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# NONLINEAR OBLIQUE BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL CURVATURE EQUATIONS

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**Abstract.** We prove the existence of smooth solutions of two-dimensional nonuniformly elliptic curvature equations subject to a nonlinear oblique boundary condition. These are equations whose principal part is given by a suitable symmetric function of the principal curvatures of the graph of the solution $u$. The types of boundary conditions we are able to treat are the same as those we considered in earlier work on Hessian equations.

**1. Introduction.** This paper is a sequel to [18], in which we studied nonlinear oblique boundary value problems for Hessian equations in two dimensions. Here we study similar boundary value problems for curvature equations, again only in two dimensions.

We consider problems of the form

$$F[u] = g(x,u) \quad \text{in} \quad \Omega\,, \tag{1.1}$$

$$b(x,u,Du) = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

on bounded, uniformly convex domains $\Omega \subset \mathbb{R}^2$. The operator $F$ is given by

$$F[u] = f(\kappa_1, \kappa_2) \tag{1.3}$$

where $\kappa_1, \kappa_2$ are the curvatures of the graph of $u$ relative to the upward unit normal $\nu$, and $f$ is a suitable symmetric function defined on an open, convex, symmetric (under interchange of $\kappa_1$ and $\kappa_2$) region $\Sigma \subset \mathbb{R}^2$. $\Sigma$ is assumed to be closed under the addition of elements of the positive cone $\Gamma_+ = \{\kappa \in \mathbb{R}^2 : \kappa_1, \kappa_2 > 0\}$. Clearly then we are interested in solutions $u \in C^2(\Omega)$ of (1.1) such that $\kappa = (\kappa_1, \kappa_2)$ belongs to $\Sigma$ at each point of $\Omega$. We shall refer to such $u$ as $\Sigma$-*admissible*, or briefly just *admissible*.

Our hypotheses on $\Sigma$ and $f$ are similar to those in [18]. We assume that $f \in C^{0,1}(\Sigma) \cap C^0(\bar{\Sigma})$ is a positive function such that

$$f_i = \frac{\partial f}{\partial \kappa_i} > 0 \quad \text{on} \quad \Sigma \quad \text{for} \quad i = 1, 2\,, \tag{1.4}$$

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and for any compact set $K \subset \Sigma$ there is a positive constant $C(K)$ such that

$$\max\{f_1, f_2\} \leq C(K) \min\{f_1, f_2\} \quad \text{on} \quad K\,. \tag{1.5}$$

This follows automatically from (1.4) if $f \in C^1(\Sigma)$. Since $f$ is generally only Lipschitz continuous, $f_i$ may exist only almost everywhere, and (1.4) and(1.5) are to be interpreted in this sense. In addition we assume

$$f \ \text{ is concave} \tag{1.6}$$

and

$$f \equiv 0 \quad \text{on} \quad \partial\Sigma\,. \tag{1.7}$$

As explained in [4, 5], (1.4) and (1.5) imply that (1.1) is elliptic on admissible solutions, while (1.4) and (1.6) imply that $F$ is a concave function of the second derivatives of $u$ if $u$ is admissible.

We also assume that $f$ satisfies the following structure conditions:

$$\sum f_i(\kappa)\,\kappa_i \geq 0 \quad \text{on} \quad \Sigma\,, \tag{1.8}$$

$$\mathcal{T} = \sum f_i(\kappa) \geq \sigma_0 \quad \text{on} \quad \Sigma_\mu = \{\kappa \in \Sigma : f(\kappa) \leq \mu\} \tag{1.9}$$

for any $\mu > 0$ and some positive constant $\sigma_0 = \sigma_0(f, \mu)$, and finally,

$$\mathcal{T} \to \infty \quad \text{as} \quad |\kappa| \to \infty \quad \text{on} \quad \Sigma_\mu\,. \tag{1.10}$$

Conditions (1.4) to (1.9) are the same as those we assumed in [18], while (1.10) (and in some cases a stronger condition) was required only when the right-hand side of the equation depended on the gradient.

It follows from (1.4) and (1.8) that $\Sigma$ does not contain the origin and, also using (1.7), that $\partial\Sigma$ is asymptotic to $(\alpha, \alpha) + \partial\Gamma$ for some number $\alpha \geq 0$ and some open, convex, symmetric cone $\Gamma$ with vertex at the origin and containing $\Gamma_+$. As explained in [18], if $\Gamma \neq \Gamma_+$, then (1.1) is uniformly elliptic with respect to the second derivatives of $u$; i.e., (1.1) is uniformly elliptic in the usual sense once the gradient of $u$ is bounded. For such equations second derivative bounds for a large class of oblique boundary conditions follow from the theory developed by Lieberman and Trudinger ([11]). We shall consider, therefore, only the case that $\Gamma = \Gamma_+$ and $\alpha = 0$; thus we assume

$$\partial\Sigma \quad \text{is asymptotic to} \quad \partial\Gamma_+\,. \tag{1.11}$$

Let us now proceed to our hypotheses on $g$ and $b$. We assume that $\Omega$ is a $C^{2,1}$ uniformly convex domain in $\mathbb{R}^2$ and $g \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$ is a positive function satisfying

$$g_z \geq 0 \quad \text{on} \quad \Omega \times \mathbb{R}\,. \tag{1.12}$$

As in [18], it is convenient to consider semilinear and fully nonlinear boundary conditions separately. For the semilinear boundary condition

$$D_\beta u + \phi(x,u) = 0 \quad \text{on} \quad \partial\Omega\,, \tag{1.13}$$

we assume that $\phi \in C^{1,1}(\partial\Omega \times \mathbb{R})$ satisfies

$$\phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}\,, \tag{1.14}$$

$$\phi(x,z) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N \tag{1.15}$$

for some constant $N$, and

$$\phi(x,z) \to \infty \quad \text{as} \quad z \to -\infty \tag{1.16}$$

uniformly for $x \in \partial\Omega$. We also assume that $\beta \in C^{1,1}(\partial\Omega;\mathbb{R}^2)$ is a unit vector field on $\partial\Omega$ satisfying the strict obliqueness condition

$$\beta \cdot \gamma > 0 \quad \text{on} \quad \partial\Omega \tag{1.17}$$

and also the structure condition

$$\big[-2\big(1 + \big(\frac{\beta\cdot\tau}{\beta\cdot\gamma}\big)^2\big)\,\delta_i\beta_j(x) - \phi_z(x,z)\,\delta_{ij}\big]\tau_i\tau_j > 0 \tag{1.18}$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$, where $\tau$ is a unit tangent vector to $\partial\Omega$ at $x$, $\gamma$ is the inner unit normal vector field to $\partial\Omega$, and $\delta = (\delta_1,\delta_2)$ denotes the tangential gradient operator relative to $\partial\Omega$ given by

$$\delta_i = (\delta_{ij} - \gamma_i\gamma_j)D_j\,.$$

Notice that (1.18) is automatically satisfied if $\beta \equiv \gamma$, or more generally if $\beta$ is a vector field with constant normal and tangential components; this follows easily from (1.14) and the uniform convexity of $\Omega$.

We then have the following result which is analogous to Theorems 1.1 and 1.2 of [18] for Hessian equations.

**Theorem 1.1.** *Suppose that the above hypotheses on $\Sigma, f, \Omega, g, \phi$ and $\beta$ are satisfied and there is an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar\Omega)$ of* (1.1)*. Then the boundary value problem* (1.1), (1.13) *has a unique admissible solution $u$ belonging to $C^{2,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$.*

Condition (1.18) seems somewhat artificial. However, in [18] we showed that for Hessian equations second derivative bounds may fail if (1.18) is not satisfied. It seems likely, therefore, that (1.18) is generally necessary in the case of curvature equations. As we shall see, however, our proof of Theorem 1.1 can be modified slightly to yield existence results in certain cases without assuming (1.18); i.e., we require only that $\beta$ be a sufficiently smooth strictly oblique vector field, as in the uniformly elliptic case.

**Theorem 1.2.** *Suppose that the hypotheses of Theorem* 1.1 *are satisfied, with the exception of* (1.18)*, and suppose also that*

$$\kappa_1\kappa_2 \to \infty \quad as \quad \kappa_2 \to \infty \quad on \quad \{\kappa \in \Sigma : \mu_1 \le f(\kappa) \le \mu_2\} \tag{1.19}$$

*for any positive constants* $\mu_1 \le \mu_2$*. Then the boundary value problem* (1.1), (1.13) *has a unique admissible solution* $u$ *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

We shall see later that an analogous result for Hessian equations follows as a consequence of Theorem 1.2. However, we do not have a direct proof of this result.

It is sufficient to assume (1.19) holds only on the set $\{\kappa \in \Sigma : f(\kappa) = \mu_1\}$, since it then also holds on $\{\kappa \in \Sigma : f(\kappa) = \mu\}$ for any $\mu > \mu_1$.

The most interesting example of a function $f$ satisfying (1.4) to (1.9) and (1.19) is the harmonic mean curvature which is given by

$$f(\kappa) = \frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2} \quad \text{on} \quad \Gamma_+ . \tag{1.20}$$

Unfortunately, this example lies outside the scope of Theorem 1.2 because it does not satisfy (1.10). We are able to handle this case only for domains with sufficiently large curvature.

**Theorem 1.3.** *Suppose that the hypotheses of Theorem* 1.2 *are satisfied, with the exception of* (1.10)*. Then there is a positive number* $\kappa_0$*, depending only on* $\Sigma, f, \Omega, g, \phi$ *and* $\beta$*, such that if the curvature of* $\partial\Omega$ *is greater than* $\kappa_0$ *at each point, then the boundary value problem* (1.1), (1.13) *has a unique admissible solution* $u$ *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

The hypothesis concerning the existence of an admissible subsolution can be dropped if $\kappa_0$ is sufficiently large, since admissible subsolutions are easily constructed on small domains.

It is not difficult to verify that condition (1.10) is equivalent to the condition

$$\lim_{t\to\infty} f(s,t) = \infty \quad \text{for any} \quad s > 0\,, \tag{1.21}$$

assuming of course that the remaining conditions on $f$ are unchanged. It follows from this that if (1.10) fails, then (1.19) necessarily holds.

We now consider fully nonlinear boundary conditions. We assume the strict obliqueness condition

$$\chi(x,z,p) = b_p(x,z,p)\cdot\gamma(x) > 0 \tag{1.22}$$

for all $(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$. It follows that (1.2) can be written in the form

$$D_\gamma u + \phi(x,u,\delta u) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.23}$$

We assume furthermore that $\phi \in C^{1,1}(\partial\Omega \times \mathbb{R} \times \mathbb{R}^2)$ satisfies the conditions

$$\phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R} \times \mathbb{R}^2\,, \tag{1.24}$$

$$\phi(x, z, 0) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N \tag{1.25}$$

for some constant $N$, and

$$\phi(x, z, p^T) \to \infty \quad \text{as} \quad z \to -\infty \tag{1.26}$$

uniformly for $(x, p)$ lying in any compact subset of $\partial\Omega \times \mathbb{R}^2$, where $p^T = p - (p \cdot \gamma(x))\, \gamma(x)$. We also assume that $\phi$ satisfies the concavity condition

$$\phi_{p_i p_j}(x, z, p^T)\, \tau_i \tau_j < 0 \tag{1.27}$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$ where $\tau$ is a unit tangent vector to $\partial\Omega$ at $x$.

**Theorem 1.4.** *Let $\Sigma, f, \Omega$ and $g$ satisfy the hypotheses of Theorem* 1.1*, let $\phi$ satisfy the hypotheses above, and assume that there is an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of* (1.1)*. Then the boundary value problem* (1.1), (1.23) *has a unique admissible solution $u$ belonging to $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$.*

As in [18], we shall consider another type of boundary condition which is natural for convex solutions. This is the condition

$$Du(\Omega) = \Omega^* \tag{1.28}$$

for a given uniformly convex domain $\Omega^* \subset \mathbb{R}^2$. This can be reformulated in a more conventional way as

$$h(Du) = 0 \quad \text{on} \quad \partial\Omega, \tag{1.29}$$

where $h$ is a uniformly concave defining function for $\Omega^*$; i.e., $\Omega^* = \{p \in \mathbb{R}^2 : h(p) > 0\}$ and $Dh \neq 0$ on $\partial\Omega^*$. If $u$ is a convex solution of (1.1), (1.29), then $H = h(Du)$ is positive in $\Omega$ and zero on $\partial\Omega$, and it follows that (1.29) is a degenerate oblique boundary condition on convex solutions. It is not immediately clear, however, that we have an *a priori* strict obliqueness estimate

$$h_{p_i}(Du)\gamma_i \geq c_0 > 0 \quad \text{on} \quad \partial\Omega\,, \tag{1.30}$$

so this type of boundary condition is not necessarily expressible in a form suitable for the application of Theorem 1.4. Nevertheless, we are able to treat this problem for a general class of curvature equations. Our hypotheses on $f$ and $g$ are now somewhat different. We assume that $\Sigma$ and $f$ satisfy (1.4) to (1.11) and in addition that

$$\kappa_1 \kappa_2 \leq G(f(\kappa)) \quad \text{for} \quad \kappa \in \Sigma \tag{1.31}$$

for some real-valued, continuous, increasing function $G$ on $[0, \infty)$ with $G(0) = 0$. It follows that $\Sigma = \Gamma_+$. Further, we assume that $h \in C^{2,1}(\mathbb{R}^2)$ is a uniformly concave defining function for some $C^{2,1}$ uniformly convex domain $\Omega^* \subset \mathbb{R}^2$, and $g \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$ is a positive function satisfying

$$g(x, z) \to \infty \quad \text{as} \quad z \to \infty\,, \tag{1.32}$$

$$g(x, z) \to 0 \quad \text{as} \quad z \to -\infty\,, \tag{1.33}$$

uniformly for all $(x, z) \in \bar{\Omega} \times \mathbb{R}$. We then have the following result.

**Theorem 1.5.** *Under the above hypotheses on $\Sigma, f, \Omega, \Omega^*, g$ and $h$, the boundary value problem* (1.1), (1.29) *has a convex solution $u$ belonging to $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. If in addition*

$$g_z > 0 \quad on \quad \Omega \times \mathbb{R}\,, \tag{1.34}$$

*the solution is unique.*

Boundary conditions of the form (1.29) were studied by Pogorelov ([13]) and Delanoë ([6]) for two-dimensional Monge–Ampère equations, with the existence of globally smooth solutions being established in [6]. More recently, the regularity of convex solutions of higher-dimensional Monge–Ampère equations subject to the boundary condition (1.29) has been studied by Caffarelli ([1, 2]). In these papers he establishes the $C^{1,\alpha}$ interior and global regularity of generalized solutions. Higher global regularity in dimensions greater than two remains an open problem.

The Dirichlet problem for curvature equations in all dimensions has been studied by Caffarelli, Nirenberg and Spruck ([5]), Ivochkina ([8, 9]) and Trudinger ([15, 16]). In particular, the Dirichlet problem for the $m$-th mean curvature equation has been solved under geometrically natural conditions on the data.

For the semilinear boundary condition it suffices to assume $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$, but in the most important case $\beta \equiv \gamma$ we automatically have $\partial\Omega \in C^{2,1}$ by virtue of the regularity hypothesis on $\beta$. Such a weakening of the regularity hypothesis on $\partial\Omega$ is not evident for the fully nonlinear boundary conditions (1.23) and (1.29).

Higher regularity of the solution obtained in each of the theorems above follows from uniformly elliptic theory ([7], Chapter 6) if the data are smooth enough. In particular, if $f, g, h, \Omega, \Omega^*, \beta$ and $\phi$ are $C^\infty$, then the solution $u$ belongs to $C^\infty(\bar{\Omega})$.

Our approach is the well-known continuity method which requires the estimation of admissible solutions of each of the above boundary value problems in the space $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha > 0$. In Section 2 we recall some technical inequalities from [18], and also prove some additional results which we will use. In Section 3 we prove the second derivative bounds, which are the central result of the paper. The main ideas for the second derivative bounds come from our earlier work [18] on Hessian equations, which in turn is based on earlier work on Monge–Ampère equations ([17]). However, for curvature equations the details are considerably more complicated. In Section 4 we discuss extensions of our results to the degenerate situation where conditions (1.4), (1.5) and the positivity of $g$ are weakened, as well as existence results without the monotonicity assumptions (1.12), (1.14) and (1.24). Some improvements of the results of [18] concerning Hessian equations are also given.

**2. Preliminary lemmas.** In this section we recall a few lemmas from [18] concerning the structure of $f$, and we derive some additional results which will be used in the proof of the second derivative bounds in Section 3.

Unless otherwise stated, we assume $f$ satisfies conditions (1.4) to (1.9) and $\Sigma$ satisfies (1.11). However, it is clear from the proofs that we do not need (1.4) and (1.5), and we could assume instead the weaker condition

$$f_i \geq 0 \quad \text{on} \quad \Sigma \quad \text{for} \quad i = 1, 2\,. \tag{2.1}$$

In fact, this follows automatically from the concavity and positivity of $f$, and from (1.11). This observation is useful in extending our results to degenerate curvature equations in Section 4.

The following two lemmas are proved in [18], Section 2.

**Lemma 2.1.** *The following are true.*

(i) $\sum f_i\kappa_i \le f(\kappa) + a\mathcal{T}$, *where* $(a,a)$ *is the point where the line* $\kappa_1 = \kappa_2$ *intersects* $\partial\Sigma$.

(ii) *For any* $\mu > 0$ *and any* $\kappa \in \Sigma_\mu = \{\kappa \in \Sigma : f(\kappa) \le \mu\}$ *with* $\kappa_1 \le \kappa_2$ *we have* $f_1 \ge \frac{1}{2}\sigma_0$ *where* $\sigma_0$ *is the constant from* (1.9).

(iii) $\lim_{t\to\infty} f(t,t) = \infty$.

(iv) $f_2/f_1 \to 0 \quad as \quad \kappa_2 \to \infty, \quad \kappa \in \Sigma_\mu \quad$ *for any* $\quad \mu > 0$.

*If in addition* $f$ *satisfies* (1.10), *then*

(v) $f_1 \to \infty \quad as \quad \kappa_2 \to \infty, \quad \kappa \in \Sigma_\mu$, *and*

(vi) $\lim_{t\to\infty} f(s,t) = \infty \quad$ *for any* $\quad s > 0$.

**Lemma 2.2.** *For any* $\mu, \epsilon > 0$ *there is a positive constant* $C(\epsilon)$, *depending only on* $f, \mu$ *and* $\epsilon$, *such that*

$$\sum f_i\kappa_i^2 \le (C(\epsilon) + \epsilon|\kappa|)\mathcal{T} \quad on \quad \Sigma_\mu\,. \tag{2.2}$$

We will also need the following refinement of Lemma 2.1(i); it is of interest only if $a \neq 0$; i.e., $\Sigma \neq \Gamma_+$.

**Lemma 2.3.** *Assume that* (1.10) *holds. Then for any* $\mu_2 \ge \mu_1 > 0$ *and any* $\epsilon > 0$ *there is a positive constant* $C(\epsilon)$, *depending only on* $f$, $\mu_1$, $\mu_2$ *and* $\epsilon$, *such that*

$$\sum f_i\kappa_i \le C(\epsilon) + \epsilon\mathcal{T} \quad on \quad \{\kappa \in \Sigma : \mu_1 \le f(\kappa) \le \mu_2\}\,. \tag{2.3}$$

**Proof.** Let $L(\mu) = \{\kappa \in \Sigma : f(\kappa) = \mu\}$ for $\mu \in [\mu_1, \mu_2]$. Then each $L(\mu)$ is the graph of a convex function $\psi_\mu : (0,\infty) \to (0,\infty)$ and

$$\psi_\mu(t)\,,\ \dot\psi_\mu(t) \to 0 \quad \text{as} \quad t \to \infty\,, \tag{2.4}$$

by Lemma 2.1(vi). The convergence is uniform with respect to $\mu \in [\mu_1, \mu_2]$. Now let $\psi$ denote any $\psi_\mu$. The normal to $L(\mu)$ is given by

$$\frac{(-\dot\psi, 1)}{\sqrt{1+\dot\psi^2}} = \frac{(f_1, f_2)}{\sqrt{f_1^2 + f_2^2}}\,,$$

so for $\kappa \in L(\mu)$ we have

$$\sum f_i\kappa_i \le \Big[\frac{-\dot\psi(\kappa_1)\,\kappa_1 + \psi(\kappa_1)}{\sqrt{1+\dot\psi(\kappa_1)^2}}\Big]\mathcal{T} = \Psi(\kappa_1)\,\mathcal{T}\,.$$

Since $\psi$ is convex, $\dot\psi$ is an increasing function, so for any $t > 0$

$$\psi(t) - \psi(t/2) = \int_{t/2}^{t} \dot\psi(s)\, ds \le \tfrac{1}{2}\, t\, \dot\psi(t)\,.$$

Thus

$$\Psi(t) \le \frac{2\psi(t/2) - \psi(t)}{\sqrt{1 + \dot\psi(t)^2}} \to 0 \quad \text{as} \quad t \to \infty\,,$$

by (2.4). The convergence is of course uniform with respect to $\mu \in [\mu_1, \mu_2]$.

We have shown that for any $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

$$\sum f_i \kappa_i \le \epsilon \mathcal{T} \quad \text{on} \quad \{\kappa \in \Sigma : \mu_1 \le f(\kappa) \le \mu_2\,,\ |\kappa| \ge C(\epsilon)\}\,. \tag{2.5}$$

On the other hand, on the set $K_\epsilon = \{\kappa \in \Sigma : \mu_1 \le f(\kappa) \le \mu_2, |\kappa| \le C(\epsilon)\}$ we have

$$\sum f_i \kappa_i \le C'(\epsilon)\,, \tag{2.6}$$

since $K_\epsilon$ is a compact subset of $\Sigma$ (here we use the fact that $\mu_1 > 0$) and

$$\sup_{K_\epsilon} |Df| \le \frac{\mu_2}{\operatorname{dist}(K_\epsilon, \partial\Sigma)}$$

by the concavity of $f$. The lemma now follows by combining (2.5) and (2.6).

**Remark.** If $f \in C^{0,1}(\bar\Sigma)$ we can allow $\mu_1 = 0$ in the lemma.

Next we recall various geometric quantities associated with the graph of a function $u \in C^2(\Omega)$. The metric of graph $u$ is given by

$$g_{ij} = \delta_{ij} + D_i u\, D_j u \tag{2.7}$$

and its inverse is

$$g^{ij} = \delta_{ij} - \frac{D_i u\, D_j u}{1 + |Du|^2}\,. \tag{2.8}$$

The second fundamental form is given by

$$h_{ij} = \frac{D_{ij} u}{\sqrt{1 + |Du|^2}}\,. \tag{2.9}$$

The principal curvatures are the eigenvalues of $[h_{ij}]$ relative to the metric $[g_{ij}]$—thus they are the eigenvalues of the (generally nonsymmetric) matrix $[h_{ij}\, g^{jk}]$. Equivalently, they are the eigenvalues of the symmetric matrix

$$a_{ij} = b^{ik}\, h_{kl}\, b^{lj}, \tag{2.10}$$

where $[b^{ij}]$ is the positive square root of $[g^{ij}]$, given explicitly by

$$b^{ij} = \delta_{ij} - \frac{D_i u D_j u}{v(1+v)}, \tag{2.11}$$

where $v = \sqrt{1+|Du|^2}$. For later use we also note that the inverse of $[b^{ij}]$ is given by

$$b_{ij} = \delta_{ij} + \frac{D_i u\, D_j u}{1+v}\,. \tag{2.12}$$

Explicitly we have

$$a_{ij} = \frac{1}{v}\big\{D_{ij}u - \frac{D_i u D_l u D_{jl} u}{v(1+v)} - \frac{D_j u D_l u D_{il} u}{v(1+v)} + \frac{D_i u D_j u D_k u D_l u D_{kl} u}{v^2(1+v)^2}\big\}\,. \tag{2.13}$$

As mentioned in the introduction, condition (1.4) implies that the function $F$ defined by

$$F(\mathcal{A}) = f(\kappa_1, \kappa_2)\,, \tag{2.14}$$

where $\kappa_1, \kappa_2$ are the eigenvalues of $\mathcal{A} = [a_{ij}]$, satisfies

$$\sum F_{ij}(\mathcal{A})\xi_i\xi_j > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^2 \tag{2.15}$$

if $u$ is admissible, where $F_{ij}(\mathcal{A}) = \frac{\partial F}{\partial a_{ij}}(\mathcal{A})$. Notice that $[F_{ij}]$ is symmetric since $\mathcal{A}$ is symmetric. Furthermore, $[F_{ij}]$ is diagonal if $\mathcal{A}$ is diagonal, and then $[F_{ij}] = \operatorname{diag}(f_1, f_2)$. In addition, (1.4) and (1.6) imply that $F$ is a concave function of $\mathcal{A}$, and hence also of $D^2u$, if $u$ is admissible; i.e.,

$$\sum F_{ij,kl}(\mathcal{A})\eta_{ij}\eta_{kl} \le 0 \tag{2.16}$$

for any symmetric matrix $[\eta_{ij}]$, where $F_{ij,kl} = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}$. These facts are proved in [4].

To prove the second derivative bounds in the following section we need a more precise result than (2.16). We need to investigate a little more closely what the convexity of $f$ implies about $F$. Differentiating (2.14) we obtain

$$F_{ij} = \sum_{l=1}^{2} \frac{\partial f}{\partial \kappa_l}\frac{\partial \kappa_l}{\partial a_{ij}}\,,$$

$$F_{ij,rs} = \sum_{l=1}^{2} \frac{\partial f}{\partial \kappa_l}\frac{\partial^2 \kappa_l}{\partial a_{ij}\partial a_{rs}} + \sum_{l,m=1}^{2} \frac{\partial^2 f}{\partial \kappa_l \partial \kappa_m}\frac{\partial \kappa_l}{\partial a_{ij}}\frac{\partial \kappa_m}{\partial a_{rs}}\,,$$

and hence, by the concavity of $f$,

$$F_{ij,rs} \le \sum_{l=1}^{2} \frac{\partial f}{\partial \kappa_l}\frac{\partial^2 \kappa_l}{\partial a_{ij}\partial a_{rs}}\,, \tag{2.17}$$

where this inequality is to be interpreted in the matrix sense of (2.16). The eigenvalues of $[a_{ij}]$ are given explicitly by

$$\kappa_l = \tfrac{1}{2}\left(a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}\, a_{21}}\right).$$

A direct computation now shows that at a diagonal matrix $[a_{ij}]$ with $\kappa_2 > \kappa_1$ (so that $\kappa_1$ and $\kappa_2$ are smooth functions of $a_{ij}$) we have

$$\frac{\partial^2 \kappa_l}{\partial a_{12} \partial a_{21}} = \pm 2|a_{11} - a_{22}|^{-1}, \tag{2.18}$$

where we choose the plus sign for the maximum eigenvalue $\kappa_2$ and the minus sign for the minimum eigenvalue $\kappa_1$. All other second derivatives of $\kappa_1$ and $\kappa_2$ are zero.

In the special case that $[a_{ij}] = D^2 w$ for a $C^3$ function $w$ with $D^2 w$ diagonal at a point $x_0$ and $D_{11} w < D_{22} w$ there, we see that (2.17) and (2.18) imply that at $x_0$

$$\begin{aligned} \sum F_{ij,rs}(D^2 w) D_{ij2} w D_{rs2} w &\leq \sum f_l \frac{\partial^2 \kappa_l}{\partial a_{ij} \partial a_{rs}} D_{ij2} w D_{rs2} w \\ &= \frac{2(f_2 - f_1)}{D_{22} w - D_{11} w} (D_{122} w)^2 . \end{aligned} \tag{2.19}$$

In the second derivative estimation we shall express the graph of a solution $u$ of (1.1) locally as the graph of some function $w$ defined over the tangent plane to graph $u$ at some point $P$. We need to know how the derivatives up to third order of the function $w$ are related to those of $u$. Since the results of this computation may be useful in dealing with higher-dimensional curvature problems in the future, and since it is no more complicated, we shall carry out the computation for any dimension $n \geq 2$.

Let $u \in C^3(\bar{\Omega})$ where $\Omega \subset \mathbb{R}^n$ and let $T$ be the hyperplane passing through the origin in $\mathbb{R}^{n+1}$ and parallel to the tangent hyperplane to graph $u$ at $P = (x_0, u(x_0))$ for some $x_0 \in \bar{\Omega}$. Then near $P$ graph $u$ can be expressed as the graph of a function defined on a suitable subset of $T$. Let $e_1, \ldots, e_{n+1}$ be the usual orthonormal basis for $\mathbb{R}^{n+1}$, so that $\Omega \subset \operatorname{span}\{e_1, \ldots, e_n\}$, and let $\hat{e}_1, \ldots, \hat{e}_{n+1}$ be a rotated orthonormal basis such that $T = \operatorname{span}\{\hat{e}_1, \ldots, \hat{e}_n\}$ and $\hat{e}_{n+1}$ points in the direction of the upward normal to graph $u$ at $P$. Clearly $\{e_i\}$ and $\{\hat{e}_i\}$ are related by

$$e_i = \sum_{j=1}^{n+1} c_{ij} \hat{e}_j\,, \qquad \hat{e}_k = \sum_{i=1}^{n+1} c_{ik}\, e_i\,, \tag{2.20}$$

where $c_{ij} = \langle e_i, \hat{e}_j \rangle$ is an orthogonal matrix. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^{n+1}$.

Let $x_1, \ldots, x_{n+1}$ denote the coordinates of a point in $\mathbb{R}^{n+1}$ relative to the basis $\{e_i\}$ and let $y_1, \ldots, y_{n+1}$ denote the coordinates of the same point in the basis $\{\hat{e}_i\}$. Then from (2.20) and the orthonormality of $\{e_i\}$ and $\{\hat{e}_i\}$ we obtain

$$x_i = \sum_{j=1}^{n+1} y_j\, c_{ij}\,, \qquad y_j = \sum_{i=1}^{n+1} x_i\, c_{ij}\,. \tag{2.21}$$

In particular, if $x_{n+1} = u(x)$ and $y_{n+1} = w(y)$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, then (2.21) can be rewritten as

$$\begin{aligned} x_i &= \sum_{j=1}^{n} y_j c_{ij} + w(y)\, c_{i,n+1}\,, \quad i = 1, \dots, n\,, \\ u(x) &= \sum_{j=1}^{n} y_j c_{n+1,j} + w(y)\, c_{n+1,n+1}\,, \end{aligned} \tag{2.22}$$

$$\begin{aligned} y_j &= \sum_{j=1}^{n} x_i c_{ij} + u(x) c_{n+1,j}\,, \quad j = 1, \dots, n\,, \\ w(y) &= \sum_{j=1}^{n} x_i c_{i,n+1} + u(x) c_{n+1,n+1}\,. \end{aligned} \tag{2.23}$$

We now make the further assumption that $|Du(x_0)| \le M$. Then, since

$$\hat{e}_{n+1} = \nu(x_0) = \frac{1}{\sqrt{1 + |Du(x_0)|^2}} \big( -\sum_{i=1}^{n} D_i\, u(x_0) e_i + e_{n+1} \big)\,, \tag{2.24}$$

we have

$$c_{n+1,n+1} = \langle e_{n+1}, \hat{e}_{n+1} \rangle \ge \frac{1}{\sqrt{1 + M^2}}\,. \tag{2.25}$$

We now compute how the derivatives of $u$ and $w$ are related at $P$ by differentiating the relations (2.22) and (2.23). At this point it is convenient to introduce the following notation: partial derivatives of $u$ and $w$ will be indicated by subscripts, with the understanding that $u$ is differentiated with respect to the $x$ variables and $w$ is differentiated with respect to the $y$ variables. Thus $u_k = \frac{\partial u}{\partial x_k}$ and $w_l = \frac{\partial w}{\partial y_l}$. Also, we write $x_{i,k}$ for $\frac{\partial x_i}{\partial y_k}$, $y_{j,k}$ for $\frac{\partial y_j}{\partial x_k}$ and similar expressions for higher derivatives.

From the first equation of (2.22) we find

$$x_{i,k} = c_{ik} + w_k c_{i,n+1}\,, \quad x_{i,kl} = w_{kl} c_{i,n+1}\,, \quad x_{i,klm} = w_{klm} c_{i,n+1}\,, \tag{2.26}$$

and from the first equation of (2.23) we obtain

$$y_{j,k} = c_{kj} + u_k c_{n+1,j}\,, \quad y_{j,kl} = u_{kl}\, c_{n+1,j}\,, \quad y_{j,klm} = u_{klm}\, c_{n+1,j}\,. \tag{2.27}$$

Next we differentiate the second equation of (2.22) to obtain

$$\begin{aligned}
u_k &= \sum_{j=1}^{n} y_{j,k} c_{n+1,j} + c_0 \sum_{p=1}^{n} w_p y_{p,k}\,, \\
u_{kl} &= \sum_{j=1}^{n} y_{j,kl} c_{n+1,j} + c_0 \sum_{p=1}^{n} w_p y_{p,kl} + c_0 \sum_{p,q=1}^{n} w_{pq} y_{p,k} y_{q,l}\,, \\
u_{klm} &= \sum_{j=1}^{n} y_{j,klm} c_{n+1,j} + c_0 \sum_{p=1}^{n} w_p y_{p,klm} \\
&\quad + c_0 \sum_{p,q=1}^{n} w_{pq}(y_{p,kl} y_{q,m} + y_{p,km} y_{q,l} + y_{p,k} y_{q,lm}) + c_0 \sum_{p,q,r=1}^{n} w_{pqr} y_{p,k} y_{q,l} y_{r,m}\,,
\end{aligned}$$

where $c_0 = c_{n+1,n+1}$. Using (2.27) in these relations, together with the orthogonality of $[c_{ij}]$, we see that

$$u_k = -\frac{c_{k,n+1} - \sum_{p=1}^{n} w_p c_{kp}}{c_0 - \sum_{p=1}^{n} w_p c_{n+1,p}}\,, \tag{2.28}$$

and hence, since $Dw(P) = 0$,

$$u_k = -\frac{c_{k,n+1}}{c_0} \quad \text{at} \quad P\,. \tag{2.29}$$

At $P$ we also have

$$u_{kl} = \frac{1}{c_0} \sum_{p,q=1}^{n} w_{pq}(c_{kp} + u_k c_{n+1,p})(c_{lq} + u_l c_{n+1,q}) \tag{2.30}$$

and

$$\begin{aligned}
u_{klm} &= \frac{1}{c_0} \sum_{p,q,r=1}^{n} w_{pqr}(c_{kp} + u_k c_{n+1,p})(c_{lq} + u_l c_{n+1,q})(c_{mr} + u_m c_{n+1,r}) \\
&\quad + \frac{1}{c_0} \sum_{p,q=1}^{n} w_{pq}\{u_{kl} c_{n+1,p}(c_{mq} + u_m c_{n+1,q}) \\
&\quad + u_{km} c_{n+1,p}(c_{lq} + u_l c_{n+1,q}) + u_{lm} c_{n+1,q}(c_{kp} + u_k c_{n+1,p})\}\,.
\end{aligned} \tag{2.31}$$

Finally, using (2.30) we can replace the second derivatives of $u$ appearing on the right-hand side of (2.31) by a suitable expression involving second derivatives of $w$. Thus at $P$ we get

$$\begin{aligned}
u_{klm} &= \frac{1}{c_0} \sum_{p,q,r=1}^{n} w_{pqr}(c_{kp} + u_k c_{n+1,p})(c_{lq} + u_l c_{n+1,q})(c_{mr} + u_m c_{n+1,r}) \\
&\quad + \frac{1}{c_0^2} \sum_{p,q,r,s=1}^{n} w_{pq} w_{rs}\{c_{n+1,p}(c_{mq} + u_m c_{n+1,q})(c_{kr} + u_k c_{n+1,r})(c_{ls} + u_l c_{n+1,s}) \\
&\quad + c_{n+1,p}(c_{lq} + u_l c_{n+1,q})(c_{kr} + u_k c_{n+1,r})(c_{ms} + u_m c_{n+1,s}) \\
&\quad + c_{n+1,q}(c_{kp} + u_k c_{n+1,p})(c_{lr} + u_l c_{n+1,r})(c_{ms} + u_m c_{n+1,s})\}\,.
\end{aligned} \tag{2.32}$$

Notice that the second derivatives of $u$ depend linearly on the second derivatives of $w$, while this is not true of the third derivatives—there is a correction term which is quadratic in second derivatives of $w$.

**3. Second derivative bounds.** To prove the theorems of Section 1 we use the well-known continuity method, which is discussed in [7], Sections 17.2 and 17.9, and also, in the case of Theorem 1.5, the Leray–Schauder fixed-point theorem, [7], Section 11.4. This involves embedding each of the problems considered in Section 1 into a suitable family of problems and proving estimates for admissible solutions of each of these. This procedure is explained in detail in Section 3 of [18] for Hessian equations, and only very minor modifications need to be made to treat curvature equations—essentially we need only replace $F(D^2u)$ by $F[u]$ given by (1.3). For this reason we shall not repeat the details here. We note only that it is sufficient to prove the *a priori* estimate

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \tag{3.1}$$

for some $\alpha \in (0,1)$ for admissible solutions of each of the boundary value problems considered in Section 1.

Solution and gradient estimates are easily proved, the former by use of the maximum principle and the appropriate structural assumptions from Section 1, together with the existence of an admissible subsolution, and the latter by an easy argument using the convexity of the solution ([12], Theorem 2.2). The arguments are virtually identical to those in the Hessian case ([18]), so we shall not repeat the details.

To prove second derivative bounds we need to assume that the solution belongs to $C^4(\Omega) \cap C^3(\bar{\Omega})$ (or at least to $u \in W^{4,2}_{loc}(\Omega) \cap C^3(\bar{\Omega})$). This regularity of the solution is not guaranteed by our regularity assumptions on the data, so we assume initially that all the data are $C^\infty$. We can then obtain the results of Section 1 under the stated regularity hypotheses by a standard approximation procedure. This is required in any case, because for functional analytic reasons the continuity method requires somewhat smoother data than assumed in Section 1.

We now proceed to the second derivative estimation which is the main result of the paper. As in [18], the two-dimensionality plays a crucial role.

Since we have already bounded $u$ and $Du$, we have

$$g(x,u) + |Dg(x,u)| + |D^2g(x,u)| \leq \mu \tag{3.2}$$

for some positive constant $\mu$. Similarly, for the function $\phi$ in (1.23) we have

$$|\phi(x,u,\delta u)| + |D\phi(x,u,\delta u)| + |D^2\phi(x,u,\delta u)| \leq \tilde{\mu} \tag{3.3}$$

for some positive constant $\tilde{\mu}$. A similar inequality of course also holds for the function $\phi$ in (1.13). Furthermore, for (1.23) we have

$$b_{p_k}(x,u,Du) = \gamma_k + \phi_{p_l}(x,u,\delta u)(\delta_{kl} - \gamma_k\gamma_l)\,. \tag{3.4}$$

Since $u$ and $Du$ have already been bounded, the vector field

$$\beta = b_p(x,u,Du)/|b_p(x,u,Du)|$$

satisfies the strict obliqueness condition

$$\beta\cdot\gamma \ge \beta_0 \quad \text{on} \quad \partial\Omega \tag{3.5}$$

for some positive constant $\beta_0$. The use of $\beta$ to denote the vector field $b_p/|b_p|$ is convenient and should cause no confusion with the $\beta$ appearing in (1.13). The structure conditions (1.18) and (1.27) can now also be rewritten as

$$\big[-2\big(1+\big(\frac{\beta\cdot\tau}{\beta\cdot\gamma}\big)^2\big)\delta_i\beta_j(x) - \phi_z(x,u)\delta_{ij}\big]\,\tau_i\tau_j \ge \theta \tag{3.6}$$

and

$$\phi_{p_ip_j}(x,u,\delta u)\,\tau_i\tau_j \le -\tilde\theta \quad \text{on} \quad \partial\Omega \tag{3.7}$$

respectively, for all $x\in\partial\Omega$ and all directions $\tau$ tangential to $\partial\Omega$ at $x$, where $\theta$ and $\tilde\theta$ are positive constants.

We begin by estimating some second derivatives on $\partial\Omega$. This part of the argument does not require the boundary condition to be written in a particular form such as (1.13) or (1.23), and does not even require the boundary condition to be oblique, but to obtain useful estimates we should assume $b_p\neq 0$. We therefore carry out this part of the proof for the general boundary condition (1.2). We assume that the function $b$ has been extended in a $C^2$ fashion to $\bar\Omega\times\mathbb{R}\times\mathbb{R}^2$.

First, differentiating (1.2) tangentially we obtain

$$0=\delta_i b = (b_{x_j}+b_zD_ju+b_{p_k}D_{jk}u)(\delta_{ij}-\gamma_i\gamma_j) \quad \text{on} \quad \partial\Omega\,,$$

so

$$|D_{\tau\beta}u(x_0)|\le C \tag{3.8}$$

for any point $x_0\in\partial\Omega$ and any direction $\tau$ tangential to $\partial\Omega$ at $x_0$, where $\beta=b_p/|b_p|$.

Next we bound $D_{\beta\beta}u$ on $\partial\Omega$. The procedure is similar to that used in [18] for Hessian equations, but considerably more complicated. First we need to show that $B=b(x,u,Du)$ satisfies a suitable differential inequality in $\Omega$. Differentiating $B$ we obtain

$$\begin{aligned} D_iB &= b_{x_i}+b_zD_iu+b_{p_k}D_{ik}u\,, \\ D_{ij}B &= b_{x_ix_j}+b_{x_iz}D_ju+b_{x_ip_k}D_{jk}u+b_{x_jz}D_iu+b_{zz}D_iuD_ju+b_{zp_k}D_iuD_{jk}u \\ &\quad + b_{x_jp_k}D_{ik}u+b_{zp_k}D_juD_{ik}u+b_{p_kp_l}D_{ik}uD_{jl}u+b_zD_{ij}u+b_{p_k}D_{ijk}u\,. \end{aligned} \tag{3.9}$$

Next, writing equation (1.1) in the form

$$G(Du,D^2u)=F(\mathcal{A}[u])=g(x,u) \tag{3.10}$$

and differentiating once we obtain

$$G_{ij}\,D_{ijk}u + G_i\,D_{ik}u = g_{x_k} + g_z D_k u, \tag{3.11}$$

where

$$G_{ij} = \frac{\partial G}{\partial r_{ij}}(Du, D^2u)\,, \quad G_i = \frac{\partial G}{\partial p_i}(Du, D^2u)\,.$$

Notice that $[G_{ij}]$ is symmetric because of the way we have chosen $\mathcal{A}$. From (3.9) and (3.11) we see that

$$\begin{aligned} LB = G_{ij}D_{ij}B + G_i\,D_iB &= G_{ij}(b_{x_ix_j} + 2b_{x_iz}D_ju + b_{zz}D_iuD_ju) \\ &+ 2G_{ij}(b_{x_ip_k}D_{jk}u + b_{zp_k}D_iuD_{jk}u) + b_{p_kp_l}G_{ij}D_{ik}uD_{jl}u + b_zG_{ij}D_{ij}u \\ &+ b_{p_k}(g_{x_k} + g_zD_ku) + G_i(b_{x_i} + b_zD_iu)\,. \end{aligned} \tag{3.12}$$

To proceed further we need to express the second derivatives of $u$ in terms of $a_{ij}$, and the coefficients $G_{ij}$ and $G_i$ in terms of $F_{ij}$ and $a_{ij}$. Recalling (2.9) and (2.10) we see that

$$D_{kl}u = vb_{km}a_{mn}b_{nl} \tag{3.13}$$

where $v = \sqrt{1+|Du|^2}$ and $b_{km}$ is given by (2.12). Next, since $G(Du, D^2u) = F(\mathcal{A}[u])$, we have

$$G_{ij} = F_{kl}\frac{\partial a_{kl}}{\partial r_{ij}} = \frac{1}{v}F_{kl}b^{ki}b^{jl} \tag{3.14}$$

and

$$G_i = F_{kl}\frac{\partial a_{kl}}{\partial p_i} = F_{kl}\frac{\partial}{\partial p_i}\left(\frac{1}{v}b^{kp}b^{ql}\right)D_{pq}u\,. \tag{3.15}$$

After some computation we see that this can be written as

$$G_i = -\frac{D_iu}{v^2}F_{kl}a_{kl} - \frac{2}{v}F_{kl}a_{lm}b^{ik}D_mu\,. \tag{3.16}$$

From (3.14) we see that $\mathcal{T}_G = \sum_{i=1}^2 G_{ii}$ is the trace of a product of three matrices, so it is invariant under orthogonal transformations. Thus we may assume that $[a_{ij}]$ (and hence also $[F_{ij}]$) is diagonal. Since the eigenvalues of $[b^{ij}]$ are bounded between two controlled positive constants, we see that $\mathcal{T}$ and $\mathcal{T}_G$ are comparable; i.e.,

$$\sigma_1\mathcal{T} \le \mathcal{T}_G \le \sigma_2\mathcal{T} \tag{3.17}$$

for some positive constants $\sigma_1$ and $\sigma_2$ depending only on $\sup_\Omega|Du|$.

We may also estimate $G_i$ under the assumption that $[a_{ij}]$ and $[F_{ij}]$ are diagonal. We obtain

$$|G_i| \le C\sum f_l|\kappa_l| = C\sum f_l\kappa_l \quad \text{for each} \quad i\,, \tag{3.18}$$

since each $\kappa_l > 0$ . We may now apply Lemma 2.1(i) or Lemma 2.3 respectively to conclude that

$$|G_i| \le C \quad \text{for each } i \text{ if } \Sigma = \Gamma_+ \,, \tag{3.19}$$

where $C$ depends on $f$, $\sup_\Omega |Du|$ and $\sup_\Omega g(x,u)$, or

$$|G_i| \le C(\theta) + \theta\mathcal{T} \quad \text{for each } i \text{ if } \Sigma \ne \Gamma_+ \tag{3.20}$$

for any $\theta > 0$ and some constant $C(\theta)$ depending on $\theta$, $f$, $\sup_\Omega |Du|$, $\sup_\Omega g(x,u)$ and $\inf_\Omega g(x,u) > 0$.

We now return to (3.12) and estimate the various terms. We obtain

$$|LB| \le C\left(1 + \sum |G_i|\right) + \left|F_{kl}A^{(1)}_{kl} + F_{kl}A^{(2)}_{km}a_{lm} + F_{kl}A^{(3)}_{mn}a_{km}a_{ln}\right| \tag{3.21}$$

for some bounded functions $A^{(1)}_{kl}, A^{(2)}_{kl}, A^{(3)}_{kl}$. Assuming as before that $[a_{ij}]$ and $[F_{ij}]$ are diagonal at the point at which we are computing, we obtain

$$|LB| \le C(1 + \mathcal{T} + \sum f_i\kappa_i + \sum f_i\kappa_i^2)\,. \tag{3.22}$$

Notice that the last terms on the right-hand sides of (3.21) and (3.22) arise from the term $b_{p_kp_l}G_{ij}D_{ik}uD_{jl}u$ in (3.12). It is important for our proof to observe that for a semilinear boundary condition this term is zero, so the term $\sum f_i\kappa_i^2$ is absent from the right-hand side of (3.22) in this case. Using Lemma 2.1(i), Lemma 2.2 and (1.9) we conclude that

$$|LB| \le C\mathcal{T} \quad \text{in} \quad \Omega \tag{3.23}$$

in the case of a semilinear boundary condition, while for a fully nonlinear boundary condition we have

$$|LB| \le (C(\epsilon) + \epsilon M)\mathcal{T} \quad \text{in} \quad \Omega \tag{3.24}$$

for any $\epsilon > 0$, where $M = \sup \kappa$ with the supremum taken over all normal curvatures of graph $u$.

We now proceed to construct a suitable barrier function. This is similar to a barrier argument used in [18], which in turn was adapted from [3], Section 7.

Suppose for convenience that $x_0$ is the origin with the positive $x_2$ axis in the direction of the inner normal to $\partial\Omega$ at 0. Near 0 we can represent $\partial\Omega$ as

$$x_2 = \omega(x_1) = \tfrac{1}{2}\,\kappa_0 x_1^2 + O(|x_1|^3)\,, \tag{3.25}$$

where $\kappa_0 > 0$ is the curvature of $\partial\Omega$ at 0. In $\Omega_\epsilon = \Omega \cap B_\epsilon(0)$ we shall use the barrier function

$$w = \tfrac{1}{2}\,(\kappa_0 - \sigma)\,x_1^2 + \tfrac{1}{2}\,Nx_2^2 - x_2 \tag{3.26}$$

with $\sigma > 0$ fixed so small that $\kappa_0 - \sigma > 0$. We first fix $N \ge 1$ so large that

$$Lw \ge c\mathcal{T} \quad \text{in} \quad \Omega_\epsilon \tag{3.27}$$

for $\epsilon > 0$ small enough, where $c$ is a positive constant. To do this, first observe that

$$G_{ij}\, D_{ij} w = (\kappa_0 - \sigma)\, G_{11} + N G_{22} = v^{-1}\{(\kappa_0 - \sigma) F_{kl} b^{1k} b^{l1} + N F_{kl} b^{2k} b^{l2}\}\,.$$

Let $\tilde{N}$ be the symmetric matrix given by

$$\tilde{N}_{kl} = \frac{1}{v}(b^{1k} b^{l1} + N b^{2k} b^{l2})\,.$$

$\tilde{N}$ is clearly a nonnegative matrix, and in fact, since $[b^{ij}]$ is positive definite with positive upper and lower bounds on its eigenvalues, we easily verify that the minimum eigenvalue of $\tilde{N}$ has a positive lower bound, say $c_1$, and that the maximum eigenvalue of $\tilde{N}$ is of order $N$. Consequently, for $N$ sufficiently large $\tilde{N}$ is admissible in the sense that its vector of eigenvalues belongs to $\Sigma$. From the concavity of $F$ we deduce that

$$F(\tilde{N}) \le F(\mathcal{A}) + F_{ij}(\mathcal{A})(\tilde{N}_{ij} - a_{ij})\,,$$

and hence

$$F_{ij}(\mathcal{A})\tilde{N}_{ij} \ge F(\tilde{N}) - C_1\,.$$

Thus

$$G_{ij}\, D_{ij} w \ge c_2(\mathcal{T} + F_{ij}\tilde{N}_{ij}) \ge c_2\mathcal{T} + c_2(F(\tilde{N}) - C_1)$$

for some positive constant $c_2$. Since $G_i$ satisfies the estimate (3.20), we see that for any $\theta > 0$ we have

$$Lw \ge c_2\mathcal{T} + c_2(F(\tilde{N}) - C_1) - (C(\theta) + \theta\mathcal{T})(C_2 N\epsilon + C_3) \quad \text{in} \quad \Omega_\epsilon\,.$$

If we now fix $\theta$ so small that $\theta(C_2 + C_3) \le \frac{1}{2}\, c_2$ , and then fix $N$ so large that $c_2(F(\tilde{N}) - C_1) \ge C(\theta)(C_2 + C_3)$ (which is possible by virtue of Lemma 2.1(vi) and the fact that the maximum eigenvalue of $\tilde{N}$ is of order $N$ and the minimum eigenvalue is greater than $c_1$), then for all $\epsilon > 0$ so small that

$$N\epsilon \le 1 \tag{3.28}$$

we obtain (3.27) with $c = \frac{1}{2}\, c_2$. If $G_i$ satisfies the estimate (3.19) rather than (3.20), there is no need to introduce $\theta$, and the above estimation is a little simpler.

Next we examine $w$ on $\partial\Omega_\epsilon$. On $\partial\Omega \cap B_\epsilon$ we have, by (3.25),

$$w \le -\tfrac{1}{2}\, \sigma x_1^2 + C|x_1|^3\,,$$

and hence

$$w \le -\tfrac{1}{4}\, \sigma x_1^2 \quad \text{on} \quad \partial\Omega \cap B_\epsilon \tag{3.29}$$

for $\epsilon$ small enough. On the remaining part of $\partial\Omega_\epsilon$, where $|x| = \epsilon$, we consider two cases.

(i) $\frac{1}{2}\, \sigma x_1^2 > N x_2^2 = N(\epsilon^2 - x_1^2)$. In this case, since $x_2 > \omega(x_1)$, we have on $\Omega \cap \partial B_\epsilon$

$$\begin{aligned} w &\le \tfrac{1}{2}\,(\kappa_0 - \sigma)\, x_1^2 + \tfrac{1}{2}\, N x_2^2 - \omega(x_1) \\ &\le -\tfrac{1}{2}\, \sigma x_1^2 + C|x_1|^3 + \tfrac{1}{2}\, N x_2^2 \le -\tfrac{1}{4}\, \sigma x_1^2 + C|x_1|^3 \le -c_3\, \epsilon^2 \end{aligned} \tag{3.30}$$

for some positive constant $c_3$ and for $\epsilon$ small enough, depending only on $\sigma$ and $N$, which have already been fixed.

(ii) $\frac{1}{2}\,\sigma x_1^2 \le N x_2^2$. On this portion of $\Omega \cap \partial B_\epsilon$ we have $x_2 \ge c_4\epsilon$ for some positive constant $c_4$, again depending only on $\sigma$ and $N$. Consequently,

$$w \le C x_1^2 + \tfrac{1}{2} N x_2^2 - x_2 \le (C+N)\epsilon^2 - c_4\epsilon \le -\tfrac{1}{2} c_4 \epsilon \tag{3.31}$$

for $\epsilon$ small enough, depending only on $\sigma$ and $N$. Finally, fixing $\epsilon > 0$ so small that (3.28), (3.29), (3.30) and (3.31) are satisfied, we conclude that $w(0) = 0$ and $w < 0$ on $\partial\Omega_\epsilon - \{0\}$.

We now return to (3.23), which we recall is valid in the case of a semilinear boundary condition. We see that for $A > 0$ chosen sufficiently large $W = Aw$ satisfies $LW \ge \pm LB$ in $\Omega_\epsilon$ and $W \le \pm B$ on $\partial\Omega_\epsilon$ with equality at 0. By the maximum principle $W \le \pm B$ in $\Omega_\epsilon$ and hence $\pm D_\gamma B \ge D_\gamma W$ at 0. It follows that

$$|D_{\gamma\beta} u(0)| \le C\,. \tag{3.32}$$

Combining this with (3.8) we obtain, since 0 can be any point of $\partial\Omega$,

$$|D_{\eta\beta} u(x_0)| \le C \tag{3.33}$$

for any $x_0 \in \partial\Omega$ and any direction $\eta$.

For a fully nonlinear boundary condition we have only the weaker estimate (3.24) instead of (3.23), and therefore, in place of (3.33) we obtain

$$|D_{\eta\beta} u(x_0)| \le C(\epsilon) + \epsilon M \tag{3.34}$$

for any $x_0 \in \partial\Omega$, any direction $\eta$ and any $\epsilon > 0$.

**Remark**. The barrier construction above depends strongly on the fact that $f$ satisfies (1.10). However, if this condition is not assumed, as in Theorem 1.3, we can still obtain a bound for $D_{\beta\beta}u$ on $\partial\Omega$ if the curvature of $\partial\Omega$ is sufficiently large at each point, say greater than $\kappa_0$ which is to be chosen. For any point $x_0 \in \partial\Omega$ we can find a ball $B = B_R(y)$ with $R = 1/\kappa_0$ such that $\Omega \subset B$ and $\partial\Omega \cap \partial B = \{x_0\}$. For the function

$$\psi(x) = A(|x-y|^2 - R^2) \tag{3.35}$$

we then have, by (3.19) or (3.20) with $\theta = 1$,

$$L\psi \ge 2A\big(\mathcal{T}_G - R\sum_{i=1}^{2} |G_i|\big) \ge 2A\big(\mathcal{T}_G - C\,(1+\mathcal{T}_G)\,R\big) \ge A\mathcal{T}_G \ge \pm LB$$

if $R$ is small enough and $A$ is large enough, by virtue of (1.9), (3.17), (3.23) and (3.24). Since we also have $\psi \le 0$ on $\partial\Omega$, with equality at $x_0$, $\psi$ is a lower barrier for $\pm B$ and consequently we obtain the estimates (3.33) and (3.34) for semilinear and fully nonlinear boundary conditions respectively.

At this point it is convenient to establish the strict obliqueness estimate (1.30) for the boundary value problem (1.1), (1.29). If $u, h$ and $\Omega^*$ are as in Theorem 1.5, then $H = h(Du)$ is positive in $\Omega$ and zero on $\partial\Omega$, so $D_\gamma H \geq 0$ and $\delta H = 0$ on $\partial\Omega$. Thus

$$D_i H = D_\gamma H \gamma_i = h_{p_k}(Du) D_{ik} u \quad \text{on} \quad \partial\Omega\,. \tag{3.36}$$

Since $D^2 u$ is invertible, we see that

$$\chi = h_{p_k}(Du)\gamma_k = D_\gamma H u^{\gamma\gamma} \geq 0\,, \tag{3.37}$$

where $u^{\gamma\gamma} = u^{ij}\gamma_i\gamma_j$ and $[u^{ij}] = [D^2 u]^{-1}$. Thus (1.29) is degenerate oblique on convex functions. However, from (3.36) we also see that

$$D_\gamma H \chi = D_{ik} u\, h_{p_i}\, h_{p_k}\,. \tag{3.38}$$

Combining this with (3.37) we obtain

$$\chi = \left[u^{\gamma\gamma}\, D_{ij} u\, h_{p_i}\, h_{p_j}\right]^{1/2}\,, \tag{3.39}$$

which is positive provided $u \in C^2(\bar\Omega)$ with $D^2 u > 0$. Thus (1.29) is in fact strictly oblique on convex $C^2(\bar\Omega)$ solutions if $g$ is positive.

To prove the estimate (1.30) we consider the function

$$\psi = \chi + AH \tag{3.40}$$

where $A$ is a positive constant to be chosen. Here $\gamma$ is assumed to have been extended in a $C^2$ fashion to $\bar\Omega$. Computations similar to those performed above lead to

$$L\psi \leq C\mathcal{T} \quad \text{in} \quad \Omega\,. \tag{3.41}$$

In deriving this we use the uniform concavity of $h$ to fix $A$ so large that the term which is quadratic in $D^2 u$, namely

$$G_{ij}(h_{p_k p_l p_m} D_{il} u D_{jm} u \gamma_k + A h_{p_k p_l} D_{ik} u D_{jl} u)\,,$$

is negative. In addition, we use the fact that $G_i$ is bounded, since $\Sigma = \Gamma_+$ (see (3.19)).

Now let $x_0$ be the point on $\partial\Omega$ at which $\psi\big|_{\partial\Omega}$ attains its minimum. We want to prove

$$\psi(x_0) = \chi(x_0) \geq c_0 \tag{3.42}$$

for some positive constant $c_0$. To do this we first show that

$$D_\gamma \psi(x_0) \geq -C\,. \tag{3.43}$$

This follows by observing that for $B$ large enough the function $Bw + \psi(x_0)$, where $w$ is given by (3.26), is a lower barrier for $\psi$ in $\Omega_\epsilon = \Omega \cap B_\epsilon(x_0)$ for $\epsilon > 0$ small enough. Recalling the definition of $\psi$ we see that (3.43) can be rewritten as

$$h_{p_k p_l} D_{il} u \gamma_i \gamma_k + h_{p_k} (D_i \gamma_k) \gamma_i + A h_{p_k} D_{ik} u \gamma_i \geq -C \quad \text{at} \quad x_0 \,. \tag{3.44}$$

We now make a rotation of coordinates so that the positive $x_2$ axis is in the direction of $\gamma(x_0)$. Then (3.44) implies at $x_0$

$$\begin{aligned} -h_{p_2 p_2} D_{22} u &\leq C_1 + A h_{p_1} D_{12} u + A h_{p_2} D_{22} u + h_{p_1 p_2} D_{12} u \\ &\leq C_1 + C_2 |D_{12} u| + A h_{p_2} D_{22} u \,. \end{aligned} \tag{3.45}$$

Since $H = h(Du) = 0$ on $\partial\Omega$, we also have

$$h_{p_1} D_{11} u + h_{p_2} D_{12} u = 0 \quad \text{at} \quad x_0 \,. \tag{3.46}$$

Assume now that $\chi(x_0) = h_{p_2}(Du(x_0))$ is small, say $0 \leq \chi(x_0) \leq \epsilon$. Since we may assume that

$$|Dh| \geq 1 \quad \text{on} \quad \partial\Omega^* \,, \tag{3.47}$$

we see from (3.46) that

$$D_{11} u \leq C\epsilon |D_{12} u| \quad \text{at} \quad x_0 \,. \tag{3.48}$$

Together with the convexity of $u$, this implies

$$|D_{12} u|^2 \leq D_{11} u D_{22} u \leq C\epsilon |D_{12} u| \, D_{22} u \quad \text{at} \quad x_0 \,, \tag{3.49}$$

which simplifies to

$$|D_{12} u| \leq C\epsilon D_{22} u \quad \text{at} \quad x_0 \,. \tag{3.50}$$

Inserting (3.50) into (3.45), using the fact that $-h_{p_2 p_2} \geq \theta_0$ for some positive constant $\theta_0$, and fixing $\epsilon > 0$ sufficiently small, we deduce

$$D_{22} u(x_0) \leq C \,. \tag{3.51}$$

But then, the estimates (3.48) and (3.50) imply

$$D^2 u(x_0) \leq CI \,, \tag{3.52}$$

and this in turn implies

$$D^2 u(x_0) \geq \sigma I \tag{3.53}$$

for some positive constant $\sigma$, since $g(x, u)$ has a positive lower bound. If we now use (3.52) and (3.53) in (3.39), we obtain the strict obliqueness estimate (1.30).

We now proceed to estimate the remaining second derivatives. To do this we consider the function

$$\tilde{h} = e^{\mathcal{X}} \, \kappa(x, \xi) \tag{3.54}$$

where $\mathcal{X}$ is a function to be chosen and $\kappa(x,\xi)$ denotes the normal curvature of graph $u$ at $(x,u(x))$ in a tangential direction $\xi$. Clearly $\tilde{h}$ achieves its maximum over all $x$ and $\xi$ as above at some $(x_0,\xi_0)$ with $x_0\in\bar{\Omega}$ and $\xi_0$ tangential to graph $u$ at $X_0=(x_0,u(x_0))$. We need to consider the two cases $x_0\in\Omega$ and $x_0\in\partial\Omega$ separately.

Let $\{e_1,e_2,e_3\}$ be the usual orthonormal basis for $\mathbb{R}^3$, with $\Omega\subset\operatorname{span}\{e_1,e_2\}$, and let $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ be a rotated basis such that the tangent plane to graph $u$ at $X_0$ is parallel to $E=\operatorname{span}\{\hat{e}_1,\hat{e}_2\}$. We can represent graph $u$ near $X_0$ as the graph of a function $w$ defined on a subdomain $\tilde{\Omega}$ of $E$. Let $y_1$, $y_2$ denote the coordinates of points in the basis $\{\hat{e}_1,\hat{e}_2\}$ and let $Y_0=(y_0,w(y_0))$ be the coordinates of $X_0$ in the basis $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$. Then $Dw(y_0)=0$. We can make a rotation of $\{\hat{e}_1,\hat{e}_2\}$ so that $\hat{e}_2=\xi_0$. What we have achieved by this change of coordinates is the following: the function

$$h=\frac{e^{\mathcal{X}}\,D_{22}w}{(1+(D_2w)^2)\sqrt{1+|Dw|^2}} \tag{3.55}$$

has a local maximum at $y_0$.

Differentiating $h$ twice and using the fact that $Dw(y_0)=0$ we see that

$$D_i\mathcal{X}+\frac{D_{i22}w}{D_{22}w}=0 \tag{3.56}$$

and

$$D_{ii}\mathcal{X}+\frac{D_{ii22}w}{D_{22}w}-\left(\frac{D_{i22}w}{D_{22}w}\right)^2-2(D_{i2}w)^2-(D_{ii}w)^2\le 0 \tag{3.57}$$

at $y_0$ for $i=1,2$.

Since $F$ is given by (1.3), $w$ satisfies the same kind of equation as $u$, namely

$$F(\mathcal{A}(w))=\tilde{g}(y,w)\quad\text{near}\quad y_0, \tag{3.58}$$

where $\mathcal{A}=[a_{ij}]$ is given by (2.10) and $\tilde{g}$ is obtained from $g$ in an obvious way, so that $\tilde{g}$ is positive and of class $C^2$. Differentiating (3.58) twice in the $\hat{e}_2$ direction and using the fact that $Dw(y_0)=0$, together with the fact that $D^2w$ (and hence also $[F_{ij}]$) is diagonal at $y_0$, we find that

$$\sum f_iD_2a_{ii}=\sum f_i\,D_{ii2}w=D_2\tilde{g}(y,w)\,, \tag{3.59}$$

$$\begin{aligned}&\sum f_iD_{22}a_{ii}+\sum F_{ij,rs}D_2\,a_{ij}D_2a_{rs}\\ &=\sum f_iD_{ii22}w-\kappa_2^2\sum f_i\kappa_i-2f_2\kappa_2^3+\sum F_{ij,rs}D_2a_{ij}D_2a_{rs}=D_{22}\tilde{g}(y,w)\end{aligned} \tag{3.60}$$

at $y_0$, where $[F_{ij}]=\operatorname{diag}(f_1,f_2)$ and $D^2w=\operatorname{diag}(\kappa_1,\kappa_2)$ at $y_0$, with $\kappa_2=D_{22}w$ the largest eigenvalue. Combining (3.57) and (3.60) we obtain, at $y_0$,

$$\begin{aligned}0\le&\sum\frac{f_i(D_{i22}w)^2}{\kappa_2}+\kappa_2\sum f_i\kappa_i^2-\kappa_2^2\sum f_i\kappa_i\\ &-\kappa_2\sum f_iD_{ii}\mathcal{X}+\sum F_{ij,rs}D_{ij2}wD_{rs2}w+C(1+\kappa_2)\,.\end{aligned} \tag{3.61}$$

We now want to show that (3.61) leads to a bound for $\kappa_2$, and hence also for $h$, provided we make a suitable choice of $\mathcal{X}$. This is more delicate for the semilinear boundary condition than for a fully nonlinear one, so we will treat this case in detail. It will be clear that a simpler choice of $\mathcal{X}$ will suffice in the fully nonlinear case. The main idea in the construction of $\mathcal{X}$ has already been used in [12, 17, 18] in connection with oblique boundary conditions for Monge-Ampère equations.

Since $u$ and $Du$ are already bounded and $g$ is positive, there is a number $\delta_0 > 0$ such that $g(x,u) \geq \delta_0$ in $\Omega$. Furthermore, since $Du$ is bounded, at any point of $\Omega$ we have $\kappa_i \leq C_0\lambda_i$ for $i = 1, 2$, for some constant $C_0 \geq 1$ depending only on $\sup_\Omega |Du|$, where $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$ are the eigenvalues of $\mathcal{A}[u]$ and $D^2u$ respectively. Thus $C_0\lambda \in \Sigma$ and

$$\tilde{F}(D^2u) = F(C_0 D^2 u) \geq F(\mathcal{A}(u)) \geq \delta_0 \quad \text{in} \quad \Omega\,. \tag{3.62}$$

By Theorem 1.1 of [18], for each $\rho \in (0,1)$ there is a unique admissible function (in the sense of [18]; i.e., the vector of eigenvalues of $C_0 D^2 v$ belongs to $\Sigma$) $v = v_\rho \in C^2(\bar{\Omega})$ solving the boundary value problem

$$\begin{aligned} \tilde{F}(D^2v) &= \tfrac{1}{2}\delta_0 \quad \text{in} \quad \Omega\,, \\ D_\beta(v+\rho\psi) + \phi(x, v+\rho\psi) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.63}$$

where $\psi \in C^{2,1}(\bar{\Omega})$ is any uniformly convex function with $D_\beta\psi < 0$ on $\partial\Omega$. In particular, if $\partial\Omega \in C^{2,1}$, $\psi$ can be chosen to be a defining function for $\Omega$. By the estimates proved in [18]

$$\sup_{\rho\in(0,1)} |v_\rho|_{2;\Omega} \leq \Lambda \tag{3.64}$$

for some positive constant $\Lambda$, independent of $\rho \in (0,1)$. Consequently we also have

$$D^2 v_\rho \geq \lambda_0 I \tag{3.65}$$

for some positive constant $\lambda_0$, independent of $\rho$. Setting $\bar{v} = v + \rho\psi$ and using the concavity of $F$ we see that

$$\tilde{F}(D^2\bar{v}) \leq \tilde{F}(D^2 v) + \rho\tilde{F}_{ij}(D^2v)D_{ij}\psi \leq \tfrac{1}{2}\delta_0 + \rho C(\Lambda, \delta_0, f, \psi)\,,$$

since (3.63) and(3.64) imply that the eigenvalues of $C_0D^2v$ lie in some compact subset of $\Sigma$. Thus we may fix $\rho > 0$ so small that

$$\tilde{F}(D^2\bar{v}) \leq \delta_0 \quad \text{in} \quad \Omega\,. \tag{3.66}$$

By the mean value theorem $u - \bar{v}$ satisfies an elliptic differential inequality

$$a^{ij}D_{ij}(u-\bar{v}) \geq 0 \quad \text{in} \quad \Omega\,,$$

together with the boundary condition

$$D_\beta(u-\bar{v}) + \sigma(u-\bar{v}) = 0 \quad \text{on} \quad \partial\Omega$$

for some negative function $\sigma$ (since we are assuming (1.14)). From the maximum principle we deduce that $u - \bar{v} \leq 0$ in $\Omega$, and hence that $D_\beta(u - \bar{v}) \leq 0$ on $\partial\Omega$. Thus

$$D_\beta(v - u) \geq -\rho D_\beta \psi \geq \rho_0 \quad \text{on} \quad \partial\Omega \tag{3.67}$$

for some positive constant $\rho_0$.

We can now make our choice of $\mathcal{X}$. We first extend $u$ and $v$ to $\Omega \times \mathbb{R}$ by making them constant in the $e_3$ direction and then choose

$$\mathcal{X} = \alpha(v - u), \tag{3.68}$$

where $\alpha > 0$ is a constant to be determined. We now need to compute the first and second derivatives of $\mathcal{X}$ with respect to the $y$ variables. We denote by $\widetilde{D}u$ and $\widetilde{D}^2 u$ the gradient and Hessian of $u$ as a function on $\Omega \times \mathbb{R}$. Thus

$$\widetilde{D}u = (D_{x_1}u, D_{x_2}u, 0) = \sum_{i=1}^{2} D_{x_i}u e_i$$

and

$$\widetilde{D}^2 u = \begin{pmatrix} D^2 u & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently, using (2.20)

$$D_{y_k}u = \langle \widetilde{D}u, \hat{e}_k \rangle = \langle \widetilde{D}u, \sum_{i=1}^{3} c_{ik}e_i \ \rangle = \sum_{i=1}^{2} D_{x_i}u c_{ik} \tag{3.69}$$

and

$$D_{y_k y_l}u = \langle \hat{e}_k(\widetilde{D}^2 u), \hat{e}_l \rangle = \sum_{i,j=1}^{2} D_{x_i x_j}u c_{ik} c_{jl}, \tag{3.70}$$

where $c_{ik} = \langle e_i, \hat{e}_k \rangle$. Of course, we also have similar expressions for $D_{y_k}v$ and $D_{y_k y_l}v$. Since (3.70) holds we have

$$\sum_{k=1}^{2} f_k D_{y_k y_k} v = \sum_{i,j,k=1}^{2} f_k c_{ik} c_{jk} D_{x_i x_j} v \geq \lambda_0 \sum_{i,k=1}^{2} f_k c_{ik}^2 = \lambda_0 c_{33}^2 \mathcal{T}\,,$$

since for each $k = 1, 2$

$$\sum_{i=1}^{2} c_{ik}^2 = 1 - c_{3k}^2 \geq c_{33}^2$$

by the orthogonality of $[c_{ij}]$. Since $Du$ is bounded, $c_{33}$ has a positive lower bound (see (2.25)), so we have

$$\sum_{i=1}^{2} f_i D_{y_i y_i} v \geq \eta_0 \mathcal{T} \tag{3.71}$$

for some positive constant $\eta_0$.

At $y_0$ we also have, by (2.30) and (3.70),

$$\begin{aligned} f_1 D_{y_1y_1}u &= \sum_{i,j=1}^{2} f_1 c_{i1}c_{j1} D_{x_ix_j}u \qquad (3.72)\\ &= \frac{1}{c_{33}} \sum_{i,j,p,q=1}^{2} f_1 c_{i1}c_{j1} D_{y_py_q}w(c_{ip}+D_{x_i}uc_{3p})(c_{jq}+D_{x_j}uc_{3q}) = \frac{1}{c_{33}} f_1\kappa_1\,, \end{aligned}$$

and similarly

$$f_2 D_{y_2y_2}u = \frac{1}{c_{33}} f_2\kappa_2\,. \qquad (3.73)$$

The last line of (3.72) follows because by the orthogonality of $[c_{ij}]$ we have

$$\sum_{i=1}^{2} c_{i1}(c_{ip}+D_{x_i}uc_{3p}) = \delta_{1p} - c_{31}c_{3p} + \sum_{i=1}^{2} c_{i1}D_{x_i}uc_{3p} = \delta_{1p}\,,$$

since

$$c_{31} = \sum_{i=1}^{2} D_{x_i}uc_{i1}\,. \qquad (3.74)$$

To prove this observe that

$$\sum_{i=1}^{2} D_{x_i}uc_{i1} = \sum_{i=1}^{2} D_{x_i}u\langle e_i,\hat{e}_1\rangle = \langle \widetilde{D}u,\hat{e}_1\rangle\,.$$

Now, since $\hat{e}_1$ is tangential to graph $u$ at $X_0$, we have

$$\hat{e}_1 = \frac{\sum_{i=1}^{2}\eta_i e_i + D_\eta u e_3}{\sqrt{1+|D_\eta u|^2}}$$

for some direction $\eta$ in span$\{e_1,e_2\}$, and hence

$$\langle \widetilde{D}u,\hat{e}_1\rangle = \frac{D_\eta u}{\sqrt{1+|D_\eta u|^2}} = \langle e_3,\hat{e}_1\rangle = c_{31}\,.$$

Thus (3.74) is proved. Combining (3.71), (3.72) and (3.73) we now obtain

$$\kappa_2 \sum f_i D_{ii}\mathcal{X} \geq \alpha\eta_0\kappa_2\mathcal{T} - C\alpha\kappa_2 \sum f_i\kappa_i\,. \qquad (3.75)$$

We now proceed to estimate the remaining terms in (3.61). First we handle the third derivative terms. Using (2.19) we see that

$$\begin{aligned} &\sum f_i \frac{(D_{i22}w)^2}{D_{22}w} + \sum F_{ij,rs}D_{2ij}wD_{2rs}w \\ &\leq \sum f_i \frac{(D_{i22}w)^2}{D_{22}w} - \frac{2(f_1-f_2)}{(\kappa_2-\kappa_1)}(D_{122}w)^2 \leq f_2\frac{(D_{222}w)^2}{D_{22}w} \end{aligned}$$

for $\kappa_2$ sufficiently large; here we have used Lemma 2.1(iv) and the fact that $\kappa_1$ remains bounded as $\kappa_2 \to \infty$ on $\{\kappa \in \Sigma : f(\kappa) \leq \mu\}$ for any $\mu > 0$. Next, at $y_0$ we have, by Lemma 2.1(iv), (3.56), (3.69) and a similar expression for $D_{y_k} v$,

$$f_2 \frac{(D_{222}w)^2}{D_{22}w} = \kappa_2 f_2 (D_2 \mathcal{X})^2 \leq C\alpha^2 \kappa_2 f_2 \leq C\alpha^2 \theta \kappa_2 \mathcal{T} \tag{3.76}$$

for any $\theta > 0$, for $\kappa_2$ large enough, depending on $\theta$. Next,

$$\kappa_2^2 \sum f_i \kappa_i - \kappa_2 \sum f_i \kappa_i^2 = \kappa_1 \kappa_2 f_1 (\kappa_2 - \kappa_1) \geq \tfrac{1}{2} \kappa_1 \kappa_2^2 \mathcal{T} \tag{3.77}$$

for $\kappa_2$ sufficiently large, for the reasons just mentioned above.

Using the above estimates in (3.61) we find that at $y_0$

$$\alpha \eta_0 \kappa_2 \mathcal{T} + \tfrac{1}{2} \kappa_1 \kappa_2^2 \mathcal{T} \leq C \left( \alpha^2 \theta \kappa_2 \mathcal{T} + \alpha \kappa_2 \sum f_i \kappa_i + \kappa_2 \right) \tag{3.78}$$

for $\kappa_2$ sufficiently large. Discarding the term $\frac{1}{2}\kappa_1\kappa_2^2\mathcal{T}$, setting $\theta = \frac{1}{4}\eta_0\alpha^{-1}C^{-1}$, and using Lemma 2.3 (more precisely, the estimate (2.5)) with $\epsilon = \frac{1}{4}\eta_0 C^{-1}$, we obtain a bound

$$\kappa_2 \leq C(\alpha) \tag{3.79}$$

at $y_0$ for any $\alpha > 2C(\eta_0\sigma_0)^{-1}$ where $\sigma_0$ is the constant from (1.9) with $\mu = \sup_\Omega g(x,u)$. A similar bound for $h$ then also follows. The constant $\alpha$ will be fixed later when we consider the case that $\tilde{h}$ attains its maximum on $\partial\Omega$. Notice that here we have used condition (1.10) only through Lemma 2.3 to control the term containing $\sum f_i \kappa_i$. If $\Sigma = \Gamma_+$ we do not need this condition at this point of the proof.

**Remarks.** (i) An examination of the above argument shows that the important properties of $\widetilde{\mathcal{X}} = v - u$ that we used were the boundedness of $D\widetilde{\mathcal{X}}$ (in (3.76)) and

$$D^2 \widetilde{\mathcal{X}} \geq \lambda_0 I - C D^2 u \quad \text{in} \quad \Omega \tag{3.80}$$

for some positive constants $\lambda_0$ and $C$; this was used to obtain the term $\alpha\eta_0\kappa_2\mathcal{T}$ on the left-hand side of (3.78). In addition, for the boundary estimation below we need

$$D_\beta \widetilde{\mathcal{X}} \geq \rho_0 \quad \text{on} \quad \partial\Omega \tag{3.81}$$

for some positive constant $\rho_0$ (see (3.67)). Evidently then, if we are prepared to dispense with (3.81) we can simply choose $\widetilde{\mathcal{X}}(x) = |x|^2$ and (3.80) will still be satisfied. We shall see below that (3.81) is not needed to handle the fully nonlinear boundary conditions (1.23) and (1.29). On the other hand, if we instead want to retain (3.81) and dispense with (3.80), we can choose $\widetilde{\mathcal{X}}$ to be a $C^2$ uniformly concave defining function for $\Omega$. After setting $\theta = 1$ in (3.76), instead of (3.78) we arrive at

$$\kappa_1 \kappa_2^2 \mathcal{T} \leq C \left( (\alpha + \alpha^2) \kappa_2 \mathcal{T} + \kappa_2 \right) \tag{3.82}$$

at $y_0$ for $\kappa_2$ sufficiently large. A bound of the form (3.79) then follows if we assume condition (1.19) of Theorem 1.3. Notice that we do not require (1.10) for this step.

We now consider the possibility that $\tilde{h}$ attains its maximum at a boundary point. We fix $e_1$ and $e_2$ so that $x_0$ is the origin, $e_1$ is tangential to $\partial\Omega$ at $x_0$ and $e_2 = \gamma(x_0)$. We also fix $\{\hat{e}_i\}$ so that $\xi_0 = \hat{e}_2$.

We define new vectors $\tilde{\tau}$ and $\tilde{\beta}$ by

$$\tilde{\tau} = \sum_{k=1}^{2} \tilde{\tau}_k \hat{e}_k \,, \qquad \tilde{\beta} = \sum_{k=1}^{2} \tilde{\beta}_k \hat{e}_k \,, \tag{3.83}$$

where

$$\tilde{\tau}_k = c_{1k} + D_1 u(x_0) c_{3k} \,, \quad \tilde{\beta}_k = \sum_{p=1}^{2} (c_{pk} + D_p u(x_0) c_{3k}) \beta_p(x_0) \,. \tag{3.84}$$

Clearly, $\tilde{\tau}$ and $\tilde{\beta}$ are tangential to graph $u$ at $X_0$. In fact, at $x_0$

$$\tilde{\tau} = e_1 + D_1 u e_3 \,, \qquad \tilde{\beta} = \sum_{i=1}^{2} \beta_i e_i + D_\beta u e_3 \,. \tag{3.85}$$

To prove this we observe that from (2.20) and the orthogonality of $[c_{ij}]$ we have, at $x_0$,

$$\begin{aligned} \tilde{\tau} &= \sum_{k=1}^{2} \sum_{j=1}^{3} (c_{1k} + D_1 u c_{3k}) c_{jk} e_j \\ &= \sum_{j=1}^{3} (\delta_{1j} - c_{13} c_{j3} + D_1 u (\delta_{3j} - c_{33} c_{j3})) e_j = e_1 + D_1 u e_3 \,, \end{aligned}$$

by virtue of (2.29). The second equality of (3.85) may be proved similarly.

It is obvious from (3.85) that $\tilde{\tau}$ is tangential to $\partial\Omega \times \mathbb{R}$ at $x_0$, and consequently, if we denote by $\tilde{\gamma}$ the inner unit normal to $\partial\tilde{\Omega}$ at $y_0$ (recall that $\tilde{\Omega}$ is the subdomain of span$\{\hat{e}_1, \hat{e}_2\}$ on which $w$ is defined), then

$$\tilde{\tau} \cdot \tilde{\gamma} = 0 \,. \tag{3.86}$$

Next we show that

$$\tilde{\beta} \cdot \tilde{\gamma} \geq \beta_0, \tag{3.87}$$

where $\beta_0$ is the obliqueness constant from (3.5). To prove this it is sufficient to show

$$\tilde{\beta} \cdot \gamma(x_0) \geq \beta_0 \,. \tag{3.88}$$

For we can express $\tilde{\beta}$ as a linear combination of $\tilde{\tau}$ and $\tilde{\gamma}$; then by (3.88) and the fact that $\tilde{\tau} \cdot \gamma(x_0) = 0$ we have

$$\beta_0 \leq \tilde{\beta} \cdot \gamma(x_0) = (\tilde{\beta} \cdot \tilde{\gamma}) \tilde{\gamma} \cdot \gamma(x_0) \leq \tilde{\beta} \cdot \tilde{\gamma} \tag{3.89}$$

since clearly $0 \le \tilde{\gamma} \cdot \gamma(x_0) \le 1$. To show (3.88) we proceed as above using the orthogonality of $[c_{ij}]$ and (2.20), (2.29) to obtain, at $x_0$,

$$\tilde{\beta} \cdot \gamma = \sum_{k,p=1}^{2} (c_{pk} + D_p u c_{3k})\beta_p c_{2k} = \beta_2 - c_{23} \sum_{p=1}^{2} (c_{p3} + D_p u c_{33})\beta_p = \beta_2 \,.$$

As before, the function $h$ given by (3.55) has a local maximum at $y_0$ where $y_0$ is now a boundary point of $\widetilde{\Omega}$. Thus $D_\eta h(y_0) \le 0$ for any direction $\eta$ which is tangential to graph $w$ at $y_0$ and such that $\tilde{\gamma} \cdot \eta \ge 0$. In particular, $\tilde{\beta}$ is such a direction (although $\tilde{\beta}$ generally is not a unit vector). Without loss of generality we may also assume that $\hat{e}_1$ and $\hat{e}_2$ satisfy $\hat{e}_1 \cdot \gamma \ge 0$ and $\hat{e}_2 \cdot \tilde{\gamma} \ge 0$ (the orientation of $\{\hat{e}_i\}$ may be changed by this, but this is unimportant since $[c_{ij}]$ will still be orthogonal). Thus

$$D_{\tilde{\beta}} h \le 0 \,, \qquad D_2 h \le 0 \quad \text{at} \quad y_0 \,. \tag{3.90}$$

To proceed further recall that $\mathcal{X} = \alpha(v - u)$ where $v$ is the unique admissible solution of (3.63), with $u$ and $v$ extended to be constant in the $e_3$ direction. Thus from the second equation of (3.85) we have

$$D_{\tilde{\beta}} \mathcal{X} = \alpha D_\beta (v - u) \ge \alpha \rho_0 \quad \text{at} \quad y_0 \tag{3.91}$$

by virtue of (3.67). Recalling the definition of $h$ (see (3.55)) we see that (3.90) can be written as

$$D_{22\tilde{\beta}} w + \alpha \rho_0 D_{22} w \le 0 \,, \qquad D_{222} w \le C_0 \alpha D_{22} w \quad \text{at} \quad y_0 \,. \tag{3.92}$$

Next we need to relate the second and third derivatives of $w$ at $y_0$ to those of $u$ at $x_0$. We have done the computations for this in Section 2. From (2.30) and (3.85) we have

$$D_{\tau\tau} u = D_{11} u = c_{33}^{-1} D_{pq} w (c_{1p} + D_1 u c_{3p})(c_{1q} + D_1 u c_{3q}) = c_{33}^{-1} D_{\tilde{\tau}\tilde{\tau}} w \tag{3.93}$$

by the definition of $\tilde{\tau}$. Here and below we adopt the convention that derivatives of $u$ and $w$ are evaluated at $x_0$ and $y_0$ respectively. Similarly

$$D_{\tau\beta} u = c_{33}^{-1} D_{\tilde{\tau}\tilde{\beta}} w \tag{3.94}$$

and

$$D_{\beta\beta} u = c_{33}^{-1} D_{\tilde{\beta}\tilde{\beta}} w \,. \tag{3.95}$$

From the bounds (3.8) and (3.33) we deduce that for any direction $\eta$ tangential to graph $w$ at $y_0$ we have

$$|D_{\eta\tilde{\beta}} w| \le C \tag{3.96}$$

in the case of a semilinear boundary condition. In the case of a fully nonlinear boundary condition we still have

$$|D_{\tilde{\tau}\tilde{\beta}} w| \le C \,, \tag{3.97}$$

by (3.8) and (3.94), but in place of (3.96) we have only the weaker estimate

$$|D_{\eta\tilde\beta}w| \le C(\epsilon) + \epsilon M \tag{3.98}$$

for any direction $\eta$ tangential to graph $w$ at $y_0$ and any $\epsilon > 0$.

Next we see how the third derivatives are related. From (2.32) and (3.85) we have

$$\begin{aligned} D_{\tau\tau\beta}u &= c_{33}^{-1}D_{\tilde\tau\tilde\tau\tilde\beta}w + c_{33}^{-2}D_{pq}wD_{rs}w(c_{3p}\tilde\beta_q\tilde\tau_r\tilde\tau_s + c_{3p}\tilde\tau_q\tilde\tau_r\tilde\beta_s + c_{3q}\tilde\tau_p\tilde\tau_r\tilde\beta_s) \\ &= c_{33}^{-1}D_{\tilde\tau\tilde\tau\tilde\beta}w + c_{33}^{-2}D_{p\tilde\beta}wc_{3p}D_{\tilde\tau\tilde\tau}w + 2c_{33}^{-2}D_{\tilde\tau q}wc_{3q}D_{\tilde\tau\tilde\beta}w\,. \end{aligned} \tag{3.99}$$

If we now express $\sum_{p=1}^{2} c_{3p}\hat e_p$ as a linear combination of $\tilde\tau$ and $\tilde\beta$ (which can be done with controlled coefficients by virtue of (3.86) and (3.87)) and then use (3.96) and (3.99), we obtain

$$D_{\tau\tau\beta}u \le C(1 + D_{\tilde\tau\tilde\tau}w + D_{\tilde\tau\tilde\tau\tilde\beta}w) \tag{3.100}$$

for a semilinear boundary condition, while for a fully nonlinear boundary condition we instead obtain

$$D_{\tau\tau\beta}u \le C\big((C(\epsilon)+\epsilon M)D_{\tilde\tau\tilde\tau}w + D_{\tilde\tau\tilde\tau\tilde\beta}w\big)\,. \tag{3.101}$$

To remove the term $D_{\tau\tau\beta}u$ from the left-hand sides of (3.100) and (3.101) we use the boundary condition. Computing the second tangential derivatives of (1.13) on $\partial\Omega$ we get

$$D_ku\delta_i\delta_j\beta_k + \delta_i\beta_k\delta_jD_ku + \delta_j\beta_k\delta_iD_ku + \beta_k\delta_i\delta_jD_ku + \delta_i\delta_j\phi = 0 \quad \text{on} \quad \partial\Omega\,. \tag{3.102}$$

Using the estimate (3.33) we see that

$$D_{\tau\tau\beta}u \ge -C(1 + D_{\tau\tau}u) \quad \text{at} \quad x_0\,. \tag{3.103}$$

Notice that we have not used the structure condition (1.18) to obtain this. Similarly, from the fully nonlinear boundary condition (1.23) we obtain, with the use of the structure condition (3.7),

$$D_{\tau\tau\beta}u \ge \tilde\theta(D_{\tau\tau}u)^2 - C(1+|D^2u|) \quad \text{at} \quad x_0\,. \tag{3.104}$$

A similar bound also holds for the boundary condition (1.29), by virtue of the uniform concavity of $h$. Combining (3.100), (3.101) with (3.103), (3.104) respectively and again using (3.93) we get

$$-C(1 + D_{\tilde\tau\tilde\tau}w) \le D_{\tilde\tau\tilde\tau\tilde\beta}w \quad \text{at} \quad y_0 \tag{3.105}$$

for the semilinear boundary condition (1.13) and, also using (3.34),

$$\theta_0(D_{\tilde\tau\tilde\tau}w)^2 \le C\big((C(\epsilon)+\epsilon M)(1 + D_{\tilde\tau\tilde\tau}w) + D_{\tilde\tau\tilde\tau\tilde\beta}w\big) \quad \text{at} \quad y_0 \tag{3.106}$$

for some positive constant $\theta_0$, for either of the fully nonlinear boundary conditions (1.23) and (1.29).

The next step is to show that the inequalities (3.92) imply an estimate

$$D_{\tilde{\tau}\tilde{\tau}\tilde{\beta}}w + c_0\alpha D_{22}w \leq C \quad \text{at} \quad y_0 \tag{3.107}$$

for $D_{22}w(y_0)$ large enough, where $C$ and $c_0$ are controlled positive constants. Once we have this we may finally fix $\alpha$ sufficiently large, in accordance with any earlier restrictions on $\alpha$, so that (3.105) and (3.106) imply a bound

$$D_{\tilde{\tau}\tilde{\tau}}w(y_0) \leq C \tag{3.108}$$

for the semilinear boundary condition, and a bound

$$D_{\tilde{\tau}\tilde{\tau}}w(y_0) \leq C(\epsilon) + \epsilon M \tag{3.109}$$

for all $\epsilon > 0$ for a fully nonlinear boundary condition. Once we have these, we also have the respective upper bounds

$$\sup \tilde{h} \leq C \tag{3.110}$$

and

$$\sup \tilde{h} \leq C(\epsilon) + \epsilon M \tag{3.111}$$

for all $\epsilon > 0$, corresponding to the two classes of boundary conditions. The second derivative bound now follows, in the second case by finally fixing $\epsilon$ sufficiently small.

The proof of (3.107) is the same as in [18] for the Hessian case, with minor modifications. However, for the convenience of the reader we provide the proof here.

It is convenient at this point to normalize $\tilde{\tau}$ and $\tilde{\beta}$ to have unit length. The estimates above still hold if we do this, with some controlled changes to various constants. We express various vectors at $y_0$ in the basis $\hat{e}_1, \hat{e}_2$. Recall that we have assumed that $\tilde{\gamma} \cdot \hat{e}_1$ and $\tilde{\gamma} \cdot \hat{e}_2$ are nonnegative at $y_0$ and $\xi_0 = \hat{e}_2$. At $y_0$ we have

$$\tilde{\gamma} = a\hat{e}_1 + b\hat{e}_2 \tag{3.112}$$

for some nonnegative constants $a, b$ satisfying $a^2 + b^2 = 1$. Then, modulo a possible change of sign,

$$\tilde{\tau} = -b\hat{e}_1 + a\hat{e}_2\,. \tag{3.113}$$

Let $c, d$ be constants satisfying $c^2 + d^2 = 1$ and such that at $y_0$ we have

$$\tilde{\beta} = c\hat{e}_1 + d\hat{e}_2\,. \tag{3.114}$$

Since $D_{12}w(y_0) = 0$ we have

$$D_{\tilde{\beta}\tilde{\beta}}w(y_0) = c^2 D_{11}w(y_0) + d^2 D_{22}w(y_0) \geq d^2 D_{22}w(y_0)\,,$$

so by (3.96),

$$D_{22}w(y_0) \le Cd^{-2}\,. \tag{3.115}$$

For a fully nonlinear boundary condition we use (3.98) in place of (3.96), so in this case the right-hand side of (3.115) should be replaced by $(C(\epsilon)+\epsilon M)d^{-2}$. We therefore need to consider only the case that $|d|$ is small, say $|d| \le d_0$ for a suitably chosen positive constant $d_0$.

To prove (3.107) we first differentiate (3.58) in the directions $\hat{e}_1$ and $\hat{e}_2$. Noting that $Dw(y_0) = 0$ and $D_{12}w(x_0) = 0$, we obtain at $y_0$

$$D_{111}w = \frac{\tilde{g}_{y_1}}{f_1} - \frac{f_2}{f_1}D_{122}w \tag{3.116}$$

and

$$D_{112}w = \frac{\tilde{g}_{y_2}}{f_1} - \frac{f_2}{f_1}D_{222}w\,. \tag{3.117}$$

Using (3.113), (3.114), (3.116) and (3.117) we then obtain at $y_0$

$$\begin{aligned}
D_{\tilde{\tau}\tilde{\tau}\tilde{\beta}}w &= b^2D_{11\tilde{\beta}}w - 2abD_{12\tilde{\beta}}w + a^2D_{22\tilde{\beta}}w \qquad (3.118)\\
&= b^2cD_{111}w + b^2dD_{112}w - 2abcD_{112}w - 2abdD_{122}w + a^2D_{22\tilde{\beta}}w\\
&= b^2c\Big(\frac{\tilde{g}_{y_1}}{f_1} - \frac{f_2}{f_1}D_{122}w\Big) + (b^2d - 2abc)\Big(\frac{\tilde{g}_{y_2}}{f_1} - \frac{f_2}{f_1}D_{222}w\Big)\\
&\quad - 2abdD_{122}w + a^2D_{22\tilde{\beta}}w\\
&= \Big(a^2 - b^2\frac{f_2}{f_1} - \frac{2abd}{c}\Big)D_{22\tilde{\beta}}w + \Big(\frac{2abd^2}{c} + 2abc\frac{f_2}{f_1}\Big)D_{222}w\\
&\quad + \frac{1}{f_1}\big(b^2c\tilde{g}_{y_1} + (b^2d - 2abc)\tilde{g}_{y_2}\big)\,.
\end{aligned}$$

Next, using the obliqueness condition

$$\tilde{\beta}\cdot\tilde{\gamma} = ac + bd \ge \beta_0$$

for some positive constant $\beta_0$, we see that

$$ac \ge \beta_0 - bd \ge \beta_0 - |d| \ge \tfrac{1}{2}\beta_0\,,$$

provided

$$|d| \le \tfrac{1}{2}\beta_0\,. \tag{3.119}$$

Assuming henceforth that (3.119) is satisfied, we obtain, since $a^2, c^2 \le 1$ and $a \ge 0$,

$$\tfrac{1}{2}\beta_0 \le a, c \le 1\,. \tag{3.120}$$

Using Lemma 2.1(iv) we see there is a constant $N_0 = N_0(f, \mu, \beta_0)$ such that if

$$D_{22}w(y_0) \geq N_0\,, \tag{3.121}$$

then

$$\frac{f_2}{f_1} \leq \frac{1}{16}\beta_0^2\,. \tag{3.122}$$

If we assume in addition that

$$|d| \leq \frac{1}{64}\beta_0^3\,, \tag{3.123}$$

we obtain

$$\frac{1}{8}\beta_0^2 \leq a^2 - b^2\frac{f_2}{f_1} - \frac{2abd}{c} \leq 2\,. \tag{3.124}$$

Next, from (3.120), (3.121), (3.122) and (3.123) we obtain

$$0 \leq \frac{2abd^2}{c} + 2abc\frac{f_2}{f_1} \leq 4\beta_0^{-1}d^2 + 2\frac{f_2}{f_1}\,. \tag{3.125}$$

If we now require, in addition to (3.119), (3.121) and (3.123), that $D_{22}w(y_0)$ is so large that

$$\frac{f_2}{f_1} \leq \frac{1}{64}C_0^{-1}\rho_0\beta_0^2\,, \tag{3.126}$$

and also

$$d^2 \leq \frac{1}{128}C_0^{-1}\rho_0\beta_0^3\,, \tag{3.127}$$

where $C_0$ and $\rho_0$ are the constants from (3.92), and then use the estimates (3.92) in (3.118), we finally arrive at (3.107) with $c_0 = \frac{1}{8}\rho_0\beta_0^2$. With this the proof of the second derivative bound is complete.

**Remark**. It is evident from the proof above that somewhat weaker inequalities than (3.92) would suffice to handle the fully nonlinear boundary condition (1.23) with $\phi$ satisfying (1.27), or the boundary condition (1.29). All we really need in these cases are the estimates

$$D_{22\tilde{\beta}}w(y_0) \leq \epsilon M^2 + C(\epsilon) \tag{3.128}$$

for $\epsilon > 0$ sufficiently small and

$$D_{222}w(y_0) \leq C(1 + M^2)\,. \tag{3.129}$$

As we have already remarked, these inequalities can be achieved with a simpler choice of $\mathcal{X}$ than (3.68).

The hypotheses (1.7) and $g > 0$, and the second derivative bound, imply that the eigenvalues of $\mathcal{A}(u)$ lie in a fixed compact subset of $\Sigma$. Thus (1.4) and (1.5) imply that equation (1.1) is uniformly elliptic. Consequently, we can apply the theory developed in [10, 11] to deduce a second derivative Hölder estimate

$$[D^2u]_{\alpha;\Omega} \leq C \tag{3.130}$$

for some $\alpha > 0$. As noted in [11] (see also [10], Theorem 3), the estimate (3.130) can be proved much more easily in two dimensions than in higher dimensions, and in the two-variable case (3.130) is in fact valid under our somewhat weaker regularity hypotheses. In particular, at this stage we require only $\partial\Omega \in C^{2,\epsilon}$ for some $\epsilon > 0$. The proofs of Theorems 1.1 to 1.5 for the case of $C^\infty$ data are now complete. However, once we have this result, the above estimates and a standard approximation argument (approximating $f, g, h, \beta, \phi, \Omega$ and $\Omega^*$ by smooth functions and domains satisfying the appropriate structure conditions of Section 1 and having uniform bounds in suitable norms) lead to Theorems 1.1 to 1.5 under our regularity hypotheses. We note that it is clearly possible to carry out this procedure in such a way that the subsolution $\underline{u}$ in Theorems 1.1 to 1.4 is also a subsolution of the approximating equations. With these observations our proofs of Theorems 1.1 to 1.5 are complete.

**4. Extensions.** In this section we indicate how our results may be extended in various directions. Most of these are similar to those for Hessian equations in [18], so we shall not provide detailed proofs. The possible variants are numerous, so we leave it to the reader to formulate precise statements of theorems.

**(i) More general equations.** By making minor modifications to our arguments we can allow $g$ to depend on $Du$ in Theorems 1.1, 1.2, 1.4 and 1.5, but not in Theorem 1.3. In the estimation of $D_{\beta\beta}u$ on $\partial\Omega$ we need to replace $G_i$ by $\tilde{G}_i = G_i - g_{p_i}(x, u, Du)$ and subtract $g_{p_i}(b_{x_i} + b_z D_i u)$ from the right-hand side of (3.12). This causes no difficulties in the subsequent barrier argument.

When we come to the interior estimation, however, we get some additional terms from $D_{22}\tilde{g}(y, w, Dw)$ in (3.60), the main ones being $\tilde{g}_{p_i}D_{i22}w$ and $\tilde{g}_{p_ip_j}D_{i2}wD_{j2}w$. The first of these is equal to $-\tilde{g}_{p_i}D_i\mathcal{X}D_{22}w$ at $y_0$, by virtue of (3.56). Thus, in place of $C\kappa_2$ on the right side of (3.78) we now get a quadratic term $C\kappa_2^2$, unless we happen to know that $\tilde{g}$ is convex with respect to $Dw$; this, however, is not apparent even if $g$ is convex with respect to $Du$, since $Du$ and $Dw$ are not related in a linear fashion (see (2.28)). To control this quadratic term we need to strengthen condition (1.10) on $f$. It suffices to assume

$$\mathcal{T}|\kappa|^{-1} \geq \sigma \quad \text{as} \quad |\kappa| \to \infty \quad \text{on} \quad \Sigma_\mu \tag{4.1}$$

for any $\mu > 0$ and some positive constant $\sigma = \sigma(f, \mu)$.

In addition, when we come to the estimate (3.118), we see that $\tilde{g}_{y_i}$ needs to be replaced by $\tilde{g}_{y_i} + \tilde{g}_{p_k}D_{ik}w$. However, the last line of (3.118) can be estimated by $C(\epsilon) + \epsilon M$ for any $\epsilon > 0$, for $D_{22}w(y_0)$ large enough, depending on $\epsilon$, by virtue of (1.10) (see Lemma 2.1(v)). This suffices to complete the proof.

For the Gauss curvature equation

$$\frac{\det D^2u}{(1 + |Du|^2)^2} = g(x) \tag{4.2}$$

(for which this paper gives no better results than those already obtained in [18]) the hypothesis concerning the existence of a convex subsolution can be replaced by the condition

$$\int_\Omega g < \pi \tag{4.3}$$

(see [12, 18]), as this leads to a bound for $\sup_\Omega |u|$ under the hypotheses of Section 1. A similar remark can also be made for other curvature equations, if $f$ has a structure which allows a suitable comparison between $f(\kappa)$ and $\sqrt{\kappa_1\kappa_2}$.

**(ii) Monotonicity assumptions.** Once we have the basic existence theorems stated in Section 1 we can obtain existence results without the monotonicity assumptions (1.12) on $g$ and (1.24) on $\phi$ for the fully nonlinear boundary condition (1.23). This is achieved by using the Leray–Schauder fixed-point theorem as was done for Hessian equations in [18]. We now need to strengthen our hypotheses on the subsolution $\underline{u}$ : it should now be a subsolution of the boundary value problem in question; i.e.,

$$F[\underline{u}] \geq g(x,u) \quad \text{in} \quad \Omega\,, \tag{4.4}$$

$$b(x,\underline{u},D\underline{u}) \geq 0 \quad \text{on} \quad \partial\Omega\,, \tag{4.5}$$

and we obtain an admissible solution $u \in C^\infty(\bar\Omega)$ with $u \geq \underline{u}$ in $\Omega$ (cf. [18], Theorem 1.5). The hypotheses (1.16) and (1.26) now become redundant.

We cannot remove the monotonicity assumption (1.14) on $\phi$ for the semilinear boundary condition because it is used to prove the crucial estimate (3.67). If, however, we can somehow find a suitable function $\widetilde{\mathcal{X}}$ to replace $v-u$ in (3.68) (see the remarks following (3.79)), then (1.14) can be dropped, provided of course that we can find an admissible subsolution of the boundary value problem (1.1), (1.13).

**(iii) Degenerate equations.** If we assume that $g$ is nonnegative rather than positive, or if conditions (1.4) and (1.5) are replaced by (2.1) (which in fact follows automatically from the positivity and concavity of $f$), then (1.1) becomes a degenerate elliptic equation on admissible solutions. We can deduce the existence of admissible solutions in $C^{1,1}(\bar\Omega)$ by straightforward approximation arguments similar to those described in the Hessian case ([18], Section 5), whenever we can obtain estimates in $C^2(\bar\Omega)$ for admissible solutions of a suitable sequence of nondegenerate approximating problems. Admissibility should now be interpreted as meaning that the principal curvatures $(\kappa_1,\kappa_2)$ belong to $\bar\Sigma$ wherever they are defined. An examination of the proof above shows that the conditions (1.4) and (1.5) are used only to get the estimate (3.130) once the second derivatives are already bounded. We can therefore get degenerate analogues, in which (1.4), (1.5) are replaced by (2.1), of each of the theorems of Section 1; all we need to know is that we can find uniform estimates in $C^0(\bar\Omega)$ for solutions of suitable approximating problems—this is the case, for example, if we can find an admissible function $\underline{u} \in C^2(\Omega) \cap C^1(\bar\Omega)$ which is a strict subsolution of (1.1); i.e., (4.4) holds with strict inequality.

The positivity of $g$, on the other hand, plays a more important role in the second derivative estimation. A positive lower bound on $f$ is required in Lemma 2.3, and so enters into the estimation of $D_{\beta\beta}u$ on $\partial\Omega$ if $\Sigma \neq \Gamma_+$. More importantly, it is used in the construction of the function $v$ in (3.68) and to obtain (3.65), which in turn is used to get (3.75). Consequently, we cannot weaken the positivity of $g$ to obtain a degenerate analogue of Theorem 1.1, unless we can find a function $\widetilde{\mathcal{X}}$ as indicated in the remarks following (3.79).

The positivity of $g$ is also used in Theorem 1.5 to derive the obliqueness estimate (1.30).

**(iv) Hessian equations.** Some of the ideas used in this paper can be used to improve some of the results of [18]. For the Hessian equation

$$F(D^2u) = g(x, u, Du) \quad \text{in} \quad \Omega \tag{4.6}$$

we can proceed similarly to the curvature case and consider the auxiliary function

$$w = e^{\alpha(v-u)} D_{\xi\xi} u\,, \tag{4.7}$$

where $v$ is obtained by solving (3.63) for a suitable $\delta_0 > 0$ (as usual, we replace $v - u$ by $|x|^2$ if we are considering a fully nonlinear boundary condition). We then obtain a bound for $D^2u(x_0)$ in the case that $w$ has its maximum at an interior point $x_0$ under the assumption (4.1) on $f$ in place of the stronger condition

$$\mathcal{T}|\kappa|^{-1} \to \infty \quad \text{as} \quad |\kappa| \to \infty \quad \text{on} \quad \Sigma_\mu \tag{4.8}$$

for any $\mu > 0$, which was required in [18]. The key point is that by handling the third derivative terms with the aid of (2.19), as in the curvature case, we are free to choose $\alpha$ as large as we wish. This was done in [17, 18] for the special case of the Monge–Ampère equation.

Finally, we are able to obtain a result analogous to Theorem 1.2 for Hessian equations, i.e., an existence result for (4.6) subject to the semilinear boundary condition (1.13) without requiring the structure condition (1.18). For convenience we assume that $\Sigma = \Gamma_+$. We consider the auxiliary function (4.7) as above, where $v = v_\rho \in C^2(\bar{\Omega})$ is a uniformly convex function such that

$$\begin{aligned} F(D^2v) &\leq \tfrac{1}{2}\,\delta_0 \quad \text{in} \quad \Omega\,, \\ D_\beta(v + \rho\psi) + \phi(x, v + \rho\psi) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.9}$$

for some $\rho \in (0, 1)$ to be chosen, where $\psi$ is as in (3.63) and $\delta_0 = \min_\Omega g(x, u, Du)$. To find such a $v$ we solve the problem

$$\begin{aligned} F(K\mathcal{A}[v]) &= \epsilon_0 \quad \text{in} \quad \Omega\,, \\ D_\beta(v + \rho\psi) + \phi(x, v + \rho\psi) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.10}$$

for a suitable choice of $\epsilon_0 \in (0, \frac{1}{2}\delta_0)$ and $K \geq 1$. First we observe that for $R$ fixed so large that $\Omega \subset B_{R/2}(0)$ we may choose $\epsilon_0$ so small that $\underline{v}(x) = -\sqrt{R^2 - |x|^2}$ is a convex subsolution of $F[v] = \epsilon_0$, and consequently also of $F(K\mathcal{A}[v]) = \epsilon_0$ for any $K \geq 1$. Thus by Theorem 1.2, (4.10) has a unique convex solution $v = v_\rho \in C^2(\bar{\Omega})$ for any $K \geq 1$ and any $\rho \in (0,1)$. Furthermore, from the proof of the $C^1$ estimates in [18] (the arguments are the same for curvature equations) and the second derivative bounds proved above, we have

$$\|v\|_{C^1(\bar{\Omega})} \leq C_1\,, \tag{4.11}$$

with $C_1$ independent of $\epsilon_0$, $K$ and $\rho$, and

$$\sup_\Omega |D^2 v| \leq \Lambda(\epsilon_0, K)\,, \tag{4.12}$$

$$D^2 v \geq \lambda_0(\epsilon_0, K) I\,, \tag{4.13}$$

for some positive constants $\Lambda$ and $\lambda_0$, independent of $\rho \in (0,1)$. Since (4.11) holds, at each point of $\Omega$ we have

$$\lambda_i \leq C_2 \kappa_i \quad \text{for} \quad i = 1, 2\,,$$

for some positive constant $C_2$ depending only on $\sup_\Omega |Du|$, where $\lambda_1 \leq \lambda_2$ and $\kappa_1 \leq \kappa_2$ are the eigenvalues of $D^2 v$ and $\mathcal{A}[v]$ respectively. If we now fix $K = C_2$ we obtain

$$F(D^2 v) \leq F(K\mathcal{A}[v]) \leq \tfrac{1}{2}\,\delta_0\,,$$

as required.

We can now proceed to fix $\rho$ as before, and we get the crucial estimate (3.67). Notice that we still need condition (1.10) to solve (4.10), unless the curvature of $\partial\Omega$ is sufficiently large. We also need (4.1) if $g$ depends on $Du$, although this can be weakened to (1.10) if $g$ is convex with respect to $Du$. It is curious that this result for Hessian equations is a consequence of our results for curvature equations—we do not have a direct proof.

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