^2(\Omega)} = 1. \qquad (41)$$

To prove (41) we proceed by contradiction. Let us suppose that

$$\liminf_{j\to\infty} \ |\hat\psi_j^0|_{L^2(\Omega)} < 1.$$

Since $|(\hat\varphi_j^0, \hat\varphi_j^1, \hat\psi_j^0)|_{*\tilde V} = 1$ for every j, in that case we deduce that

$$\liminf_{j\to\infty}[|\hat\varphi_j^0|_{H_0^1} + \|\tilde C_0\hat\varphi_j^1 + \beta_3\Delta\hat\psi_j^0\|_{H^{-2}}] > 0. \qquad (42)$$

From (42) and by the fact that $\hat\psi_j^0$ is bounded in $L^2(\Omega)$, as a consequence of Proposition 1, we deduce that

$$\liminf_{j\to\infty}\int_0^T\!\!\int_\Omega \rho|\nabla\hat\varphi_j^0|^2 > 0,$$

but this contradicts (40). Then necessarily (41) holds and consequently, (38) holds, too. □

As a consequence of the coercivity property (38) it can be verified that the infimum of J_ρ over $\tilde V$ is achieved at some $(\hat\varphi^0, \hat\varphi^1, \hat\psi^0) \in \tilde V$. At this minimizer $(\hat\varphi^0, \hat\varphi^1, \hat\psi^0)$ we have the following optimality condition:

$$\left|\int_0^T\!\!\int_\Omega \rho\nabla\hat\varphi\cdot\nabla\varrho + \langle\tilde C_0\varrho^1 + \beta_3\Delta\eta^0, v^0\rangle - \langle\tilde C_0\varrho^0, v^1\rangle - \beta_2\int_\Omega \xi^0\eta^0\right| \le \varepsilon\beta_2|\eta^0|_{L^2(\Omega)} \qquad (43)$$

for all $(\varrho^0, \varrho^1, \eta^0) \in \tilde V$ where $(\hat\varphi, \hat\psi)$ denotes the solution of (5) corresponding to the minimizer $(\hat\varphi^0, \hat\varphi^1, \hat\psi^0)$ and (ϱ, η) the solution of (5) with data $(\varrho^0, \varrho^1, \eta^0)$.

Lemma 5. *Let* (w,θ) *be the solution of* (3) *with* $(w^0,w^1,\theta^0)\equiv 0$ *and* $f=-\rho\Delta\hat{\varphi}-\nabla\rho\cdot\nabla\hat{\varphi}$. *Then*

$$\int_0^T\int_\Omega \rho\nabla\hat{\varphi}\cdot\nabla\varrho\,dx\,dt \tag{44}$$
$$=\langle\tilde{C}_0\varrho^0,w_t(T)\rangle-\langle\tilde{C}_0\varrho^1,w(T)\rangle+\beta_2\int_\Omega\theta(T)\eta^0-\beta_3\langle w(T),\Delta\eta^0\rangle$$

for every final data $(\varrho^0,\varrho^1,\eta^0)\in\tilde{V}$, (ϱ,η) *being the corresponding solution of* (5).

We give the proof of this lemma at the end of the section.

Assume for the moment that Lemma 5 holds. Let w be the solution of (3) with control $f=-\rho\Delta\hat{\varphi}-\nabla\rho\cdot\nabla\hat{\varphi}$, $(\hat{\varphi},\hat{\psi})$ being the solution of (5) associated to the minimizer $(\hat{\varphi}^0,\hat{\varphi}^1,\hat{\psi}^0)$ of the functional J_ρ in $\tilde{V}$. Then, we can substitute (44) in (43). Taking $\eta_0=0$ we see immediately that

$$w(T)=v^0,\quad w_t(T)=v^1. \tag{45}$$

From (45) we obtain

$$\left|\beta_2\int_\Omega\theta(T)\eta^0-\beta_2\int_\Omega\xi^0\eta^0\right|\le\varepsilon\beta_2|\eta^0|_{L^2(\Omega)},$$

which is equivalent to

$$\|\theta(T)-\xi^0\|_{L^2(\Omega)}\le\varepsilon.$$

In consequence, the proof of Theorem 1 will be completed if we prove Lemma 5.

Proof of Lemma 5. We consider a sequence $\{\varrho_n^0,\varrho_n^1,\eta_n^0\}_n\subset H_0^2(\Omega)\times H_0^1(\Omega)\times H_0^2(\Omega)$ such that

$$\varrho_n^0\to\varrho^0 \text{ strongly in } H_0^1(\Omega);\ \varrho_n^1\to\varrho^1 \text{ strongly in } \tilde{H};\ \eta_n^0\to\eta^0 \text{ strongly in } L^2(\Omega).$$

Let (ϱ_n,η_n) be the corresponding solutions to (5). Then, (see [10], Theorem 2.5 a)),

$$\varrho_n\in C([0,T];H_0^2(\Omega)),\quad \varrho_{n,t}\in C([0,T];H_0^1(\Omega)),\quad \varrho_{n,tt}\in C([0,T];\tilde{H})$$

$$\eta_n\in C([0,T];H_0^2(\Omega)),\quad \eta_{n,t}\in C([0,T];L^2(\Omega)).$$

Moreover,

$$\|(\varrho_n(t),\varrho_{n,t}(t),\eta_n(t))\|_{H_0^2\times H_0^1\times H_0^2}\le(C_1(T-t)+C_2)\|(\varrho_n^0,\varrho_n^1,\eta_n^0)\|_{H_0^2\times H_0^1\times H_0^2} \tag{46}$$

where C_1,C_2 are positive constants independent of n.

We have that ϱ_n,η_n are solutions of (5) in the following sense:

$$\langle\tilde{C}_0\varrho_{n,tt}(t),\varphi\rangle_{H^{-2},H_0^2}=-\int_\Omega\Delta\varrho_n(t)\Delta\varphi-\beta_3\int_\Omega\eta_{n,t}(t)\Delta\varphi,\quad\forall\varphi\in H_0^2(\Omega)$$

$$-\beta_2\int_\Omega\eta_{n,t}\xi=\int_\Omega\Delta\eta_n\xi-\beta_1\int_\Omega\Delta\varrho_n(t)\xi,\quad\forall\xi\in L^2(\Omega).$$

We have, thanks to (46), that there exists a constant $C > 0$ independent of n such that

$$|\varrho_{n,tt}(t)|_{\tilde{H}} \leq C, \quad \forall n; \quad \forall t \in [0,T] \tag{47}$$

$$|\eta_{n,t}(t)|_{L^2(\Omega)} \leq C, \quad \forall n; \ \forall t \in [0,T]. \tag{48}$$

In view of (46), (47), (48) and classical compactness results (see J. Simon, [21], Theorems 3 and 5), we obtain the existence of $\{\varrho, \varrho_t, \eta\} \in C([0,T]; H_0^1(\Omega)) \times C([0,T]; \tilde{H}) \times C([0,T]; H_0^1(\Omega))$ and a subsequence (denoted by the index n to simplify notation), such that

$$\varrho_n \to \varrho \ \text{ strongly in } C([0,T]; H_0^1(\Omega)) \tag{49}$$

$$\varrho_{n,t} \to \varrho_t \ \text{ strongly in } C([0,T]; \tilde{H}) \tag{50}$$

$$\eta_n \to \eta \ \text{ strongly in } C([0,T]; H_0^1(\Omega)). \tag{51}$$

Moreover, we have that $\{\varrho(T), \varrho_t(T), \eta_t(T)\} = \{\varrho^0, \varrho^1, \eta^0\}$.

We multiply (3) by (ϱ_n, η_n) (which is possible to do because of the regularity of the solutions) and we multiply (5) by (w, θ). Integrating by parts, we see that (44) is satisfied for $(\varrho_n^0, \varrho_n^1, \eta_n^0)$. In view of (49), (50) and (51) we can pass to the limit in (44) and we observe that (44) is satisfied by $(\varrho, \varrho_t, \eta)$, the limit of $(\varrho_n, \varrho_{n,t}, \eta_n)$.

We need to see that $(\varrho, \varrho_t, \eta)$ is a solution of (5) in the sense of Proposition 2. We observe that, thanks to the convergences just obtained, we only need to prove the convergences

$$\Delta \int_t^T \varrho_n \to \Delta \int_t^T \varrho \ \text{ strongly in } C([0,T]; L^2(\Omega)), \tag{52}$$

and

$$\eta_{n,t} \to \eta_t \ \text{ strongly in } L^2([0,T]; H^{-1}(\Omega)). \tag{53}$$

To obtain (52) we proceed as in Remark 3. That is, if (ϱ_n, η_n) is a solution of (5), we have that $(\tilde{\varrho}_n, \eta_n)$, with

$$\tilde{\varrho}_n(x,t) = -\int_t^T \varrho_n + \xi_n(x)$$

is a solution of (20).

We can extract a subsequence such that $\xi_n \to \xi$ strongly in $H_0^2(\Omega)$. Since the semigroup generated by (20) is continuous, the convergence in the data implies the convergence of the solutions and then (for a subsequence) we have that $\tilde{\varrho}_n \to \tilde{\varrho}$ strongly in $C([0,T]; H_0^2(\Omega))$. Therefore, thanks to the strong convergence of ξ_n and to the uniqueness of the weak limit we obtain (52).

To prove the convergence (53), we see that $\{\varrho_n, \vartheta_n\}$ with $\vartheta_n = \eta_{n,t}$ is a solution of (20) with $\vartheta_{n,t} \in L^2(0,T; H^{-1}(\Omega))$ and therefore we can apply classical compactness results (see Simon [21], Theorem 5), obtaining (53).

In view of the convergences above, we can pass to the limit in (8) and (9) and we observe that (44) holds for $\{\varrho, \eta\}$ the solution of (5) corresponding to the final data $(\varrho^0, \varrho^1, \eta^0)$.

6. Comments.

6.1. The case $\gamma = 0$. In this section we comment the different nature of the control problem when $\gamma = 0$. For this value of γ the system of thermoelastic plates can be written in the following way:

$$\begin{cases} w_{tt} + \Delta^2 w + \beta_1 \Delta\theta = 0 & \text{in } Q \\ \beta_2\theta_t - \Delta\theta - \beta_3\Delta w_t = 0 & \text{in } Q \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma \\ \theta = 0 & \text{on } \Sigma \\ w(0) = w^0;\ w_t(0) = w^1;\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{54}$$

It can be proved that the system is well posed in $H = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. More precisely, for every initial data $(w^0, w^1, \theta^0) \in H$ there exists a unique solution $(w, w_t, \theta) \in C([0,T], H)$. We consider now the uncoupled system:

$$\begin{cases} \tilde{w}_{tt} + \Delta^2 \tilde{w} - \beta_3\beta_1 \Delta\tilde{w}_t = 0 & \text{in } Q \\ \beta_2\tilde{\theta}_t - \Delta\tilde{\theta} - \beta_3\Delta \tilde{w}_t = 0 & \text{in } Q \\ \tilde{w} = \dfrac{\partial \tilde{w}}{\partial \nu} = 0 & \text{on } \Sigma \\ \tilde{\theta} = 0 & \text{on } \Sigma \\ \tilde{w}(0) = w^0;\ \tilde{w}_t(0) = w^1;\ \tilde{\theta}(0) = \theta^0 & \text{in } \Omega \end{cases} \tag{55}$$

and the plate equation satisfied by $\tilde{w}$:

$$\begin{cases} \tilde{w}_{tt} + \Delta^2 \tilde{w} - \beta_3\beta_1 \Delta\tilde{w}_t = 0 & \text{in } Q \\ \tilde{w} = \dfrac{\partial \tilde{w}}{\partial \nu} = 0 & \text{on } \Sigma\,. \\ \tilde{w}(0) = w^0;\ \tilde{w}_t(0) = w^1 & \text{in } \Omega \end{cases} \tag{56}$$

This equation has a very different nature from the one obtained when $\gamma > 0$; that is,

$$\tilde{w}_{tt} - \gamma\Delta\tilde{w}_{tt} + \Delta^2\tilde{w} - \beta_3\beta_1\Delta\tilde{w}_t = 0.$$

Actually, when $\gamma > 0$ the dissipative term $-\beta_3\beta_1\Delta\tilde{w}_t$ is a bounded perturbation of the conservative system

$$\tilde{w}_{tt} - \gamma\Delta\tilde{w}_{tt} + \Delta^2\tilde{w} = 0$$

and therefore it is natural to expect observability inequalities of Proposition 6 type and exact controllability results.

Nevertheless, when $\gamma = 0$ in system (56) the dissipative term $-\beta_1\beta_3\Delta\tilde{w}_t$ is not bounded in the energy space (in which $\tilde{w}_t$ belongs to $L^2(\Omega)$ and not to $H_0^1(\Omega)$ as when $\gamma > 0$) and that introduces regularizing effects and irreversibility phenomena. Actually, multiplying by $\tilde{w}_t$ in (56) we obtain that the energy

$$E(t) = \frac{1}{2}\int_\Omega |\tilde{w}_t|^2 + |\Delta\tilde{w}|^2$$

satisfies

$$\frac{dE}{dt}(t) = -\beta_3\beta_1 \int_\Omega |\nabla \tilde{w}_t|^2 dx.$$

Then, when $(w_0, w_1) \in H_0^2(\Omega) \times L^2(\Omega)$ the solution $\tilde{w}$ of (56) is in $C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ but has the additional regularity property $\tilde{w}_t \in L^2(0,T; H_0^1(\Omega))$, that is a typically parabolic behaviour.

With these kind of arguments it is easy to see that (56) is not reversible with respect to time. Therefore we cannot expect for (56) exact controllability or observability results of the type of Proposition 6.

Irreversibility effects like those described for equation (56) can be found in the wave equation with strong dissipation: $u_{tt} - \Delta u - \Delta u_t = 0$. See, for instance, Rodríguez-Bernal ([19]).

6.2. On the geometry of the support of the control. We have proved the exact-approximate controllability of system (1) when the control is supported in a neighborhood of the whole boundary.

The development of the proof of Theorem 1 shows that it is very natural to expect that exact-approximate controllability of system (1) holds as soon as we have an estimate like Proposition 7 for the uncoupled plate equation (37). The proof we give of Proposition 7 is valid for any neighborhood ω of a subset of the boundary of the form

$$\Gamma(x^0) = \{x \in \partial\Omega : (x - x^0) \cdot \nu(x) > 0\},$$

x^0 being any point in $\mathbb{R}^2$. One can expect also the techniques of C. Bardos, G. Lebeau and J. Rauch ([2]) to apply for system (37) in order to give sharp sufficient conditions on the control region ω and the control time T, so that the inequality of Proposition 7 holds.

However, in order to guarantee that Proposition 7 implies immediately Theorem 1, one has to show the analog of the uniqueness result of Proposition 3. The Holmgren Uniqueness Theorem is of local nature and therefore applies for any open subset ω of Ω. However, one has also to prove a backward uniqueness result showing that if (w, θ) solves (1) and it is such that $(w, \theta) = 0$ for $t \geq \sqrt{\gamma}\ \mathrm{diam}(\Omega \backslash \omega)/2$ then necessarily $(w, \theta) \equiv 0$. At this level we have used the fact that, in our case, Δw vanishes identically on the whole boundary of Ω and therefore, both w and θ can be expanded in Fourier series in the base of the Dirichlet Laplacian.

Obviously, when ω is not a neighborhood of the whole boundary this argument has to be modified to show that backward uniqueness holds.

7. Appendix: Some technical results.

7.1. Backward uniqueness. In this section we complete the proof of Proposition 3. More precisely, we are going to prove the following lemma:

Lemma 6. *Let $(w, w_t, \theta) \in C([0,T]; V)$ be a solution of (1) such that $\Delta w = 0$ on $\partial\Omega \times (0,T)$ and $(w(x,t), \theta(x,t)) = 0$ for every $t_1 < t \leq T$, $x \in \Omega$, with $t_1 > 0$. Then $(w(x,t), \theta(x,t)) = 0$ for every $0 \leq t \leq T$, $x \in \Omega$.*

Proof. Let $\varphi_k \in L^2(\Omega)$ be the eigenfunctions of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions and λ_k the corresponding eigenvalues; that is,

$$-\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \ \text{ on } \partial\Omega.$$

We can choose $\{\varphi_k\}_k$ forming an orthonormal basis of $L^2(\Omega)$. We observe that if w satisfies the hypothesis of Lemma 6, then we can think of (w, θ) as solving the thermoelastic system (1) with boundary conditions $w = \Delta w = \theta = 0$ on $\partial\Omega \times (0, T)$, instead of $w = \partial w/\partial\nu = \theta = 0$.

We consider the Fourier's series expansion of (w, θ); i.e., we write

$$w(x,t) = \sum a_k(t)\varphi_k, \quad a_k(t) = \int_\Omega w(x,t)\varphi_k(x)\,dx;$$

$$\theta(x,t) = \sum b_k(t)\varphi_k, \quad b_k(t) = \int_\Omega \theta(x,t)\varphi_k(x)\,dx,$$

with $a_k(0) = a_k^0,\ a_k'(0) = a_k^1,\ b_k(0) = b_k^0,\ (a_k^0, a_k^1, b_k^0)$ being the Fourier coefficients of the initial data (w^0, w^1, θ^0). We have that $a_k,\ b_k$ satisfy

$$\sum \varphi_k \left[(1+\gamma\lambda_k)a_k'' + \lambda_k^2 a_k - \beta_1\lambda_k b_k\right] = 0 \quad \text{in } \Omega \times (0,T)$$
$$\sum \varphi_k \left[\beta_2 b_k' + \lambda_k b_k + \beta_3\lambda_k a_k'\right] = 0 \quad \text{in } \Omega \times (0,T),$$

or equivalently,

$$(1+\gamma\lambda_k)a_k'' + \lambda_k^2 a_k - \beta_1\lambda_k b_k = 0; \quad \beta_2 b_k' + \lambda_k b_k + \beta_3\lambda_k a_k' = 0 \ \ t \in (0,T)$$

for every k. We note that for every k, $a_k(t) = b_k(t) = 0$ for every $t \in [t_1, T]$. Then, by uniqueness of solutions of the ordinary differential system, we have that $a_k(t) = b_k(t) = 0$ for every $0 \le t \le T$.

7.2. Proof of Proposition 7. For the proof of Proposition 7 we need the following estimates:

Lemma 7. *Assume that ω is a neighborhood of the boundary $\partial\Omega$ in Ω and that $T > 0$. Then, for any $0 < \varepsilon < T/2$ and $\omega_0 \subset \Omega$ open and nonempty subset such that $\bar\omega_0 \cap \Omega \subset \omega$, there exists a constant $C > 0$ and a seminorm $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}^+$, such that*

$$\int_\varepsilon^{T-\varepsilon}\int_{\omega_0} |\Delta v|^2 \le C\Big(\int_0^T\int_\Omega \rho|\nabla v_t|^2 + X^2(v^0, v^1)\Big) \quad \forall (v^0, v^1) \in H_0^2(\Omega) \times H_0^1(\Omega), \quad (57)$$

where v denotes the solution of (37). *Moreover, X is continuous and compact.*

Proof. Let $q = \rho(x)h(t)$ with $h \in C^\infty(0,T)$ such that $h(t) = 1$ for $t \in [\varepsilon, T-\varepsilon]$ and $h(0) = h(T) = 0$ where ρ is given by (21)–(22). We multiply (37) by qv and integrate in Q:

$$\int_0^T\int_\Omega v_{tt}qv - \gamma\int_0^T\int_\Omega \Delta v_{tt}qv + \int_0^T\int_\Omega \Delta^2 vqv + \sqrt{A}\int_0^T\int_\Omega v_t qv$$
$$+ \frac{A\gamma}{4}\int_0^T\int_\Omega \Delta vqv + \frac{A}{4}\int_0^T\int_\Omega qv^2 = 0.$$

Integrating by parts we obtain

$$-\int_0^T\int_\Omega q|v_t|^2 - \int_0^T\int_\Omega h_t\rho v_t v + \frac{A}{4}\int_0^T\int_\Omega qv^2 - \frac{\sqrt{A}}{2}\int_0^T\int_\Omega \rho h_t|v|^2 - \gamma\int_0^T\int_\Omega q|\nabla v_t|^2$$
$$-\gamma\int_0^T\int_\Omega hv_t\nabla\rho\cdot\nabla v_t - \gamma\int_0^T\int_\Omega h_t v\nabla\rho\cdot\nabla v_t - \gamma\int_0^T\int_\Omega \rho h_t\nabla v_t\cdot\nabla v - \frac{A\gamma}{4}\int_0^T\int_\Omega q|\nabla v|^2$$
$$-\frac{A\gamma}{4}\int_0^T\int_\Omega hv\nabla\rho\cdot\nabla v + \int_0^T\int_\Omega q|\Delta v|^2 + 2\int_0^T\int_\Omega \Delta v\nabla q\nabla v + \int_0^T\int_\Omega v\Delta v\Delta q = 0.$$

Therefore,

$$\int_0^T\int_\Omega q|\Delta v|^2 \le C\int_0^T\int_\Omega \rho|\nabla v_t|^2 + \frac{1}{2}\int_0^T\int_\Omega q|\Delta v|^2 + C(q)\left(\int_0^T\int_\Omega |v_t|^2 + \int_0^T\int_\Omega |v|^2 + \int_0^T\int_\Omega |\nabla v|^2\right).$$

This implies that

$$\frac{1}{2}\int_0^T\int_\Omega q|\Delta v|^2 \le C\left(\int_0^T\int_\Omega \rho|\nabla v_t|^2 + X^2(v^0,v^1)\right),$$

where $X : H_0^2(\Omega)\times H_0^1(\Omega)\to \mathbb{R}^+$ is continuous and compact.

On the other hand, from the construction of q we easily see that there exists a positive constant $\alpha > 0$ such that $q(x,t)\ge\alpha$ for every $(x,t)\in\omega_0\times(\varepsilon, T-\varepsilon)$.

In consequence,

$$\frac{1}{2}\int_\varepsilon^{T-\varepsilon}\int_{\omega_0}|\Delta v|^2 \le C\left(\int_0^T\int_\Omega \rho|\nabla v_t|^2 + X^2(v^0,v^1)\right).$$

Lemma 8. *Let Ω be a bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega$ of class C^3. Let $q(x,t) = (q_k(x,t))_{k=1,2}$ be a vector field such that $q\in[C^2(\overline{\Omega}\times(0,T))]^2$. Then, for every weak solution*

$$v\in C([0,T];H_0^2(\Omega))\cap C^1([0,T];H_0^1(\Omega))\cap C^2([0,T];\tilde{H})$$

of problem (37)*, we have the identity*

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_{\partial\Omega} q_k\nu_k|\Delta v|^2\\
&= \frac{-1}{2}\int_0^T\int_\Omega \mathrm{div} q|\Delta v|^2 + \int_0^T\int_\Omega \Delta v\Delta q_k\frac{\partial v}{\partial x_k} + 2\int_0^T\int_\Omega \Delta v[\frac{\partial q_k}{\partial x_j}\frac{\partial^2 v}{\partial x_j\partial x_k}]\\
&- \frac{A\gamma}{4}\int_0^T\int_\Omega \nabla v\cdot\nabla q_k\frac{\partial v}{\partial x_k} + \frac{A\gamma}{8}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v|^2 - \gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla(q_{k,t}\frac{\partial v}{\partial x_k}) \qquad (58)\\
&+ \frac{\gamma}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v_t|^2 - \gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla q_k\frac{\partial v_t}{\partial x_k} + Y - \frac{A}{8}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}v^2\\
&+ \sqrt{A}\int_0^T\int_\Omega v_t q_k\frac{\partial v}{\partial x_k} - \int_0^T\int_\Omega v_t q_{k,t}\frac{\partial v}{\partial x_k} + \frac{1}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|v_t|^2,
\end{aligned}$$

where

$$Y = \int_\Omega v_t q_k\frac{\partial v}{\partial x_k}\Big|_0^T + \gamma\int_\Omega \nabla v_t\cdot\nabla(q_k\frac{\partial v}{\partial x_k})\Big|_0^T.$$

Proof. As in Lemma 1, we prove the identity for regular solutions. By density it can be extended for every solution of finite energy. We multiply (37) by $q_k\partial v/\partial x_k$ and integrate on Q:

$$\int_0^T\int_\Omega(v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + \sqrt{A}v_t + \frac{A}{4}v + \frac{\gamma A}{4}\Delta v)q_k\frac{\partial v}{\partial x_k}\,dx\,dt = 0.$$

Integrating by parts we obtain

$$\int_0^T\int_\Omega v_{tt}q_k\frac{\partial v}{\partial x_k}=-\int_0^T\int_\Omega v_tq_{k,t}\frac{\partial v}{\partial x_k}-\frac{1}{2}\int_0^T\int_\Omega q_k\frac{\partial|v_t|^2}{\partial x_k}+\int_\Omega v_tq_k\frac{\partial v}{\partial x_k}\Big|_0^T. \tag{59}$$

We observe that

$$-\frac{1}{2}\int_0^T\int_\Omega q_k\frac{\partial|v_t|^2}{\partial x_k}=\frac{1}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|v_t|^2 \tag{60}$$

since $v_t=0$ on Σ. From (59) and (60) we obtain

$$\int_0^T\int_\Omega v_{tt}q_k\frac{\partial v}{\partial x_k}=-\int_0^T\int_\Omega v_tq_{k,t}\frac{\partial v}{\partial x_k}+\frac{1}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|v_t|^2+\int_\Omega v_tq_k\frac{\partial v}{\partial x_k}\Big|_0^T. \tag{61}$$

On the other hand

$$\frac{A}{4}\int_0^T\int_\Omega vq_k\frac{\partial v}{\partial x_k}=\frac{A}{8}\int_0^T\int_\Omega q_k\frac{\partial v^2}{\partial x_k}=-\frac{A}{8}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}v^2. \tag{62}$$

Moreover, we have that

$$\begin{aligned}-\gamma\int_0^T\int_\Omega\Delta v_{tt}q_k\frac{\partial v}{\partial x_k}=&\gamma\int_0^T\int_\Omega\Delta v_tq_{k,t}\frac{\partial v}{\partial x_k}+\gamma\int_0^T\int_\Omega\Delta v_tq_k\frac{\partial v_t}{\partial x_k}-\gamma\int_\Omega\Delta v_tq_k\frac{\partial v}{\partial x_k}\Big|_0^T\\ =&-\gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla(q_{k,t}\frac{\partial v}{\partial x_k})-\gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla q_k\frac{\partial v_t}{\partial x_k}\\ &-\frac{\gamma}{2}\int_0^T\int_\Omega q_k\frac{\partial|\nabla v_t|^2}{\partial x_k}+\gamma\int_\Omega\nabla v_t\cdot\nabla(q_k\frac{\partial v}{\partial x_k})\Big|_0^T\end{aligned} \tag{63}$$

since $q_k\partial v/\partial x_k=0$ on Σ. We note that

$$-\frac{\gamma}{2}\int_0^T\int_\Omega q_k\frac{\partial|\nabla v_t|^2}{\partial x_k}=\frac{\gamma}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v_t|^2. \tag{64}$$

In view of (63) and (64), we obtain

$$\begin{aligned}-\gamma\int_0^T\int_\Omega\Delta v_{tt}q_k\frac{\partial v}{\partial x_k}=&-\gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla(q_{k,t}\frac{\partial v}{\partial x_k})-\gamma\int_0^T\int_\Omega\nabla v_t\cdot\nabla q_k\frac{\partial v_t}{\partial x_k}\\ &+\frac{\gamma}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v_t|^2-\gamma\int_\Omega\nabla v_t\cdot\nabla(q_k\frac{\partial v}{\partial x_k})\Big|_0^T.\end{aligned} \tag{65}$$

On the other hand

$$\frac{\gamma A}{4}\int_0^T\int_\Omega\Delta vq_k\frac{\partial v}{\partial x_k}=-\frac{A\gamma}{4}\int_0^T\int_\Omega\nabla v\cdot\nabla q_k\frac{\partial v}{\partial x_k}+\frac{A\gamma}{8}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v|^2. \tag{66}$$

Finally, since $v\in H_0^2(\Omega)$, we have $\frac{\partial v}{\partial\nu\partial x_k}=\frac{\partial^2v}{\partial\nu^2}\nu_k$ and $\frac{\partial^2 v}{\partial x_k^2}=\frac{\partial^2 v}{\partial\nu^2}\nu_k^2$ on Σ. Integrating by parts we obtain:

$$\begin{aligned}\int_0^T\int_\Omega\Delta^2vq_k\frac{\partial v}{\partial x_k}=&-\frac{1}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\Delta v|^2+\int_0^T\int_\Omega\Delta v\Delta q_k\frac{\partial v}{\partial x_k}\\ &+2\int_0^T\int_\Omega\Delta v[\nabla q_k\cdot\nabla\frac{\partial v}{\partial x_k}]-\frac{1}{2}\int_0^T\int_\Sigma q\cdot\nu|\Delta v|^2.\end{aligned} \tag{67}$$

Adding (61), (62), (65), (66), (67) with the term $\sqrt{A}\int_0^T\int_\Omega v_t q_k \frac{\partial v}{\partial x_k}$, we obtain (58).

Proof of Proposition 7. Along this proof we will denote by $X(v^0, v^1)$ a generic term which is continuous and compact from $H_0^2(\Omega)\times H_0^1(\Omega)$ in $\mathbb{R}^+$ and that may change from line to line. We proceed in several steps.

Step 1: For $x^0 \in \mathbb{R}^2$ given we define

$$m(x) = x - x^0; \quad R(x^0) = \max_{x\in\bar{\Omega}} |x - x^0|;$$

$$\Gamma(x^0) = \{x \in \partial\Omega / m(x)\cdot\nu(x) > 0\}, \quad \Sigma(x^0) = \Gamma(x^0)\times(0,T).$$

To simplify the proof we suppose that $T > \sqrt{\gamma}\mathrm{diam}(\Omega)$. At the end of this Appendix, we go back to the case $T > \sqrt{\gamma}\mathrm{diam}(\Omega\backslash\omega)$. Then, there exist $\varepsilon, \delta > 0$ and x^ε such that

$$T > \sqrt{\gamma}(\mathrm{diam}(\Omega)+\varepsilon) > 2\sqrt{\gamma}(R(x^\varepsilon)+\delta). \tag{68}$$

We set $q(x,t) = m(x)$ (with $x^0 = x^\varepsilon$) in identity (58) and we obtain

$$\int_0^T\int_{\partial\Omega}\frac{m(x)\cdot\nu}{2}|\Delta v|^2 = \int_0^T\int_\Omega |v_t|^2 + \sqrt{A}\int_0^T\int_\Omega v_t(m(x)\cdot\nabla v) + \int_0^T\int_\Omega|\Delta v|^2 - \frac{A}{4}\int_0^T\int_\Omega |v|^2 + Z, \tag{69}$$

where

$$Z = \int_\Omega v_t m(x)\cdot\nabla v\Big|_0^T + \gamma\int_\Omega \nabla v_t\cdot\nabla(m(x)\cdot\nabla v)\Big|_0^T.$$

Now we multiply (37) by v and integrate by parts:

$$\begin{aligned} &-\int_0^T\int_\Omega |v_t|^2 - \gamma\int_0^T\int_\Omega|\nabla v_t|^2 + \int_0^T\int_\Omega|\Delta v|^2 \\ &= -\int_\Omega v_t v\Big|_0^T - \gamma\int_\Omega \nabla v_t\cdot\nabla v\Big|_0^T - \sqrt{A}\int_0^T\int_\Omega v_t v - \frac{A}{4}\int_0^T\int_\Omega v^2 + \frac{A\gamma}{4}\int_0^T\int_\Omega|\nabla v|^2. \end{aligned} \tag{70}$$

On the other hand, for v a solution of (37) we define the energy:

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_\Omega |v_t|^2 + \gamma\int_\Omega|\nabla v_t|^2 + \int_\Omega|\Delta v|^2 - \frac{A\gamma}{4}\int_\Omega|\nabla v|^2 + \frac{A}{4}\int_\Omega |v|^2\right).$$

Multiplying (37) by v_t and integrating by parts in Ω, we note that $\mathcal{E}(t)$ satisfies:

$$\frac{d\mathcal{E}}{dt} = -\sqrt{A}\int_\Omega |v_t|^2;$$

that is,

$$\mathcal{E}(T) = -\sqrt{A}\int_t^T\int_\Omega |v_t|^2 + \mathcal{E}(t) \qquad \forall\ 0\le t<T. \tag{71}$$

We rewrite (69) in the following way:

$$\begin{aligned} &\int_0^T\mathcal{E}(t) + \int_0^T\int_\Omega|v_t|^2 + \frac{1}{2}\int_0^T\int_\Omega|\Delta v|^2 - \frac{1}{2}\int_0^T\int_\Omega|v_t|^2 - \frac{\gamma}{2}\int_0^T\int_\Omega|\nabla v_t|^2 + \frac{\gamma A}{8}\int_0^T\int_\Omega|\nabla v|^2 \\ &- \frac{3A}{8}\int_0^T\int_\Omega|v|^2 + \sqrt{A}\int_0^T\int_\Omega v_t(m(x)\cdot\nabla v) + Z = \int_0^T\int_{\partial\Omega}\frac{m(x)\cdot\nu}{2}|\Delta v|^2. \end{aligned} \tag{72}$$

Multiplying (70) by 1/2 and substituting in (72) we obtain

$$\begin{aligned}&\int_0^T \mathcal{E}(t) + \int_0^T\int_\Omega |v_t|^2 + \frac{\gamma A}{4}\int_0^T\int_\Omega |\nabla v|^2 - \frac{A}{4}\int_0^T\int_\Omega |v|^2\\ &+\sqrt{A}\int_0^T\int_\Omega v_t(m(x)\cdot\nabla v) - \frac{\sqrt{A}}{2}\int_0^T\int_\Omega v_t v + Z - \tilde{Z} = \int_0^T\int_{\partial\Omega}\frac{m(x)\cdot\nu}{2}|\Delta v|^2,\end{aligned} \tag{73}$$

where

$$Z - \tilde{Z} = \int_\Omega v_t(m(x)\cdot\nabla v - \frac{1}{2}v)\Big|_0^T + \gamma\int_\Omega \nabla v_t\cdot\nabla(m(x)\cdot\nabla v - \frac{1}{2}v)\Big|_0^T.$$

In consequence,

$$\begin{aligned}T\mathcal{E}(T) \leq& \frac{R(x^\varepsilon)}{2}\int_0^T\int_{\Gamma(x^\varepsilon)}|\Delta v|^2 + \frac{A\gamma T}{8}\int_\Omega |\nabla v(T)|^2\\ &+\frac{A}{4}\int_0^T\int_\Omega v^2 + \sqrt{A}\int_0^T\int_\Omega v_t(\frac{v}{2} - m(x)\cdot\nabla v) + \tilde{Z} - Z.\end{aligned} \tag{74}$$

By Young's inequality

$$\sqrt{A}\int_0^T\int_\Omega v_t(\frac{v}{2} - m(x)\cdot\nabla v) \leq \frac{\sqrt{A}}{2}\int_0^T\int_\Omega |v_t|^2 + \frac{\sqrt{A}}{2}\int_0^T\int_\Omega |\frac{v}{2} - m(x)\cdot\nabla v|^2.$$

On the other hand

$$\int_0^T\int_\Omega |\frac{v}{2} - m(x)\cdot\nabla v|^2 = \int_0^T\int_\Omega \frac{v^2}{4} + \int_0^T\int_\Omega |m(x)\cdot\nabla v|^2 - \int_0^T\int_\Omega v m(x)\cdot\nabla v,$$

and we note that

$$-\int_0^T\int_\Omega v m(x)\cdot\nabla v = -\frac{1}{2}\int_0^T\int_\Omega \mathrm{div}(m(x)v^2) + \int_0^T\int_\Omega v^2 = \int_0^T\int_\Omega v^2.$$

Therefore

$$\sqrt{A}\int_0^T\int_\Omega v_t(\frac{v}{2} - m(x)\cdot\nabla v) \leq \frac{\sqrt{A}}{2}\int_0^T\int_\Omega |v_t|^2 + \frac{5\sqrt{A}}{8}\int_0^T\int_\Omega v^2 + \frac{R^2\sqrt{A}}{2}\int_0^T\int_\Omega |\nabla v|^2. \tag{75}$$

We have obtained

$$T\mathcal{E}(T) \leq \frac{R(x^\varepsilon)}{2}\int_0^T\int_{\Sigma(x^\varepsilon)}|\Delta v|^2 + X^2(v^0, v^1) + Y, \tag{76}$$

with

$$Y = \gamma\int_\Omega \nabla v_t\cdot\nabla(m(x)\cdot\nabla v - \frac{1}{2}v)\Big|_0^T.$$

We define

$$Y(t) = \gamma\int_\Omega \nabla v_t\cdot\nabla(m(x)\cdot\nabla v - \frac{1}{2}v).$$

By Young's inequality

$$|Y(t)| \leq \frac{\gamma\alpha}{2}\int_\Omega |\nabla v_t(t)|^2 + \frac{\gamma}{2\alpha}\int_\Omega |\nabla(m(x)\cdot\nabla v - \frac{1}{2}v)|^2. \tag{77}$$

We observe that

$$\tilde{Y}(t) = \int_\Omega |\nabla(m(x)\cdot\nabla v - \frac{1}{2}v)|^2 = \int_\Omega |\nabla(m\cdot\nabla v)|^2 - \int_\Omega \nabla(m\cdot\nabla v)\cdot\nabla v + \frac{1}{4}\int_\Omega |\nabla v|^2.$$

Developing the squares and integrating by parts we obtain:

$$\begin{aligned}\tilde{Y}(t) \leq & R^2(x^\varepsilon)\int_\Omega |\Delta v(t)|^2 + \int_\Omega m_1\partial^2_{1,1}v\partial_1 v + \int_\Omega m_2\partial_1 v\partial^2_{1,2}v \\ & + \int_\Omega m_2\partial_2 v\partial^2_{2,2}v + \int_\Omega m_1\partial_2 v\partial^2_{1,2}v + \frac{1}{4}\int_\Omega |\nabla v|^2,\end{aligned}$$

where ∂_j denotes $\partial/\partial x_j$, m_j denotes the j-th coordinate of the vector m, $j = 1, 2$ and $\partial^2_{j,k}$ denotes $\partial^2/\partial x_j\partial x_k$. Integrating by parts and applying Young's inequality we obtain

$$\begin{aligned}\tilde{Y}(t) \leq & R^2(x^\varepsilon)\int_\Omega |\Delta v|^2 + 3\int_\Omega m_1\partial_1 v\partial^2_{2,2}v + 3\int_\Omega m_2\partial_2 v\partial^2_{1,1}v \\ & + \int_\Omega m_1\partial_2 v\partial^2_{1,1}v + \int_\Omega m_2\partial_2 v\partial^2_{2,2}v + \frac{1}{4}\int_\Omega |\nabla v|^2 \\ \leq & (R(x^\varepsilon)+\delta)^2\int_\Omega |\Delta v(t)|^2 + C\int_\Omega |\nabla v(t)|^2.\end{aligned}$$

Therefore,

$$|Y(t)| \leq \frac{\alpha\gamma}{2}\int_\Omega |\nabla v_t(t)|^2 + \frac{\gamma(R+\delta)^2}{2\alpha}\int_\Omega |\Delta v(t)|^2 + C\int_\Omega |\nabla v(t)|^2,$$

where $C > 0$ is a constant depending on δ and on $R(x^\varepsilon)$. We set $\alpha(\gamma) = (R+\delta)\sqrt{\gamma}$. Therefore,

$$|Y(t)| \leq \alpha(\gamma)\mathcal{E}(t) + C\int_\Omega |\nabla v(t)|^2$$

and in consequence

$$|Y(t)| \leq \alpha(\gamma)\left(\mathcal{E}(T) + \sqrt{A}\int_0^T\int_\Omega |v_t|^2\right) + CX^2(v^0, v^1).$$

That is,

$$|Y| \leq 2\alpha(\gamma)\left(\mathcal{E}(T) + \sqrt{A}\int_0^T\int_\Omega |v_t|^2\right) + CX^2(v^0, v^1). \tag{78}$$

In view of (76) and (78) we obtain

$$(T - 2\alpha(\gamma))\,\mathcal{E}(T) \leq C\int_0^T\int_{\Gamma(x^\varepsilon)} |\Delta v|^2 + X^2(v^0, v^1).$$

By construction, $T > 2\alpha(\gamma)$ and therefore

$$|v^0|^2_{H_0^2(\Omega)} + |v^1|^2_{H_0^1(\Omega)} \leq C\int_0^T\int_{\Gamma(x^\varepsilon)} |\Delta v|^2 + X^2(v^0, v^1), \tag{79}$$

where $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}^+$ is continuous and compact.

Since the system is invariant by translations with respect to time we obtain

$$\mathcal{E}(T-\varepsilon_1) \leq C\int_{\varepsilon_1}^{T-\varepsilon_1}\int_{\Gamma(x^\varepsilon)} |\Delta v|^2 + X^2(v^0, v^1) \quad \forall\, T > \sqrt{\gamma}\mathrm{diam}(\Omega) + 2\varepsilon_1. \tag{80}$$

Step 2: Let $T - 2(\varepsilon_1+\delta) > 2\sqrt{\gamma}R(x^\varepsilon)$, $\omega_0 \subset \Omega$ a neighborhood of $\partial\Omega$ such that $\bar{\omega}_0 \cap \Omega \subset \omega$ and x^ε satisfying (68). We can construct $q(x,t) \in C^2(\overline{\Omega} \times (0,T))$ such that

$$q(x,t) = \nu(x) \text{ in } \Gamma(x^\varepsilon) \times (\varepsilon_1, T-\varepsilon_1), \quad q(x,t)\cdot\nu(x) \geq 0\ \forall\ (x,t) \text{ on } \Sigma,$$
$$q(x,0) = q(x,T) = 0\ \forall x \in \Omega, \quad q(x,t) = 0\ \forall (x,t) \in [\Omega\backslash\omega_0] \times (0,T).$$

Applying identity (58) to this function $q(x,t)$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^T\int_{\partial\Omega} q_k\nu_k|\Delta v|^2 &= \frac{-1}{2}\int_0^T\int_\Omega \mathrm{div}q|\Delta v|^2 + \int_0^T\int_\Omega \Delta v\Delta q_k\frac{\partial v}{\partial x_k}\\
&+ 2\int_0^T\int_\Omega \Delta v[\frac{\partial q_k}{\partial x_j}\frac{\partial^2 v}{\partial x_j\partial x_k}] - \frac{A\gamma}{4}\int_0^T\int_\Omega \nabla v\nabla q_k\frac{\partial v}{\partial x_k} + \frac{A\gamma}{8}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v|^2\\
&- \gamma\int_0^T\int_\Omega \nabla v_t\cdot\nabla(q_{k,t}\frac{\partial v}{\partial x_k}) + \frac{\gamma}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|\nabla v_t|^2 - \gamma\int_0^T\int_\Omega \nabla v_t\nabla q_k\frac{\partial v_t}{\partial x_k}\\
&- \frac{A}{8}\int_0^T\int_\Omega \frac{\partial q_k}{\partial x_k}v^2 + \sqrt{A}\int_0^T\int_\Omega v_t q_k\frac{\partial v}{\partial x_k} - \int_0^T\int_\Omega v_t q_{k,t}\frac{\partial v}{\partial x_k} + \frac{1}{2}\int_0^T\int_\Omega\frac{\partial q_k}{\partial x_k}|v_t|^2,
\end{aligned} \tag{81}$$

and then

$$\frac{1}{2}\int_\Sigma q\cdot\nu|\Delta v|^2 \leq C\left(\int_0^T\int_{\omega_0}|\Delta v|^2 + \int_0^T\int_{\omega_0}|\nabla v_t|^2\right) + X^2(v^0, v^1).$$

Let ρ be like in Proposition 7. Since $\bar{\omega}_0 \cap \Omega \subset \omega$ we have

$$\frac{1}{2}\int_{\varepsilon_1}^{T-\varepsilon_1}\int_{\Gamma(x^\varepsilon)}|\Delta v|^2 \leq C\left(\int_0^T\int_{\omega_0}|\Delta v|^2 + \int_0^T\int_\omega \rho|\nabla v_t|^2\right) + X^2(v^0, v^1).$$

Since the system is invariant by translations with respect to time, we obtain

$$\frac{1}{2}\int_{(\varepsilon_1+\delta)}^{T-(\varepsilon_1+\delta)}\int_{\Gamma(x^\varepsilon)}|\Delta v|^2 \leq C\left(\int_\delta^{T-\delta}\int_{\omega_0}|\Delta v|^2 + \int_0^T\int_\omega \rho|\nabla v_t|^2\right) + X^2(v^0, v^1). \tag{82}$$

Then, in view of (80) and (82), we have

$$E(T-(\varepsilon_1+\delta)) \leq C\left(\int_\delta^{T-\delta}\int_{\omega_0}|\Delta v|^2+\int_0^T\int_\omega \rho|\nabla v_t|^2\right)+X^2(v^0,v^1).$$

But since we have (71), we obtain that

$$E(T-(\varepsilon_1+\delta)) = E(T)+X^2(v^0,v^1).$$

In consequence

$$|v^0|^2_{H_0^2(\Omega)}+|v^1|^2_{H_0^1(\Omega)} \leq C\left(\int_\delta^{T-\delta}\int_{\omega_0}|\Delta v|^2+\int_0^T\int_\omega \rho|\nabla v_t|^2\right)+X^2(v^0,v^1).$$

Combining this with Lemma 7 we conclude the proof.

7.3. Further estimates. Let us now give the estimates we avoided in the proof of Proposition 7 supposing $T>\sqrt{\gamma}$ diam (Ω). We remember that the proof was divided in two parts. It is precisely in the first part where the restrictions on the control time appeared and so we are going to show how to modify it to obtain the inequality for any $T>\sqrt{\gamma}\mathrm{diam}(\Omega\backslash\omega)$.

We consider first $\omega_0\subset\omega$, an open nonempty set such that $\Omega\backslash\omega_0$ is connected, $\bar{\omega}_0\subset\omega\cup\partial\Omega$ and such that $T>\sqrt{\gamma}$ $\mathrm{diam}(\Omega\backslash\omega_0)$. For $x^0\in\mathbb{R}^2$ we define

$$R(x^0)=\max_{x\in\overline{\Omega\backslash\omega_0}}|x-x^0|.$$

Since $\Omega\backslash\omega_0$ is connected, there exists $\delta>0,\ \varepsilon>0$ and x^ε such that

$$T>\sqrt{\gamma}(\mathrm{diam}(\Omega\backslash\omega_0)+\varepsilon)>2\sqrt{\gamma}(R(x^\varepsilon)+\delta).$$

We define

$$M_\varepsilon=\max_{x\in\bar{\Omega}}|x-x^\varepsilon|,\quad m(x)=x-x^\varepsilon$$

and

$$\Gamma(x^\varepsilon)=\{x\in\partial\Omega : m(x)\cdot\nu(x)>0\}.$$

Let $\varepsilon_1>0$ and define

$$\psi(t)=\begin{cases}1 & t\in[\varepsilon_1,T-\varepsilon_1]\\ \frac{t}{\varepsilon_1} & t\in[0,\varepsilon_1)\\ \frac{-t}{\varepsilon_1}+\frac{T}{\varepsilon_1} & t\in(T-\varepsilon_1,T].\end{cases}$$

Let us put $q(x,t)=\psi(t)m(x)$ in identity (58). We obtain

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_{\partial\Omega}\psi(t)(m(x)\cdot\nu)|\Delta v|^2\\ &=\int_0^T\int_\Omega\psi(t)|v_t|^2+\sqrt{A}\int_0^T\int_\Omega\psi(t)v_t(m(x)\cdot\nabla v)+\int_0^T\int_\Omega\psi(t)|\Delta v|^2\\ &-\frac{A}{4}\int_0^T\int_\Omega\psi(t)|v|^2-\gamma\int_0^T\int_\Omega\psi_t(t)\nabla v_t\cdot\nabla(m\cdot\nabla v)-\int_0^T\int_\Omega\psi_t(t)v_t(m\cdot\nabla v).\end{aligned} \tag{83}$$

We multiply (37) by $\frac{1}{2}\psi(t)v$ and integrate by parts in Q. We obtain

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\!\!\int_\Omega \psi(t)|v_t|^2 - \frac{\gamma}{2}\int_0^T\!\!\int_\Omega \psi(t)|\nabla v_t|^2 + \frac{1}{2}\int_0^T\!\!\int_\Omega \psi(t)|\Delta v|^2 = \frac{1}{2}\int_0^T\!\!\int_\Omega \psi_t(t)v_t v \\
&+\frac{\gamma}{2}\int_0^T\!\!\int_\Omega \psi_t(t)\nabla v_t\cdot\nabla v - \frac{\sqrt{A}}{2}\int_0^T\!\!\int_\Omega \psi(t)v_t v + \frac{A\gamma}{8}\int_0^T\!\!\int_\Omega \psi(t)|\nabla v|^2 - \frac{A}{8}\int_0^T\!\!\int_\Omega \psi(t)|v|^2.
\end{aligned} \tag{84}
$$

On the other hand we remember that the energy

$$
\mathcal{E}(t) = \frac{1}{2}\left(\int_\Omega |v_t|^2 + \gamma\int_\Omega |\nabla v_t|^2 + \int_\Omega |\Delta v|^2 - \frac{A\gamma}{4}\int_\Omega |\nabla v|^2 + \frac{A}{4}\int_\Omega |v|^2\right)
$$

satisfies

$$
\frac{d\mathcal{E}}{dt} = -\sqrt{A}\int_\Omega |v_t|^2.
$$

That is,

$$
\mathcal{E}(T) = -\sqrt{A}\int_t^T\int_\Omega |v_t|^2 + \mathcal{E}(t)\ \forall 0\le t<T.
$$

In view of (84) and the definition of $\mathcal{E}(t)$, we can rewrite (83) in the following way:

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_{\partial\Omega}\psi(t)m(x)\cdot\nu|\Delta v|^2 = \int_0^T \psi(t)\mathcal{E}(t) + \int_0^T\!\!\int_\Omega \psi(t)|v_t|^2 \\
&+\sqrt{A}\int_0^T\!\!\int_\Omega \psi_t(t)v_t(m(x)\cdot\nabla v) - \frac{3A}{8}\int_0^T\!\!\int_\Omega \psi(t)|v|^2 + \int_0^T\!\!\int_\Omega \psi_t(t)v_t(\frac{v}{2} - m\cdot\nabla v) \\
&-\sqrt{A}\int_0^T\!\!\int_\Omega \psi(t)v_t v + \gamma\int_0^T\!\!\int_\Omega \psi_t(t)\nabla v_t\cdot\nabla(\frac{v}{2} - m\cdot\nabla v).
\end{aligned} \tag{85}
$$

We also have that

$$
\int_0^T \psi(t)\mathcal{E}(t) = \int_0^T \psi(t)\mathcal{E}(T) - \sqrt{A}\int_t^T\int_\Omega |v_t|^2 \ge \int_{\varepsilon_1}^{T-\varepsilon_1}\mathcal{E}(T) = T\mathcal{E}(T).
$$

In consequence

$$
\begin{aligned}
T\mathcal{E}(T) \le{}& \frac{M_\varepsilon}{2}\int_0^T\int_{\partial\Omega}|\Delta v|^2 - \sqrt{A}\int_0^T\!\!\int_\Omega \psi_t(t)v_t(m(x)\cdot\nabla v) + \sqrt{A}\int_0^T\!\!\int_\Omega \psi(t)v_t v \\
&+\frac{3A}{8}\int_0^T\!\!\int_\Omega \psi(t)|v|^2 - \gamma\int_0^T\!\!\int_\Omega \psi_t(t)\nabla v_t\cdot\nabla(\frac{v}{2} - m\cdot\nabla v) - \int_0^T\!\!\int_\Omega \psi_t(t)v_t(\frac{v}{2} - m\cdot\nabla v).
\end{aligned}
$$

Proceeding like in (75) we obtain

$$
\begin{aligned}
&\sqrt{A}\int_0^T\!\!\int_\Omega \psi_t(t)v_t(\frac{v}{2} - m(x)\cdot\nabla v) \\
&\le \frac{C\sqrt{A}}{2}\int_0^T\!\!\int_\Omega |v_t|^2 + \frac{5C\sqrt{A}}{8}\int_0^T\!\!\int_\Omega v^2 + \frac{C{M_\varepsilon}^2\sqrt{A}}{2}\int_0^T\!\!\int_\Omega |\nabla v|^2
\end{aligned}
$$

and then we have

$$T\mathcal{E}(T) \leq \frac{M_\varepsilon}{2}\int_0^T \int_{\Sigma(x^\varepsilon)} |\Delta v|^2 + X^2(v^0, v^1) + Y,$$

where $X : H_0^2(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is continuous and compact and

$$Y = \gamma \int_0^T\int_\Omega \psi_t \nabla v_t \cdot \nabla(m(x)\cdot\nabla v - \frac{1}{2}v).$$

We consider now $\omega_1 \subset \omega_0$, nonempty, with measure satisfying

$$|\omega_1| < \frac{\delta^2 R(x_\varepsilon)}{M_\varepsilon^2(R(x_\varepsilon)+2\delta)}$$

and such that $\overline{\omega_0\backslash\omega_1} \subset \omega$.

As in the proof of Proposition 7 we consider

$$\tilde{Y}(t) = \int_\Omega |\nabla(m(x)\cdot\nabla v - \frac{1}{2}v)|^2 = \int_\Omega |\nabla(m\cdot\nabla v)|^2 - \int_\Omega \nabla(m\cdot\nabla v)\cdot\nabla v + \frac{1}{4}\int_\Omega |\nabla v|^2.$$

Developing the squares and integrating by parts we obtain

$$\begin{aligned}\tilde{Y}(t) \leq &\int_\Omega |m(x)|^2|\Delta v(t)|^2 + \int_\Omega m_1\partial_{1,1}^2 v\partial_1 v + \int_\Omega m_2\partial_1 v\partial_{1,2}^2 v + \int_\Omega m_2\partial_2 v\partial_{2,2}^2 v\\ &+ \int_\Omega m_1\partial_2 v\partial_{1,2}^2 v + \frac{1}{4}\int_\Omega |\nabla v|^2\\ \leq &R^2(x^\varepsilon)\int_{\Omega\backslash\omega_0} |\Delta v(t)|^2 + {M_\varepsilon}^2\int_{\omega_1} |\Delta v(t)|^2 + {M_\varepsilon}^2\int_{\omega_0\backslash\omega_1} |\Delta v(t)|^2 + A(t),\end{aligned}$$

where

$$A(t) = \int_\Omega m_1\partial_{1,1}^2 v\partial_1 v + \int_\Omega m_2\partial_1 v\partial_{1,2}^2 v + \int_\Omega m_2\partial_2 v\partial_{2,2}^2 v + \int_\Omega m_1\partial_2 v\partial_{1,2}^2 v + \frac{1}{4}\int_\Omega |\nabla v|^2.$$

Let $\alpha > 0$ be chosen later. Integrating by parts and applying Young's inequality we obtain

$$\begin{aligned}A(t) = &\frac{1}{4}\int_\Omega |\nabla v|^2 + 3\int_\Omega m_1\partial_1 v\partial_{2,2}^2 v + 3\int_\Omega m_2\partial_2 v\partial_{1,1}^2 v + \int_\Omega m_1\partial_2 v\partial_{1,1}^2 v + \int_\Omega m_2\partial_1 v\partial_{2,2}^2 v\\ \leq &\frac{1}{4}\int_\Omega |\nabla v|^2 + \frac{3\alpha}{2}\int_\Omega m_1^2|\partial_{2,2}^2 v|^2 + \frac{3}{2\alpha}\int_\Omega |\partial_1 v|^2 + \frac{3\alpha}{2}\int_\Omega m_2^2|\partial_{1,1}^2 v|^2\\ &+ \frac{3}{2\alpha}\int_\Omega |\partial_2 v|^2 + \frac{3\alpha}{2}\int_\Omega m_2^2|\partial_{2,2}^2 v|^2 + \frac{1}{6\alpha}\int_\Omega |\partial_2 v|^2 + \frac{3\alpha}{2}\int_\Omega m_1^2|\partial_{1,1}^2 v|^2 + \frac{1}{6\alpha}\int_\Omega |\partial_1 v|^2\\ \leq &\frac{3\alpha}{2}\int_\Omega |m|^2|\Delta v|^2 + C\int_\Omega |\nabla v|^2.\end{aligned}$$

We write

$$\begin{aligned}\frac{3\alpha}{2}\int_\Omega |m|^2|\Delta v|^2 &= \frac{3\alpha}{2}\int_{\Omega\setminus\omega_0}|m|^2|\Delta v|^2 + \frac{3\alpha}{2}\int_{\omega_1}|m|^2|\Delta v|^2 + \frac{3\alpha}{2}\int_{\omega_0\setminus\omega_1}|m|^2|\Delta v|^2\\ &\le \frac{3\alpha}{2}R^2(x_\varepsilon)\int_{\Omega\setminus\omega_0}|\Delta v|^2 + \frac{3\alpha}{2}M_\varepsilon^2\int_{\omega_1}|\Delta v|^2 + \frac{3\alpha}{2}M_\varepsilon^2\int_{\omega_0\setminus\omega_1}|\Delta v|^2\\ &\le [R^2(x_\varepsilon) + M_\varepsilon^2|\omega_1|]\frac{3\alpha}{2}\int_\Omega |\Delta v|^2 + C\int_{\omega_0\setminus\omega_1}|\Delta v|^2.\end{aligned}$$

Let $\alpha = 4\delta/3R(x_\varepsilon)$. Then

$$\tilde{Y}(t) \le (R(x_\varepsilon)+\delta)^2\int_\Omega |\Delta v(t)|^2 + C\int_{\omega_0\setminus\omega_1}|\Delta v(t)|^2 + C\int_\Omega |\nabla v(t)|^2.$$

Therefore, by Young's inequality we have

$$Y \le \frac{\alpha\gamma}{2\varepsilon_1}\int_0^{\varepsilon_1}\int_\Omega |\nabla v_t|^2 + \frac{\alpha\gamma}{2\varepsilon_1}\int_{T-\varepsilon_1}^T\int_\Omega |\nabla v_t|^2 + \frac{\gamma}{2\alpha\varepsilon_1}\int_0^{\varepsilon_1}\int_\Omega \tilde{Y} + \frac{\gamma}{2\alpha\varepsilon_1}\int_{T-\varepsilon_1}^T\int_\Omega \tilde{Y}.$$

We make

$$\alpha(\gamma) = (R(x_\varepsilon)+\delta)\sqrt{\gamma}.$$

Therefore

$$\begin{aligned}Y &\le \alpha(\gamma)\frac{1}{\varepsilon_1}\int_0^{\varepsilon_1}\mathcal{E}(t) + \alpha(\gamma)\frac{1}{\varepsilon_1}\int_{T-\varepsilon_1}^T\mathcal{E}(t) + C\int_0^T\int_{\omega_0\setminus\omega_1}|\Delta v|^2 + CX^2(v^0,v^1)\\ &\le 2\alpha(\gamma)\mathcal{E}(T) + C\int_0^T\int_{\omega_0\setminus\omega_1}|\Delta v|^2 + CX^2(v^0,v^1).\end{aligned}$$

By construction of $\alpha(\gamma)$ we obtained

$$\mathcal{E}(T) \le \frac{CM_\varepsilon}{2}\int_0^T\int_{\Sigma(x^\varepsilon)}|\Delta v|^2 + C\int_0^T\int_{\omega_0\setminus\omega_1}|\Delta v|^2 + X^2(v^0,v^1).$$

In view of Lemma 7, we have

$$\mathcal{E}(T) \le \frac{CM_\varepsilon}{2}\int_0^T\int_{\Sigma(x^\varepsilon)}|\Delta v|^2 + C\int_0^T\int_\Omega \rho|\nabla v_t|^2 + X^2(v^0,v^1)$$

where $X : H_0^2\times H_0^1 \to \mathbb{R}$ is continuous and compact.

The end of the proof does not need to be changed.

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