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FINITE SPEED OF PROPAGATION AND CONTINUITY OF THE INTERFACE FOR THIN VISCOUS FLOWS

Francisco Bernis[1]
Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

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Abstract. We consider the fourth-order nonlinear degenerate parabolic equation

$$u_t + (|u|^n u_{xxx})_x = 0$$

which arises in lubrication models for thin viscous films and spreading droplets as well as in the flow of a thin neck of fluid in a Hele-Shaw cell. We prove that if $0 < n < 2$ this equation has finite speed of propagation for nonnegative "strong" solutions and hence there exists an interface or free boundary separating the regions where $u > 0$ and $u = 0$. Then we prove that the interface is Hölder continuous if $1/2 < n < 2$ and right-continuous if $0 < n \leq 1/2$. Finally we study the Cauchy problem and obtain optimal asymptotic rates as $t \to \infty$ for the solution and for the interface when $0 < n < 2$; these rates exactly match those of the source-type solutions. If $0 < n < 1$ the property of finite speed of propagation is also proved for changing sign solutions.

1. Introduction. In this paper we prove that if $0 < n < 2$ the fourth-order nonlinear degenerate parabolic equation

$$u_t + (|u|^n u_{xxx})_x = 0 \tag{1.1}$$

has finite speed of propagation for nonnegative "strong" solutions. Then we consider the interface or free boundary $\zeta(t)$ separating the regions where $u > 0$ and $u = 0$ and prove continuity results for $\zeta(t)$. Once having established these properties for bounded intervals, we study the Cauchy problem and obtain optimal asymptotic rates as $t \to \infty$ for the solution and for the interface. Changing sign solutions are also considered. (For nonnegative solutions $|u|^n$ can of course be replaced by u^n.) These results are described in more detail below, in the paragraphs I to VI of this introduction.

Equation (1.1) arises in *lubrication* models for thin viscous films and spreading droplets. In this context $u = u(x,t)$ is the height of the droplet or the thickness of the film and the interface is the boundary of the region occupied by the liquid. A guide to the physical literature on these models can be found e.g. in [3], [7] and [8]. For $n = 1$ Equation (1.1) also describes the flow of a thin neck of fluid of width $2u$ in a Hele-Shaw cell; see [12], [13] and [17].

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A mathematical theory for Equation (1.1) has been developed by this author and Friedman ([4]), Bertozzi, Brenner, Dupont and Kadanoff ([8]), Beretta, Bertsch and Dal Passo ([1]), and Bertozzi and Pugh ([9]). Some aspects of these papers are summarized in Section 2 and elsewhere as needed. Bertozzi ([6, 7] and again [8]) study Equation (1.1) from the points of view of numerical simulations, matched asymptotics and similarity structures. The papers [21], [5] and [11] deal with significant special solutions. Elliott and Garcke ([14]) and Grün ([16]) consider the equation in higher dimensions. Fourth-order parabolic equations with a similar type of degeneracy and/or additional lower-order terms are analyzed in [10], [14], [15], [16], [17] and [22].

In this paper we consider two initial-boundary value problems. The first one has homogeneous boundary conditions for u_x and u_{xxx}:

$$\begin{cases} u_t + (|u|^n u_{xxx})_x = 0 & \text{in } Q := \Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \Omega := (-a,a) \\ u_x = u_{xxx} = 0 & \text{for } x = -a \text{ and } x = a\ ,\ t > 0\,, \end{cases} \tag{I}$$

where Ω is a bounded open interval of $\mathbf{R}$, $a > 0$ and

$$u_0 \in H^1(\Omega) \equiv W^{1,2}(\Omega)\ , \quad u_0 \not\equiv 0\,. \tag{1.2}$$

The second problem we consider has periodic boundary conditions:

$$\begin{cases} u_t + (|u|^n u_{xxx})_x = 0 & \text{in } Q \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \\ \dfrac{\partial^j u}{\partial x^j}(-a,t) = \dfrac{\partial^j u}{\partial x^j}(a,t) & \text{for } t > 0\ , \quad j = 0,1,2,3\,, \end{cases} \tag{II}$$

where

$$u_0 \in H^1(\Omega)\ ,\ u_0 \not\equiv 0 \quad \text{and} \quad u_0(-a) = u_0(a)\,. \tag{1.3}$$

We introduce the hypothesis

$$u_0 = 0 \ \text{ in the nonempty open subinterval } \ \omega = (b - r_0, b + r_0) \ \text{ of } \ \Omega\,. \tag{1.4}$$

Definition 1.1. Let $u : \overline{Q} \to \mathbf{R}$ be a function such that $u(\cdot,0) = u_0$ in Ω with $u_0 = 0$ in some nonempty open subset of Ω. We say that u has *finite speed of propagation* if for all ω satisfying (1.4) there exist a number $T^* > 0$ and two continuous functions $b_-(t)$, $b_+(t)$ such that $b_-(t) < b_+(t)$ in $(0,T^*)$, $b_-(0) = b - r_0$, $b_+(0) = b + r_0$ and $u(\cdot,t) = 0$ in $(b_-(t), b_+(t))$ for all $t \in (0,T^*)$.

We define the *interface* or *free boundary* $\zeta(t)$ of the function u associated to the extreme $b - r_0$ of ω in the following way.

Definition 1.2. Under the conditions of Definition 1.1, for each $t \in [0,T^*)$ we set

$$\zeta(t) = \sup\{x \in \text{support } u(\cdot,t) \,:\, x \le b_-(t)\} \tag{1.5}$$

if the set considered in (1.5) is nonempty; otherwise we set $\zeta(t) = -a$. We say that $\zeta(t)$ *exists* if $0 \le t < T^*$ for some T^* satisfying Definition 1.1.

The concepts of weak and strong solutions will be explained in Section 2 (Definitions 2.1–2.3). We proceed to describe our results in more detail. In the results I to III below u is a nonnegative strong solution of Problem (I) or of Problem (II).

I. If $0 < n < 2$ the solution u has finite speed of propagation (Theorem 5.1).

II. If $1/2 < n < 2$ the free boundary $\zeta(t)$ of u is Hölder continuous; and if $0 < n \leq 1/2$ $\zeta(t)$ is right-continuous (Theorem 6.1).

III. If $1/2 < n < 3/2$ the solution u has finite backward speed of propagation (Theorem 6.2). (For $n \geq 3/2$ it is known ([1]) that the spatial support of $u(\cdot, t)$ is nonshrinking, which implies this backward speed property.)

In the next result u is a nonnegative strong solution of the Cauchy problem with initial datum of compact support, and $\zeta(t)$ is the right interface of u.

IV. If $0 < n < 2$ the growth exponent of $\zeta(t)$ as $t \to \infty$ is $1/(n+4)$ and the decay exponent of $\|u(\cdot, t)\|_\infty$ is $-1/(n+4)$ (Theorem 7.1); these exponents are optimal (Theorem 7.2) and equal to the exponents of the source-type solutions obtained in [5]. (The source-type solutions are briefly recalled in Section 7.)

V. The results I to III also hold for "pressure" boundary conditions (Section 8.1), a type of nonhomogeneous conditions introduced and analyzed in [12], [13] and [8] for their physical significance.

VI. If $0 < n < 1$ Equation (1.1) also has finite speed of propagation for changing sign strong solutions (Section 8.2).

The above results I to VI are new as far as we know. It seems that the continuity of the interface and the behavior of the Cauchy problem as $t \to \infty$ have not been treated in the mathematical literature up to now. The study of the Cauchy problem remains open if $n \geq 2$. With regard to the finite speed of propagation property, the work of Beretta, Bertsch and Dal Passo ([1]) includes two interesting results: 1) If $n \geq 4$ the spatial support of any nonnegative weak solution remains constant for all $t > 0$. (Hence the question of finite speed of propagation is still open when $2 \leq n < 4$.) 2) For any $n > 0$ there exists a nonnegative weak solution with nonexpanding support; this solution does not satisfy the regularity and asymptotic properties explained in Remark 2.3 below and it is not known to be a strong solution. (In [1] these results are stated for Problem (I), but the proofs also apply to Problem (II) and Problem (P) of Section 8.1; notice that no "flatness" condition on u_0 is assumed.) On the other hand, all known special solutions of Equation (1.1) (see [21], [5], [11]) exhibit finite speed of propagation. In [2] we proved the finite speed of propagation property for a different class of fourth- and higher-order degenerate parabolic equations.

The *plan of the paper* is as follows. In Section 2 we summarize the ideas of [4], [1] and [9] on the construction of nonnegative solutions and collect some properties and estimates. In Sections 3 and 4 we obtain global and local estimates which are used in the rest of the paper. In Section 5 we prove the finite speed of propagation property by means of an *energy method*. Section 6 deals with the continuity of the free boundary and the property of finite backward speed of propagation. Section 7 is on the Cauchy problem. In Section 8.1 we consider the "pressure" (nonhomogeneous) boundary conditions and in Section 8.2 changing sign solutions. Section 9 is devoted to some "calculus" inequalities which simplify the exposition of Sections 3 and 4. Finally, Sections 10 and 11 contain interpolation and differential inequalities used to implement the energy method.

Notation.
$\Omega = (-a, a)$, $\quad Q = \Omega \times (0, \infty)$, $\quad Q_T = \Omega \times (0, T)$.
$\mathcal{P} = \overline{Q} \setminus (\{u = 0\} \cup \{t = 0\})$, $\quad \mathcal{P}_T = \mathcal{P} \cap \overline{Q}_T$.
support of $u(\cdot, t)$: closure in $\overline{\Omega}$ of the set $\{u(\cdot, t) \neq 0\}$.
$\|u\|_p$: L^p norm of u in Ω.
$u \in C^{1/2,1/8}(\overline{Q})$: $u \in C(\overline{Q}) \cap L^\infty(Q)$ and there exists a constant M such that for all $x, y \in \overline{\Omega}$ and all $t, T \in [0, \infty)$

$$|u(x,t) - u(y,t)| \leq M\,|x - y|^{1/2} \quad \text{and} \quad |u(x,T) - u(x,t)| \leq M\,|T - t|^{1/8}\,.$$

$u \in C^{4,1}(S)$, where $S \subset \overline{Q}$: u, u_t and all the spatial derivatives up to fourth order are continuous in S.

2. Overview on the existence of nonnegative solutions. In these section we summarize the ideas of [4], [1] and [9] on the construction of a solution of Problems (I) and (II) by means of positive smooth approximations. First we present the concepts of weak and strong solutions. The concept of weak solution is taken from [4], while that of strong solution is suggested by the results of [1] and [9]. Notice that strong solutions are weaker than classical solutions.

Definition 2.1. Let $n > 0$. A *weak solution* of Problem (I) is a function u satisfying the following relations (2.1) to (2.6):

$$u \in C^{1/2,1/8}(\overline{Q}) \cap L^\infty(0, \infty; H^1(\Omega)), \tag{2.1}$$

$$u \in C^{4,1}(\mathcal{P}) \quad \text{and} \quad |u|^{n/2} u_{xxx} \in L^2(\mathcal{P}), \tag{2.2}$$

where $\mathcal{P} = \overline{Q} \setminus (\{u = 0\} \cup \{t = 0\})$; u satisfies (1.1) in the following sense:

$$\iint_Q u\psi_t\,dx\,dt + \iint_{\mathcal{P}} |u|^n u_{xxx}\psi_x\,dx\,dt = 0 \tag{2.3}$$

for all $\psi \in \mathrm{Lip}(\overline{Q})$ with compact support in $\overline{\Omega} \times (0, \infty)$;

$$u(\cdot, 0) = u_0 \text{ in } \overline{\Omega}, \tag{2.4}$$

$$u_x(\cdot, t) \to u_x(\cdot, 0) \text{ strongly in } L^2(\Omega) \text{ as } t \to 0\,, \tag{2.5}$$

and

$$u \text{ satisfies the boundary conditions at the points where } u \neq 0. \tag{2.6}$$

Definition 2.2. Let $n > 0$. A weak solution of Problem (II) is a function u satisfying the same relations as in Definition 2.1, except that we require, in addition, that

$$u(-a, t) = u(a, t) \quad \text{and} \quad \psi(-a, t) = \psi(a, t) \quad \text{for all} \quad t \geq 0\,. \tag{2.7}$$

Any weak solution of Problems (I) and (II) enjoys the property of *mass conservation*:

$$\int_\Omega u(x,t)\,dx = \text{ constant } = \int_\Omega u_0(x)\,dx \quad \text{for all} \quad t \geq 0\,. \tag{2.8}$$

This follows by taking in (2.3) $\psi = \theta(t)$ with $\theta \in C_c^1((0,\infty))$. (Notice that ψ may be different from zero at the lateral boundary of the cylinder $\overline{\Omega} \times (0,\infty)$.)

Remark 2.1. Since the function $f(s) = |s|^n$ is C^∞ for $s \neq 0$, it follows from (2.2)–(2.3) and linear parabolic theory that any weak solution u of Problem (I) or of Problem (II) satisfies $u \in C^\infty(\mathcal{P})$.

Remark 2.2. Let u be any weak solution of Problem (I) (respectively, of Problem (II)) for any $n > 0$. Consider the function h defined by $h = |u|^n u_{xxx}$ if $u \neq 0$ and $h = 0$ if $u = 0$. From $u \in L^\infty(Q)$ and (2.2) it follows that $h \in L^2(Q)$, while (2.3) implies that $u_t = -h_x$ in the sense of weak derivatives in Q. Hence $u_t \in L^2(0,\infty; H^{-1}(\Omega))$. This and (2.1) imply, by a standard result (see e.g. [18, Lemma 8.1, page 297]), that $u \in C([0,\infty); H^1 \text{ weak })$. In particular, $u(\cdot,t) \in H^1(\Omega)$ and $u_x(\cdot,t)$ is well defined for *all* $t \geq 0$ (and not only for almost every t); furthermore, $\|u_x(\cdot,t)\|_2$ is lower semicontinuous in $[0,\infty)$.

Definition 2.3. A *strong solution* of Problem (I) (respectively, of Problem (II)) is a weak solution u such that $u(\cdot,t) \in C^1(\overline{\Omega})$ for almost every $t > 0$.

Next we introduce

$$f_\varepsilon(s) = \frac{s^4 s^n}{\varepsilon s^n + s^4} = \frac{s^4}{\varepsilon + s^{4-n}} \quad \text{if } 0 < n < 4\,, \quad f_\varepsilon(s) = s^n \quad \text{if } n \geq 4\,, \tag{2.9}$$

and consider the approximating problems

$$\begin{cases} u_t + (f_\varepsilon(u)u_{xxx})_x = 0 & \text{in } Q \\ u(x,0) = u_0(x) + \varepsilon^\theta \quad (0 < \theta < 2/5) & \text{for } x \in \Omega \\ u_x = u_{xxx} = 0 & \text{for } x = -a \text{ and } x = a\ ,\ t > 0\,, \end{cases} \tag{I$_\varepsilon$}$$

$$\begin{cases} u_t + (f_\varepsilon(u)u_{xxx})_x = 0 & \text{in } Q \\ u(x,0) = u_0(x) + \varepsilon^\theta \quad (0 < \theta < 2/5) & \text{for } x \in \Omega \\ \dfrac{\partial^j u}{\partial x^j}(-a,t) = \dfrac{\partial^j u}{\partial x^j}(a,t) & \text{for } t > 0\ ,\ j = 0,1,2,3 \end{cases} \tag{II$_\varepsilon$}$$

(see Remark 2.3 below for the role of the condition $0 < \theta < 2/5$). Since for $\varepsilon > 0$ $f_\varepsilon(s)$ behaves near $s = 0$ as s^4/ε if $n < 4$ (and as s^n if $n \geq 4$), Problems I_ε and II_ε have a unique *positive* solution, smooth for $t > 0$. More precisely, the following proposition is proved as in [4, Theorem 4.1].

Proposition 2.1. *Let $u_0 \geq 0$, $n > 0$ and $\varepsilon > 0$. Under the hypotheses* (1.2) *for* (I_ε) *and* (1.3) *for* (II_ε), *Problems* (I_ε) *and* (II_ε) *have a unique positive solution*

$$u_\varepsilon \in C^{1/2,1/8}(\overline{Q}) \cap C^\infty(\overline{\Omega} \times (0,\infty))$$

such that u_ε *satisfies* (2.5).

By passing to the limit as $\varepsilon \to 0$, in [4] a weak solution of Problem (I) is obtained for $n \geq 1$. The papers [1] and [9] extend this result to all $n > 0$ and prove that the solution is strong if $0 < n < 3$. More specifically, in these references the following proposition is proved.

Proposition 2.2. *Under the conditions of Proposition* 2.1, *there exists* $u \in C^{1/2,1/8}(\overline{Q})$ *and a sequence* $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, *such that*

$$u_{\varepsilon_k} \to u \text{ in } C^{1/2,1/8}(\overline{Q}_T) \quad \text{for all } T > 0 \quad \text{as } k \to \infty \tag{2.10}$$

and any limit function u *obtained in this way is a nonnegative weak solution of Problem* (*I*) (*respectively, of Problem* (*II*)). *Furthermore, the function* u *is a strong solution if* $0 < n < 3$.

(In [1] are proved very general results on nonuniqueness of weak solutions; the uniqueness of strong solutions is not known.) In the process of proving Proposition 2.2 the above references establish the estimates

$$\frac{1}{2}\int_\Omega u_{\varepsilon x}^2(x,T)\,dx + \iint_{Q_T} f_\varepsilon(u_\varepsilon)\, u_{\varepsilon xxx}^2\,dx\,dt = \frac{1}{2}\int_\Omega u_{0x}^2(x)\,dx \quad \text{for all} \quad T > 0\,, \tag{2.11}$$

$$\frac{1}{2}\int_\Omega u_x^2(x,T)\,dx + \iint_{\mathcal{P}_T} u^n u_{xxx}^2\,dx\,dt \leq \frac{1}{2}\int_\Omega u_{0x}^2\,dx \quad \text{for all} \quad T > 0\,, \tag{2.12}$$

$$\sup_Q u_\varepsilon \leq C\,, \tag{2.13}$$

where C is a constant independent of $\varepsilon \in \{\varepsilon_k\}$. It is also known ([1]) that for all $n \geq 3/2$ the support of the above solution $u(\cdot,t)$ is nonshrinking in t and hence the interface $\zeta(t)$ of u (Definition 1.2) satisfies that

$$\zeta(t) \ \text{ is nondecreasing if } \ n \geq 3/2\,. \tag{2.14}$$

Remark 2.3. Let $0 < n < 3$. The above strong solution u enjoys several additional properties ([1, 9]):

1) It satisfies an additional family of integral estimates; in particular, $u_{xx} \in L^2(Q)$ if $0 < n < 2$. The boundary conditions for u_x are satisfied for almost every $t > 0$.

2) The regularity of u matches for almost every t (in the sense explained in [1, 9]) the regularity of the source-type solutions obtained in [5].

3) $u(\cdot,t)$ converges uniformly in $\overline{\Omega}$ to the mean value of u_0 as $t \to \infty$ and hence we have $u(\cdot,t) > 0$ for all t large enough.

If the condition $0 < \theta < 2/5$ is dropped then (2.10) still defines a weak solution but it may not satisfy the above properties 1) to 3) and in general is not a strong solution. This condition is needed in the proof of (3.7)–(3.8) below.

Remark 2.4. We do not assume that u_0 satisfies the boundary conditions. (In fact, for a function in $H^1(\Omega)$ the boundary conditions for u_x, u_{xx} and u_{xxx} are in general meaningless.) Therefore, the solution of Problems (I_ε) and (II_ε) may not be smooth up to $t=0$. Some computations of this paper involve $t=0$; they can be justified by taking $t=\delta>0$ and then letting $\delta\to 0$.

Remark 2.5. If we consider Problem (I) in the space interval $(0,a)$ and perform an even extension (into the interval $(-a,a)$) we obtain Problem (II) with an even initial datum u_0. Nevertheless we think that Problem (I) deserves a separate study because it enjoys some specific properties and has a natural generalization to higher dimensions.

3. Global estimates and identities. Consider Problems (I) and (II). If we multiply the equation by $u^{1-n+\lambda}/(1-n+\lambda)$ and perform "formal" computations we obtain that u satisfies an integral identity and some estimates for each λ in a certain range, provided that $0<n<3$. In order to prove these identities and estimates we consider first the approximating problems I_ε and II_ε. Let

$$\lambda\notin\{n-1,n-2,2,3\} \tag{3.1}$$

and introduce the functions

$$g_\varepsilon(s)=g_{\varepsilon\lambda}(s):=-\frac{\varepsilon\, s^{\lambda-3}}{3-\lambda}+\frac{s^{1-n+\lambda}}{1-n+\lambda}\,, \tag{3.2}$$

$$G_\varepsilon(s)=G_{\varepsilon\lambda}(s):=\frac{\varepsilon\, s^{\lambda-2}}{(3-\lambda)(2-\lambda)}+\frac{s^{2-n+\lambda}}{(1-n+\lambda)(2-n+\lambda)}\,. \tag{3.3}$$

Notice that $G'_\varepsilon(s)=g_\varepsilon(s)$ and (for $0<n<4$)

$$g'_\varepsilon(s)=\frac{s^\lambda}{f_\varepsilon(s)}=\varepsilon s^{\lambda-4}+s^{\lambda-n}\,. \tag{3.4}$$

In the sequel in this section we use the notation and assume the hypotheses of Propositions 2.1–2.2. When we say that $\varepsilon\to 0$ it is to be understood that $\varepsilon\to 0$ along the sequence $\{\varepsilon_k\}$ of Proposition 2.2.

Lemma 3.1. *Assume* (3.1) *and* $0<n<4$*. Then for all* $T>0$

$$\int_\Omega G_\varepsilon(u_\varepsilon(x,T))\,dx+\iint_{Q_T}u_\varepsilon^\lambda u_{\varepsilon xx}^2\,dx\,dt+\frac{\lambda(1-\lambda)}{3}\iint_{Q_T}u_\varepsilon^{\lambda-2}u_{\varepsilon x}^4\,dx\,dt =\int_\Omega G_\varepsilon(u_0(x)+\varepsilon^\theta)\,dx\,. \tag{3.5}$$

Proof. This lemma follows by multiplying the equation of Problem I_ε or II_ε (with $u=u_\varepsilon$) by $g_\varepsilon(u_\varepsilon)$ and integrating by parts in Q_T. Here the condition $0<n<4$ is introduced because of the definition (2.9) of $f_\varepsilon(s)$. We comment that the precise value of the constant $\lambda(1-\lambda)/3$ is important.

Lemma 3.2. *Assume that*

$$0 < n < 3 \quad , \quad \max\{-1/2, n-2\} < \lambda < 1 \quad \textit{and} \quad \lambda \neq n-1\,. \tag{3.6}$$

Then there exists a constant C (independent of ε and T) such that for all $\varepsilon \in \{\varepsilon_k\}$ and all $T > 0$

$$(1) \quad \varepsilon \int_\Omega u_\varepsilon^{\lambda-2}(x,T)\,dx \le C, \quad (2) \quad \iint_{Q_T} u_\varepsilon^\lambda u_{\varepsilon xx}^2 \le C, \quad (3) \quad \iint_{Q_T} u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 \le C.$$

Proof. Notice that the set of values of λ defined by (3.6) is *nonempty*. The term of the form (1) in (3.5) is nonnegative. The term containing $u_\varepsilon^{2-n+\lambda}$ is bounded independently of ε and T because $\lambda > n-2$ and (2.13) holds. Since $\lambda > -1/2$, $\lambda - 2 < 0$, $u_0 \ge 0$ and $\theta < 2/5$ we have that

$$\varepsilon(u_0(x) + \varepsilon^\theta)^{\lambda-2} \le \varepsilon^{1-\theta(2-\lambda)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.7}$$

and hence

$$\int_\Omega G_\varepsilon(u_0(x) + \varepsilon^\theta)\,dx \to \frac{1}{(1-n+\lambda)(2-n+\lambda)} \int_\Omega u_0^{2-n+\lambda}(x)\,dx \quad \text{as} \quad \varepsilon \to 0\,. \tag{3.8}$$

For $0 < \lambda < 1$ the terms in (3.5) containing integrals over Q_T are nonnegative and the proof is easily completed. For $-1/2 < \lambda \le 0$ from the calculus inequality (9.5) it follows that these terms are greater than or equal to

$$\frac{2\lambda+1}{1-\lambda} \iint_{Q_T} u_\varepsilon^\lambda u_{\varepsilon xx}^2\,,$$

and hence we obtain the estimates (1) and (2). Finally, (3) is implied by (2) and the calculus inequality (9.3).

The purpose of the next lemma on strong convergence is to obtain the *equality* (3.20) below (see also Remark 3.2), which will be used in Section 7. (Weak convergence would imply an inequality rather than an equality.) Relation (3.10) is needed in several places in order to extend to any time $T > 0$ the arguments performed for $t = 0$ (see e.g. Remark 4.1).

Lemma 3.3. *Assume that*

$$0 < n < 3 \quad \textit{and} \quad \max\{-1/2, n-2\} < \lambda\,. \tag{3.9}$$

Let $T > 0$. Then as $\varepsilon \to 0$ along the sequence $\{\varepsilon_k\}$

$$\varepsilon \int_\Omega u_\varepsilon^{\lambda-2}(x,T)\,dx \to 0, \tag{3.10}$$

$$u_\varepsilon^{\lambda/2} u_{\varepsilon xx} \to u^{\lambda/2} u_{xx} \, \chi_{\mathcal{P}} \quad \textit{strongly in} \quad L^2(Q_T)\,, \tag{3.11}$$

$$u_\varepsilon^{(\lambda-2)/4} u_{\varepsilon x} \to u^{(\lambda-2)/4} u_x \, \chi_{\mathcal{P}} \quad \textit{strongly in} \quad L^4(Q_T)\,, \tag{3.12}$$

$$\left(u_\varepsilon^{(\lambda+2)/2}\right)_{xx} \to \left(u^{(\lambda+2)/2}\right)_{xx} \quad \textit{strongly in} \quad L^2(Q_T)\,, \tag{3.13}$$

where $\chi_{\mathcal{P}}$ is the characteristic function of the set $\mathcal{P}$.

Proof. First we observe that (3.13) follows from (3.11)–(3.12) and the identity

$$\frac{2}{\lambda+2}\left(v^{(\lambda+2)/2}\right)_{xx} = v^{\lambda/2} v_{xx} + \frac{\lambda}{2} v^{(\lambda-2)/2} v_x^2\,. \tag{3.14}$$

The proofs of (3.10)–(3.12) are very similar and we only consider (3.11). For each λ satisfying (3.9) there exists $\overline{\lambda}$ such that $\overline{\lambda} < \lambda$ and $\overline{\lambda}$ satisfies (3.6). Hence, by Lemma 3.2,

$$\iint_{Q_T} u_\varepsilon^{\overline{\lambda}} u_{\varepsilon xx}^2 \le C\,. \tag{3.15}$$

Since $\overline{\lambda} < \lambda$ by (2.13) we have that (3.15) also holds with $\overline{\lambda}$ replaced by λ; hence there exists z such that, for a subsequence,

$$u_\varepsilon^{\lambda/2} u_{\varepsilon xx} \to z \quad \text{weakly in} \quad L^2(Q_T)\,. \tag{3.16}$$

Take any $\delta > 0$ and any $\tau \in (0,T)$. For simplicity of notation we write $\{u > \delta\}$, $\{u \le \delta\}$, $\{u = 0\}$ and $\{t > \tau\}$ for the intersection of Q_T and these sets. From (2.10) and Hölder estimates for uniformly parabolic equations it follows that

$$u_{\varepsilon xx} \to u_{xx} \quad \text{uniformly on} \quad \{u > \delta\} \cap \{t > \tau\} \tag{3.17}$$

and therefore $z = u^{\lambda/2} u_{xx}$ on $\mathcal{P}_T$. (In (3.17) it is necessary to stay away from $t = 0$ because u_0 is not smooth enough and no compatibility conditions are assumed.) From (3.15) and the uniform convergence $u_\varepsilon \to u$ we obtain

$$\iint_{\{u=0\}} u_\varepsilon^{\lambda} u_{\varepsilon xx}^2 \le C \sup_{u=0} u_\varepsilon^{\lambda-\overline{\lambda}} \to 0\,.$$

This shows that $u_\varepsilon^{\lambda/2} u_{\varepsilon xx} \to 0$ strongly in $L^2(\{u = 0\})$ and completes the proof of

$$z = u^{\lambda/2} u_{xx} \, \chi_{\mathcal{P}} \quad \text{on} \quad Q_T\,. \tag{3.18}$$

(The uniqueness of z also implies that (3.16) holds for the whole sequence $\{\varepsilon_k\}$.) If ε is sufficiently small, depending on δ,

$$\iint_{Q_T} u_\varepsilon^{\lambda} u_{\varepsilon xx}^2 - \iint_{\{u>\delta\}} u_\varepsilon^{\lambda} u_{\varepsilon xx}^2 \le \Big(\sup_{u\le\delta} u_\varepsilon^{\lambda-\overline{\lambda}}\Big) \iint_{\{u\le\delta\}} u_\varepsilon^{\overline{\lambda}} u_{\varepsilon xx}^2 \le C(\delta/2)^{\lambda-\overline{\lambda}}\,,$$

where (3.15) was used. This and (3.17) imply

$$\limsup_{\varepsilon\to 0} \iint_{Q_T} u_\varepsilon^\lambda u_{\varepsilon xx}^2 \le C(\delta/2)^{\lambda-\overline{\lambda}} + \iint_{\{u>\delta\}\cap\{t>\tau\}} u^\lambda u_{xx}^2 + \limsup_{\varepsilon\to 0} \iint_{Q_\tau} u_\varepsilon^\lambda u_{\varepsilon xx}^2 \,.$$

From (3.5) and the proof of Lemma 3.2 it follows that

$$\limsup_{\varepsilon\to 0} \iint_{Q_\tau} u_\varepsilon^\lambda u_{\varepsilon xx}^2 \le C_{\lambda n} \int_\Omega \left| u_0^{2-n+\lambda}(x) - u^{2-n+\lambda}(x,\tau) \right| dx \,.$$

Letting $\delta \to 0$ and $\tau \to 0$ we obtain

$$\limsup_{\varepsilon\to 0} \iint_{Q_T} u_\varepsilon^\lambda u_{\varepsilon xx}^2 \le \iint_{\mathcal{P}_T} u^\lambda u_{xx}^2 \,. \tag{3.19}$$

Finally, (3.16), (3.18) and (3.19) imply (3.11).

Remark 3.1. For $\lambda = \max\{-1/2, n-2\}$ we expect that $u^\lambda u_{xx}^2 \notin L^1(\mathcal{P}_T)$ and $u^{\lambda-2} u_x^4 \notin L^1(\mathcal{P}_T)$ when u is zero in some nonempty open subset of Q_T; otherwise u would be more regular in this respect than the source-type solutions obtained in [5].

Lemma 3.4. *Assume* (3.1) *and* (3.9). *Then for all $T > 0$ the solution u defined in Proposition 2.2 satisfies*

$$\frac{1}{(1-n+\lambda)(2-n+\lambda)} \int_\Omega u^{2-n+\lambda}(x,T)\,dx + \iint_{\mathcal{P}_T} u^\lambda u_{xx}^2 \,dx\,dt + \frac{\lambda(1-\lambda)}{3} \iint_{\mathcal{P}_T} u^{\lambda-2} u_x^4 \,dx\,dt = \frac{1}{(1-n+\lambda)(2-n+\lambda)} \int_\Omega u_0^{2-n+\lambda}(x)\,dx \tag{3.20}$$

and hence the function $t \mapsto \int_\Omega u^{2-n+\lambda}(x,t)\,dx$ is absolutely continuous in $[0,\infty)$.

Proof. The lemma follows at once from (3.5), (3.8) and Lemma 3.3.

Remark 3.2. There are two simpler particular cases of Lemma 3.4. I) Case $\lambda = 0$: If $0 < n < 2$ and $n \ne 1$ then for almost every $t > 0$

$$\int_\Omega u_{xx}^2(x,t)\,dx = -\frac{1}{(1-n)(2-n)} \frac{d}{dt} \int_\Omega u^{2-n}(x,t)\,dx \,. \tag{3.21}$$

II) Case $\lambda = 1$: If $0 < n < 3$ and $n \ne 2$ then for almost every $t > 0$

$$\int_{\Omega\cap\{u>0\}} u(x,t)\, u_{xx}^2(x,t)\,dx = -\frac{1}{(2-n)(3-n)} \frac{d}{dt} \int_\Omega u^{3-n}(x,t)\,dx \,. \tag{3.22}$$

This relation will be used in Section 7. (For $0 < n < 2$ we have that $u_{xx} \in L^2(Q)$ and hence in (3.22) we can replace $\Omega \cap \{u > 0\}$ by Ω.)

Lemma 3.5. *Assume that*

$$0 < n < 2 \quad \textit{and} \quad \max\{-1/2, n-1\} < \lambda < 1\,. \tag{3.23}$$

Let u be the solution defined in Proposition 2.2. *Then there exists a positive constant C depending only on λ and n such that if $0 \le T_1 < T_2$*

$$\int_{T_1}^{T_2}\int_\Omega \left(u^{(\lambda+2)/2}\right)_{xx}^2 dx\,dt \le C\int_\Omega u^{2-n+\lambda}(x,T_1)\,dx\,.$$

Proof. Notice that the set of values of λ defined by (3.23) is *nonempty*. Applying the calculus inequalities (9.3) and (9.5) to u_ε and letting $\varepsilon \to 0$ as in Lemma 3.3 it follows that

$$\int_{T_1}^{T_2}\int_\Omega u^{\lambda-2}u_x^4\,\chi_{\mathcal{P}} \le \frac{9}{(1-\lambda)^2}\int_{T_1}^{T_2}\int_\Omega u^\lambda u_{xx}^2\,\chi_{\mathcal{P}}\,, \tag{3.24}$$

$$\int_{T_1}^{T_2}\int_\Omega u^\lambda u_{xx}^2\,\chi_{\mathcal{P}} + \frac{\lambda(1-\lambda)}{3}\int_{T_1}^{T_2}\int_\Omega u^{\lambda-2}u_x^4\,\chi_{\mathcal{P}} \ge C_\lambda\int_{T_1}^{T_2}\int_\Omega u^\lambda u_{xx}^2\,\chi_{\mathcal{P}}\,, \tag{3.25}$$

where $C_\lambda = 1$ if $0 \le \lambda < 1$ and $C_\lambda = (2\lambda+1)/(1-\lambda)$ if $-1/2 < \lambda < 0$. Applying the identity (3.14) to u_ε, letting $\varepsilon \to 0$ and using (3.24) we obtain

$$\int_{T_1}^{T_2}\int_\Omega \left(u^{(\lambda+2)/2}\right)_{xx}^2 \le C_\lambda^*\int_{T_1}^{T_2}\int_\Omega u^\lambda u_{xx}^2\,\chi_{\mathcal{P}}\,. \tag{3.26}$$

Now the lemma follows from Lemma 3.4, (3.25), (3.26) and $1-n+\lambda > 0$.

Lemma 3.6. *Assume that*

$$0 < n \le 2 \quad \textit{and} \quad \max\{(3/2)-n, 1\} \le p \le 3-n\,. \tag{3.27}$$

Then the function $\int_\Omega u^p(x,t)\,dx$ is nonincreasing in t for all $t \ge 0$.

Proof. Set $p = 2-n+\lambda$. First assume $0 < n < 2$ and $\max\{(3/2)-n, 1\} < p < 3-n$, which is equivalent to (3.23). Then the conclusion of the lemma follows from (3.25), $1-n+\lambda > 0$ and Lemma 3.4. Keeping $0 < n < 2$ the cases $p = \max\{(3/2)-n, 1\}$ and $p = 3-n$ follow by continuity (arguing for fixed n). Finally, the case $n = 2$ is just the property (2.8) of conservation of mass.

Remark 3.3. In a similar way, $\int_\Omega u^p$ is nondecreasing if $1/2 \le n < 3$ and $\max\{(3/2)-n, 0\} \le p \le \min\{3-n, 1\}$, where $\int_\Omega u^p$ for $p = 0$ is to be understood as the measure of the set $\{u(\cdot,t) > 0\}$.

4. Local estimates. In this section we again use the notation and assume the hypotheses of Propositions 2.1–2.2. We also recall that f_ε, g_ε and G_ε are defined by (2.9), (3.2) and (3.3), respectively. Furthermore, we consider a cut-off function $\xi = \xi(x)$ satisfying

$$\xi \in C^2(\mathbf{R}), \quad \xi \ge 0 \quad \text{and} \quad \xi(x) = 0 \text{ if } x \notin \Omega. \tag{4.1}$$

Lemma 4.1. *Assume* (3.1), $0 < n < 4$ *and* (4.1). *Then for all* $T > 0$ *and all* $\varepsilon > 0$

$$\int_\Omega \xi(x) G_\varepsilon(u_\varepsilon(x,T))\,dx - \int_\Omega \xi(x) G_\varepsilon(u_0(x)+\varepsilon^\theta)\,dx + \iint_{Q_T} \xi u_\varepsilon^\lambda u_{\varepsilon xx}^2$$
$$+ \frac{\lambda(1-\lambda)}{3} \iint_{Q_T} \xi u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 = \frac{\lambda}{3} \iint_{Q_T} \xi' u_\varepsilon^{\lambda-1} u_{\varepsilon x}^3 - 2 \iint_{Q_T} \xi' u_\varepsilon^\lambda u_{\varepsilon x} u_{\varepsilon xx}$$
$$- \iint_{Q_T} \xi' f_\varepsilon'(u_\varepsilon) g_\varepsilon(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon xx} - \iint_{Q_T} \xi'' f_\varepsilon(u_\varepsilon) g_\varepsilon(u_\varepsilon) u_{\varepsilon xx}\,.$$

Proof. This follows by multiplying the equation of Problem I_ε or II_ε (with $u = u_\varepsilon$) by $\xi g_\varepsilon(u_\varepsilon)$ and integrating by parts in Q_T. Notice that $f_\varepsilon(s) g_\varepsilon'(s) = s^\lambda$.

Lemma 4.2. *Assume* (3.6). *Then for all* $\varepsilon > 0$ *and all* $s > 0$

$$|f_\varepsilon(s) g_\varepsilon(s)| \le \frac{s^{1+\lambda}}{|1-n+\lambda|} \quad \text{and} \quad |f_\varepsilon'(s) g_\varepsilon(s)| \le \frac{4\,s^\lambda}{|1-n+\lambda|}\,.$$

Proof. Taking into account that $|1-n+\lambda| < 2 < 3-\lambda$, from (3.2) it follows that

$$|g_\varepsilon(s)| \le \frac{\varepsilon|1-n+\lambda| + (3-\lambda) s^{4-n}}{(3-\lambda)|1-n+\lambda| s^{3-\lambda}} \le \frac{\varepsilon + s^{4-n}}{|1-n+\lambda| s^{3-\lambda}}\,.$$

Also, by explicitly computing $f_\varepsilon'(s)$ we find that

$$0 \le f_\varepsilon'(s) = \frac{s^3(4\varepsilon + n s^{4-n})}{(\varepsilon + s^{4-n})^2} \le \frac{4s^3}{\varepsilon + s^{4-n}}\,.$$

The lemma follows from these relations and the definition of $f_\varepsilon(s)$ given by (2.9).

Lemma 4.3. *Assume* (3.6) *and* (4.1). *Then there exists a positive constant* C *depending only on* λ *and* n *such that for all* $T > 0$ *and all* $\varepsilon > 0$

$$\int_\Omega \xi(x)\big(G_\varepsilon(u_\varepsilon(x,T)) - G_\varepsilon(u_0(x)+\varepsilon^\theta)\big)\,dx + \iint_{Q_T} \xi u_\varepsilon^\lambda u_{\varepsilon xx}^2 + \iint_{Q_T} \xi u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4$$
$$\le C\left(\iint_{Q_T} |\xi' u_\varepsilon^{\lambda-1} u_{\varepsilon x}^3| + \iint_{Q_T} |\xi' u_\varepsilon^\lambda u_{\varepsilon x} u_{\varepsilon xx}| + \iint_{Q_T} |\xi'' u_\varepsilon^{\lambda+1} u_{\varepsilon xx}|\right).$$

Proof. If $0 < \lambda < 1$ this lemma follows at once from Lemmas 4.1 and 4.2. If $-1/2 < \lambda \le 0$ the lemma is implied by Lemmas 4.1–4.2 and the calculus inequalities (9.6) and (9.4).

Now we introduce a cut-off function $\xi = \varphi_r^4$ (depending on a parameter r) of the form

$$\varphi_r(x) := r\varphi_1(x/r), \quad r > 0, \quad \varphi_1 \ge 0, \quad \varphi_1 \in C_c(\mathbf{R}) \cap C^2(\operatorname{support}_{\mathbf{R}} \varphi_1)\,, \tag{4.2}$$

where φ_1 is a given function and $\operatorname{support}_{\mathbf{R}}$ stands for the support in $\mathbf{R}$, not in $\overline{\Omega}$. (See Remark 4.2 below for a relevant example.)

Lemma 4.4. *Set* $\xi = \varphi_r^4$ *and assume* (3.6), (4.1) *and* (4.2). *Then there exists a positive constant* C_1 *depending only on* λ, n *and* φ_1 *(hence independent of* r*) such that for all* $T > 0$ *and all* $\varepsilon > 0$

$$\int_\Omega \varphi_r^4(x) G_\varepsilon(u_\varepsilon(x,T))\,dx - \int_\Omega \varphi_r^4(x) G_\varepsilon(u_0(x)+\varepsilon^\theta)\,dx + \iint_{Q_T} \varphi_r^4 u_\varepsilon^\lambda u_{\varepsilon xx}^2 + \iint_{Q_T} \varphi_r^4 u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 \le C_1 \iint_{Q_T \cap \{\varphi_r > 0\}} u_\varepsilon^{\lambda+2}\,.$$

Proof. Observing that $\xi'(x) = \xi''(x) = 0$ if $x \notin \operatorname{support}_{\mathbf{R}} \varphi_r$ and that φ_1, φ_1' and φ_1'' are bounded in $\operatorname{support}_{\mathbf{R}} \varphi_1$ it follows that

$$|\xi'| \le A_1 \varphi_r^3 \quad \text{and} \quad |\xi''| \le A_2 \varphi_r^2\,, \tag{4.3}$$

where A_1 and A_2 are constants depending only on φ_1 (hence independent of r). Therefore, the right-hand side of the inequality in Lemma 4.3 is bounded by

$$C^* \left(\iint_{Q_T} |\varphi_r^3 u_\varepsilon^{\lambda-1} u_{\varepsilon x}^3| + \iint_{Q_T} |\varphi_r^3 u_\varepsilon^\lambda u_{\varepsilon x} u_{\varepsilon xx}| + \iint_{Q_T} |\varphi_r^2 u_\varepsilon^{\lambda+1} u_{\varepsilon xx}| \right), \tag{4.4}$$

where C^* depends only on λ, n and φ_1. We proceed to estimate each integral of (4.4). In the next three relations δ is a positive number to be chosen below, B_1, B_2 and B_3 are positive constants depending only on δ, and the integrals are taken over $Q_T \cap \{\varphi_r > 0\}$. Applying Hölder's inequality with exponents 4/3 and 4 we obtain

$$\iint |\varphi_r^3 u_\varepsilon^{\lambda-1} u_{\varepsilon x}^3| \le \left(\iint \varphi_r^4 u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 \right)^{3/4} \left(\iint u_\varepsilon^{\lambda+2} \right)^{1/4} \le \delta \iint \varphi_r^4 u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 + B_1 \iint u_\varepsilon^{\lambda+2}\,.$$

From Hölder's inequality with exponents 2, 4 and 4 it follows that

$$\iint |\varphi_r^3 u_\varepsilon^\lambda u_{\varepsilon x} u_{\varepsilon xx}| \le \left(\iint \varphi_r^4 u_\varepsilon^\lambda u_{\varepsilon xx}^2 \right)^{1/2} \left(\iint \varphi_r^4 u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 \right)^{1/4} \left(\iint u_\varepsilon^{\lambda+2} \right)^{1/4} \le \delta \iint \varphi_r^4 u_\varepsilon^\lambda u_{\varepsilon xx}^2 + \delta \iint \varphi_r^4 u_\varepsilon^{\lambda-2} u_{\varepsilon x}^4 + B_2 \iint u_\varepsilon^{\lambda+2}\,.$$

The third integral of (4.4) is estimated by means of Schwarz's and Young's inequalities:

$$\iint |\varphi_r^2 u_\varepsilon^{\lambda+1} u_{\varepsilon xx}| \le \delta \iint \varphi_r^4 u_\varepsilon^\lambda u_{\varepsilon xx}^2 + B_3 \iint u_\varepsilon^{\lambda+2}\,.$$

Now the proof is completed by choosing δ small enough (e.g. setting $2\delta C^* = 1/2$), inserting the last three relations in (4.4) and taking into account Lemma 4.3.

We proceed with the main lemma of this section.

Lemma 4.5. *Set $\xi = \varphi_r^4$ and assume* (3.6), (4.1) *and* (4.2). *Let u be the solution defined in Proposition* 2.2. *Then there exists a positive constant C_2 depending only on λ, n and φ_1 (hence independent of r) such that for all $T > 0$*

$$\frac{1}{(1-n+\lambda)(2-n+\lambda)}\left[\int_\Omega \varphi_r^4(x)u^{2-n+\lambda}(x,T)\,dx - \int_\Omega \varphi_r^4(x)u_0^{2-n+\lambda}(x)\,dx\right] + \iint_{Q_T} \varphi_r^4\left(u^{(\lambda+2)/2}\right)_{xx}^2 \le C_2 \iint_{Q_T\cap\{\varphi_r>0\}} u^{\lambda+2}. \quad (4.5)$$

Proof. Consider the inequality of Lemma 4.4. First we use the identity (3.14). Then we let $\varepsilon \to 0$ along the sequence $\{\varepsilon_k\}$. The lemma follows from (3.8), (3.10), (3.13) and the uniform convergence (2.10).

Remark 4.1. Due to the convergence result (3.10), Lemma 4.5 still holds if we replace 0 by T_1 and T by T_2. More specifically, we have that if $0 \le T_1 < T_2$ then

$$\frac{1}{(1-n+\lambda)(2-n+\lambda)}\left[\int_\Omega \varphi_r^4(x)u^{2-n+\lambda}(x,t)\,dx\right]_{t=T_1}^{t=T_2} + \int_{T_1}^{T_2}\int_\Omega \varphi_r^4\left(u^{(\lambda+2)/2}\right)_{xx}^2 \le C_2\int_{T_1}^{T_2}\int_{\Omega\cap\{\varphi_r>0\}} u^{\lambda+2}. \quad (4.6)$$

Remark 4.2. The combination of the conditions (4.1) for $\xi = \varphi_r^4$ and (4.2) for φ_r is perhaps a little subtle. A simple example (to be used in Section 5) is the function $\varphi_r(x) = (r^2 - x^2)_+/r$. This function satisfies (4.2) with $\varphi_1(x) = (1-x^2)_+$ and $\xi = \varphi_r^4$ satisfies (4.1) if $a \ge r$. Notice that in this example $\varphi_1 \in C^\infty([-1,1])$ but $\varphi_1 \notin C^1(\mathbf{R})$.

Remark 4.3. The hypothesis (4.1) on ξ can be relaxed; all the lemmas of this section still hold if the condition $\xi(x) = 0$ if $x \notin \Omega$ is replaced by: 1) $\xi'(-a) = \xi'(a) = 0$ for Problem (I), or 2) $\xi(-a) = \xi(a)$ and $\xi'(-a) = \xi'(a)$ for Problem (II). This remark does not apply to the nonhomogeneous problem of Section 8.1.

5. Finite speed of propagation.

Let $\omega = (b - r_0, b + r_0)$ be as in Definition 1.1. We consider the interface or free boundary $\zeta(t)$ (Definition 1.2) of the solution u associated to the extreme $b - r_0$ of ω. Without loss of generality we may assume (and do assume) that

$$\zeta(0) = b - r_0 > -a\,, \text{ i.e. that } b - r_0 \in \operatorname{support} u_0\,. \quad (5.1)$$

We also consider the hypothesis

$$\max\{-1/2, n-1\} < \lambda < 1\,. \quad (5.2)$$

Theorem 5.1. *Let* $0 < n < 2$. *Assume that* $u_0 = 0$ *in a nonempty open subset of* Ω. *Then the nonnegative strong solution* u *of Problem* (I) (*respectively, of Problem* (II)) *defined in Proposition* 2.2 *has finite speed of propagation in the sense of Definition* 1.1. *Furthermore, let* λ *satisfy* (5.2). *Then there exists a positive constant* $T_* = T_*(\lambda, n, u_0, r_0)$ *such that if* $0 < T < T_*$ *the interface* $\zeta(T)$ *of Definition* 1.2 *exists and*

$$\zeta(T) - \zeta(0) \le A_0 T^\alpha \left(\int_0^T \int_{\zeta(0)}^b \left(u^{(\lambda+2)/2} \right)_{xx}^2 dx\, dt \right)^\beta , \tag{5.3}$$

where A_0 *is a positive constant depending only on* λ *and* n *and*

$$\alpha = \frac{2 - n + \lambda}{8 - 3n + 4\lambda} , \qquad \beta = \frac{n}{8 - 3n + 4\lambda} . \tag{5.4}$$

The proof of Theorem 5.1 is based on Lemma 4.5 and uses an *energy method*. We set

$$w := u^{(\lambda+2)/2} , \qquad q := 2 - \frac{2n}{\lambda + 2} \tag{5.5}$$

(notice that $0 < q < 2$), and consider a function φ_r satisfying the hypotheses of Lemma 4.5 and such that

$$\varphi_r(x) = 0 \quad \text{if} \quad x \notin (b - r_0, b + r_0) . \tag{5.6}$$

Observing that $1 - n + \lambda > 0$ by (5.2) and that $\varphi_r(x) u_0(x) \equiv 0$, we rewrite (4.5) in the following way:

$$\sup_{0<t<T} \int_\Omega \varphi_r^4(x) w^q(x,t)\, dx + \iint_{Q_T} \varphi_r^4 w_{xx}^2 \le C_3 \iint_{Q_T \cap \{\varphi_r > 0\}} w^2 , \tag{5.7}$$

where the constant C_3 depends only on λ, n and φ_1.

We organize the proof in two stages. In the first stage we prove that u has finite speed of propagation. In the second stage we use this information to obtain the estimate (5.3). The constant T_* is defined by (5.22) below. In the sequel in this section C_i will stand for a positive constant depending *only* on λ and n.

First stage. We translate the origin so that $b = 0$ and therefore $\omega = (-r_0, r_0)$. Then we take $\varphi_1(x) = (1 - x^2)_+$ and hence, by (4.2), $\varphi_r(x) = (r^2 - x^2)_+/r$. For $0 < r \le r_0$ this function φ_r satisfies (5.6) and is admissible in (5.7). (Recall Remark 4.2.) After inserting this function in (5.7) we use the inequalities

$$(r - |x|)_+ \le \varphi_r(x) \le 2(r - |x|)_+ \tag{5.8}$$

to obtain

$$\sup_{0<t<T} \int_{-r}^r (r - |x|)^4 w^q(x,t)\, dx + \int_0^T \int_{-r}^r (r - |x|)^4 w_{xx}^2 \le C_4 \int_0^T \int_{-r}^r w^2 . \tag{5.9}$$

(We do not insert $(r-|x|)_+$ directly in (5.7) because $r-|x|$ is not C^1 at $x=0$.) We introduce the notation

$$E_s = E_s(r,T) := \int_0^T \int_{-r}^{r} (r-|x|)^s w_{xx}^2 \,, \tag{5.10}$$

$$F = F(r,T) := \sup_{0<t<T} \int_{-r}^{r} (r-|x|)^4 w^q(x,t)\,dx \,. \tag{5.11}$$

Integrating in t the inequality (10.1) with $v=w$ (see Section 10) and using the notation (5.10)–(5.11) it follows that

$$C_5 \int_0^T \int_{-r}^{r} w^2 \le T^{1-d} E_0^d F^{2(1-d)/q} + T\, r^{-2\nu} F^{2/q} \,, \tag{5.12}$$

where d and ν are given by (10.2). From (5.9)–(5.12) we obtain

$$F + E_4 \le C_6 T^{1-d} E_0^d F^{2(1-d)/q} + C_7 T\, r^{-2\nu} F^{2/q} \,. \tag{5.13}$$

Now we take a positive number T_0 and require that $T \le T_0$. Then

$$(F(r,T))^{2/q} \le (F(r_0,T_0))^{(2-q)/q} F(r,T) := MF(r,T) \,.$$

Furthermore, we also require that T satisfy

$$C_7 T (r_0/2)^{-2\nu} M \le 1/2$$

and hence, if $r_0/2 \le r \le r_0$,

$$C_7 T\, r^{-2\nu} F^{2/q} \le F/2 \,.$$

Inserting this relation in (5.13), checking that $2(1-d)/q = 8/(10+3q) < 1$ and applying Young's inequality with exponents $p = q/(2(1-d))$ and $p' = q/(q-2(1-d))$ it follows that, under the above restrictions on T,

$$E_4 \le C_8 T^\eta E_0^\theta \quad \text{if} \quad r_0/2 \le r \le r_0 \,, \tag{5.14}$$

where

$$\eta = \frac{q(1-d)}{q-2(1-d)} \,, \qquad \theta = \frac{qd}{q-2(1-d)} = 1 + \frac{(2-q)(1-d)}{q-2(1-d)} \,. \tag{5.15}$$

Notice that $\theta > 1$. Considering T as a parameter, (5.14) is a differential inequality in the variable r of the form (11.1) (see Section 11) with $K = C_9 T$ and $r_m = r_0/2$. Consider $r_1 = r_1(T)$ defined by (11.4). Since $r_1 > r_0/2$ for all T small enough and $r_1 \to r_0$ as $T \to 0$, it follows from Lemma 11.1 that u has finite speed of propagation. This completes the first stage of the proof.

Second stage. We keep the origin as in the first stage; hence, $b = 0$ and, by (5.1), $\zeta(0) = -r_0$. We define

$$\widehat{T} = \sup\{\, t \in (0,\infty) \,:\, \exists\delta = \delta(t) \text{ such that } u = 0 \text{ in } (-\delta,\delta)\times(0,t)\,\}\,. \tag{5.16}$$

From the first stage we know that this set is nonempty. Let $0 < T < \widehat{T}$; thus $\zeta(T) < b = 0$. In the sequel δ stands for $\delta(T)$ in the sense of (5.16). We want to introduce $(x-z)_+^4$ as cut-off function, where z is a parameter. We proceed to show that this can be done with an appropriate choice of φ_r in (5.7). Let $\varphi_1(x) = (x+1)_+\theta(x)$ with $\theta \in C^\infty(\mathbf{R})$, $\theta(x) = 1$ if $x < 0$ and $\theta(x) = 0$ if $x > \delta/r_0$. Taking $0 < r \le r_0$ and recalling (4.2) we have

$$\varphi_r(x) = (x+r)_+\theta(x/r) = \begin{cases} (x+r)_+ & \text{if } x < 0\,, \\ 0 & \text{if } x > \delta\,. \end{cases}$$

Inserting this function in (5.7) and setting $z = -r$, $-r_0 \le z < 0$, it follows that

$$\sup_{0<t<T} \int_z^0 (x-z)^4 w^q(x,t)\,dx + \int_0^T \int_z^0 (x-z)^4 w_{xx}^2 \le C_{10} \int_0^T \int_z^0 w^2\,. \tag{5.17}$$

Setting

$$\widehat{E}_s(z,T) := \int_0^T \int_z^0 (x-z)^s w_{xx}^2\,, \quad \widehat{F}(z,T) := \sup_{0<t<T} \int_z^0 (x-z)^4 w^q(x,t)\,dx\,, \tag{5.18}$$

we repeat the arguments used to deduce (5.14) from (5.9), except that now we apply the inequality (10.3) and, hence, we have $C_7 = 0$ in the relation of the form (5.13). So we obtain

$$\widehat{E}_4 \le C_{11} T^\eta \widehat{E}_0^\theta \quad \text{if} \quad -r_0 \le z < 0 \quad \text{and} \quad 0 < T < \widehat{T}\,, \tag{5.19}$$

where η and θ are given by (5.15) and d by (10.2). Inserting (10.2) into (5.15) we obtain the explicit formulas

$$\eta = 4q/(2+3q)\,, \qquad \theta = (10-q)/(2+3q)\,. \tag{5.20}$$

The differential inequality (5.19) in the variable z is of the form (11.6) with $K = C_{12}T$ and $a_0 = -r_0 = \zeta(0)$. Applying Lemma 11.2 it follows that $\zeta(T) \le z_1 = z_1(T)$, where z_1 is given by (11.9). Recalling (5.5) and computing the exponents $\alpha = \eta/4$ and $\beta = (\theta-1)/4$, we obtain (5.3)–(5.4) for $0 < T < \widehat{T}$. Notice that if $z_1(T) \ge b = 0$ then the relation holds trivially because $\zeta(T) < 0$ for all $T \in (0, \widehat{T})$.

Definition of T_* and conclusion of the proof. From (5.3) and Lemma 3.5 it follows that for $0 < T < \widehat{T}$

$$\zeta(T) - \zeta(0) \le AT^\alpha \left(\int_\Omega u_0^{2-n+\lambda}(x)\,dx \right)^\beta\,, \tag{5.21}$$

where $A = A(\lambda, n)$. Observe that $\zeta(T) < b$ is equivalent to $\zeta(T) - \zeta(0) < r_0$. Taking into account that a similar relation (with the same constants) holds for the interface associated to $b + r_0$, it follows that if we define T_* by the relation

$$A(T_*)^\alpha \left(\int_\Omega u_0^{2-n+\lambda}(x)\, dx \right)^\beta = r_0 \,, \tag{5.22}$$

then $\widehat{T} \geq T_*$ and we can choose $T^* = T_*$ in Definition 1.1. The proof of Theorem 5.1 is now complete.

Remark 5.1. For Problem (I) if $b + r_0 = a$ the "interface" associated to $b + r_0$ is just $x = a$. For Problem (II) this is true if, in addition, the support of u_0 is a compact subset of Ω. These assertions can be proved at once by considering the even extension with respect to $x = a$ for Problem (I) and the periodic extension of period $2a$ for Problem (II). This remark does not apply to the nonhomogeneous problem of Section 8.1.

As explained in Remark 4.1, we can repeat the above arguments with 0 replaced by T_1 and T replaced by T_2. Therefore the following theorem holds.

Theorem 5.2. *Under the hypotheses of Theorem* 5.1, *assume that* $u(x, T_1) = 0$ *if* $\zeta(T_1) < x < \zeta(T_1) + 2r_0$, *where* $T_1 \geq 0$ *and* $r_0 > 0$. *Consider the constant* $T_* = T_*(\lambda, n, u_0, r_0)$ *defined by* (5.22). *If* $T_1 < T_2 < T_1 + T_*$ *then* $\zeta(T_2)$ *exists and*

$$\zeta(T_2) - \zeta(T_1) \leq A_0 (T_2 - T_1)^\alpha \left(\int_{T_1}^{T_2} \int_{\zeta(T_1)}^{b} \left(u^{(\lambda+2)/2} \right)_{xx}^2 dx\, dt \right)^\beta , \tag{5.23}$$

where $b = \zeta(T_1) + r_0$ *and* A_0, α *and* β *are the same constants as in Theorem* 5.1.

Remark 5.2. In Theorems 5.1 and 5.2 we have $4\alpha + \beta = 1$, $0 < \alpha < 1/4$ and $0 < \beta < 1$. For fixed n the exponent α increases with λ. Hence the best information on the regularity of $\zeta(t)$ is obtained as $\lambda \to 1$, i.e., as $\alpha \to (3 - n)/(3(4 - n))$, but $\lambda = 1$ is not allowed. The best (i.e., the smallest) growth exponent of $\zeta(t)$ for large t will be discussed in Section 7. See also Remark 6.1.

6. Continuity of the free boundary.

Theorem 6.1. *Let* u *be the nonnegative strong solution of Problem* (I) (*respectively, of Problem* (II)) *defined in Proposition* 2.2 *and let* J *be a compact interval in which the interface* $\zeta(t)$ *exists. Then*:

(1) ζ *is right-continuous in* J *if* $0 < n \leq 1/2$,
(2) $\zeta \in C^\gamma(J)$ *for all* $\gamma \in (0, 1/(n + 4))$ *if* $1/2 < n < 3/2$,
(3) $\zeta \in C^\gamma(J)$ *for all* $\gamma \in (0, (3 - n)/(12 - 3n))$ *if* $3/2 \leq n < 2$,

where C^γ *stands for the space of uniformly Hölder-continuous functions with exponent* γ.

At the end of the section we show that this theorem is a consequence of Theorem 5.2 and the results on finite backward speed of propagation and lower semicontinuity of the interface obtained below.

Let $T_2 > 0$ and $u(\cdot, T_2)$ be zero in a nonempty open set of Ω. Taking T_2 as the origin of time and replacing t by $-t$ in Definitions 1.1 and 1.2, we define the concept of *finite backward speed of propagation* and the corresponding interface $\zeta(t)$. If $n \geq 3/2$ it is clear that u has finite backward speed of propagation (because $\zeta(t)$ is nondecreasing). The next theorem states that for $1/2 < n < 3/2$ this backward propagation property is also true and provides an estimate to complete the proof of the Hölder continuity of the interface. We introduce the condition

$$1/2 < n < 3/2 \quad \text{and} \quad -1/2 < \lambda < n-1 \tag{6.1}$$

which implies

$$1 - n + \lambda < 0 \quad \text{and} \quad 2 - n + \lambda > 0\,. \tag{6.2}$$

Theorem 6.2. *Let n and λ satisfy* (6.2). *Let u be the nonnegative strong solution of Problem* (I) (*respectively, of Problem* (II)) *defined in Proposition* 2.2. *Assume that $u(x, T_2) = 0$ if $\zeta(T_2) < x < \zeta(T_2) + 2r_0$, where $T_2 > 0$ and $r_0 > 0$. Then there exist positive constants $\widetilde{T} = \widetilde{T}(\lambda, n, u_0, r_0, a)$ and $B_0 = B_0(\lambda, n)$ such that if $T_1 \geq 0$ and $T_2 - \widetilde{T} < T_1 < T_2$ we have that $\zeta(T_1)$ exists and*

$$\zeta(T_1) - \zeta(T_2) \leq B_0(T_2 - T_1)^{\alpha} \left(\int_{T_1}^{T_2} \int_{\zeta(T_2)}^{b} \left(u^{(\lambda+2)/2} \right)_{xx}^{2} dx\, dt \right)^{\beta}, \tag{6.3}$$

where $b = \zeta(T_2) + r_0$ and α and β are given by (5.4).

Proof. Consider the relation (4.6), in Remark 4.1. Because of (6.2) the integral involving $u^{2-n+\lambda}(x, T_2)$ appears now with positive sign in the right-hand side. Arguing for fixed T_2 and considering T_1 as a variable, the proof is performed as the proof of Theorems 5.1–5.2, except for the definition of T_*, now replaced by $\widetilde{T}$, which requires the following minor modification. Observe that, again by (6.2), the inequality of Lemma 3.5 holds now with T_1 replaced by T_2 in the right-hand side; then by Hölder's inequality and (2.8) we have

$$\int_{\Omega} u^{2-n+\lambda}(x, T_2)\, dx \leq (2a)^{n-1-\lambda} \left(\int_{\Omega} u_0(x)\, dx \right)^{2-n+\lambda}$$

and hence we define $\widetilde{T}$ by the relation

$$B(\widetilde{T})^{\alpha} (2a)^{\beta(n-1-\lambda)} \left(\int_{\Omega} u_0(x)\, dx \right)^{\beta(2-n+\lambda)} = r_0\,, \tag{6.4}$$

where B is a positive constant depending only on λ and n.

Remark 6.1. As in Remark 5.2, in Theorem 6.2 we have $0 < \alpha < 1/4$ and $0 < \beta < 1$. (These relations follow from $4\alpha + \beta = 1$, $n > 0$ and $2 - n + \lambda > 0$.) Now the best information on the regularity of $\zeta(t)$ is obtained as $\lambda \to n-1$, i.e., as $\alpha \to 1/(n+4)$.

Next we present a simple result on the lower semicontinuity of the interface $\zeta(t)$.

Proposition 6.3. *Let u be any weak solution* (*without assuming* $u \geq 0$) *of Problem* (I) *or of Problem* (II) *for any* $n > 0$. *Assume that the interface* $\zeta(t)$ *of* u *exists. Then* $\zeta(t)$ *is lower semicontinuous in its domain of existence.*

Proof. We have to prove that

$$\zeta(t_0) \leq \liminf_{t \to t_0} \zeta(t) \tag{6.5}$$

for all t_0 in the domain of existence. Let $\{t_m\}$ be a sequence such that $t_m \to t_0$ and $\zeta(t_m) \to z_0$. Recalling Definitions 1.1–1.2 we have that $\zeta(t_m) \leq b_-(t_m)$ and hence $z_0 \leq b_-(t_0)$. Furthermore, for any smooth function φ

$$0 = \int_{\zeta(t_m)}^{b_-(t_m)} \varphi(x) u(x, t_m)\, dx \to \int_{z_0}^{b_-(t_0)} \varphi(x) u(x, t_0)\, dx\,; \tag{6.6}$$

therefore $u(x, t_0) = 0$ for $z_0 \leq x \leq b_-(t_0)$ and $\zeta(t_0) \leq z_0$. This proves (6.5). (If $u \geq 0$ it is enough to take $\varphi \equiv 1$.)

Corollary 6.4. *Let $n \geq 2$ and let u be the nonnegative solution of Problem* (I) *or of Problem* (II) *defined in Proposition* 2.2. *Assume that the interface* $\zeta(t)$ *of* u *exists. Then* $\zeta(t)$ *is left-continuous in its domain of existence.*

Proof. This corollary holds because $\zeta(t)$ is nondecreasing ([1]) and lower semicontinuous.

Proof of Theorem 6.1. Assertion (1) follows from Theorem 5.2 and Proposition 6.3. Assertion (2) is a consequence of Theorems 5.2 and 6.2, taking into account Remarks 5.2 and 6.1 on the exponent α. Finally, if $3/2 \leq n < 2$ the interface $\zeta(t)$ is nondecreasing and Assertion (3) is implied by Theorem 5.2 and Remark 5.2.

Remark 6.2. Theorems 5.2, 6.1 and 6.2 clearly apply also to a free boundary $\zeta(t)$ starting at $t = t_0 > 0$ instead of starting at $t = 0$. An example of this situation would be a solution with a "dead core," i.e., a solution strictly positive for $t = 0$ which is zero in a nonempty open subset of Q. Bertozzi ([6]) reports numerical evidence on the formation of dead cores for small n ($n \sim 0.5$) for Problem (II). If $n \geq 3/2$ there are no dead cores because of (2.14), which holds for Problems (I)–(II) and for the problem of Section 8.1. The dead core phenomenon is closely related to the question of finite time singularities (see [8, 1, 6]), i.e., the question of the existence of zeros for solutions with $u_0 > 0$ in $\overline{\Omega}$. If $n \geq 7/2$ it is known ([8,1]) that there are no finite time singularities for the three above-mentioned problems.

Remark 6.3. If $0 < n \leq 1/2$ Theorem 6.1 does not exclude the possibility of a "backwards" jump discontinuity of the interface at some instant t_0; such a jump would satisfy $\zeta(t_0^-) > \zeta(t_0^+) = \zeta(t_0)$. Similarly, if $4 > n \geq 2$ Corollary 6.4 and [1] leave open the possibility of a "forwards" jump satisfying $\zeta(t_0^+) > \zeta(t_0^-) = \zeta(t_0)$.

7. The Cauchy problem. In this section $\|v\|_p$ stands for the L^p norm of v in $\mathbf{R}$ and we set

$$\zeta(t) = \sup \operatorname{supp} u(\cdot, t), \quad \zeta_-(t) = \inf \operatorname{supp} u(\cdot, t), \quad |\zeta|(t) = \zeta(t) - \zeta_-(t). \tag{7.1}$$

We consider the Cauchy problem

$$\begin{cases} u_t + (|u|^n u_{xxx})_x = 0 & \text{in } Q := \mathbf{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbf{R}\,, \end{cases} \tag{C}$$

where

$$u_0 \in H^1(\mathbf{R})\ ,\ u_0 \geq 0\ ,\ u_0 \not\equiv 0 \quad \text{and} \quad \text{support}\, u_0 \text{ compact}\,. \tag{7.2}$$

The definitions of *weak solution* and *strong solution* of Problem (C) are as Definitions 2.1 and 2.3 except that Ω is replaced by $\mathbf{R}$ and the relation (2.6) is dropped. ($\overline{\Omega}$ is also to be replaced by $\mathbf{R}$.)

Let $0 < n < 2$. Given $T > 0$, let $\widehat{u}$ be the solution of Problem (I) defined in Proposition 2.2 and choose a so that $-a < \zeta_-(t) < \zeta(t) < a$ for all $t \in [0,T]$; notice that this choice is possible because the estimate (5.21) is independent of a. (Recall also Theorem 5.1 on finite speed of propagation.) For $t \in [0,T]$ we set

$$u(x,t) = \begin{cases} \widehat{u}(x,t) & \text{if } \zeta_-(t) \leq x \leq \zeta(t) \\ 0 & \text{if } x < \zeta_-(t) \text{ or } x > \zeta(t)\,. \end{cases}$$

Performing a similar construction in $[T,2T]$, ... , $[mT,(m+1)T]$, ... , we obtain a nonnegative strong solution u of Problem (C) that satisfies the same identities and estimates as the solution of Problem (I) and clearly $u(\cdot,t)$ has compact support for all $t \geq 0$. Notice that the estimates depending on a for Problem (I) may depend on T for Problem (C).

Theorem 7.1. *Assume* (7.2) *and* $0 < n < 2$. *Let u be the nonnegative strong solution of Problem* (C) *defined above. Then for all* $t > 0$

$$\|u_x(\cdot,t)\|_2 \leq A_1 \|u_0\|_1^{(8-n)/(2n+8)}\, t^{-3/(2n+8)}\,, \tag{7.3}$$

$$\|u(\cdot,t)\|_\infty \leq A_2 \|u_0\|_1^{4/(n+4)}\, t^{-1/(n+4)}\,, \tag{7.4}$$

$$\zeta(t) \leq \zeta(0) + A_3 \|u_0\|_1^{n/(n+4)}\, t^{1/(n+4)}\,, \tag{7.5}$$

where the positive constants A_1, A_2 and A_3 depend only on n.

Notice that (7.5) and a similar relation for ζ_- imply

$$|\zeta|(t) \leq |\zeta|(0) + 2A_3 \|u_0\|_1^{n/(n+4)}\, t^{1/(n+4)}\,. \tag{7.6}$$

On the other hand, if $1 < p < \infty$, from (7.4), (2.8) and the Hölder inequality $\|u\|_p \leq \|u\|_1^{1/p} \|u\|_\infty^{1-1/p}$ we deduce that for all $t > 0$

$$\|u(\cdot,t)\|_p \leq A_2^{1-1/p} \|u_0\|_1^{(n+4p)/(np+4p)}\, t^{-(p-1)/(np+4p)}\,. \tag{7.7}$$

The bounds (7.3), (7.4) and (7.7) tend to ∞ as $t \to 0$, but we also have the bounds supplied by the monotonicity properties of Lemma 3.6 and Lemma 7.4 below. We recall that the nonnegative C^1 source-type solutions obtained in [5] exist for $0 < n < 3$ and have the form

$$u(x,t) = t^{-b}\, U(xt^{-b})\,, \quad b = 1/(n+4)\,,$$

where $U = U(y)$ is even and has bounded support. Hence the exponents of t in (7.3)–(7.7) are *equal to the exponents of the source-type solutions*. The following theorem confirms that these exponents are optimal as $t \to \infty$.

Theorem 7.2. *Under the hypotheses of Theorem* 7.1 *we have*

$$|\zeta|(t) \geq \frac{1}{A_2}\,\|u_0\|_1^{n/(n+4)}\,t^{1/(n+4)} \quad \textit{for all} \quad t>0\,,$$

$$\|u_x(\cdot,t)\|_2^{-1} = O(t^{3/(2n+8)}) \quad \textit{and} \quad \|u(\cdot,t)\|_\infty^{-1} = O(t^{1/(n+4)}) \quad \textit{as} \quad t\to\infty\,.$$

Before proving Theorems 7.1–7.2 we present some lemmas. We will use the Gagliardo-Nirenberg inequality (see e.g. [19])

$$\|v\|_p \leq C_p\|v'\|_2^{(2p-2)/3p}\|v\|_1^{(p+2)/3p} \quad \text{if} \quad 1<p\leq\infty\,. \tag{7.8}$$

In the sequel in this section we assume the hypotheses of Theorem 7.1 and C_n stands for a positive constant which depends only on n and may be different in different occurrences.

Lemma 7.3. *For all* $t>0$

$$\Phi(t) := \int_{\mathbf{R}} u^{3-n}(x,t)\,dx \leq C_n\|u_0\|_1^{(12-3n)/(n+4)}\,t^{-(2-n)/(n+4)}\,. \tag{7.9}$$

Proof. Since $0<n<2$ by Lemma 3.4 $u_{xx}\in L^2(Q_T)$ for all $T>0$ and Φ is absolutely continuous; furthermore, by (3.22) we have for almost every $t>0$

$$\int_{\mathbf{R}} u(x,t)\,u_{xx}^2(x,t)\,dx = -\frac{1}{(2-n)(3-n)}\Phi'(t)\,. \tag{7.10}$$

Notice that $(2-n)(3-n)>0$ and $\Phi'\leq 0$. In the sequel in this proof all the statements are for almost every $t>0$. We also have

$$\int_{\mathbf{R}} u_x^2(x,t)\,dx = -\int_{\mathbf{R}} u(x,t)\,u_{xx}(x,t)\,dx \leq \|u_0\|_1^{1/2}\left(\int_{\mathbf{R}} u(x,t)\,u_{xx}^2(x,t)\,dx\right)^{1/2}, \tag{7.11}$$

where (2.8) was used. From $n<2$, (7.8) and (2.8) it follows that

$$\Phi(t) \leq C_n\|u_0\|_1^{(5-n)/3}\|u_x(\cdot,t)\|_2^{(4-2n)/3}\,.$$

The last three relations imply

$$\Phi(t) \leq C_n\|u_0\|_1^{(4-n)/2}|\Phi'(t)|^{(2-n)/6}\,.$$

Integration of this differential inequality yields (7.9).

Lemma 7.4. $\|u_x(\cdot,t)\|_2$ *is equal almost everywhere in* $(0,\infty)$ *to a nonincreasing function.*

Proof. By the definition of the solution of Problem (C) it is enough to prove the lemma when u is a solution of Problem (I). So in the sequel in this proof u stands for a solution of Problem (I). By (2.11) $\|u_{\varepsilon x}(\cdot,t)\|_2$ is nonincreasing and hence $\iint_{Q_T} u_{\varepsilon x}^2$ is concave in T. By Lemma 3.3 for all $T>0$ $u_{\varepsilon x}\to u_x$ *strongly* in $L^4(Q_T)$ and hence in $L^2(Q_T)$. Therefore $\iint_{Q_T} u_{\varepsilon x}^2 \to \iint_{Q_T} u_x^2$ and the latter is concave in T. This proves the lemma.

Lemma 7.5. (7.3) *holds.*

Proof. Setting $\Psi(t) := \|u_x(\cdot,t)\|_2^4$ it follows from (7.10)–(7.11) that for almost every $t > 0$

$$\Psi(t) \le C_n \|u_0\|_1 |\Phi'(t)| .$$

From Lemma 7.4 we have for almost every $t > 0$

$$\Phi(t) = \int_t^\infty |\Phi'| \ge M \int_t^{2t} \Psi \ge Mt\Psi(2t) ,$$

where $M = C_n^{-1}\|u_0\|_1^{-1}$. This and Lemma 7.3 imply (7.3) for almost every $t > 0$. Finally, (7.3) holds for all $t > 0$ because $\|u_x(\cdot,t)\|_2$ is lower semicontinuous (recall Remark 2.2).

Proof of Theorem 7.1. (7.3) was proved in the former lemma; (7.4) follows from (7.3), (7.8) and (2.8). Recall that (7.7) is implied by (7.4) and (2.8). From (5.23), Lemma 3.5 and (7.7) we obtain

$$\zeta(T_2) - \zeta(T_1) \le C(T_2 - T_1)^\alpha \|u_0\|_1^{\beta(n+4p)/(n+4)} \, T_1^{-\beta(p-1)/(n+4)} ,$$

where $p = 2 - n + \lambda$, α and β are given by (5.4), and the positive constant C depends only on λ and n. Now (7.5) follows from the following calculus lemma. Recall that $\zeta(t)$ is right-continuous for all $n \in (0,2)$ (Theorem 6.1). Notice that for $p = 2 - n + \lambda$

$$\beta \, \frac{n+4p}{n+4} = \frac{n}{n+4} \quad \text{and} \quad \alpha - \beta \, \frac{p-1}{n+4} = \frac{1}{n+4}$$

and hence the exponents of $\|u_0\|_1$ and t obtained from Lemma 7.6 are independent of λ. For the constant A_3 of (7.5) we may choose for each n the smallest value when λ varies in the admissible range.

Lemma 7.6. *Let* $f : [0,\infty) \to \mathbf{R}$ *be a function such that* f *is* (*right-*) *continuous at* 0 *and*

$$f(t) - f(s) \le M(t-s)^a s^{-b} \quad \textit{for all} \quad t > s > 0 , \tag{7.12}$$

where M*,* a *and* b *are real numbers satisfying* $M > 0$ *and* $a > b > 0$*. Then*

$$f(T) - f(0) \le M(1 - 2^{b-a})^{-1} T^{a-b} \quad \textit{for all} \quad T > 0 . \tag{7.13}$$

Proof. Let $r \in (0,T)$ and set $t_k = 2^k r$. From (7.12) we have

$$f(t_{k+1}) - f(t_k) \le M t_k^{a-b} .$$

Summing up from $k = 0$ to $k = m - 1$ we obtain

$$f(t_m) - f(r) \le M(2^{a-b} - 1)^{-1} \, (t_m^{a-b} - r^{a-b}) . \tag{7.14}$$

Choosing m such that $t_m \le T < t_{m+1}$, from (7.12) it follows that

$$f(T) - f(t_m) \le M(t_{m+1} - t_m)^a t_m^{-b} = M t_m^{a-b} \le M T^{a-b} . \tag{7.15}$$

From (7.14), $t_m \le T$, $a - b > 0$ and (7.15) we deduce that

$$f(T) - f(r) \le M(1 - 2^{b-a})^{-1} T^{a-b} .$$

Finally, this and the (right-) continuity of f at 0 imply (7.13).

Proof of Theorem 7.2. From $\|u_0\|_1 = \|u(\cdot,t)\|_1 \le |\zeta|(t)\, \|u(\cdot,t)\|_\infty$, (7.4) and (7.6) we deduce the statements of the theorem on $|\zeta|(t)$ and $\|u(\cdot,t)\|_\infty$. Then the assertion on $\|u_x(\cdot,t)\|$ follows from (7.8) with $p = \infty$.

8.1 Pressure boundary conditions. We consider the problem

$$\begin{cases} u_t + (|u|^n u_{xxx})_x = 0 & \text{in } Q = \Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \Omega = (-a,a) \\ u = 1 \,, \quad u_{xx} = p & \text{for } x = -a \text{ and } x = a \,,\ t > 0 \end{cases} \tag{P}$$

where $p \in \mathbf{R}$,

$$u_0 \in H^1(\Omega) \,,\ u_0 \ge 0 \quad \text{and} \quad u_0(-a) = u_0(a) = 1 \,. \tag{8.1}$$

The nonhomogeneous conditions of Problem (P) are known as pressure boundary conditions and were introduced and analyzed in [12], [13] and [8] for their physical significance.

The definition of *weak solution* of Problem (P) is as Definition 2.1 except that now the test function ψ has compact support in Q and the boundary conditions are satisfied for *all* $t > 0$. Mass conservation (2.8) is not true for Problem (P). The concept of *strong solution* is as in Definition 2.3. In [8] is sketched a proof of the existence of weak solution for Problem (P), based on the estimates (8.2)–(8.3) below. We proceed to combine this proof with the results of [1]. Consider the approximating problems

$$\begin{cases} u_t + (f_\varepsilon(u) u_{xxx})_x = 0 & \text{in } Q \\ u(x,0) = u_0(x) + \varepsilon^\theta (a^2 - x^2) & \text{for } x \in \Omega \qquad (0 < \theta < 2/5) \\ u = 1 \,, \quad u_{xx} = p & \text{for } x = -a \text{ and } x = a \,,\ t > 0 \,, \end{cases} \tag{P$_\varepsilon$}$$

where f_ε is defined by (2.9). The solution u of Problem (P) is obtained as in Propositions 2.1–2.2. In this process the local estimates of [1, Proposition 2.1] and their consequences on regularity, positivity and behavior of the support remain true. Notice that near $x = -a$ and $x = a$ the solution u is smooth because of the boundary condition $u = 1$ and uniformly parabolic theory. Hence the following proposition holds.

Proposition 8.1 ([8, 1])**.** *Assume* (8.1) *and let u be the function described in the preceding paragraph. Then u is a nonnegative weak solution of Problem* (P) *for all $n > 0$ and is a strong solution if $0 < n < 3$. If $n \ge 3/2$ the support of $u(\cdot,t)$ is nonshrinking with t and hence* (2.14) *holds.*

The estimate (2.13) remains true (it follows from (8.2) below), while (2.11)–(2.12) are replaced by

$$\frac{1}{2}\int_\Omega v_{\varepsilon x}^2(x,T)\,dx + \iint_{Q_T} f_\varepsilon(u_\varepsilon)\,u_{\varepsilon xxx}^2\,dx\,dt = \frac{1}{2}\int_\Omega v_{\varepsilon x}^2(x,0)\,dx \quad \text{for all} \quad T>0\,, \tag{8.2}$$

$$\frac{1}{2}\int_\Omega v_x^2(x,T)\,dx + \iint_{\mathcal{P}_T} u^n u_{xxx}^2\,dx\,dt \le \frac{1}{2}\int_\Omega v_x^2(x,0)\,dx \quad \text{for all} \quad T>0\,, \tag{8.3}$$

where

$$v = u - \overline{u}\,, \quad v_\varepsilon = u_\varepsilon - \overline{u}\,, \quad \overline{u}(x) = 1 - \frac{p}{2}a^2 + \frac{p}{2}x^2\,. \tag{8.4}$$

Notice that $\overline{u}$ is a stationary solution of Equation (1.1) and satisfies the boundary conditions of Problem (P); $\overline{u}$ has negative values if and only if $pa^2 > 2$.

Theorem 8.2. *Let* $0 < n < 2$ *and let* u *be the nonnegative strong solution of Problem* (P) *defined in Proposition* 8.1. *Then*

(1) *u has finite speed of propagation and satisfies the conclusions of Theorems* 5.1 *and* 5.2, *except for the definition and properties of* T_*.
(2) *u satisfies the conclusions of Theorem* 6.1 (*continuity properties of the interface*).
(3) *u satisfies Theorem* 6.2, *except for the definition and properties of* $\widetilde{T}$; *hence, u has finite backward speed of propagation if* $n > 1/2$.

Sketch of proof. Checking the arguments of Sections 3 to 6 we observe that the nonhomogeneous boundary conditions require a modification of Section 3. We begin with the local estimates of Lemmas 4.1 to 4.4, which clearly still hold in the present situation, except that ε^θ is to be replaced by $\varepsilon^\theta(a^2 - x^2)$. Taking $\varphi_r(x) = a^2 - x^2$ in Lemma 4.4 and using (2.13) we deduce the global estimates of Lemma 3.2, except that now the constant C depends on T. (Notice that near the lateral boundary the estimates hold because of the boundary condition $u = 1$ and parabolic theory.) Lemma 3.3 is stated and proved exactly as in Section 3. Using Lemma 3.3 we prove Lemma 4.5 by passing to the limit in Lemma 4.4. Finally, Sections 5 and 6 are based on Lemma 4.5, except that the definitions of T_* and $\widetilde{T}$ involve global estimates. Now we just assert the existence of T_* and $\widetilde{T}$.

8.2 Changing sign solutions. We consider Problems (I) and (II) without requiring $u_0 \ge 0$. Replacing $f_\varepsilon(s)$ of (2.9) by $f_\varepsilon(s) = (\varepsilon + s^2)^{n/2}$ we construct for any $n > 0$ a weak solution u as in [4]. (This choice of f_ε assures that the solutions of the approximating problems are smooth where $u = 0$ even for $n < 1$ and hence the proof of [4] applies also for $0 < n < 1$.) In the approximating problems we set $u(x,0) = u_0(x)$, without adding ε^θ.

Theorem 8.3. *Assume* (1.2) *for Problem* (I) *and* (1.3) *for Problem* (II). *Let* $0 < n < 1$ *and let u be the weak solution described in the preceding paragraph. Then*

(1) $u_{xx} \in L^2(Q)$ *and therefore u is a strong solution.*
(2) *u has finite speed of propagation;* (5.3), (5.4) *and* (5.23) *hold with* $\lambda = 0$.
(3) *The interface* $\zeta(t)$ *of u is right-continuous.*

Sketch of proof. The general idea is to follow the arguments of Sections 3 to 5 with $\lambda = 0$ and replacing the functions $g_\varepsilon(s)$ and $G_\varepsilon(s)$ of (3.2)–(3.3) by

$$g_\varepsilon(s) = \int_0^s \frac{dr}{f_\varepsilon(r)} = \int_0^s \frac{dr}{(\varepsilon + r^2)^{n/2}}\,, \quad G_\varepsilon(s) = \int_0^s g_\varepsilon(r)\,dr\,. \tag{8.5}$$

Notice that these integrals are convergent even for $\varepsilon = 0$ when $n < 1$. Now we do not prove results on strong convergence as in Lemma 3.3 and hence we obtain (3.20) for $\lambda = 0$ with the $=$ symbol replaced by $\leq$. (We do not use (3.7) and (3.10)—now the convergence of the terms in G_ε follows easily from the inequality $|g_\varepsilon(s)| \leq |s|^{1-n}/(1-n)$.) In particular, this proves that $u_{xx} \in L^2(Q)$. Lemma 4.2 is replaced by

$$|f_\varepsilon(s) g_\varepsilon(s)| \leq \frac{2-n}{1-n}|s| \quad \text{and} \quad |f'_\varepsilon(s) g_\varepsilon(s)| \leq \frac{n}{1-n}\,,$$

where we use $(\varepsilon + s^2)^{n/2} \leq \varepsilon^{n/2} + |s|^n$, $|g_\varepsilon(s)| \leq |s|^{1-n}/(1-n)$, $|g_\varepsilon(s)| \leq |s|/\varepsilon^{n/2}$ and $|f'_\varepsilon(s)| \leq n|s|^{n-1}$. In Lemmas 4.3 and 4.4 the term containing $u^4_{\varepsilon x}$ is dropped. With regard to the proof of Lemma 4.4, the first integral of (4.4) does not appear because $\lambda = 0$. The second integral of (4.4) is bounded in the following way:

$$\iint |\varphi_r^3 u_{\varepsilon x} u_{\varepsilon xx}| \leq \left(\iint \varphi_r^4 u^2_{\varepsilon xx}\right)^{1/2} \left(\iint \varphi_r^2 u^2_{\varepsilon x}\right)^{1/2}\,,$$

and then an integration by parts shows that

$$\iint \varphi_r^2 u^2_{\varepsilon x} \leq C \iint \varphi_r^4 u^2_{\varepsilon xx} + C \iint u^2_\varepsilon\,.$$

The third integral of (4.4) is bounded as in Section 4. Next we consider the proof of Lemma 4.5 and Relation (4.6) (with $\lambda = 0$ and u^{2-n} replaced by $|u|^{2-n}$): we pass to the limit in the terms in G_ε as indicated above; the strong convergence (3.13) is not necessary here—the weak convergence is enough. Finally, the arguments of Section 5 are exactly the same, taking into account that now $w = u$, $q = 2 - n$ and w^q is to be replaced by $|u|^q$. This completes the proof of Assertion (2).

Assertion (3) follows from (5.23) with $\lambda = 0$ and Proposition 6.3.

When considering *the Cauchy problem* with $\operatorname{supp} u_0$ compact and $0 < n < 1$, but without the hypothesis $u_0 \geq 0$, we obtain for the interface the bound (5.3)–(5.4) with $\lambda = 0$, i.e.

$$\zeta(t) \leq \zeta(0) + Ct^{(2-n)/(8-3n)}\,. \tag{8.6}$$

In this context both the optimal bound (7.5) and the relation $\sup_{t>0} \int |u(x,t)|\,dx < \infty$ are open questions.

9. Calculus inequalities. In this section Ω is again the bounded open interval $(-a, a)$, $a > 0$, and $v, \xi : \overline{\Omega} \to \mathbf{R}$ are functions such that

$$v \in H^2(\Omega) \equiv W^{2,2}(\Omega)\,, \quad v > 0 \text{ in } \overline{\Omega}\,, \tag{9.1}$$

$$\xi \in C^1(\overline{\Omega}) , \quad \xi \geq 0 \text{ in } \Omega . \tag{9.2}$$

In the following two lemmas we state "global" and "local" versions of a calculus inequality used in Sections 3 and 4. The particular case $\lambda = -1/2$ appears already in [8] to improve the positivity results for Problems (I) and (II). We comment that the precise value $9/(1-\lambda)^2$ of the constant is essential for this type of application of the inequality.

Lemma 9.1. *Let* $\lambda \in \mathbf{R}$, $\lambda \neq 1$, *and let* v *satisfy* (9.1). *Assume that either* (1) $v'(-a) = v'(a) = 0$ *or* (2) $v(-a) = v(a)$ *and* $v'(-a) = v'(a)$. *Then*

$$\int_\Omega v^{\lambda-2}(v')^4\,dx \leq \frac{9}{(1-\lambda)^2}\int_\Omega v^\lambda (v'')^2\,dx . \tag{9.3}$$

Lemma 9.2. *Let* $\lambda \in \mathbf{R}$, $\lambda \neq 1$, *and let* v *and* ξ *satisfy* (9.1)–(9.2). *Assume that either* (1) $\xi(-a) = \xi(a) = 0$ *or* (2) $v'(-a) = v'(a) = 0$ *or* (3) $\xi(-a) = \xi(a)$, $v(-a) = v(a)$ *and* $v'(-a) = v'(a)$. *Then*

$$\int_\Omega \xi v^{\lambda-2}(v')^4\,dx \leq \frac{9}{(1-\lambda)^2}\int_\Omega \xi v^\lambda (v'')^2\,dx + \frac{2}{1-\lambda}\int_\Omega \xi' v^{\lambda-1}(v')^3\,dx . \tag{9.4}$$

Proof of Lemma 9.1. The lemma follows from $\int_\Omega v^{\lambda-2}(v')^4 = \frac{3}{1-\lambda}\int_\Omega v^{\lambda-1}(v')^2 v'' \leq \frac{3}{|1-\lambda|}\left(\int_\Omega v^{\lambda-2}(v')^4\right)^{1/2}\left(\int_\Omega v^\lambda (v'')^2\right)^{1/2}$.

Proof of Lemma 9.2. Integrating by parts we obtain

$$\int_\Omega \xi v^{\lambda-2}(v')^4 = \frac{3}{1-\lambda}\int_\Omega \xi v^{\lambda-1}(v')^2 v'' + \frac{1}{1-\lambda}\int_\Omega \xi' v^{\lambda-1}(v')^3 .$$

The lemma is implied by this relation and the following inequality.

$$\begin{aligned}\frac{3}{|1-\lambda|}\int_\Omega \xi v^{\lambda-1}(v')^2|v''| &\leq \frac{3}{|1-\lambda|}\left(\int_\Omega \xi v^{\lambda-2}(v')^4\right)^{1/2}\left(\int_\Omega \xi v^\lambda (v'')^2\right)^{1/2}\\ &\leq \frac{1}{2}\int_\Omega \xi v^{\lambda-2}(v')^4 + \frac{1}{2}\frac{9}{(1-\lambda)^2}\int_\Omega \xi v^\lambda (v'')^2 .\end{aligned}$$

Corollary 9.3. *Under the hypotheses of Lemmas* 9.1 *and* 9.2 *if* $\lambda \leq 0$ *we have*

$$\int_\Omega v^\lambda (v'')^2\,dx + \frac{\lambda(1-\lambda)}{3}\int_\Omega v^{\lambda-2}(v')^4\,dx \geq \frac{2\lambda+1}{1-\lambda}\int_\Omega v^\lambda (v'')^2\,dx , \tag{9.5}$$

$$\begin{aligned}&\int_\Omega \xi v^\lambda (v'')^2\,dx + \frac{\lambda(1-\lambda)}{3}\int_\Omega \xi v^{\lambda-2}(v')^4\,dx\\ &\qquad \geq \frac{2\lambda+1}{1-\lambda}\int_\Omega \xi v^\lambda (v'')^2\,dx + \frac{2}{3}\lambda\int_\Omega \xi' v^{\lambda-1}(v')^3\,dx .\end{aligned} \tag{9.6}$$

The corollary is a straightforward consequence of the lemmas.

10. Interpolation inequalities. In this section we adhere to the following conventions. Derivatives are in the weak (or distribution) sense. When an inequality is stated it is understood that the existence and finiteness of the right-hand side imply the existence of the left-hand side. The symbols K_1 to K_5 and $\widehat{K}_3$ stand for positive constants depending *only* on q; in particular, they are *independent of* r, z *and* b.

First we state the two lemmas used in Sections 5 and 6; these lemmas will be proved at the end of the section. The motivation of these lemmas will be explained in Remark 10.1 below.

Lemma 10.1. *Let* $0 < q < 2$ *and* $0 < r < \infty$. *Then*

$$K_1 \int_{-r}^{r} |v|^2 \leq \left(\int_{-r}^{r} |v''|^2 \right)^d \left(\int_{-r}^{r} (r - |x|)^4 |v|^q \, dx \right)^{2(1-d)/q} + r^{-2\nu} \left(\int_{-r}^{r} (r - |x|)^4 |v|^q \, dx \right)^{2/q} , \quad (10.1)$$

where

$$d = (10 - q)/(10 + 3q) , \qquad \nu = (10 - q)/(2q) . \quad (10.2)$$

Lemma 10.2. *Let* $0 < q < 2$ *and* $-\infty < z < b \leq \infty$. *Assume that either* $b = \infty$ *or* $v(b) = v'(b) = 0$. *Then*

$$K_2 \int_{z}^{b} |v|^2 \leq \left(\int_{z}^{b} |v''|^2 \right)^d \left(\int_{z}^{b} (x - z)^4 |v|^q \, dx \right)^{2(1-d)/q} , \quad (10.3)$$

where d *is given by* (10.2).

Lemma 10.3 (Gagliardo-Nirenberg inequalities)**.** *Assume that* $0 < q < 2$ *and* $0 < r < \infty$. *Let* I *be a real interval and set* $\|v\|_p^p = \int_I |v|^p$. *If the length of* I *is* $2r$ *then*

$$\|v\|_2 \leq K_3 \|v''\|_2^a \|v\|_q^{1-a} + \widehat{K}_3 r^{-\mu} \|v\|_q , \quad (10.4)$$

where

$$a = (2 - q)/(2 + 3q) , \qquad \mu = (2 - q)/(2q) . \quad (10.5)$$

If $I = \mathbf{R}$ *or* I *is a half-line, then* (10.4) *holds with* $\widehat{K}_3 = 0$.

The case of unbounded intervals is proved e.g. in [19], while the case of a bounded interval can be found in [20]. The fact that K_3 and $\widehat{K}_3$ do not depend on r follows by considering first $r = 1$ and then performing a scaling transformation.

Lemma 10.3 is frequently stated only for $1 \leq q < 2$, but Hölder's inequality implies that the lemma also holds for $0 < q < 1$. In fact, let $0 < q < 1$ and assume that I is bounded. Then the lemma follows by applying (10.4)–(10.5) with $q = 1$, inserting $\|v\|_1 \leq \|v\|_q^{q/(2-q)} \|v\|_2^{(2-2q)/(2-q)}$ and using Young's inequality. The case of unbounded intervals (for $0 < q < 1$) follows by letting $r \to \infty$.

Lemma 10.4. *Let $0 < q < 2$ and $z \in \mathbf{R}$. Then*

$$K_4 \int_z^\infty |v|^q \le \left(\int_z^\infty (x-z)^4 |v|^q \, dx \right)^c \left(\int_z^\infty |v|^2 \right)^{q(1-c)/2} ,$$

where $c = (2-q)/(10-q)$.

Proof. It is enough to consider the case $z = 0$. For each $s > 0$ we have

$$\int_0^\infty |v|^q = \int_0^s |v|^q + \int_s^\infty |v|^q \le s^{(2-q)/2} \left(\int_0^\infty |v|^2 \right)^{q/2} + s^{-4} \int_0^\infty x^4 |v|^q \, dx \, .$$

The proof is completed by minimizing in s.

Lemma 10.5. *Let $0 < q < 2$ and $0 < r < \infty$. Then*

$$K_5 \int_{-r}^r |v|^q \le \left(\int_{-r}^r (r - |x|)^4 |v|^q \, dx \right)^c \left(\int_{-r}^r |v|^2 \right)^{q(1-c)/2}$$

with c as in Lemma 10.4.

Proof. Consider the function v_1 defined by $v_1(x) = v(x)$ if $-r < x < 0$ and $v_1(x) = 0$ if $x \ge 0$. Applying Lemma 10.4 with $z = -r$ to v_1 we have

$$K_4 \int_{-r}^0 |v|^q \le \left(\int_{-r}^0 (r+x)^4 |v|^q \, dx \right)^c \left(\int_{-r}^0 |v|^2 \right)^{q(1-c)/2} , \tag{10.6}$$

and applying (10.6) to the function $v(-x)$ we obtain

$$K_4 \int_0^r |v|^q \le \left(\int_0^r (r-x)^4 |v|^q \, dx \right)^c \left(\int_0^r |v|^2 \right)^{q(1-c)/2} . \tag{10.7}$$

The lemma follows from (10.6)–(10.7) and standard inequalities for sums of powers.

Proof of Lemma 10.1. It follows from Lemma 10.3, Lemma 10.5 and Young's inequality.

Proof of Lemma 10.2. If $b < \infty$ we consider the zero extension v_1 of v to the interval (z, ∞). Since $v(b) = v'(b) = 0$, the weak derivative v_1'' belongs to $L^2((z,\infty))$ if $v \in L^2((z,b))$. Hence, it is enough to consider the case $b = \infty$. Then the lemma follows from Lemma 10.3, Lemma 10.4 and Young's inequality.

Remark 10.1. In Section 5 we apply an energy method without using any kind of boundary conditions for the function u (because we work in regions away from the lateral boundary). In this context a direct application of the Gagliardo-Nirenberg inequality (10.4) leads to important difficulties with the term $\widehat{K}_3 r^{-\mu} \|v\|_q$. This is the motivation for introducing Lemmas 10.1 and 10.2.

11. Differential inequalities. This section is devoted to the two ordinary differential inequalities used in Sections 5 and 6. These inequalities imply that the unknown function is zero in a nonempty open interval, mainly because of the condition $\theta > 1$. In the sequel K, η, θ, r_m and r_0 are given real numbers. We consider the relation

$$R_4(r) \le K^\eta (R_0(r))^\theta \quad \text{if} \quad r_m \le r \le r_0\,, \tag{11.1}$$

where

$$R_s(r) = \int_{-r}^{r} (r - |x|)^s \Phi(x)\,dx\,, \tag{11.2}$$

$$\Phi \in L^1((-r_0, r_0))\,, \qquad \Phi \ge 0 \text{ a.e. in } (-r_0, r_0)\,. \tag{11.3}$$

Notice that $4!\,R_0(r) = d^4 R_4(r)/dr^4$ and, hence, (11.1) is a fourth-order differential inequality.

Lemma 11.1. *Let $K > 0$, $\eta \in \mathbf{R}$, $\theta > 1$ and $0 \le r_m < r_0$. Under the hypotheses* (11.2)–(11.3), *assume that the differential inequality* (11.1) *holds. If $r_1 \ge r_m$ and $r_1 > 0$ then $R_0(r) = 0$ for $0 < r \le r_1$, where r_1 is defined by*

$$r_1 = r_0 - \frac{\theta + 3}{\theta - 1} K^{\eta/4} (R_0(r_0))^{(\theta-1)/4}\,. \tag{11.4}$$

Proof. Since $\Phi \ge 0$ it follows from Hölder's inequality that $R_1 \le (R_4)^{1/4}(R_0)^{3/4}$. This and (11.1) imply that

$$R_1(r) \le K^{\eta/4} (R_0(r))^{(\theta+3)/4}\,. \tag{11.5}$$

Since $R_0 = R_1'$, (11.5) is a first-order differential inequality for the nondecreasing function $R_1(r)$. We take the power $4/(\theta+3)$ and, if $R_1(r) > 0$, multiply by $R_1^{-4/(\theta+3)}$. Then an explicit integration completes the proof.

Remark 11.1. The function R_0 corresponds to the "energy" E_0 of Section 5, while R_1 corresponds to the weighted energy E_1. That is why we present formula (11.4), which involves R_0 rather than R_1.

The second differential inequality reads

$$Z_4(z) \le K^\eta (Z_0(z))^\theta \quad \text{if} \quad a_0 \le z < b\,, \tag{11.6}$$

where

$$Z_s(r) = \int_{z}^{b} (x - z)^s \Phi(x)\,dx\,, \tag{11.7}$$

$$\Phi \in L^1((a_0, b))\,, \qquad \Phi \ge 0 \text{ a.e. in } (a_0, b)\,. \tag{11.8}$$

Lemma 11.2. *Let $K > 0$, $\eta \in \mathbf{R}$, $\theta > 1$, $a_0 \in \mathbf{R}$ and $-\infty < b \leq \infty$. Under the hypotheses* (11.7)–(11.8), *assume that* (11.6) *holds. If $z_1 < b$ then $Z_0(z) = 0$ for $z_1 \leq z < b$, where z_1 is defined by*

$$z_1 = a_0 + \frac{\theta+3}{\theta-1} K^{\eta/4} (Z_0(a_0))^{(\theta-1)/4} . \tag{11.9}$$

The proof is almost identical to the former one. The only difference is that now $Z_s(z)$ is nonincreasing and $Z_1' = -Z_0$.

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