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HIGHER-ORDER MELNIKOV FUNCTIONS FOR DEGENERATE CUBIC HAMILTONIANS

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Abstract. It is shown that, in general, the first four Melnikov functions have to be taken into account in order to obtain definitive results concerning the limit cycles in quadratic perturbations of Hamiltonian systems in the plane with degenerate cubic Hamiltonians. An application is done in completing the proof that no more than two limit cycles can bifurcate out of homoclinic loops of quadratic Hamiltonian systems.

Introduction. It has been proved in [8] for small quadratic perturbations of generic quadratic Hamiltonian systems in the plane, that if the first Melnikov function is zero, then the perturbation is Hamiltonian (for a general result in this direction, see Il'yashenko, [11]). In this paper we consider the degenerate cases. The standard elliptic Hamiltonian $H = y^2 - x^3 + x$, which falls within these cases, was extensively studied in many papers ([3], [12], [13], [16]). For this reason we discuss in brief this class and concentrate our efforts on the other nongeneric cubic Hamiltonians.

Recall that the integrable quadratic systems (i.e., having a center) are divided into four classes ([17]): Q_3^{LV}, Q_3^H, Q_3^R and Q_4, depending on the algebraic invariants; Q_3^H is the Hamiltonian class. Then nongeneric are those systems (and the corresponding Hamiltonians) from Q_3^H, which in addition belong to another integrable class. Consider an arbitrary nongeneric cubic Hamiltonian H with a center. It is known that in suitable coordinates H has an axis of symmetry. As is easy to see ([9]), each nongeneric Hamiltonian (except for the standard elliptic Hamiltonian) also has a straight line as a level curve. Taking the axis of symmetry and the invariant line to be $y = 0$ and $x = 0$ respectively, we obtain the following normal form of H:

$$H = x(y^2 + Ax^2 + Bx + C) = x(y^2 + P_2(x)). \tag{0.1}$$

Further we can place a center of H at point $(1, 0)$ and rescale the y-variable to obtain explicitly the one-parameter family of all nongeneric Hamiltonians except the standard one:

$$H = x\left[y^2 + Ax^2 - 3(A-1)x + 3(A-2)\right], \quad A \in \mathbb{R}. \tag{0.2}$$

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Using this normal form, we can classify the separatrix contours appearing in the class of nongeneric Hamiltonians as follows (see Figure 1):

1) saddle-loop: $A \in (-\infty, -1) \cup (2, \infty)$;
2) hyperbolic segment: $A \in (-1, 0)$;
3) elliptic segment: $A \in (0, 2)$;
4) parabolic segment: $A = 0$;
5) triangle: $A = -1$;
6) non-Morsean point: $A = 2$.

In fact it is known ([17]) that all cases listed above are in $Q_3^H \cap Q_3^R$ and, in addition, the triangle is in $Q_3^{LV} \cap Q_3^H \cap Q_3^R$.

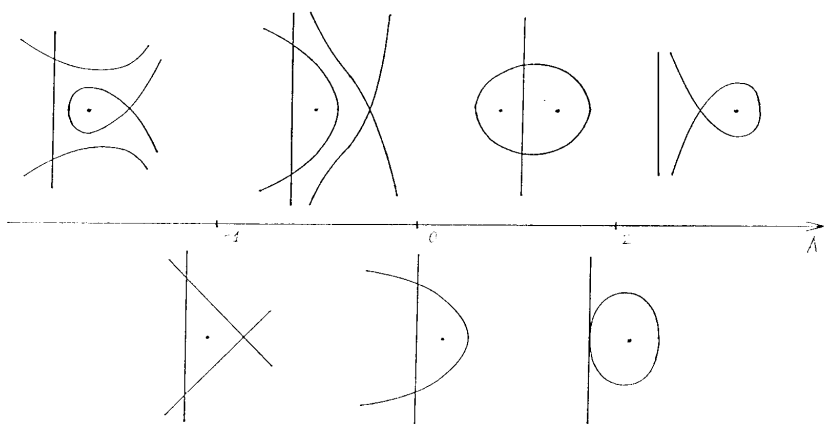

Figure 1

The purpose of this paper is to compute explicitly the first several Melnikov functions needed to study the bifurcations of limit cycles in quadratic perturbations of Hamiltonian systems with degenerate Hamiltonians. The obtained formulas are then applied to solve three particular bifurcation problems. Let H be any Hamiltonian from the one-parameter family (0.2). Consider the corresponding perturbed quadratic system

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y) \\ \dot{y} &= -H_x + \varepsilon g(x, y). \end{aligned} \tag{0.3}$$

Denote $\omega_1 = -f(x, y)\, dy + g(x, y)\, dx$. Take a value of h for which the level curve $H = h$ has a closed compact component in the half-plane $x > 0$ which surrounds the center

$(1, 0)$. The corresponding displacement function has the form

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \varepsilon^4 M_4(h) + \cdots ,$$

where, as is well known,

$$M_1(h) = \int_{H=h} \omega_1$$

(the integral is taken along the compact component $\subset \{H = h\}$). This integral is known as the Melnikov function. An easy computation yields

$$M_1(h) = -\int_{H=h} \alpha xy\, dx + \gamma y\, dx = \int\int_{H<h} (\alpha x + \gamma)\, dx\, dy \tag{0.4}$$

with constants α, γ depending on the coefficients of $f,\ g$. It is well known that contrary to the generic cases, in the degenerate cases the (first) Melnikov function $M_1(h)$ does not suffice to study the limit cycles of the perturbed system, since $M_1(h) \equiv 0$ does not yield necessarily that the perturbation is Hamiltonian (or even integrable). Provided the first Melnikov function M_1 is identically zero, one needs to inspect the next term M_2 in the expansion of the displacement function, sometimes called a second Melnikov function, and so on. Recall that the limit cycles in (0.3) for small ε are determined by the zeros of the first nonvanishing Melnikov function. We show that, in general, the first four Melnikov functions have to be taken into account in order to obtain definitive bifurcation results in the degenerate cases. Except for the triangle, a perturbation for which the first four Melnikov functions vanish, is integrable; that is, the perturbed system has a center. Certainly, in this situation $d(h, \varepsilon) \equiv 0$.

Theorem 1. *Assume H is a nongeneric cubic Hamiltonian with a center. Then the perturbed system* (0.3) *is integrable and belongs to the union $Q_3^H \cup Q_3^R$ provided*:

(i) $M_1(h) = M_2(h) = M_3(h) \equiv 0$ *and H is a parabolic segment* ($A = 0$ *in* (0.2)).
(ii) $M_1(h) = M_2(h) = M_3(h) = M_4(h) \equiv 0$ *and H is not a triangle* ($A \neq -1$ *in* (0.2)).

We obtain quite simple formulas for $M_k(h)$, $k = 2, 3, 4$ in terms of the coefficients of an arbitrary perturbation of the original system (see equations (1.3), (1.8) and (1.9) below). They can be useful in situations where, given a concrete perturbation, one wants to determine the *exact number* of limit cycles in the particular system, as well as when the *exact upper bound* for the number of limit cycles within the entire class of perturbations (0.3) has to be found. As an application of the derived formulas, it is established that at most two limit cycles can appear near the separatrix loop of any nongeneric Hamiltonian system. Note that for generic cubic Hamiltonians the problem was solved in [8]. Thus we complete the proof of the conjecture of Guckenheimer et al. in [7] about the number of the limit cycles bifurcating from a Hamiltonian loop under quadratic perturbations.

Theorem 2. *In any quadratic Hamiltonian system the cyclicity of the saddle-loop under quadratic perturbation is two. The total cyclicity of two Hamiltonian loops provided they exist also is two.*

Note that the other integrable class possessing systems with a saddle-loop is Q_3^R. The problem of finding the cyclicity of the loop in Q_3^R under nonconservative quadratic perturbations (which means $M_1(h) \not\equiv 0$) was considered recently in [15].

Another result we prove is concerned with the standard elliptic Hamiltonian $H = y^2 - x^3 + x$. Given a quadratic perturbation, either no more than two limit cycles exist in the finite part of the (x, y)-plane or the system is integrable. It should be noted about this case that there are perturbations for which a second focus appears in (0.3) (coming from infinity). Therefore the bifurcations of large limit cycles near infinity should be considered separately which is out of the context of this paper.

Theorem 3. *Suppose that in* (0.3)*, H is the standard elliptic Hamiltonian. Then for small ε and for arbitrary quadratic perturbations (f, g) the system has no more than two limit cycles in the finite part of the plane.*

A similar result holds for the parabolic segment case $A = 0$. It should be noted however that contrary to the homoclinic bifurcation, the polycycle bifurcation cannot be studied by using Melnikov functions. The reason is that one of the Abelian integrals appearing in the higher-order Melnikov functions tends to infinity as the oval approaches the polycycle. Therefore the expansion in ε of the displacement function is useless in investigation of this bifurcation as the series is not a good approximation close to polycycles. Thus we are unable to estimate the number of limit cycles bifurcated out of the polycycle itself (see in this connection the recent paper of Żołądek, [18]). We also mention that this is the other case which may have large limit cycles escaping to infinity as $\varepsilon \to 0$. For the parabolic segment we establish the following result.

Theorem 4. *Suppose that the Hamiltonian in* (0.3) *has a parabolic segment as a level set. Then any small quadratic perturbation has at most two limit cycles born simultaneously from the center and the set of periodic orbits of the unperturbed system.*

For the computation of the higher-order Melnikov functions, we use the scheme proposed recently by Françoise ([6]), who considered arbitrary polynomial perturbations of a Hamiltonian system with Hamiltonian $H = \frac{1}{2}(x^2 + y^2)$.

We also note that another concept for the notion of Melnikov function (namely linear and nonlinear Melnikov integrals) was used in [17] in order to study systems from the intersection $Q_3^R \cap Q_3^{LV}$.

1. Melnikov functions. In this section we use the algorithm [6] proposed for computing the first nonzero derivative of the displacement function. It is easy to see that the restrictions from [6] are not essential and the algorithm applies for a larger class of Hamiltonians than the case considered there.

1.1. *Evaluation of $M_2(h)$ in the case where $M_1(h) \equiv 0$.* We begin by noticing that the Hamiltonian triangle $A = -1$ as well as the standard elliptic Hamiltonian require a different approach and because of this we consider these cases separately at the end of the section. Suppose now that the Melnikov function $M_1(h)$ is identically zero. Then $\alpha = \gamma = 0$ in (0.4), so we can write $\omega_1 = -f dy + g dx = dH_1 + \beta xy dy$, where

$$\begin{aligned} H_1 = a_{10}x + a_{01}y + \frac{a_{20}}{2}x^2 + a_{11}xy + \frac{a_{02}}{2}y^2 \\ + \frac{a_{30}}{3}x^3 + a_{21}x^2y + a_{12}xy^2 + \frac{a_{03}}{3}y^3. \end{aligned} \tag{1.1}$$

Obviously if $\beta = 0$ then the perturbation is Hamiltonian. Next we compute the second Melnikov function $M_2(h)$ for the nontrivial case where $\beta \neq 0$. Following [6], we start with

Lemma 1. *The one-form ω_1 can be written as $\omega_1 = q_1 dH + dQ_1$, where*

$$q_1 = \frac{\beta}{2}\ln x, \quad Q_1 = H_1 + \frac{\beta}{2}[xy^2 + P_3(x) - H\ln x], \quad P_3' = P_2.$$

Proof. We have by (0.1)

$$\begin{aligned} 2xy\,dy = dxy^2 - y^2dx = dxy^2 + \big(P_2(x) - \frac{H}{x}\big)dx \\ = dxy^2 + P_2(x)dx - dH\ln x + \ln x\, dH. \end{aligned}$$

Therefore $\beta xy dy = \frac{1}{2}\beta \cdot d[xy^2 + P_3(x) - H\ln x] + \frac{1}{2}\beta \ln x\, dH$ and hence the lemma follows.

Now define ([6]) $\omega_2 = q_1.\omega_1$. Then the second-order Melnikov function is given by

$$M_2(h) = \int_{H=h} \omega_2.$$

We recall the construction from [6] in brief. Given a perturbation ω_1 with $M_1(h) \equiv 0$, let γ_ε be the trajectory of $dH - \varepsilon\omega_1 = 0$ connecting a point $P(0)$ and the point $P(T)$ determined by the first return map. Then $\omega_1 = q_1 dH + dQ_1$ yields

$$(1 + \varepsilon q_1)(dH - \varepsilon\omega_1) = d(H - \varepsilon Q_1) - \varepsilon^2 q_1\omega_1.$$

After integrating the above identity over γ_ε and using that $dH - \varepsilon\omega_1 = 0$ on γ_ε, we obtain

$$\int_{\gamma_\varepsilon} d(H - \varepsilon Q_1) = \varepsilon^2 \int_{\gamma_\varepsilon} \omega_2.$$

Since $M_1(h) \equiv 0$, we have $\varepsilon \int_{\gamma_\varepsilon} dQ_1 = O(\varepsilon^3)$ whenever Q_1 is a Lipschitz-continuous function. Therefore

$$d(h,\varepsilon) = \varepsilon \int_{\gamma_\varepsilon} dQ_1 + \varepsilon^2 \int_{\gamma_\varepsilon} \omega_2 = \varepsilon^2 \int_{H=h} \omega_2 + O(\varepsilon^3).$$

The higher-order Melnikov functions can be computed similarly. Note that in our case the above construction is valid in any compact subset of $\{x>0\}$ as far as Q_1 and q_1 are uniformly Lipschitz continuous there. Clearly this construction fails in a neighborhood of the segment in the segment cases listed above.

We need an explicit formula for $M_2(h)$ in terms of the coefficients of the Hamiltonian and the perturbation. For this purpose we obtain

$$\begin{aligned} M_2(h) &= \int_{H=h} q_1\cdot\omega_1 = \frac{\beta}{2}\int_{H=h}\ln x\,(dH_1+\beta xydy) = -\frac{\beta}{2}\int_{H=h}\frac{H_1}{x}dx \\ &= -\frac{\beta}{2}\int_{H=h}\big[a_{01}\frac{y}{x}+a_{11}y+a_{21}xy+\frac{a_{03}}{3}\frac{y^3}{x}\big]\,dx. \end{aligned}$$

In order to express the last term in the integrand in an appropriate form, we observe that on the curve $H=h$ one has

$$y^2=\frac{h}{x}-Ax^2-Bx-C\equiv\mathcal{R}(x,h).$$

Then

$$\int_{H=h} y\mathcal{R}'\,dx=\int_{H=h} y\,d\mathcal{R}=\int_{H=h} y\,dy^2=0.$$

Hence

$$\int_{H=h}\frac{y^3}{x}\,dx=\int_{H=h}\big(\frac{y}{x}\mathcal{R}+y\mathcal{R}'\big)\,dx=-\int_{H=h}\big(3Axy+2By+C\frac{y}{x}\big)\,dx. \tag{1.2}$$

Denoting for each integer k

$$J_k(h)=\int_{H=h}x^k y\,dx,$$

we finally get the needed formula for the second-order Melnikov function:

$$M_2(h)=-\frac{\beta}{2}\big[(a_{01}-\frac{C}{3}a_{03})J_{-1}+(a_{11}-\frac{2B}{3}a_{03})J_0+(a_{21}-Aa_{03})J_1\big]. \tag{1.3}$$

Lemma 2. *The functions $J_k(h)$, $k=-1,0,1$ are linearly independent except for the Hamiltonian triangle $A=-1$, in which case $J_0(h)\equiv J_1(h)$.*

Proof. The last claim in the formulation follows from identity (1.10) below. We are going to prove the linear independence of J_k by inspecting the first three coefficients in their expansions near the critical value $h=h_c$, which corresponds to the center $(1,0)$. The needed values can be most easily obtained from the Picard-Fuchs system, satisfied by $J_{-1},\ J_0,\ J_1$. For this, the identity

$$\int_{H=h}x^k\mathcal{R}'y\,dx=-\frac{2k}{3}\int_{H=h}x^{k-1}\mathcal{R}\,y\,dx$$

and the formula for the derivative

$$J_k'(h) = \int_{H=h} \frac{x^{k-1}}{2y}\, dx$$

yield for each integer k

$$\begin{aligned} (2k-3)hJ_{k-2} &= (2k+6)AJ_{k+1} + (2k+3)BJ_k + 2kCJ_{k-1}, \\ kJ_{k-1} &= hJ_{k-1}' + 2AJ_{k+2}' + BJ_{k+1}', \\ \frac{1}{2}J_k &= hJ_k' - AJ_{k+3}' - BJ_{k+2}' - CJ_{k+1}'. \end{aligned} \tag{1.4}$$

Then via standard computations we obtain the following Picard-Fuchs system (which also is of its own interest),

$$\begin{aligned} 3hJ_{-1}' - 2CJ_0' - BJ_1' &= J_{-1} \\ BhJ_{-1}' + 6AhJ_0' + (B^2 - 4AC)J_1' &= 4AJ_0 \\ 2ChJ_{-1}' + 4BhJ_0' + 6AhJ_1' &= 3BJ_0 + 6AJ_1. \end{aligned} \tag{1.5}$$

We choose the coordinate system in which H has a normal form (0.2). Next use system (1.5) to determine the coefficients c_{kl} at $(h-h_c)^l$, $1 \le l \le 3$, in the expansion of $J_k(h)$ near $h_c = A - 3$. Computing the corresponding determinant $\det\|c_{kl}\|$ we find that it equals $(A+1)^2$ times an absolute constant. This proves the lemma.

Example 1. Consider the system

$$\begin{aligned} \dot{x} &= 2xy + \varepsilon(3 - 3x^2 - xy - 3y^2) \\ \dot{y} &= 3 - 3x^2 - y^2 + \varepsilon(6xy - y^2). \end{aligned} \tag{1.6}$$

For (1.6) one has $M_1(h) = M_2(h) = 0$, but the system is not in Q_3^R since it has no invariant line. This example is somewhat surprising (cf. [17]) and it shows that the second-order Melnikov function cannot solve the problem of limit cycles even if the quadratic perturbation contains only first-order terms in ε.

Example 2. Consider an arbitrary non-Hamiltonian perturbation with $a_{03} = 0$ in (1.1), which yields zero first- and second-order Melnikov functions. Then the perturbation is integrable. Indeed, in this case H_1 takes the form

$$H_1 = a_{10}x + \frac{a_{20}}{2}x^2 + \frac{a_{02}}{2}y^2 + \frac{a_{30}}{3}x^3 + a_{12}xy^2$$

and it is easy to check that the system has an integrating factor of the form $M(x) = |x+\mu|^\nu$, where

$$\mu = -\frac{\varepsilon a_{02}}{2 - \varepsilon\beta - 2\varepsilon a_{12}}, \qquad \nu = \frac{\varepsilon\beta}{2 - \varepsilon\beta - 2\varepsilon a_{12}}. \tag{1.7}$$

1.2. *The higher-order terms of the displacement function.* Suppose that $M_1(h) = M_2(h) \equiv 0$, but $\beta \neq 0$. Then H_1 takes the form

$$H_1 = a_{10}x + \frac{a_{20}}{2}x^2 + \frac{a_{02}}{2}y^2 + \frac{a_{30}}{3}x^3 + a_{12}xy^2 + \frac{a_{03}}{3}(Cy + 2Bxy + 3Ax^2y + y^3).$$

After direct calculations, we obtain the representation $\omega_2 = q_1\omega_1 = q_2dH + dQ_2$, where

$$q_2 = \frac{\beta}{2}\Big(\frac{\beta}{4}\ln^2 x + (a_{12} + \beta/2)\ln x - \frac{a_{02}}{2x} - \frac{a_{03}}{3}\frac{y}{x}\Big).$$

Indeed, we have (here and below the computations are performed modulo exact forms)

$$q_1\omega_1 = q_1(q_1dH + dQ_1) = q_1^2dH - Q_1dq_1 = q_1^2dH - \frac{\beta}{2}\frac{Q_1}{x}dx$$

and further get (we omit the detailed calculations),

$$\frac{Q_1}{x}dx = \Big[\frac{a_{02}}{2x} - \big(a_{12} + \frac{\beta}{2}\big)\ln x + \frac{a_{03}y}{3x} + \frac{\beta}{4}\ln^2 x\Big]dH.$$

Together this yields the needed formula of ω_2.

Repeating the same procedure as in the computation of $M_2(h)$, this time using the identity

$$(1 + \varepsilon q_1 + \varepsilon^2 q_2)(dH - \varepsilon\omega_1) = d(H - \varepsilon Q_1 - \varepsilon^2 Q_2) - \varepsilon^3 q_2\omega_1,$$

yields that $M_3(h) = \int_{H=h}\omega_3$ with $\omega_3 = q_2 \cdot \omega_1$. Further, to obtain an explicit formula for the integral $M_3(h)$, we perform long but straightforward computations consisting mainly in integration by parts and making use of the formulas derived above. Thus we get

$$\begin{aligned} M_3(h) &= \int_{H=h} q_2(dH_1 + \beta xy\,dy) = -\int_{H=h} H_1\,dq_2 \\ &= \frac{\beta a_{03}}{6}\int_{H=h}\big(\beta\ln x + \frac{a_{02}}{x}\big)y^2dy - \frac{\beta a_{03}}{6}\int_{H=h}\frac{y}{x}(a_{10} + a_{20}x + a_{30}x^2)\,dx \\ &\quad + \frac{\beta a_{03}}{6}\int_{H=h}\big(\frac{a_{02}}{2} + a_{12}x\big)\big(\frac{y^2}{x}\,dy - \frac{y^3}{x^2}\,dx\big). \end{aligned}$$

Consider the contribution coming from the first and the third integral in the last sum. Applying the identity

$$\frac{3y^2}{2x}dy - \frac{y^3}{2x^2}dx = d\frac{y^3}{2x}$$

we see that the coefficient at a_{02} vanishes. The sum of the remaining terms is

$$\frac{\beta a_{03}}{6}\int_{H=h}\big(\beta y^2\ln x\,dy - a_{12}\frac{y^3}{x}dx\big) = -\frac{\beta a_{03}}{6}\big(a_{12} + \frac{\beta}{3}\big)\int_{H=h}\frac{y^3}{x}\,dx.$$

By (1.2) we finally get the expression for the third Melnikov function

$$M_3(h) = -\frac{\beta}{6}a_{03}\Big\{[a_{10} - C(a_{12} + \frac{\beta}{3})]J_{-1} + [a_{20} - 2B(a_{12} + \frac{\beta}{3})]J_0 + [a_{30} - 3A(a_{12} + \frac{\beta}{3})]J_1\Big\}. \tag{1.8}$$

Indeed the last formula is of interest only if $\beta a_{03} \neq 0$. Supposing this, we observe that provided $M_1 = M_2 = M_3 = 0$, the function H_1 becomes

$$H_1 = (a_{12} + \beta/3)(Cx + Bx^2 + Ax^3) + \frac{a_{02}}{2}y^2 + a_{12}xy^2 + \frac{a_{03}}{3}(Cy + 2Bxy + 3Ax^2y + y^3).$$

Example 3. Consider the case of parabolic segment $A = 0$ with non-Hamiltonian perturbation satisfying $M_1 = M_2 = M_3 = 0$, $a_{03} \neq 0$. Then a direct computation yields that the corresponding system has an integrating factor of the form

$$M(x, y) = |x + \lambda y + \mu|^\nu,$$

where

$$\lambda = \frac{\varepsilon a_{03}}{3[(a_{12} + \beta/3)\varepsilon - 1]}, \quad \mu = \frac{1}{2 - \varepsilon\beta - 2\varepsilon a_{12}}\Big\{ -\varepsilon a_{02} + \frac{2B\varepsilon^2 a_{03}^2}{9[(a_{12} + \beta/3)\varepsilon - 1]}\Big\}$$

and ν is given by (1.7).

Example 4. The nonintegrable system from Example 1 annihilates also $M_3(h)$.

In order to compute the term at ε^4 of the displacement function, we follow the same line of investigation. The computations are as much straightforward as above, but quite long and for this reason we omit them. We obtain successively $\omega_3 = q_3 dH + dQ_3$ with

$$\begin{aligned} q_3 = {} & \frac{\beta^3}{48}\ln^3 x + \frac{\beta^2}{4}\big(a_{12} + \frac{\beta}{2}\big)\ln^2 x + \frac{\beta}{2}\big[\big(a_{12} + \frac{\beta}{2}\big)^2 + \frac{Aa_{03}^2}{3} - \frac{\beta a_{02}}{4x}\big]\ln x \\ & - \frac{\beta a_{03}^2}{36}\big(\frac{y}{x}\big)^2 - \frac{\beta a_{03}}{6}\big(\frac{y}{x}\big)\big(2a_{12} + \frac{5\beta}{6} + \frac{a_{02}}{2x} + \frac{\beta}{2}\ln x\big) \\ & - \frac{\beta a_{02}^2}{16x^2} - \frac{\beta}{2}\big[a_{02}\big(a_{12} + \frac{\beta}{2}\big) + \frac{Ba_{03}^2}{9}\big]\frac{1}{x} \end{aligned}$$

and then putting $\omega_4 = q_3 \cdot \omega_1$, $M_4(h) = \int_{H=h}\omega_4$, we come to

$$M_4(h) = \frac{\beta A a_{03}^3}{27}(CJ_{-1} + 2BJ_0 + 3AJ_1). \tag{1.9}$$

Proof of Theorem 1. The formulas for $M_k(h)$ (1.3), (1.8), (1.9) we have derived yield immediately the assertion of Theorem 1 (concerning the standard Hamiltonian, see the next section). Indeed, part (i) follows from Examples 2 and 3 given above. To prove (ii), we note that $M_k(h) \equiv 0$ for $k \leq 4$ is equivalent to $\alpha = \gamma = 0$ and either $\beta = 0$ or $a_{03} = 0$. In the first case (0.3) is Hamiltonian and in the second one, by Example 2, it is in class Q_3^R.

The theorem just proved shows that in order to investigate the number of limit cycles in (0.3) in the degenerate cases, we have to examine the first nonzero Melnikov function among $M_k(h)$, $k = 1, 2, 3, 4$. It is well known that the first Melnikov function $M_1(h)$ yields at most one limit cycle since $J_1(h)/J_0(h)$ is monotone in h. We next observe that, by (1.2),

$$CJ_{-1} + 2BJ_0 + 3AJ_1 = -\int_{H=h} \frac{y^3}{x}\, dx = 3 \int\int_{H<h} \frac{y^2}{x}\, dx\, dy > 0.$$

Therefore $M_4(h)$ cannot create limit cycles because it has no isolated zeros. In particular this implies that systems with $M_1(h) = M_2(h) = M_3(h) \equiv 0$ have for ε small no limit cycles at all. The complete investigation of $M_2(h)$ and $M_3(h)$ appears to be in general a difficult problem and we are not going to try this in the present paper. Below we consider only the parabolic segment case ($A = 0$) which allows an easy treatment.

It should also be noted that in the segment cases the integral $J_{-1}(h)$ tends to infinity as $h \to 0$. This means that $M_k(h)$, $k = 2, 3, 4$ are not good approximations of the displacement function in a small neighborhood of the segment and hence cannot be used in studying the limit cycles bifurcating from the segments. On the contrary, in the loop case the expansion of $M_k(h)$ near the loops allows to obtain definitive results for the saddle-loop bifurcations; see Section 4.

1.3. *The Hamiltonian triangle.* In this paragraph we compute the second Melnikov function for the case of the Hamiltonian triangle. The triangle case is more complicated for studying than the other degenerate cases and we are not able to compute explicitly all needed Melnikov functions (probably up to an order ε^6), which will assure that the displacement function is zero provided they vanish identically. Therefore the considerations that follow are intended to give some idea about the difficulties which occur in investigating this Hamiltonian (see the remarks below). In the triangle case $A = -1$ in (0.2) and hence H becomes

$$H = x[y^2 - (x-3)^2].$$

First we recall that the integrals $J_0(h)$ and $J_1(h)$ are equal, which as we already mentioned, is an immediate consequence of the identity (1.10) below. Hence

$$M_1(h) = -(\alpha + \gamma) \int_{H=h} y dx = (\alpha + \gamma) \int\int_{H<h} dx\, dy.$$

This yields that the (first) Melnikov function $M_1(h)$ is identically zero if and only if $\alpha + \gamma = 0$. Therefore the one-form ω_1 in this case has to be taken as follows:

$$\omega_1 = -f dy + g dx = dH_1 + [\beta xy + \gamma(x - \frac{1}{2}x^2)]\, dy.$$

Lemma 3. *The one-form* ω_1 *can be written as* $\omega_1 = q_1 dH + dQ_1$*, where*

$$q_1 = \frac{\beta}{2}\ln x - \frac{\gamma}{6}\ln\frac{3-x-y}{3-x+y},$$
$$Q_1 = H_1 + \frac{\beta}{2}xy^2 - \frac{\beta}{6}(x-3)^3 - \frac{\beta}{2}H\ln x - \frac{\gamma}{6}x^2y + \frac{\gamma}{6}H\ln\frac{3-x-y}{3-x+y}.$$

Proof. We obtain just as in the proof of Lemma 1 that

$$\beta xydy = \frac{\beta}{2}\ln xdH + \frac{\beta}{2}d[xy^2 - \frac{1}{3}(x-3)^3 - H\ln x].$$

Further, we have

$$\gamma(x - \tfrac{1}{2}x^2)dy = \gamma d(x - \tfrac{1}{2}x^2)y + \gamma(x-1)ydx.$$

To compute the contribution of the last term in the above formula, we use the identity

$$(x-1)ydx = d\Big[\frac{H}{6}\ln\frac{3-x-y}{3-x+y} + \frac{xy(x-3)}{3}\Big] - \frac{1}{6}\ln\frac{3-x-y}{3-x+y}dH, \tag{1.10}$$

which is verified by a direct computation. Since (1.10) looks somewhat strange, next we explain how it was guessed. Suppose that $y > 0$ (the case $y < 0$ is dealt with similarly). Then using the expression for H we obtain

$$(x-1)ydx = (x-1)\sqrt{\frac{H}{x} + (x-3)^2}\,dx = (x-1)(3-x)\sqrt{\frac{H}{x(x-3)^2} + 1}\,dx$$
$$= -\frac{1}{3}\sqrt{\frac{H}{x(x-3)^2} + 1}\,dx(x-3)^2 = -\frac{1}{3}\sqrt{\frac{H}{z} + 1}\,dz$$

with $z = x(x-3)^2$. Take a function $G(z,H)$ with $G_z(z,H) = -\frac{1}{3}\sqrt{H/z+1}$. Then

$$(x-1)ydx = G_z(z,H)dz = dG(z,H) - G_H(z,H)dH.$$

Computing G explicitly, we obtain (1.10). The lemma is proved.

To evaluate $M_2 = \int_{H=h} q_1 \cdot \omega_1$, we express it as

$$M_2 = \beta I_{10} + \gamma I_{01} + \beta^2 I_{20} + \beta\gamma I_{11} + \gamma^2 I_{02}.$$

Next we obtain as in paragraph 1 above

$$I_{10} = \frac{1}{2}\int_{H=h}\ln xdH_1 = -\frac{1}{2}[(a_{01} + 3a_{03})J_{-1} + (a_{11} + a_{21} - 3a_{03})J_0]$$

and

$$I_{20} = \frac{1}{2}\int_{H=h} xy\ln x\,dy = 0.$$

Further we have by (1.10) with $H = h$ and using the formula $(3-x)^2 - y^2 = -h/x$ that

$$I_{02} = -\frac{1}{6}\int_{H=h} \ln\frac{3-x-y}{3-x+y}\big[d\big(x-\frac{x^2}{2}\big)y + d\big(\frac{h}{6}\ln\frac{3-x-y}{3-x+y} + \frac{xy(x-3)}{3}\big)\big] = 0.$$

In a similar way we obtain

$$\begin{aligned} I_{01} &= \frac{1}{6}\int_{H=h} H_1\frac{-2y\,dx + (2x-6)\,dy}{(3-x)^2 - y^2} \\ &= \frac{1}{3h}\int_{H=h} [a_{10}(4x^2-6x) + \frac{a_{20}}{2}(5x^3 - 9x^2) + a_{30}(2x^4 - 4x^3) \\ &\quad + \frac{a_{02}}{6}(5x-3)y^2 + a_{12}(2x^2-2x)y^2]y\,dx. \end{aligned}$$

We use again $y^2 = h/x + (x-3)^2$ and the first equation in (1.4) in order to express J_k, $2 \le k \le 4$ in terms of J_{-1} and J_0. After simple computations we get

$$I_{01} = \frac{1}{12}(3a_{02} - a_{20} - 4a_{30})J_0 + \frac{1}{12}(2a_{10} + 3a_{20} + 6a_{30} - 3a_{02})J_{-1}.$$

At the end, on $H = h$ we have

$$xydy = \frac{x}{2}dy^2 = \big(x^2 - 3x - \frac{h}{2x}\big)dx = d\big(\frac{x^3}{3} - \frac{3x^2}{2} - \frac{h}{2}\ln x\big)$$

and hence as above

$$\begin{aligned} I_{11} &= \int_{H=h} \big[\frac{1}{2}\big(x - \frac{x^2}{2}\big)\ln x - \frac{1}{6}xy\ln\frac{3-x-y}{3-x+y}\big]\,dy \\ &= \frac{1}{12h}\int_{H=h} (hx + 54x^2 - 46x^3 + 8x^4)y\,dx = -\frac{1}{4}J_{-1}. \end{aligned}$$

Combining all formulas obtained above, we finally get the needed expression for $M_2(h)$ in the case of Hamiltonian triangle:

$$\begin{aligned} M_2(h) = &\big[-\frac{\beta}{2}(a_{01} + 3a_{03}) + \frac{\gamma}{12}(2a_{10} + 3a_{20} + 6a_{30} - 3a_{02}) - \frac{\beta\gamma}{4}\big]J_{-1} \\ + &\quad \big[-\frac{\beta}{2}(a_{11} + a_{21} - 3a_{03}) + \frac{\gamma}{12}(3a_{02} - a_{20} - 4a_{30})\big]J_0. \end{aligned}$$

We end this paragraph with some remarks concerning the triangle case. The above considerations show that the computation of M_3, M_4 etc. will be an extremely elaborate and technically difficult problem. The value of M_3 can still be computed in this way; the result is

$$M_3(h) = k_{-1}J_{-1} + \big(k_0 + \frac{k_1}{h}\big)J_0 + \frac{k_*}{h}J_*, \quad J_* = \int_{H=h} y(x-1)\ln x\,dx \tag{1.11}$$

with certain constants k_*, k_m expressed explicitly through the coefficients of the perturbation. From the treatment of this case in [17], one can expect that for $n > 3$, M_n will have the same form. Note that (1.11) contains only three integrals and therefore is more useful in studying of limit cycles than the corresponding formula from [17]. The third Melnikov function (1.11) for the triangle case will be investigated elsewhere ([10]).

The computations above show that even evaluated, the higher-order Melnikov functions in this case can be used only for investigating the cycles bifurcating from periodic orbits but not for those bifurcating from the separatrix triangle itself. It is easy to prove that $M_2(h)$ can produce at most one limit cycle. Indeed, using (1.5) we immediately get the Riccati equation for $R(h) = J_{-1}(h)/J_0(h)$:

$$-3h(h+4)R' = 8 + hR + hR^2$$

which easily yields that $R(h)$ is strictly increasing in the interval $(-4, 0)$ corresponding to the periodic annulus surrounded by the triangle. On the other hand the cyclicity of the center in the triangle case is three ([4]) and (probably) the same is the cyclicity of the separatrix contour. Therefore in general we can expect that the annulus also has cyclicity three and consequently the partial result given by $M_2(h)$ is far from the needed one. (See also the recent papers [10], [18].)

2. The standard elliptic Hamiltonian. In this section we consider in brief the standard elliptic Hamiltonian, by which is meant every cubic Hamiltonian with a center, taking after appropriate *affine* change of the coordinates the form $H = y^2 + P_3(x)$. We can place the saddle and the center at $(0, 0)$ and $(1, 0)$ respectively to obtain H in the form

$$H = \frac{y^2 - x^2}{2} + \frac{x^3}{3}. \tag{2.1}$$

The closed trajectories exist for levels $h \in (-\frac{1}{6}, 0)$ and terminate at a saddle loop $\gamma \subset \{H = 0\}$ (also known as the Bogdanov-Takens loop).

The evaluation of the Melnikov functions for Hamiltonian (2.1) repeats the considerations from Section 1 and is in fact technically easier. For this reason we confine ourselves in giving only the final formulas, obtained in the successive computation of M_2, M_3 and M_4. Suppose that $M_1(h) \equiv 0$. Taking $\omega_1 = dH_1 + \beta xydy$, where H_1 is given by (1.1), we have

$$\omega_1 = q_1 dH + dQ_1 = \beta x dH + d[H_1 + \beta(x^3/3 - x^4/4)].$$

Then computing $M_2 = \int_{H=h} q_1 \cdot \omega_1$ we get

$$M_2(h) = -\beta\big[\big(\frac{6}{11}ha_{03} + a_{01}\big)J_0 + \big(a_{11} + a_{21} + \frac{a_{03}}{11}\big)J_1\big].$$

As the functions $J_0(h)$, $hJ_0(h)$ and $J_1(h)$ are linearly independent, $M_2(h) \equiv 0$ provided $\beta \neq 0$ is equivalent to $a_{03} = a_{01} = a_{11} + a_{21} = 0$, which yields

$$H_1 = a_{10}x + \frac{a_{20}}{2}x^2 + \frac{a_{30}}{3}x^3 + a_{11}(xy - x^2y) + a_{12}xy^2 + \frac{a_{02}}{2}y^2.$$

It is easily seen that in the case $a_{11} = 0$ the system (0.3) has an integrating factor depending only on x:

$$M(x) = \begin{cases} |x+\mu|^\nu, & \beta + 2a_{12} \neq 0, \quad \mu = \frac{\varepsilon a_{02} - 1}{\varepsilon(\beta + 2a_{12})}, \quad \nu = -\frac{\beta}{\beta + 2a_{12}}, \\ \exp\left(\frac{\varepsilon\beta x}{1 - \varepsilon a_{02}}\right), & \beta + 2a_{12} = 0. \end{cases}$$

Further for $\beta a_{11} \neq 0$ we have that if $M_1 = M_2 = 0$ then $\omega_2 = q_1 \cdot \omega_1$ can be written as $\omega_2 = q_2 dH + dQ_2$ with

$$q_2 = \beta[(\beta + a_{12})x^2 + a_{02}x + a_{11}y].$$

Computing $M_3 = \int_{H=h} q_2.\omega_1$ we obtain that it equals

$$M_3(h) = \frac{\beta a_{11}}{11}\{[11a_{10} + 6(\beta + 3a_{12})h]J_0 + [11(a_{20} + a_{30}) + \beta + 3a_{12}]J_1\}.$$

Thus for nonzero β and a_{11}, the third Melnikov function $M_3(h)$ vanishes identically if and only if $a_{10} = a_{20} + a_{30} = 0$, $a_{12} = -\frac{1}{3}\beta$. Now H_1 becomes

$$H_1 = a_{30}\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + a_{11}(xy - x^2y) - \frac{\beta}{3}xy^2 + \frac{a_{02}}{2}y^2$$

and $\omega_3 = q_2\omega_1$ can be written as $\omega_3 = q_3 dH + dQ_3$ with

$$q_3 = \frac{10}{27}\beta^3x^3 + (\beta a_{11}^2 + \frac{4}{3}\beta^2 a_{02})x^2 + \frac{4}{3}\beta^2 a_{11}xy + \beta(a_{02}^2 - a_{11}^2)x + \beta a_{11}(a_{30} + a_{02})y.$$

Finally computing $M_4 = \int_{H=h} q_3 \cdot \omega_1$ we get

$$M_4(h) = \frac{2}{11}\beta a_{11}^3(6hJ_0 + J_1).$$

To show that $M_4(h)$ does not vanish for $h \neq -\frac{1}{6}$, we apply the identities

$$(3x^3 - 3x^2 - y^2)y\,dx = -d(xy^3) + 3xy\,dH, \qquad J_2(h) = J_1(h),$$

which were also used in computation of $M_k(h)$. We have successively

$$\begin{aligned} 6hJ_0 + J_1 &= \int_{H=h}(6h + x)y\,dx = \int_{H=h}[3y^2 - 3x^2 + 2x^3 + x]y\,dx \\ &= \frac{2}{3}\int_{H=h}\left[\frac{9}{2}y^2 + 3x^3 - 3x^2\right]y\,dx = \frac{11}{3}\int_{H=h} y^3\,dx \\ &= -11\int\int_{H<h} y^2\,dx\,dy < 0. \end{aligned}$$

From the formulas we derived it is easy to complete the proof of Theorem 1 concerning the standard Hamiltonian case. Below we proceed to prove Theorem 3.

Proof of Theorem 3. If $M_k(h) \equiv 0$ for $k \le 4$, then (0.3) is integrable since $\beta = 0$ or $a_{11} = 0$. Suppose that some of $M_1,\ M_2,\ M_3$ are not identically zero. As the case $M_1 \not\equiv 0$ is known to yield at most one cycle, let us consider the other possibilities. It is well known ([5]) that the function $R(h) = J_1(h)/J_0(h)$ is strictly concave in $[-\frac{1}{6}, 0)$. Writing for short the expressions for $M_2,\ M_3$ in a form

$$I(h) = J_0(h)(aR(h) + bh + c), \qquad h \in [-1/6, 0), \tag{2.2}$$

we see that the system (0.3) has at most two limit cycles which tend to both the center or periodic orbit of the unperturbed system as $\varepsilon \to 0$. On the other hand, near $h = 0$ each of M_2, M_3 has the expansion ([2])

$$I(h) \equiv aJ_1(h) + (bh + c)J_0(h) = c_1 + c_2 h \ln|h| + c_3 h + \cdots .$$

At the same time, near $h = 0$ we have

$$J_0(h) = J_0(0) + h \ln|h| + \cdots , \quad J_1(h) = J_1(0) + J_1'(0)h + \cdots$$

and $R(0) = J_1(0)/J_0(0) = 6/7$. From here we conclude that $c_1 = c_2 = c_3 = 0$ implies $a = b = c = 0$. Hence at most two limit cycles can bifurcate from saddle-loop of the unperturbed system, by the result of Roussarie ([14]). Moreover $c_1 = 0$ is equivalent to $6a + 7c = 0$ and $c_1 = c_2 = 0$ is equivalent to $a = c = 0$. Clearly in the first case the function $aR(h) + bh + c$ in (2.2) has at most one zero in $[-\frac{1}{6}, 0)$ and in the second one it has no zeros there. By the result in [14] and by the above considerations we conclude that the total number of limit cycles bifurcating simultaneously from the center, from the periodic annulus and from the saddle-loop, cannot exceed two. The theorem is proved.

We recall that in the case we consider, there exist perturbations for which a second focus appears in (0.3) (coming from infinity). For this reason we are not able to conclude that (0.3) has at most two limit cycles in the entire plane.

3. The parabolic segment. In this section we make an easy application of the formulas derived in Section 1 to the parabolic segment case. We show that together the center and the periodic annulus surrounding it can produce at most two limit cycles. The precise statement of the result is formulated in Theorem 4. We point out that the expected total bound for the number of limit cycles in small perturbations of the parabolic segment case is two. However, to prove such a statement, we have to overcome two obstacles. The first one is to investigate the number of limit cycles produced by the parabolic polycycle itself; see [18]. As we mentioned above, the formulas for M_2, M_3 are useless in studying this problem. The second one is to check whether there are limit cycles which tend to infinity as $\varepsilon \to 0$. Such cycles could be produced from perturbations for which (0.3) has two foci. In this case the second focus will tend to infinity as $\varepsilon \to 0$

together with the eventual limit cycles surrounding it. Clearly the direct approach using Melnikov functions is not applicable to this problem too.

Proof of Theorem 4. We intend to use the Picard-Fuchs system derived in Section 1. Take the coordinate system in which H has the form (0.2) with $A = 0$. Then Picard-Fuchs system (1.5) reduces to

$$\begin{aligned} &hJ_{-1}' + 4J_0' - J_1' = \frac{1}{3}J_{-1} \\ &hJ_{-1}' + 3J_1' = 0 \\ &-hJ_{-1}' + hJ_0' = \frac{3}{4}J_0. \end{aligned} \tag{3.1}$$

To prove the theorem, it suffices to show that any linear combination

$$k_0J_0'(h) + k_1J_1'(h) + k_2J_{-1}'(h) \tag{3.2}$$

has at most two zeros in $[h_c, h_s) = [-3, 0)$. This will imply that for any k_j the linear combination $k_0J_0(h) + k_1J_1(h) + k_2J_{-1}(h)$ has at most three zeros. One of these zeros occurs always at $h = h_c$ and it produces no limit cycle provided it is a simple zero. We use the second equation in (3.1) in order to eliminate J_1' from (3.1) and (3.2). Noticing that $J_{-1}' < 0$ we come to the problem of finding the number of zeros of the function

$$F(h) = k_0R(h) - \frac{k_1}{3}h + k_2, \quad h \in [-3, 0), \tag{3.3}$$

where $R(h) = J_0'(h)/J_{-1}'(h)$ satisfies the corresponding Riccati equation easily derived from (3.1):

$$4h(h+3)R' = h + (2h+12)R - 3R^2. \tag{3.4}$$

We can write (3.4) also as a system with respect to (h, R):

$$\begin{aligned} \dot{h} &= 4h(h+3) \\ \dot{R} &= h + (2h+12)R - 3R^2. \end{aligned} \tag{3.5}$$

We will use (3.4) and (3.5) in proving that $R(h)$ is a strictly concave function. To do this, we first observe that by (3.4)

$$R(h) = 1 - \frac{1}{4}(h+3) - \frac{5}{384}(h+3)^2 + \cdots \quad \text{near } h = -3$$

and respectively

$$R(h) = \frac{1}{12}h\ln|h| + ah + o(h), \quad R'(h) = \frac{1}{12}\ln|h| + o(1) \quad \text{near } h = 0.$$

Therefore the graph of $R(h)$ coincides with the separatrix of (3.5) connecting the saddle-node $(-3,1)$ and the node $(0,0)$. Further, the zero isocline $\dot{R}=0$ is given by the hyperbola $h+(2h+12)R-3R^2=0$. The right branch of this hyperbola goes through $(0,0)$, $(0,4)$ and $(-3,1)$ and has the line $h=-3$ as its tangent. All this together with the asymptotics above implies that $R(h)$ is a strictly decreasing function satisfying $1>R(h)>0$ for $-3<h<0$. To finish the proof, it remains to show that $R''(h)<0$. Differentiating once (3.4) with respect to h and expressing in the obtained equation $R'(h)$ in terms of h and R using again (3.4), we get

$$8h^2(h+3)^2R''=6h-h^2-(27h+2h^2)R-(36-3h)R^2+9R^3.$$

Therefore the points in the (h,R) plane with $R''(h)=0$ lie on the algebraic curve

$$\Gamma:\ (2R+1)h^2-(3R^2-27R+6)h-(9R^3-36R^2)=0.$$

It is easy to see that Γ consists of the isolated point $\Gamma_0=(-3,1)$ and of three branches $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ contained respectively in the half-planes $h\geq 0,\quad R<-\frac{1}{2}$ and $R\geq\frac{2}{9}(7+2\sqrt{10})$. Consequently the graph $(h,R(h))$, $-3<h<0$, lies in the part of the (h,R) plane where $R''<0$. This yields $R(h)$ is strictly concave and hence $F(h)$ has at most two zeros (counted with multiplicity). Therefore each of $M_2(h)$ and $M_3(h)$ can produce at most two limit cycles in the parabolic segment case. Clearly the claimed limit cycles tend to periodic orbits or to the center of the unperturbed system. Theorem 4 is proved.

4. Saddle-loop bifurcations. In this section we study the saddle-loop bifurcations in the perturbed system (0.3), based on a result of Roussarie ([14]) and some facts from Picard-Lefschetz theory ([2]) (see also the classical monograph [1]). Recall that the formulas for the higher-order Melnikov functions derived in Section 1 are expressed in terms of linear combinations of the integrals J_k, $k=0,\pm1$:

$$I(h)=k_1J_1+k_0J_0+k_{-1}J_{-1}=\int_{H=h}\big(k_1x+k_0+\frac{k_{-1}}{x}\big)y\,dx. \tag{4.1}$$

In this formula h is taken in an interval where $H=h$ has closed compact components, corresponding to a periodic annulus of the unperturbed Hamiltonian system. Considering the loop case, we denote by h_s and by h_c the critical levels of H corresponding respectively to the saddle point $(x_s,0)$ and the center $(x_c,0)$. Using the normal form (0.2) for the Hamiltonian, we can write these values as

$$x_c=1,\quad x_s=\frac{A-2}{A},\quad h_c=A-3,\quad h_s=\frac{(A+1)(A-2)^2}{A^2},\quad A\notin[-1,2]. \tag{4.2}$$

Thus we obtain that in the loop case one has $h\in[h_c,h_s]$ and respectively $0<x_s<1$ for $A>2$, $1<x_s$ for $A<-1$.

It is well known ([2]) that for values near $h=h_s$ each function of type (4.1) has the expansion

$$I(h)=c_1+c_2(h-h_s)\ln|h-h_s|+c_3(h-h_s)+\cdots \tag{4.3}$$

in which

$$c_1 = I(h_s) \ \text{ and } \ c_2 = \big(k_1 x_s + k_0 + \frac{k_{-1}}{x_s}\big)(\text{Hess})^{-1/2}, \tag{4.4}$$

where Hess $= 12x_s$ is the value of the Hesseian $H_{xy}^2 - H_{xx}H_{yy}$ at the saddle point.

The central result in this section is the following one:

Theorem 5. *The condition* $c_1 = c_2 = c_3 = 0$ *implies* $k_1 = k_0 = k_{-1} = 0$.

Proof. The proof consists of long and routine computations. Using $c_2 = 0$ and the second formula in (4.4), we obtain $k_0 = -k_1 x_s - k_{-1}/x_s$ which replaced in (4.1) gives

$$I(h) = \int_{H=h} (x - x_s)\big(k_1 - \frac{k_{-1}}{x x_s}\big) y\, dx. \tag{4.5}$$

Provided $c_2 = 0$ in (4.3), the third coefficient c_3 is given by $c_3 = I'(h_s)$. Hence the theorem will be proved if we establish that $I(h_s) = I'(h_s) = 0$ leads to $k_1 = k_{-1} = 0$. From $H = xy^2 + xP_2(x) = h$ we obtain $2xyy_h = 1$. Also for $h = h_s$ we have $h_s - xP_2(x) = -A(x - x_s)^2(x - x_0)$, hence the equation of the saddle-loop is

$$y^2 = -\frac{A}{x}(x - x_s)^2(x - x_0).$$

Taking into account these formulas, we obtain differentiating $I(h)$:

$$\frac{dI}{dh}_{|h=h_s} = \int_{H=h_s} (x - x_s)\big(k_1 - \frac{k_{-1}}{x x_s}\big)\frac{dx}{2xy} = -\frac{1}{2A}\int_{H=h_s} \frac{k_1 - \frac{k_{-1}}{x x_s}}{(x - x_s)(x - x_0)} y\, dx.$$

Consider first the case $A > 0$. Then the area inside the loop can be determined by

$$x_s < x < x_0, \quad |y| < (x - x_s)\sqrt{\frac{A(x_0 - x)}{x}}.$$

Using the Stokes formula we write $c_1 = I(h_s)$ and $c_3 = I'(h_s)$ as double integrals and then solve them, obtaining the expressions

$$\begin{aligned} c_1 &= 4A^{1/2} x_0 [D_1(\sigma)k_1 + (x_0 x_s)^{-1} D_2(\sigma) k_{-1}], \\ c_3 &= 2A^{-1/2} [D_3(\sigma)k_1 + (x_0 x_s)^{-1} D_4(\sigma) k_{-1}], \end{aligned} \quad \text{where} \quad \sigma = \big(\frac{x_0}{x_s} - 1\big)^{1/2}.$$

It becomes clear now that the assertion in the theorem we prove is equivalent to showing that the determinant $\Delta_+(\sigma) = D_1(\sigma)D_4(\sigma) - D_2(\sigma)D_3(\sigma) \neq 0$ for $\sigma > 0$, or in explicit form,

$$\begin{aligned} \Delta_+(\sigma) &= (\sigma^4 - 6\sigma^2 - 15)\big(\frac{\arctan\sigma}{\sigma}\big)^2 \\ &\quad - \frac{1}{2}(\sigma^4 - 4\sigma^2 - 25)\big(\frac{\arctan\sigma}{\sigma}\big) - \frac{3\sigma^4 - 4\sigma^2 - 15}{6(1+\sigma^2)} \neq 0. \end{aligned} \tag{4.6}$$

Following the same way, in the case $A < 0$ instead of (4.6) we obtain

$$\begin{aligned} \Delta_-(\sigma) &= (\sigma^4 + 6\sigma^2 - 15)\big(\frac{1}{2\sigma}\ln\frac{1+\sigma}{1-\sigma}\big)^2 \\ &\quad - \frac{1}{2}(\sigma^4 + 4\sigma^2 - 25)\big(\frac{1}{2\sigma}\ln\frac{1+\sigma}{1-\sigma}\big) - \frac{3\sigma^4 + 4\sigma^2 - 15}{6(1-\sigma^2)} \neq 0 \end{aligned} \tag{4.7}$$

for $0 < \sigma < 1$. Therefore the next lemma finishes the proof.

Lemma 4. (i) $\Delta_+(\sigma) < 0$ *for any* $\sigma > 0$*;* (ii) $\Delta_-(\sigma) > 0$ *for any* $\sigma \in (0, 1)$*.*

Proof. Note that $\Delta_+(\sigma) = \Delta_-(i\sigma)$ for $\sigma \in (0, 1)$. To prove (ii), we expand $\Delta_-(\sigma)$ in a power series:

$$\Delta_-(\sigma) = \sum_{l=3}^{\infty} a_l \sigma^{2l+4},$$

where

$$a_l = \frac{2}{(l+2)(l+3)}\Big[-\frac{(2l+1)(2l+3)}{l+1}\Big(1 + \frac{1}{3} + \cdots + \frac{1}{2l+1}\Big) + \frac{(8l^3 + 86l^2 + 176l + 45)(l+1)}{3(2l+1)(2l+5)}\Big].$$

It is easily seen that the sequence $\{a_l\}$ is monotone increasing and its elements satisfy $0 < a_l < 4/3$. This proves (ii) as well as (i) for small $\sigma > 0$. Clearly $\Delta_+(\sigma) \to -\infty$ as $\sigma \to \infty$. For intermediate values of σ one can verify (i) by a direct computation.

As a consequence of Theorem 3 (the part concerning the Bogdanov-Takens loop) and Theorem 5, by the result of Roussarie ([14]) we obtain the following corollary which completes the result from [8] about saddle-loop bifurcations in perturbations of quadratic Hamiltonian vector fields.

Corollary. *For any nongeneric cubic Hamiltonian H with a saddle-loop, the perturbed system* (0.3) *has for small* ε *at most two limit cycles in a neighborhood of the saddle-loop.*

Proof of Theorem 2. The above corollary together with the main result in [8] prove Theorem 2 and answer the question about the cyclicity of a Hamiltonian loop under quadratic perturbations.

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