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BIFURCATION FOR NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS III

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Abstract. This paper is devoted to global static bifurcation theory for a class of *degenerate* boundary value problems for nonlinear second-order elliptic differential operators which includes as particular cases the Dirichlet and Neumann problems. In the previous paper [13] we treated the asymptotic linear case, for example, such nonlinear terms as u^p, $p > 1$, near $u = 0$ but $u + 1/u$ near $u = +\infty$. The purpose of this paper is to study the asymptotic nonlinear case, for example, such nonlinear terms as u^p also near $u = +\infty$. First we prove a general existence and uniqueness theorem of positive solutions for our nonlinear boundary value problems, by using the super-subsolution method, and then we study in great detail the asymptotic nonlinear case.

0. Introduction and results. Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with C^∞ boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact C^∞ manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real C^∞ coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$, $x \in \overline{D}$, $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \overline{D},\ \xi \in \mathbf{R}^N.$$

(2) $c \geq 0$ on $\overline{D}$.

In this paper we consider the following nonlinear elliptic boundary value problem: Given a function $f(x, \xi)$ defined on $\overline{D} \times [0, \infty)$, find a nonnegative function u in D such that

$$\begin{cases} Au = f(x, u) & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases} \tag{$*$}$$

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Here

(1) $a \in C^\infty(\partial D)$ and $a \geq 0$ on ∂D.

(2) $b \in C^\infty(\partial D)$ and $b \geq 0$ on ∂D.

(3) $\partial/\partial\nu$ is the conormal derivative associated with the operator A:

$$\partial/\partial\nu = \sum_{i,j=1}^{N} a^{ij} n_j \partial/\partial x_i,$$

where $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit exterior normal to the boundary ∂D.

A solution $u \in C^2(\overline{D})$ of problem $(*)$ is said to be *nontrivial* if it does not identically equal zero on $\overline{D}$. We call a nontrivial solution u of problem $(*)$ a *positive solution* if $u(x) \geq 0$ on $\overline{D}$.

A nonnegative function $\psi \in C^2(\overline{D})$ is called a *supersolution* of problem $(*)$ if it satisfies the conditions

$$\begin{cases} A\psi - f(x,\psi) \geq 0 & \text{in } D, \\ B\psi \geq 0 & \text{on } \partial D. \end{cases}$$

Similarly, a nonnegative function $\phi \in C^2(\overline{D})$ is called a *subsolution* of problem $(*)$ if it satisfies the conditions

$$\begin{cases} A\phi - f(x,\phi) \leq 0 & \text{in } D, \\ B\phi \leq 0 & \text{on } \partial D. \end{cases}$$

In this paper we study problem $(*)$ under the following two conditions on the functions a, b and c:

$$b > 0 \text{ on } M = \{x' \in \partial D : a(x') = 0\}. \tag{H.1}$$

$$c > 0 \text{ in } D. \tag{H.2}$$

Intuitively condition (H.1) implies that, at any point of the set M where no reflection phenomenon occurs, a Markovian particle may "disappear."

First, in order to state our existence theorem of positive solutions of problem $(*)$, we introduce a fundamental condition (*slope condition*) on the nonlinear term $f(x,\xi)$:

For a positive number σ, there exists a constant $\omega = \omega(\sigma) > 0$, independent of $x \in \overline{D}$, such that

$$f(x,\xi) - f(x,\eta) > -\omega\,(\xi - \eta)\,, \quad x \in \overline{D},\ 0 \leq \eta < \xi \leq \sigma. \tag{R$_\sigma$}$$

Now we can state our existence theorem for problem $(*)$ which is a generalization of [1, Theorem 9.4] to the degenerate case:

Theorem 1. *Assume that $f(x,\xi)$ belongs to the Hölder space $C^\theta(\overline{D}\times[0,\sigma])$, $0 < \theta < 1$, and satisfies condition $(R)_\sigma$ for some $\sigma > 0$. If ψ and ϕ are respectively super- and subsolutions of problem $(*)$ satisfying $0 \leq \phi(x) \leq \psi(x) \leq \sigma$ on $\overline{D}$, then there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem $(*)$ such that $\phi(x) \leq u(x) \leq \psi(x)$ on $\overline{D}$.*

Secondly, in order to state our uniqueness theorem of positive solutions of problem $(*)$, we introduce another fundamental condition (*sublinearity*) on the nonlinear term $f(x,\xi)$:

We have for all $0 < \tau < 1$

$$f(x, \tau\xi) \geq \tau f(x, \xi), \quad x \in \overline{D},\ \xi > 0, \tag{S1}$$

and

$$f(x, 0) \geq 0, \quad x \in \overline{D}. \tag{S2}$$

Our uniqueness theorem for problem $(*)$ is stated as follows:

Theorem 2. *Assume that $f(x,\xi)$ belongs to the Hölder space $C^\theta(\overline{D} \times [0,\sigma])$, $0 < \theta < 1$, and satisfies condition $(R)_\sigma$ for every $\sigma > 0$, and also satisfies condition (S). Then problem $(*)$ has at most one positive solution.*

Finally, as an application of Theorems 1 and 2, we consider the following semilinear elliptic boundary value problem:

$$\begin{cases} Au - \lambda u + h(x)\, u^p = 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases} \tag{$**$}$$

where $p > 1$, λ is a real parameter and $h(x)$ is a real-valued function on $\overline{D}$. It is worth pointing out here that the equation $Au - \lambda u + h(x)\, u^p = 0$ is related to the so-called Yamabe problem which is a basic problem in Riemannian geometry if we take $p = (N+2)/(N-2) > 1$ where $N \geq 3$ (cf. [7], [8]).

We linearize problem $(**)$ and introduce an unbounded linear operator $\mathcal{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathcal{A})$ of $\mathcal{A}$ is the space

$$D(\mathcal{A}) = \left\{ u \in H^{2,2}(D) : Bu = 0 \text{ on } \partial D \right\}.$$

(b) $\mathcal{A}u = Au$, $u \in D(\mathcal{A})$.

It is known (see [12, Theorem 1]) that the first eigenvalue λ_1 of $\mathcal{A}$ is positive and *simple* and the associated eigenfunction is positive everywhere in D.

Assume that $h(x)$ is a function in the Hölder space $C^\theta(\overline{D})$, $0 < \theta < 1$, such that

$$h(x) \geq 0 \quad \text{on } \overline{D}.$$

We let

$$\mathcal{D}_+(h) = \{x \in D : h(x) > 0\},$$

and

$$\mathcal{D}_0(h) = D \setminus \overline{\mathcal{D}_+(h)}.$$

We consider the case when $h > 0$ near the boundary ∂D. Our fundamental hypothesis on the function h is the following:

$$\begin{cases} \text{The open set } \mathcal{D}_0(h) \text{ consists of a finite number of connected} \\ \text{components } \mathcal{D}_i(h),\ 1 \leq i \leq \ell, \text{ with piecewise-smooth boundary} \\ \partial\mathcal{D}_i(h), \text{ which are bounded away from the boundary } \partial D. \end{cases} \tag{Z}$$

We consider the Dirichlet eigenvalue problem in each connected component $\mathcal{D}_i(h)$:

$$\begin{cases} A\varphi = \lambda\varphi & \text{in } \mathcal{D}_i(h), \\ \varphi = 0 & \text{on } \partial\mathcal{D}_i(h). \end{cases} \tag{D$_i$}$$

We let

$$\lambda_1(\mathcal{D}_i(h)) = \text{ the first eigenvalue of problem } (D_i),$$

and

$$\widetilde{\lambda}_1(\mathcal{D}_0(h)) = \min_{1\le i\le \ell} \lambda_1(\mathcal{D}_i(h)).$$

The next theorem is a generalization of [8, Theorems 2 and 3] to the degenerate case:

Theorem 3. *Assume that $h(x)$ belongs to the Hölder space $C^\theta(\overline{D})$, $0 < \theta < 1$, such that $h(x) \ge 0$ on $\overline{D}$ and that condition (Z) is satisfied. Then problem $(**)$ has a unique positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda \in (\lambda_1, \widetilde{\lambda}_1(\mathcal{D}_0(h)))$. For any $\lambda \ge \widetilde{\lambda}_1(\mathcal{D}_0(h))$, there exists no positive solution of problem $(**)$. Furthermore, the uniform norm $\|u(\lambda)\|_{C(\overline{D})}$ tends to ∞ as $\lambda \to \widetilde{\lambda}_1(\mathcal{D}_0(h))$.*

Remark. It is known (cf. [4]) that the minimum eigenvalue $\widetilde{\lambda}_1(\mathcal{D}_0(h))$ is monotone decreasing with respect to the set $\mathcal{D}_0(h)$. More precisely, it tends to ∞ if $\mathcal{D}_0(h) \to \emptyset$ and tends to $\lambda_1(D)$ if $\mathcal{D}_0(h) \to D$, where $\lambda_1(D)$ is the first eigenvalue of the Dirichlet problem in the whole domain D. In the particular case when $\mathcal{D}_0(h) = \emptyset$, we have $\widetilde{\lambda}_1(\mathcal{D}_0(h)) = \infty$ (cf. [8, Theorem 2]).

The rest of this paper is organized as follows. In Section 1 we prove an existence and uniqueness theorem for the linearized boundary value problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = \varphi & \text{on } \partial D \end{cases} \tag{$\dagger$}$$

in the framework of *Hölder spaces* (Theorem 1.1) which plays an important role in the proof of Theorems 1 and 2. We reduce the study of problem (†) to that of a first-order pseudo-differential operator on the boundary (Proposition 1.2), just as in [11]. In Section 2 we study problem $(*)$ and prove Theorems 1 and 2. First, problem $(*)$ is reduced to a nonlinear operator equation for the resolvent K of problem (†) in the space $C(\overline{D})$ (equation (2.3)). This operator equation is solved by using Schauder's fixed-point theorem, although it may be solved by an iteration method as in [1]. The essential step in the proof of Theorem 1 is Lemma 2.4 which allows us to formulate the operator equation as a mapping of a bounded, closed and convex set into itself in the space $C(\overline{D})$. The proof of Theorem 2 is based on a uniqueness theorem of fixed points of strongly increasing and strongly sublinear mappings in ordered Banach spaces (Theorem 2.5) due to Amann ([1]). Proposition 2.2 due to [13], where the strong positivity of the resolvent K is proved, plays an important role in the proof. Section 3 is devoted to the proof of Theorem 3. Theorem 3 follows from an application of Theorems 1 and 2. More precisely, by using the implicit function theorem, we can find a critical value

$\overline{\lambda}(h) \in (\lambda_1, \infty]$ such that there exists a positive solution $u(\lambda)$ of problem $(**)$ for all $\lambda \in (\lambda_1, \overline{\lambda}(h))$ (Lemma 3.2). The formula: $\overline{\lambda}(h) = \widetilde{\lambda}_1(\mathcal{D}_0(h))$ is proved by making use of Theorem 1 if we construct super- and subsolutions of problem $(**)$ for every $\lambda \in (\lambda_1(D), \widetilde{\lambda}_1(\mathcal{D}_0(h)))$ (Lemma 3.5).

1. Existence and uniqueness theorem for problem (†)**.** In this section, we prove an existence and uniqueness theorem for the linearized boundary value problem (†) in the framework of Hölder spaces, which will play an important role in the proof of Theorems 1, 2 and 3.

First we introduce a subspace of the Hölder space $C^{1+\theta}(\partial D)$, $0 < \theta < 1$, which is associated with the boundary condition

$$Bu = a\,\frac{\partial u}{\partial \nu} + bu$$

in the following way: We let

$$C_*^{1+\theta}(\partial D) = \left\{\varphi = a\varphi_1 + b\varphi_2 :\ \varphi_1 \in C^{1+\theta}(\partial D),\ \varphi_2 \in C^{2+\theta}(\partial D)\right\},$$

and define a norm

$$|\varphi|_{C_*^{1+\theta}(\partial D)} = \inf\left\{|\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} :\ \varphi = a\varphi_1 + b\varphi_2\right\}.$$

Then it is easy to verify that the space $C_*^{1+\theta}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C_*^{1+\theta}(\partial D)}$. We remark that the space $C_*^{1+\theta}(\partial D)$ is an "interpolation space" between the spaces $C^{2+\theta}(\partial D)$ and $C^{1+\theta}(\partial D)$.

The purpose of this section is to prove the following:

Theorem 1.1. *If hypotheses* (H.1) *and* (H.2) *are satisfied, then the mapping*

$$(A, B) : C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \oplus C_*^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism for all $0 < \theta < 1$.

Proof. The proof is divided into four steps.

(i) Let g be an arbitrary element of $C^{\theta}(\overline{D})$, and φ an arbitrary element of $C_*^{1+\theta}(\partial D)$ such that

$$\varphi = a\varphi_1 + b\varphi_2, \quad \varphi_1 \in C^{1+\theta}(\partial D),\ \varphi_2 \in C^{2+\theta}(\partial D).$$

First we show that the boundary value problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = \varphi & \text{on } \partial D \end{cases} \tag{†}$$

can be reduced to the study of an operator on the boundary.

To do so, we consider the Neumann problem

$$\begin{cases} Av = g & \text{in } D, \\ \frac{\partial v}{\partial \nu} = \varphi_1 & \text{on } \partial D. \end{cases} \tag{N}$$

By [5, Theorem 6.31], one can find a unique solution v in the space $C^{2+\theta}(\overline{D})$ of problem (N). Then it is easy to see that a function u in $C^{2+\theta}(\overline{D})$ is a solution of problem (†) if and only if the function $w = u - v \in C^{2+\theta}(\overline{D})$ is a solution of the problem

$$\begin{cases} Aw = 0 & \text{in } D, \\ Bw = \varphi - Bv & \text{on } \partial D. \end{cases}$$

Here we remark that

$$Bv = a \frac{\partial v}{\partial \nu} + bv = a\varphi_1 + bv,$$

so that

$$Bw = \varphi - Bv = b(\varphi_2 - v) \in C^{2+\theta}(\partial D).$$

But we know that every solution $w \in C^{2+\theta}(\overline{D})$ of the homogeneous equation $Aw = 0$ in D can be expressed as follows (cf. [11, Theorem 4.3]):

$$w = \mathcal{P}\psi, \quad \psi \in C^{2+\theta}(\partial D).$$

Here the operator $\mathcal{P} : C^{2+\theta}(\partial D) \longrightarrow C^{2+\theta}(\overline{D})$ is the Poisson operator; that is, the function $w = \mathcal{P}\psi$ is the unique solution of the Dirichlet problem

$$\begin{cases} Aw = 0 & \text{in } D, \\ w = \psi & \text{on } \partial D. \end{cases}$$

Thus we have the following:

Proposition 1.2. *For given functions $g \in C^{\theta}(\overline{D})$ and $\varphi = a\varphi_1 + b\varphi_2 \in C^{1+\theta}_{*}(\partial D)$, there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem (†) if and only if there exists a solution $\psi \in C^{2+\theta}(\partial D)$ of the equation*

$$T\psi := B\mathcal{P}\psi = b(\varphi_2 - v) \quad \text{on } \partial D. \tag{‡}$$

Furthermore the solutions u and ψ are related as follows:

$$u = v + \mathcal{P}\psi,$$

where $v \in C^{2+\theta}(\overline{D})$ is the unique solution of problem (N).

(ii) We study the operator T in question. It is known (cf. [6, Chapter XX], [9, Chapter 3]) that the operator

$$T\psi = B\mathcal{P}\psi = a \frac{\partial}{\partial \nu} (\mathcal{P}\psi) + b\psi$$

is a first-order, *pseudo-differential operator* on the boundary ∂D.

The next proposition is an essential step in the proof of Theorem 1.1:

Lemma 1.3. *If hypothesis* (H.1) *is satisfied, then there exists a parametrix E in the Hörmander class $L^0_{1,1/2}(\partial D)$ for T which maps $C^{k+\theta}(\partial D)$ continuously into itself for any nonnegative integer k.*

Proof. By making use of [6, Theorem 22.1.3] just as in the proof of [11, Lemma 5.2], one can construct a parametrix $E \in L^0_{1,1/2}(\partial D)$ for T:

$$ET \equiv TE \equiv I \mod L^{-\infty}(\partial D).$$

The boundedness of $E : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$ follows from an application of a Besov-space boundedness theorem due to Bourdaud ([3, Theorem 1]), since we have $C^{k+\theta}(\partial D) = B^{k+\theta}_{\infty,\infty}(\partial D)$. □

(iii) We consider problem (†) in the framework of Sobolev spaces of L^p style, and prove an L^p version of Theorem 1.1.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$$\begin{aligned} H^{k,p}(D) = &\text{the space of functions } u \in L^p(D) \text{ such that} \\ &D^\alpha u \in L^p(D) \text{ for all } |\alpha| \le k, \end{aligned}$$

and let

$$\begin{aligned} B^{k-1/p,p}(\partial D) = &\text{the space of the boundary values } \varphi \text{ of functions} \\ &u \in H^{k,p}(D). \end{aligned}$$

In the space $B^{k-1/p,p}(\partial D)$ we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf\left\{\|u\|_{H^{k,p}(D)} : u \in H^{k,p}(D),\ u|_{\partial D} = \varphi\right\}.$$

The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$ (cf. [2], [14]).

We introduce a subspace of $B^{1-1/p,p}(\partial D)$ which is an L^p version of $C^{1+\theta}_*(\partial D)$. We let

$$B^{1-1/p,p}_*(\partial D) = \left\{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in B^{1-1/p,p}(\partial D),\ \varphi_2 \in B^{2-1/p,p}(\partial D)\right\},$$

and define a norm

$$|\varphi|_{B^{1-1/p,p}_*(\partial D)} = \inf\left\{|\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)} :\ \varphi = a\varphi_1 + b\varphi_2\right\}.$$

It is easy to verify that the space $B^{1-1/p,p}_*(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B^{1-1/p,p}_*(\partial D)}$.

Then we can obtain the following L^p version of Theorem 1.1 (cf. [15, Theorem 1]):

Theorem 1.4. *If hypotheses* (H.1) *and* (H.2) *are satisfied, then the mapping*

$$(A, B) : H^{2,p}(D) \longrightarrow L^p(D) \oplus B_*^{1-1/p,p}(\partial D)$$

is an algebraic and topological isomorphism for all $1 < p < \infty$.

(iv) Now we remark that

$$\begin{cases} C^\theta(\overline{D}) \subset L^p(D), \\ C_*^{1+\theta}(\partial D) \subset B_*^{1-1/p,p}(\partial D). \end{cases}$$

Thus we find from Theorem 1.4 that problem (†) has a unique solution $u \in H^{2,p}(D)$ for any $g \in C^\theta(\overline{D})$ and any $\varphi = a\varphi_1 + b\varphi_2 \in C_*^{1+\theta}(\partial D)$. Furthermore, by virtue of Proposition 1.2, it follows that the solution u can be written in the form

$$u = v + \mathcal{P}\psi, \quad v \in C^{2+\theta}(\overline{D}),\ \psi \in B^{2-1/p,p}(\partial D).$$

But Lemma 1.3 tells us that

$$\psi \in C^{2+\theta}(\partial D),$$

since we have by equation (‡)

$$\psi \equiv E(T\psi) = E\left(b(\varphi_2 - v)\right) \quad \text{mod } C^\infty(\partial D).$$

Therefore we obtain that

$$u = v + \mathcal{P}\psi \in C^{2+\theta}(\overline{D}).$$

The proof of Theorem 1.1 is complete. □

2. Proof of Theorems 1 and 2. Theorems 1 and 2 are proved by using the theory of positive mappings in ordered Banach spaces due to Amann ([1]).

2.1. Reduction to an operator equation. First we let

$$C_B^{2+\theta}(\overline{D}) = \{u \in C^{2+\theta}(\overline{D}) : Bu = 0 \text{ on } \partial D\}.$$

By Theorem 1.1, we can introduce a continuous linear operator

$$K : C^\theta(\overline{D}) \longrightarrow C_B^{2+\theta}(\overline{D})$$

as follows: For any $v \in C^\theta(\overline{D})$, the function $u = Kv \in C^{2+\theta}(\overline{D})$ is the unique solution of the problem

$$\begin{cases} Au = v & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

Then it follows that the spaces $C_B^{2+\theta}(\overline{D})$ and $C^\theta(\overline{D})$ are isomorphic in such a way that

$$C_B^{2+\theta}(\overline{D}) \xrightarrow{A} C^\theta(\overline{D});$$

$$C_B^{2+\theta}(\overline{D}) \underset{K}{\longleftarrow} C^\theta(\overline{D}).$$

Since $f \in C^\theta(\overline{D} \times [0, \sigma])$ and $u \in C^\theta(\overline{D})$, we find that problem $(*)$ is equivalent to the following operator equation:

$$u = K(F(u)) \quad \text{in } C^\theta(\overline{D}), \tag{2.2}$$

where F is the Nemytskii operator of f defined by the formula

$$Fu(x) = f(x, u(x)), \quad x \in \overline{D}.$$

Furthermore, by using Theorem 1.4, we can extend uniquely the operator K to a *compact* linear operator

$$K : C(\overline{D}) \longrightarrow C^1(\overline{D}),$$

since $C(\overline{D}) \subset L^p(D)$. Indeed, it follows from an application of Sobolev's imbedding theorem that $H^{2,p}(D)$ is continuously imbedded into $C^{2-N/p}(\overline{D})$ for all $N < p < \infty$; hence the compactness of K is an immediate consequence of the Ascoli and Arzelà theorem. Then it is easy to see that u is a solution of operator equation (2.2) if and only if it satisfies the operator equation

$$u = K(F(u)) \quad \text{in } C(\overline{D}). \tag{2.3}$$

Summing up, we have proved that problem $(*)$ is equivalent to operator equation (2.3).

2.2. Theory of positive mappings in ordered Banach spaces. We shall make use of the theory of positive operators in ordered Banach spaces to study positive solutions of equation (2.3).

An ordering $\leq$ in an ordered set X is said to be *linear* if the following two conditions are satisfied:

(i) If x, $y \in X$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in X$.

(ii) If x, $y \in X$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A Banach space X with a linear ordering $\leq$ is called an *ordered Banach space* if the set $Q = \{x \in X : x \geq 0\}$ is closed in X. The set Q is called the *positive cone* of the linear ordering $\leq$.

For two functions u, $v \in C(\overline{D})$, we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \overline{D}$. Then it is easy to verify that the space $C(\overline{D})$ is an ordered Banach space with the linear ordering $\leq$ and the positive cone

$$P = \{u \in C(\overline{D}) : u \geq 0 \text{ on } \overline{D}\}.$$

The next lemma ([13, Lemma 2.1]) will play an important role in the proof of Theorem 1 in Subsection 2.3:

Lemma 2.1. *Assume that hypotheses* (H.1) *and* (H.2) *are satisfied. If* $v \in C^{\theta}(\overline{D})$ *with* $0 < \theta < 1$ *and if* $v \geq 0$ *but* $v \not\equiv 0$ *on* $\overline{D}$, *then the function* $u = Kv \in C^{2+\theta}(\overline{D})$ *satisfies the following conditions:*

(1) $u = 0$ *on* $M = \{x' \in \partial D : a(x') = 0\}$.
(2) $u > 0$ *on* $\overline{D} \setminus M$.
(3) *For the conormal derivative* $\partial u/\partial \nu$ *of* u, *we have*

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } M.$$

Furthermore the operator $K : C(\overline{D}) \to C(\overline{D})$ *is* positive*; that is,* $K(P) \subset P$.

Now we introduce an ordered Banach subspace of $C(\overline{D})$ which is associated with the operator $K : C(\overline{D}) \to C^1(\overline{D})$.

If we let

$$e = K1,$$

it follows from an application of Lemma 2.1 that the function $e \in C^{2+\theta}(\overline{D})$ satisfies the conditions

$$\begin{cases} e > 0 & \text{on } \overline{D} \setminus M, \\ e = 0 & \text{on } M, \\ \frac{\partial e}{\partial \nu} < 0 & \text{on } M. \end{cases}$$

Moreover we let

$$C_e(\overline{D}) = \{u \in C(\overline{D}) : \text{ there is a constant } c > 0 \text{ such that } -ce \leq u \leq ce\}.$$

Then the space $C_e(\overline{D})$ is given a norm by the formula

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

If we let

$$P_e = C_e(\overline{D}) \cap P,$$

it is easy to verify that the space $C_e(\overline{D})$ is an ordered Banach space having the positive cone P_e with nonempty interior.

The next proposition (see [13, Proposition 2.2]) will play an important role in the proof of Theorem 2 in Subsection 2.4:

Proposition 2.2. *The operator* K *maps* $C(\overline{D})$ *compactly into* $C_e(\overline{D})$. *Moreover,* K *is strongly positive; that is, if* $v \in P$ *and* $v \not\equiv 0$ *on* $\overline{D}$, *then the function* Kv *is an interior point of* P_e.

2.3. Proof of Theorem 1. (1) First we replace the function $c(x)$ by the function $c(x) + \omega$, where $\omega > 0$ is the same constant as in condition $(\mathrm{R})_\sigma$, and consider instead of problem $(*)$ the following problem:

$$\begin{cases} (A + \omega)u = \omega u + F(u) & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{$*'$}$$

It is clear that problem $(*)$ is equivalent to problem $(*')$. Furthermore, since $f \in C^\theta(\overline{D} \times [0, \sigma])$, it is easy to verify that problem $(*')$ is equivalent to an operator equation

$$u = K_\omega(\omega u + F(u)) \quad \text{in } C(\overline{D}),$$

just as in Subsection 2.1. Here $K_\omega : C(\overline{D}) \to C^1(\overline{D})$ is the compact operator introduced in Subsection 2.1 with c replaced by $c + \omega$.

(2) We let

$$G_\omega(u) = K_\omega(\omega u + F(u)), \quad u \in C(\overline{D}).$$

Then we have the following:

Lemma 2.3. *The operator $G_\omega : [\phi, \psi] \to C(\overline{D})$ is increasing. Here $[\phi, \psi]$ is the order interval defined by the formula*

$$[\phi, \psi] = \{u \in C(\overline{D}) : \phi \le u \le \psi \ \text{on } \overline{D}\}.$$

Proof. Let u and v be arbitrary functions in $C(\overline{D})$ satisfying $\phi \le u \le v \le \psi$ on $\overline{D}$. Then we have

$$\begin{aligned} &\omega\,(v(x) - u(x)) + (Fv(x) - Fu(x)) \\ &= \begin{cases} 0 & \text{if } v(x) = u(x), \\ \left(\omega + \frac{Fv(x) - Fu(x)}{v(x) - u(x)}\right)(v(x) - u(x)) & \text{if } v(x) > u(x), \end{cases} \end{aligned}$$

and so by condition $(\mathrm{R})_\sigma$

$$\omega(v - u) + (Fu - Fv) \ge 0 \quad \text{on } \overline{D}.$$

But Lemma 2.1 tells us that $K_\omega : C(\overline{D}) \to C(\overline{D})$ is positive. Thus it follows that

$$G_\omega(v) - G_\omega(u) = K_\omega\,(\omega(v - u) + (F(v) - F(u))) \ge 0 \quad \text{on } \overline{D},$$

or equivalently,

$$G_\omega(u) \le G_\omega(v) \quad \text{on } \overline{D}.$$

This proves that G_ω is increasing. $\square$

Moreover we have the following:

Lemma 2.4. *The operator G_ω maps the order interval $[\phi, \psi]$ into itself.*

Proof. Let u be an arbitrary function $C(\overline{D})$ satisfying $\phi \le u \le \psi$ on $\overline{D}$. Then it follows from an application of Lemma 2.3 that

$$G_\omega(\phi) \le G_\omega(u) \le G_\omega(\psi) \quad \text{on } \overline{D}.$$

Hence, in order to prove the lemma, it suffices to show that

$$\phi \le G_\omega(\phi), \quad G_\omega(\psi) \le \psi \quad \text{on } \overline{D}.$$

If we let

$$v = G_\omega(\psi) = K_\omega(\omega\psi + F(\psi)),$$

then we have

$$\begin{cases} (A+\omega)v = \omega\psi + F(\psi) & \text{in } D, \\ Bv = 0 & \text{on } \partial D. \end{cases}$$

But, since ψ is a supersolution of problem $(*)$, it follows that

$$\begin{aligned} (A+\omega)(v-\psi) &= \omega\psi + F(\psi) - (A+\omega)\psi \\ &= -(A\psi - F(\psi)) \le 0 \quad \text{in } D, \end{aligned}$$

and

$$B(v-\psi) = -B\psi \le 0 \quad \text{on } \partial D.$$

Thus, using the maximum principle, we find that

$$G_\omega(\psi) = v \le \psi \quad \text{on } \overline{D}.$$

Indeed, if the function $v - \psi$ takes its positive maximum m at an interior point $x_0 \in D$, then we have

$$(A+\omega)(v-\psi)(x_0) \ge \omega m > 0,$$

which contradicts the condition: $(A+\omega)(v-\psi) \le 0$ in D. On the other hand, if $v - \psi$ takes the maximum m at a boundary point $x_0' \in \partial D$, then we have by the boundary-point lemma

$$\frac{\partial}{\partial \nu}(v-\psi)(x_0') > 0,$$

so that by condition (H.1)

$$B(v-\psi)(x_0') = a(x_0')\frac{\partial}{\partial \nu}(v-\psi)(x_0') + b(x_0')m > 0,$$

which contradicts the condition: $B(v-\psi) \le 0$ on ∂D.

Similarly we can prove that

$$\phi \le G_\omega(\phi) \quad \text{on } \overline{D}.$$

The proof of Lemma 2.4 is complete. □

(3) Since $K_\omega : C(\overline{D}) \to C^1(\overline{D})$ is compact, it follows from Lemma 2.4 that the mapping $G_\omega : [\phi, \psi] \to [\phi, \psi]$ is compact. Furthermore the order interval $[\phi, \psi]$ is bounded, closed and convex in the space $C(\overline{D})$. Therefore, applying Schauder's fixed-point theorem (see [10, Proposition 3.60]), we can find a function $u \in [\phi, \psi]$ such that

$$u = G_\omega(u) = K_\omega(\omega u + F(u)) \quad \text{in } C(\overline{D}).$$

Now the proof of Theorem 1 is complete. □

2.4. Proof of Theorem 2. (1) The proof of Theorem 2 is essentially based on the following (see [1, Theorem 24.2]):

Theorem 2.5. *Let (X, Q) be an ordered Banach space having the positive cone Q with nonempty interior. If σ is a positive number, we let*

$$\overline{Q}_\sigma = \{u \in Q : \|u\| \leq \sigma\}.$$

Assume that a mapping $f : \overline{Q}_\sigma \longrightarrow X$ satisfies the following two conditions:

(A) *f is strongly increasing; that is, if u, $v \in \overline{Q}_\sigma$ and if $u \leq v$ and $v \neq u$, then $f(v) - f(u)$ is an interior point of Q.*

(B) *f is strongly sublinear; that is, $f(0) \geq 0$ and if $u \in \overline{Q}_\sigma$ and $u \neq 0$, then $f(\tau u) - \tau f(u)$ is an interior point of Q for every $0 < \tau < 1$.*

Then the mapping f has at most one positive fixed point.

In the proof of Theorem 2, we shall apply Theorem 2.5 with

$$X = C_e(\overline{D}),$$
$$Q = P_e = C_e(\overline{D}) \cap P = \left\{u \in C_e(\overline{D}) : u \geq 0 \text{ on } \overline{D}\right\},$$
$$f = G_\omega.$$

(2) If σ is a positive number, we let

$$\left(\overline{P_e}\right)_\sigma = \{u \in P_e : \|u\|_e \leq \sigma\}.$$

We have only to prove Theorem 2 in the space $\left(\overline{P_e}\right)_\sigma$ for every $\sigma > 0$. Indeed, if u_1 and u_2 are two positive solutions of problem $(*)$, then one can find a constant $\sigma > 0$ such that $\|u_1\|_e$, $\|u_2\|_e \leq \sigma$, so that u_1, $u_2 \in \left(\overline{P_e}\right)_\sigma$.

If we take a constant $\omega = \omega(\sigma) > 0$ given in condition $(\mathrm{R})_\sigma$, then we have the following:

Lemma 2.6. *The operator G_ω maps $\left(\overline{P_e}\right)_\sigma$ into P_e.*

Proof. Let u be an arbitrary function in $\left(\overline{P_e}\right)_\sigma$. Then we have, by condition $(\mathrm{R})_\sigma$ with $\xi = u$ and $\eta = 0$ and condition (S2),

$$F(u) \geq F(0) - \omega u \geq -\omega u \quad \text{on } \overline{D},$$

so that

$$\omega u + F(u) \geq 0 \quad \text{on } \overline{D}.$$

Hence it follows from an application of Proposition 2.2 that

$$G_\omega(u) = K_\omega(\omega u + F(u)) \in P_e. \quad \square$$

Moreover we have the following:

Lemma 2.7. *The operator $G_\omega : \left(\overline{P_e}\right)_\sigma \longrightarrow P_e$ is strongly increasing.*

Proof. Lemma 2.7 follows by combining Lemma 2.3 and Proposition 2.2. $\square$

Lemma 2.8. *The operator* $G_\omega : \left(\overline{P_e}\right)_\sigma \longrightarrow P_e$ *is strongly sublinear.*

Proof. Let u be an arbitrary function in $\left(\overline{P_e}\right)_\sigma$ but $u \neq 0$. Then we have by condition (S)

$$\begin{cases} f(x, \tau u(x)) \geq \tau f(x, u(x)) & \text{if } u(x) > 0, \\ f(x, \tau u(x)) = f(x, 0) \geq 0 & \text{if } u(x) = 0. \end{cases}$$

This implies that

$$\omega \tau u + F(\tau u) - \tau(\omega u + F(u)) = F(\tau u) - \tau F(u) \geq 0 \text{ and } \not\equiv 0 \quad \text{on } \overline{D}.$$

Hence it follows from an application of Proposition 2.2 that the function

$$G_\omega(\tau u) - \tau G_\omega(u) = K_\omega\left(\omega \tau u + F(\tau u) - \tau(\omega u + F(u))\right)$$

is an interior point of P_e. $\square$

(3) Combining Lemmas 2.6, 2.7 and 2.8, we have proved that the mapping $G_\omega : \left(\overline{P_e}\right)_\sigma \to P_e$ satisfies conditions (A) and (B) of Theorem 2.5 with $X = C_e(\overline{D})$ and $Q = P_e$. Therefore, Theorem 2 follows from an application of the same theorem. The proof of Theorem 2 is complete. $\square$

3. Proof of Theorem 3. We let

$$f(x, \xi) = \lambda \xi - h(x)\, \xi^p, \quad x \in \overline{D},\ \xi \geq 0.$$

(i) First it is easy to verify that the function $f(x, \xi)$ satisfies condition $(\mathrm{R})_\sigma$ for every $\sigma > 0$.

Indeed, we have, for all $x \in \overline{D}$ and $0 \leq \eta < \xi \leq \sigma$,

$$\begin{aligned} f(x, \xi) - f(x, \eta) &= \lambda(\xi - \eta) - h(x)\,(\xi^p - \eta^p) \geq \lambda(\xi - \eta) - \max_{\overline{D}} h \cdot (\xi^p - \eta^p) \\ &= \big(\lambda - \max_{\overline{D}} h \cdot \big(\frac{\xi^p - \eta^p}{\xi - \eta}\big)\big)\big(\xi - \eta\big) > \big(\lambda - \max_{\overline{D}} h \cdot p\sigma^{p-1}\big)(\xi - \eta). \end{aligned}$$

Thus, if we take a positive constant

$$\omega = \omega(\sigma, \lambda) = \max\left\{\max_{\overline{D}} h \cdot p\sigma^{p-1} - \lambda, 1\right\},$$

then condition $(\mathrm{R})_\sigma$ is satisfied.

(ii) Secondly we show that the function $f(x, \xi)$ satisfies condition (S).

It is clear that $f(x, 0) = 0$ on $\overline{D}$, which verifies condition (S2). Furthermore, since $h \geq 0$ on $\overline{D}$, we have, for all $x \in \overline{D}$, $\xi > 0$ and $0 < \tau < 1$,

$$\begin{aligned} f(x, \tau\xi) &= \lambda(\tau\xi) - h(x)\,(\tau\xi)^p = \tau(\lambda\xi - h(x)\,\tau^{p-1}\,\xi^p) \\ &\geq \tau(\lambda\xi - h(x)\,\xi^p) = \tau f(x, \xi). \end{aligned}$$

This verifies condition (S1).

(iii) Now we construct a positive solution $u(\lambda)$ of problem $(**)$ for every $\lambda \in (\lambda_1, \widetilde{\lambda}_1(\mathcal{D}_0(h)))$, and further show that, for any $\lambda \geq \widetilde{\lambda}_1(\mathcal{D}_0(h))$, there exists no positive solution of problem $(**)$. We remark that the uniqueness of positive solutions of problem $(**)$ is an immediate consequence of Theorem 2.

We associate with problem $(**)$ a nonlinear mapping G of $\mathbf{R} \times C_B^{2+\theta}(\overline{D})$ into $C^\theta(\overline{D})$ as follows:

$$G : \mathbf{R} \times C_B^{2+\theta}(\overline{D}) \longrightarrow C^\theta(\overline{D}) \qquad (\lambda, u) \longmapsto Au - \lambda u + h\, u^p.$$

It is clear that a function $u \in C^{2+\theta}(\overline{D})$ is a solution of problem $(**)$ if and only if $G(\lambda, u) = 0$.

(iii-a) First the next lemma proves the existence of positive solutions of problem $(**)$ near the point $(\lambda_1, 0)$:

Lemma 3.1. *There exists a positive bifurcation solution curve* $(\lambda, u(\lambda))$ *of the equation* $G(\lambda, u) = 0$ *starting at* $(\lambda_1, 0)$.

Lemma 3.1 follows from an application of [12, Theorem 2], since the eigenvalue λ_1 is simple.

(iii-b) Secondly, by using the implicit function theorem, we show that there exists a critical value $\overline{\lambda}(h) \in (\lambda_1, \infty]$ such that one can extend the above bifurcation solution curve $(\lambda, u(\lambda))$ to all $\lambda \in (\lambda_1, \overline{\lambda}(h))$:

Lemma 3.2. *There exists a constant* $\overline{\lambda}(h) \in (\lambda_1, \infty]$ *such that we have a positive solution* $(\lambda, u(\lambda))$ *of the equation* $G(\lambda, u) = 0$ *for all* $\lambda \in (\lambda_1, \overline{\lambda}(h))$.

Proof. By applying Theorem 1.1 to our situation, we obtain that the Fréchet derivative

$$G_u(\lambda, u(\lambda)) : C_B^{2+\theta}(\overline{D}) \longrightarrow C^\theta(\overline{D}), \qquad v \longmapsto Av - \lambda v + phu(\lambda)^{p-1}\, v$$

is a Fredholm operator with index *zero*. Hence, in order to prove the lemma, it suffices to show that $G_u(\lambda, u(\lambda))$ is injective. Indeed, by using the implicit function theorem, one can find a constant $\overline{\lambda}(h) \in (\lambda_1, \infty]$ such that $G(\lambda, u(\lambda)) = 0$ and $G_u(\lambda, u(\lambda))$ is an algebraic and topological isomorphism for all $\lambda \in (\lambda_1, \overline{\lambda}(h))$.

The next claim proves the injectivity and hence surjectivity of $G_u(\lambda, u(\lambda))$:

Claim 3.3. *We define a densely defined, closed linear operator* $\mathfrak{A}(\lambda) : L^2(D) \to L^2(D)$ *as follows.*

(a) *The domain of definition* $D(\mathfrak{A}(\lambda))$ *is the space*

$$D(\mathfrak{A}(\lambda)) = \left\{ v \in H^{2,2}(D) : a\frac{\partial v}{\partial \nu} + bv = 0 \text{ on } \partial D \right\}.$$

(b) $\mathfrak{A}(\lambda)v = Av + phu(\lambda)^{p-1}\, v,\ v \in D(\mathfrak{A}(\lambda))$.

Then the first eigenvalue $\mu_1(\lambda)$ *of* $\mathfrak{A}(\lambda) - \lambda I$ *is* positive *for all* $\lambda \in (\lambda_1, \overline{\lambda}(h))$.

Proof. Let $\mu_1(\lambda)$ and $v_1(\lambda)$ be the first eigenvalue and associated eigenfunction of $\mathfrak{A}(\lambda) - \lambda I$, respectively:

$$(\mathfrak{A}(\lambda) - \lambda I)v_1(\lambda) = \mu_1(\lambda)\, v_1(\lambda),$$

or equivalently,

$$\begin{cases} \left(A - \lambda + phu(\lambda)^{p-1}\right) v_1(\lambda) = \mu_1(\lambda)\, v_1(\lambda) & \text{in } D, \\ a\frac{\partial v_1(\lambda)}{\partial \nu} + bv_1(\lambda) = 0 & \text{on } \partial D. \end{cases}$$

By [12, Theorem 1], one may assume that $v_1(\lambda) > 0$ in D. Then we have by Green's formula

$$\begin{aligned} \mu_1(\lambda) \int_D u(\lambda)\, v_1(\lambda)\, dx &= \int_D \left(Av_1(\lambda) - \lambda\, v_1(\lambda) + p\, h\, u(\lambda)^{p-1}\, v_1(\lambda)\right) u(\lambda)\, dx \\ &= \int_D v_1(\lambda)\, (A - \lambda)\, u(\lambda)\, dx + p \int_D h\, v_1(\lambda)\, u(\lambda)^p\, dx \\ &\quad - \int_{\partial D} \frac{\partial v_1(\lambda)}{\partial \nu}\, u(\lambda)\, d\sigma + \int_{\partial D} v_1(\lambda)\, \frac{\partial u(\lambda)}{\partial \nu}\, d\sigma \\ &= -\int_D h\, u(\lambda)^p\, v_1(\lambda)\, dx + p \int_D h\, v_1(\lambda)\, u(\lambda)^p\, dx \\ &\quad + \int_{\partial D} \left(v_1(\lambda)\, \frac{\partial u(\lambda)}{\partial \nu} - \frac{\partial v_1(\lambda)}{\partial \nu}\, u(\lambda) \right) d\sigma, \end{aligned}$$

where $d\sigma$ is the surface element of ∂D. But we recall that the functions $u(\lambda)$ and $v_1(\lambda)$ satisfy the following boundary conditions:

$$\begin{pmatrix} \frac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \frac{\partial v_1(\lambda)}{\partial \nu} & v_1(\lambda) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D.$$

Thus it follows that

$$\begin{vmatrix} \frac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \frac{\partial v_1(\lambda)}{\partial \nu} & v_1(\lambda) \end{vmatrix} = 0 \quad \text{on } \partial D,$$

since $(a, b) \neq (0, 0)$ on ∂D.

Therefore we obtain that

$$\mu_1(\lambda) \int_D u(\lambda)\, v_1(\lambda)\, dx = (p - 1) \int_D h\, u(\lambda)^p\, v_1(\lambda)\, dx.$$

This proves that

$$\mu_1(\lambda) = \frac{(p-1) \int_D h\, u(\lambda)^p\, v_1(\lambda)\, dx}{\int_D u(\lambda)\, v_1(\lambda)\, dx} > 0,$$

since $p > 1$ and $h \geq 0$ in D. $\square$

The proof of Lemma 3.2 is complete. □

By Lemma 3.2, we have a positive bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $G(\lambda, u) = 0$ for all $\lambda \in (\lambda_1, \overline{\lambda}(h))$. Then it is easy to see that the solution curve $u(\lambda)$ is of class C^1 with respect to λ, and further that it is increasing in λ and also blows up as $\lambda \to \overline{\lambda}(h)$, just as in [1, Theorem 25.4].

(iii-c) Finally we prove that there exists no positive solution of problem $(**)$ for any $\lambda \geq \widetilde{\lambda}_1(\mathcal{D}_0(h))$, and further that

$$\overline{\lambda}(h) = \widetilde{\lambda}_1(\mathcal{D}_0(h)).$$

First we begin with the following:

Lemma 3.4. *If $u(\lambda) \in C^2(\overline{D})$ is a positive solution of problem $(**)$ for $\lambda > \lambda_1$, then we have*

$$\lambda < \lambda_1(\mathcal{D}_i(h)), \quad 1 \leq i \leq \ell. \tag{3.1}$$

In particular, we have

$$\overline{\lambda}(h) \leq \widetilde{\lambda}_1(\mathcal{D}_0(h)). \tag{3.2}$$

Proof. Let φ_1 be an eigenfunction corresponding to the first eigenvalue $\lambda_1(\mathcal{D}_i(h))$ of the Dirichlet problem

$$\begin{cases} A\varphi_1 = \lambda_1(\mathcal{D}_i(h))\,\varphi_1 & \text{in } \mathcal{D}_i(h), \\ \varphi_1 = 0 & \text{on } \partial\mathcal{D}_i(h). \end{cases} \tag{D$_i$}$$

One may assume (cf. [16, Section 24.6, Theorem]) that $\varphi_1 > 0$ in $\mathcal{D}_i(h)$.

On the other hand, it follows that

$$Au(\lambda) = \lambda u(\lambda) + h\, u(\lambda)^p = \lambda u(\lambda) \quad \text{in } \mathcal{D}_i(h),$$

since $h = 0$ in $\mathcal{D}_i(h)$.

Then we have, by a direct calculation,

$$\begin{aligned} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u(\lambda)^2 \sum_{j=1}^{N} a^{ij} \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) \right) &= -u(\lambda) \cdot A\varphi_1 + \varphi_1 \cdot Au(\lambda) \\ &= -u(\lambda) \cdot \lambda_1(\mathcal{D}_i(h))\varphi_1 + \varphi_1 \cdot \lambda u(\lambda) \\ &= (\lambda - \lambda_1(\mathcal{D}_i(h)))\, u(\lambda) \cdot \varphi_1 \quad \text{in } \mathcal{D}_i(h), \end{aligned}$$

so that

$$(\lambda - \lambda_1(\mathcal{D}_i(h)))\, \varphi_1 = \frac{1}{u(\lambda)} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u(\lambda)^2 \sum_{j=1}^{N} a^{ij} \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) \right) \quad \text{in } \mathcal{D}_i(h).$$

Therefore, by integration by parts, it follows that

$$\begin{aligned}
&(\lambda - \lambda_1(\mathcal{D}_i(h))) \int_{\mathcal{D}_i(h)} \varphi_1^2 \, dx \\
&= \int_{\mathcal{D}_i(h)} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(u(\lambda)^2 \sum_{j=1}^{N} a^{ij} \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) \right) \cdot \frac{\varphi_1}{u(\lambda)} \, dx \\
&= - \int_{\mathcal{D}_i(h)} u(\lambda)^2 \sum_{i,j=1}^{N} a^{ij} \frac{\partial}{\partial x_i} \left(\frac{\varphi_1}{u(\lambda)} \right) \frac{\partial}{\partial x_j} \left(\frac{\varphi_1}{u(\lambda)} \right) dx < 0.
\end{aligned}$$

This proves inequality (3.1).

The proof of Lemma 3.4 is complete. □

The next lemma proves the reverse inequality of inequality (3.2):

Lemma 3.5. *Problem* $(**)$ *has a positive solution* $u(\lambda) \in C^{2+\theta}(\overline{D})$ *for every* $\lambda \in (\lambda_1, \widetilde{\lambda}_1(\mathcal{D}_0(h)))$. *In particular, we have*

$$\widetilde{\lambda}_1(\mathcal{D}_0(h)) \leq \overline{\lambda}(h). \tag{3.3}$$

Proof. First it follows from an application of [8, Theorems 2 and 3] that, for every $\lambda \in (\lambda_1(D), \widetilde{\lambda}_1(\mathcal{D}_0(h)))$, one can find a positive solution $\phi(\lambda)$ of the semilinear Dirichlet problem

$$\begin{cases} A\phi(\lambda) - \lambda\phi(\lambda) + h(x)\,\phi(\lambda)^p = 0 & \text{in } D, \\ \phi(\lambda) = 0 & \text{on } \partial D, \end{cases}$$

and also a positive solution $\psi(\lambda)$ of the semilinear Neumann problem

$$\begin{cases} A\psi(\lambda) - \lambda\psi(\lambda) + h(x)\,\psi(\lambda)^p = 0 & \text{in } D, \\ \frac{\partial \psi(\lambda)}{\partial \nu} = 0 & \text{on } \partial D. \end{cases}$$

Then we obtain that the function $\psi(\lambda)$ is a supersolution of problem $(**)$, since we have

$$\begin{cases} A\psi(\lambda) - \lambda\psi(\lambda) + h(x)\,\psi(\lambda)^p = 0 & \text{in } D, \\ B\psi(\lambda) = b\psi(\lambda) \geq 0 & \text{on } \partial D. \end{cases}$$

Moreover it follows that the function $\phi_\varepsilon(\lambda) = \varepsilon\phi(\lambda)$ is a subsolution of problem $(**)$ for all $0 < \varepsilon < 1$. Indeed, we have

$$\begin{aligned}
A\phi_\varepsilon(\lambda) - \lambda\phi_\varepsilon(\lambda) + h(x)\,\phi_\varepsilon(\lambda)^p &= \varepsilon \left(A\phi(\lambda) - \lambda\phi(\lambda) + h(x)\,\varepsilon^{p-1}\,\phi(\lambda)^p \right) \\
&= \varepsilon h(x) \left(\varepsilon^{p-1} - 1 \right) \phi(\lambda)^p \leq 0 \quad \text{in } D,
\end{aligned}$$

and also by the boundary-point lemma

$$B\phi_\varepsilon(\lambda) = a\,\varepsilon\,\frac{\partial \phi(\lambda)}{\partial \nu} \leq 0 \quad \text{on } \partial D.$$

Here we may choose a constant $0 < \varepsilon(\lambda) < 1$ so small that

$$0 < \varepsilon(\lambda)\phi(\lambda) \le \psi(\lambda) \quad \text{in } D.$$

Therefore, applying Theorem 1 to our situation, we can find a solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ of problem $(**)$ such that

$$0 < \varepsilon(\lambda)\phi(\lambda) \le u(\lambda) \le \psi(\lambda) \quad \text{in } D.$$

This proves that problem $(**)$ has a positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda \in (\lambda_1(D), \widetilde{\lambda}_1(\mathcal{D}_0(h)))$.

On the other hand, arguing as in the proof of [1, Proposition 18.1], we find that a positive solution curve $(\lambda, u(\lambda))$ of the equation $G(\lambda, u) = 0$ may bifurcate only at the point $(\lambda_1, 0)$. Thus it is easy to see that

$$\lambda_1 \le \lambda_1(D) < \overline{\lambda}(h).$$

Summing up, we have proved that problem $(**)$ has a positive solution $u(\lambda) \in C^{2+\theta}(\overline{D})$ for every $\lambda \in (\lambda_1, \widetilde{\lambda}_1(\mathcal{D}_0(h)))$. This proves the desired inequality (3.3). □

The proof of Theorem 3 is now complete. □

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