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ABSTRACT PERIODIC HAMILTONIAN SYSTEMS

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1. Introduction. This work concerns existence via variational arguments for abstract Hamiltonian systems

$$\begin{aligned} \mathcal{A}^*p &\in \partial_y\mathcal{H}(y,p) + f \\ \mathcal{A}y &\in \partial_p\mathcal{H}(y,p) + g \end{aligned} \tag{1.1}$$

in the product Hilbert space $\mathcal{X} \times \mathcal{X}$, where $\mathcal{H} : \mathcal{X} \times \mathcal{X} \to R$ is a convex continuous function, $\partial\mathcal{H} = (\partial_y\mathcal{H}, \partial_p\mathcal{H})$ is its subdifferential, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is a linear, densely defined closed operator with closed range $R(\mathcal{A})$ and $\mathcal{A}^*$ is its adjoint; f, g are fixed elements of $\mathcal{X}$.

A prototype of this system is the periodic Hamiltonian system

$$\begin{aligned} &y'(t) + Ay(t) \in \partial_p H(y(t), p(t)) + g(t), \quad t \in (0,T) \\ &p'(t) - A^*p(t) \in -\partial_y H(y(t), p(t)) - f(t) \\ &y(0) = y(T), \quad p(0) = p(T), \end{aligned} \tag{1.2}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup on the Hilbert space X, $H : X \times X \to R$ is a continuous convex function and $f, g \in L^2(0,T;X)$. Other problems which can be written in this form and will be considered below are second-order Hamiltonian systems with periodic conditions and elliptic Hamiltonian systems. The variational approach to be used below in existence theory of system (1.1) is inspired by the duality theory developed for finite-dimensional periodic Hamiltonian systems by Clarke and Ekeland ([6]). In a preliminary form some of these results were presented in [2].

Throughout in the sequel we assume familiarity with basic results and concepts of convex analysis and infinite-dimensional convex optimization (see e.g. [1]).

2. The main existence result. Here $\mathcal{X}$ is a real Hilbert space with the norm denoted $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$.

Hypotheses:

(i) *$\mathcal{H} : \mathcal{X} \times \mathcal{X} \to R$ is convex, continuous and satisfies the growth condition*

$$\gamma_1\|y\| + \gamma_2\|p\| + C_1 \le \mathcal{H}(y,p) \le \omega(\|y\|^2 + \|p\|^2) + C_2, \quad \forall\,(y,p) \in \mathcal{X} \times \mathcal{X} \tag{2.1}$$

where $\gamma_1, \gamma_2, \omega > 0$.

(ii) *$\mathcal{A} : \mathcal{X} \to \mathcal{X}$ is a linear, densely defined closed operator with closed range $R(\mathcal{A})$.*

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In particular, it follows by (ii) that $R(\mathcal{A}^*)$ is closed, too, and

$$\mathcal{A}^{-1} \in L(R(\mathcal{A}), \mathcal{X}), \quad (\mathcal{A}^*)^{-1} \in L(R(\mathcal{A}^*), \mathcal{X}) \tag{2.2}$$

$$\mathcal{X} = R(\mathcal{A}) \oplus N(\mathcal{A}^*) = R(\mathcal{A}^*) \oplus N(\mathcal{A}). \tag{2.3}$$

Here $\mathcal{A}^*$ is the dual of $\mathcal{A}$ and $N(\mathcal{A})$ is the null space of $\mathcal{A}$. We shall also assume that

(iii) $\mathcal{A}^{-1}$ *is compact from* $R(\mathcal{A})$ *to* $\mathcal{X}$.

We shall denote by $P : \mathcal{X} \to R(\mathcal{A})$ and $\widetilde{P} : \mathcal{X} \to R(\mathcal{A}^*)$ the projection operators on $R(\mathcal{A})$ and $R(\mathcal{A}^*)$, respectively. The last hypothesis refers to f and g.

(iv) $\|(I-P)g\| < \gamma_2, \quad \|(I-\widetilde{P})f\| < \gamma_1$.

Theorem 1. *Assume that Hypotheses* (i)–(iv) *hold and that*

$$\|\mathcal{A}^{-1}\|_{L(R(\mathcal{A}),\mathcal{X})} < (2\omega)^{-1}. \tag{2.4}$$

Then system (1.1) *has at least one solution* $(y,p) \in D(\mathcal{A}) \times D(\mathcal{A}^*)$.

Proof. Let $\mathcal{G} = \mathcal{H}^*$ be the conjugate of $\mathcal{H}$; i.e.,

$$\mathcal{G}(v,u) = \sup\{\langle v,y\rangle + \langle u,p\rangle - \mathcal{H}(y,p) : (y,p) \in \mathcal{X} \times \mathcal{X}\}. \tag{2.5}$$

Then (1.1) can be equivalently written as

$$(y,p) \in \partial\mathcal{G}(\mathcal{A}^*p - f, \mathcal{A}y - g). \tag{2.6}$$

Equivalently

$$(\mathcal{A}^{-1}u + \xi,\ (\mathcal{A}^*)^{-1}v + \eta) \in \partial\mathcal{G}(v-f, u-g) \tag{2.7}$$

where

$$u \in R(\mathcal{A}), \quad v \in R(\mathcal{A}^*); \quad \xi \in N(\mathcal{A}^*), \quad \eta \in N(\mathcal{A}). \tag{2.8}$$

Here $\partial\mathcal{G}$ is the subdifferential of $\mathcal{G}$.

We are led therefore to consider the minimization problem

$$\text{Min}\,\{\mathcal{G}(v-f, u-g) - \langle \mathcal{A}^{-1}u, v\rangle : u \in R(\mathcal{A}),\ v \in R(\mathcal{A}^*)\}. \tag{2.9}$$

We shall prove first that problem (2.9) has at least one solution. By (2.1) and (2.5) we have

$$\begin{aligned}\mathcal{G}(v-f,u-g) &\geq \langle v-f,p\rangle + \langle u-g,q\rangle - \mathcal{H}(p,q)\\ &\geq \langle v-f,p\rangle + \langle u-g,q\rangle - \omega(\|p\|^2 + \|q\|^2) - C_2,\ \forall\,(p,q) \in R.\end{aligned}$$

This yields

$$\mathcal{G}(v-f,u-g) \geq (4\omega)^{-1}(\|v\|^2 + \|u\|^2) - C_3, \quad \forall\,u,v \in \mathcal{X}. \tag{2.10}$$

On the other hand, by (2.4) we have

$$\langle \mathcal{A}^{-1}u, v\rangle \leq ((4\omega)^{-1} - \delta)(\|u\|^2 + \|v\|^2), \quad \forall\, (u,v) \in R(\mathcal{A}) \times R(\mathcal{A}^*)$$

where $\delta > 0$. Hence

$$\mathcal{G}(v-f, u-g) - \langle \mathcal{A}^{-1}u, v\rangle \geq \delta(\|u\|^2 + \|v\|^2) + C_4, \quad \forall\, (u,v) \in R(\mathcal{A}) \times R(\mathcal{A}^*). \quad (2.11)$$

Since $\mathcal{G}$ is convex, lower semicontinuous, and by hypothesis (iii) the function $(u, v) \to \langle \mathcal{A}^{-1}u, v\rangle$ is weakly lower semicontinuous we infer by (2.11) that problem (2.9) has at least one solution (u^*, v^*). To conclude the proof it suffices to show that (u^*, v^*) satisfies (2.7), (2.8). To this end consider the approximation problem

$$\begin{aligned} \text{Min}\ \{&\mathcal{G}_\lambda(v-f, u-f) - \langle \mathcal{A}^{-1}u, v\rangle + 2^{-1}\|u-u^*\|^2 + \\ &+ 2^{-1}\|v-v^*\|^2 : u \in R(\mathcal{A}),\ v \in R(\mathcal{A}^*)\} \end{aligned} \quad (2.12)$$

where

$$\mathcal{G}_\lambda(v,u) = \inf\{(2\lambda)^{-1}(\|v-v^*\|^2 + \|u-u^*\|^2) + \mathcal{G}(\bar{v}, \bar{u}) : (\bar{v}, \bar{u}) \in \mathcal{X} \times \mathcal{X}\},\ \lambda > 0. \quad (2.13)$$

We recall that $\mathcal{G}_\lambda$ is Fréchet differentiable and

$$\nabla \mathcal{G}_\lambda = \lambda^{-1}(I - (I + \lambda\partial\mathcal{G})^{-1}).$$

We have also by (2.10) and (2.13)

$$\begin{aligned} \mathcal{G}_\lambda(v-f, u-g) &= \mathcal{G}((I+\lambda\partial\mathcal{G})^{-1}(v-f, u-g)) \\ &\quad + (2\lambda)^{-1}\|(I+\lambda\partial\mathcal{G})^{-1}(v-f,u-g) - (v-f,u-g)\|^2 \\ &\geq (4\omega)^{-1}(\|v-f\|^2 + \|u-g\|^2) - C_5\lambda\mathcal{G}(v-f,u-g) \\ &\geq (8\omega)^{-1}(\|v\|^2 + \|v\|^2) + C_6, \quad \forall\, \lambda > 0, \end{aligned}$$

because $\mathcal{G}_\lambda \leq \mathcal{G}$ and $\mathcal{G}$ is bounded from below by an affine function. Hence problem (2.12) has a solution $(u_\lambda, v_\lambda) \in R(\mathcal{A}) \times R(\mathcal{A}^*)$. Moreover, arguing as in [1] it follows that

$$u_\lambda \to u^*, \quad v_\lambda \to v^* \ \text{ strongly in } \mathcal{X}. \quad (2.14)$$

Now since $\mathcal{G}_\lambda$ is differentiable, the optimal pair (u_λ, v_λ) in problem (2.12) satisfies the first-order optimality system

$$\nabla\mathcal{G}_\lambda(v_\lambda - f, u_\lambda - g) - (\mathcal{A}^{-1}u_\lambda, (\mathcal{A}^*)^{-1}v_\lambda) + (v_\lambda - v^*, u_\lambda - u^*) = (\eta_\lambda, \xi_\lambda) \quad (2.15)$$

$$\eta_\lambda \in N(\mathcal{A}) = R(\mathcal{A}^*)^\perp,\ \xi_\lambda \in N(\mathcal{A}^*) = R(\mathcal{A})^\perp. \quad (2.16)$$

By (2.14) we have

$$\left.\begin{array}{r}\mathcal{A}^{-1}u_\lambda \to y = \mathcal{A}^{-1}u^*,\\ (\mathcal{A}^*)^{-1}v_\lambda \to p = (\mathcal{A}^*)^{-1}v^*\end{array}\right. \text{strongly in } \mathcal{X} \text{ as } \lambda \to 0. \tag{2.17}$$

Next by (2.15) we have

$$\begin{aligned}&\langle \eta_\lambda + \mathcal{A}^{-1}u_\lambda + v^* - v_\lambda, v_\lambda - w_1\rangle + \langle \xi_\lambda + (\mathcal{A}^*)^{-1}v_\lambda + u^* - u_\lambda, u_\lambda - w_2\rangle\\ &\quad \geq \mathcal{G}_\lambda(v_\lambda - f, u_\lambda - g) - \mathcal{G}_\lambda(w_1 - f, w_2 - g)\end{aligned}$$

for all $w_1 \in N(\mathcal{A}),\ w_2 \in N(\mathcal{A}^*)$, $\|w_1\| = \|w_2\| = \varepsilon$. We get

$$\varepsilon(\|\eta_\lambda\| + \|\xi_\lambda\|) \leq \mathcal{G}_\lambda(w_1 - f, w_2 - g) + C_7, \quad \forall\, \lambda > 0. \tag{2.18}$$

On the other hand, by (2.1) and(2.13) we have

$$\begin{aligned}\mathcal{G}_\lambda(w_1 - f, w_2 - g) &\leq \mathcal{G}(w_1 - f, w_2 - g) \leq \langle w_1 - f, p\rangle + \langle w_2 - g, q\rangle - H(p,q)\\ &\leq \varepsilon(\|p\| + \|q\|) + \|f_2\|\,\|p\| + \|g_2\|\,\|q\| - \gamma_1\|p\| - \gamma_2\|q\| + C_8,\\ &\quad \forall\,(p,q) \in N(\mathcal{A}) \times N(\mathcal{A}^*).\end{aligned}$$

Here $f_2 = (I - \widetilde{P})f,\ g_2 = (I - P)g$. Then for ε sufficiently small we have by (iv) that

$$\mathcal{G}_\lambda(w_1 - f, w_2 - g) \leq C_8. \tag{2.19}$$

By (2.18) and (2.19) it follows that

$$\|\eta_\lambda\| + \|\xi_\lambda\| \leq C_9, \quad \forall\, \lambda > 0$$

and therefore on a subsequence, again denoted λ, we have

$$\eta_\lambda \to \eta, \quad \xi_\lambda \to \xi \ \text{ weakly in } \mathcal{X}. \tag{2.20}$$

Since $\partial\mathcal{G}$ is strongly–weakly closed in $\mathcal{X} \times \mathcal{X}$ and $\partial\mathcal{G}_\lambda = \partial\mathcal{G}(I + \lambda\partial G)^{-1}$ by virtue of (2.17), (2.20) we may pass to the limit in (2.15) to conclude that (u^*, v^*) satisfies (2.7), (2.8) as claimed. This completes the proof.

Remark 1. Theorem 1 applies neatly to boundary value problems of the form

$$\begin{aligned}(A^*p)(x) &\in \alpha(y(x), p(x)) + f(x), \quad x \in \Omega\\ (Ay)(x) &\in \beta(y(x), p(x)) + g(x), \quad x \in \Omega\end{aligned} \tag{2.21}$$

where A is an elliptic linear operator in $\Omega \subset R^n$ with usual boundary conditions on $\partial\Omega$ whilst $(\alpha, \beta) = \partial h$ and $h : R \times R \to R$ is a continuous convex function. More will be said about this problem in Section 3 below.

3. First-order periodic Hamiltonian systems. In particular, Theorem 1 applies to system (1.2) where $\mathcal{X} = L^2(0,T;X)$,

$$\mathcal{H}(y,p) = \int_0^T H(y(t),p(t))\,dt, \quad \forall\, y,p \in \mathcal{X} \tag{3.1}$$

and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned} &\mathcal{A}y = f \ \text{ if and only if } \int_0^T \big((f(t),\varphi(t)) + (y(t),\varphi'(t) - A^*\varphi(t))\big)\,dt = 0 \\ &\forall\, \varphi \in Y = \{\varphi \in W^{1,2}([0,T];X) : A^*\varphi \in L^2([0,T];X), \varphi(0) = \varphi(T)\}. \end{aligned} \tag{3.2}$$

Here $W^{1,2}([0,T];X) = \{\Psi \in L^2([0,T];X),\ \Psi' \in L^2([0,T];X)\}$ and A^* is the adjoint of A.

The linear operator $\mathcal{A}$ is closed, densely defined and can be equivalently defined as ([3])

$$\begin{aligned} &\mathcal{A}y = f \ \text{ if and only if } \ y \in C([0,T];X) \ \text{ and} \\ &y(t) = e^{-At}y(T) + \int_0^t e^{-A(t-s)}f(s)\,ds, \quad \forall\, t \in [0,T]. \end{aligned} \tag{3.3}$$

This implies that

$$R(\mathcal{A}) = \Big\{ f \in L^2([0,T];X) : \int_0^T e^{-A(T-s)}f(s)\,ds \in R(I - e^{-AT}) \Big\}, \tag{3.4}$$

$$N(\mathcal{A}) = \{ y(t) = e^{-At}y_0 : y_0 \in N(I - e^{-AT})\}. \tag{3.5}$$

Hence if $R(I - e^{-AT})$ is closed in X then $R(\mathcal{A})$ is closed in $\mathcal{X}$. In particular, this happens if e^{-AT} is compact.

The adjoint operator $\mathcal{A}^*$ can be equivalently defined as

$$\mathcal{A}^*z = g \quad \text{if and only if } \ z(t) = e^{-A^*(T-t)}z(0) + \int_t^T e^{-A^*(s-t)}g(s)\,ds, \quad t \in [0,T]. \tag{3.6}$$

We shall denote by $|\cdot|$ the norm of X and by $\langle \cdot,\cdot \rangle_0$ its scalar product.

In terms of $\mathcal{A}, \mathcal{A}^*$ and $\mathcal{H}$ we may write system (1.2) as (1.1). However, in order to apply Theorem 1 we must assume that

(j) *$H : X \times X \to R$ is convex, continuous and*

$$H(y,p) \le \omega(|y|^2 + |p|^2) + C_1, \quad \forall\, (y,p) \in X \times X \tag{3.7}$$

for some $\omega > 0$ and $C_1 \in R$.

(jj) *$-A$ generates a C_0-semigroup e^{-At} on X and e^{-AT} is compact. Moreover,*

$$\|\mathcal{A}^{-1}\|_{L(R(\mathcal{A}),\mathcal{X})} < (2\omega)^{-1}. \tag{3.8}$$

Then by Theorem 1 we have

Corollary 1. *Let Hypotheses* (j), (jj) *hold and*

$$H(y,p) \geq \gamma(|y|^2 + |p|^2) + C_2, \quad \forall\, (y,p) \in X \times X \tag{3.9}$$

for some $\gamma > 0$ *and* $C_2 \in R$. *Then for all* $f, g \in L^2([0,T];X)$ *problem* (1.2) *has at least one solution* $(y,p) \in C([0,T];X) \times C([0,T];X)$.

The solution (y,p) to (1.2) is considered of course in the weak sense (3.3), (3.6); i.e.,

$$\begin{aligned}(\mathcal{A}y)(t) &\in \partial_p H(y(t),p(t)) + g(t), \quad \text{a.e.} \quad t \in (0,T)\\ (\mathcal{A}^* p)(t) &\in \partial_y H(y(t),p(t)) + f(t).\end{aligned} \tag{1.8$'$}$$

The condition (3.9) can be weakened to

$$H(y,p) \geq \gamma_1 |y| + \gamma_2 |p| + C_3, \quad \forall\, (y,p) \in X \times X \tag{3.10}$$

if

$$f, g \in C([0,T];X) \tag{3.11}$$

and

$$\sup\{|f_2(t)| : t \in [0,T]\} < \gamma_1, \tag{3.12}$$

$$\sup\{|g_2(t)| : t \in [0,T]\} < \gamma_2. \tag{3.13}$$

Here $f_2 = (I - \widetilde{P})f$, $g_2 = (I - P)g$.

Theorem 2. *Assume that Hypotheses* (j), (jj), (3.10)–(3.13) *hold. Then system* (1.2) *has at least one weak solution* $(y,p) \in C([0,T];X) \times C([0,T];X)$.

Proof. According to Corollary 1, for each $\varepsilon > 0$ the system

$$\begin{aligned}(\mathcal{A}y)(t) &\in \partial_p H(y(t),p(t)) + \varepsilon p(t) + g(t), \quad \text{a.e.} \quad t \in (0,T)\\ (\mathcal{A}^* p)(t) &\in \partial_y H(y(t),p(t)) + \varepsilon y(t) + f(t)\end{aligned} \tag{3.14}$$

has at least one solution $(y_\varepsilon, p_\varepsilon) \in C([0,T];X) \times C([0,T];X)$. We set $G = H^*$ (the conjugate of H). Then we may rewrite (3.14) as

$$(y_\varepsilon(t), p_\varepsilon(t)) \in \partial G((\mathcal{A}^* p_\varepsilon)(t) - \varepsilon y_\varepsilon(t) - f(t), (\mathcal{A}y_\varepsilon)(t) - \varepsilon p_\varepsilon(t) - g(t))$$

for almost every $t \in (0,T)$. Equivalently,

$$\begin{aligned}(\eta_\varepsilon(t) + (\mathcal{A}^{-1}u_\varepsilon)(t), \xi_\varepsilon(t) + (\mathcal{A}^*)^{-1} p_\varepsilon(t)) &\in \partial G(v_\varepsilon(t) - \varepsilon(\mathcal{A}^{-1}u_\varepsilon)(t)\\ &- \varepsilon\eta_\varepsilon(t) - f(t), u_\varepsilon(t) - \varepsilon(\mathcal{A}^{-1}v_\varepsilon)(t) - \varepsilon\xi_\varepsilon(t) - g(t))\end{aligned} \tag{3.15}$$

$$\eta_\varepsilon \in N(\mathcal{A}), \quad \xi_\varepsilon \in N(\mathcal{A}^*); \quad u_\varepsilon \in R(\mathcal{A}), \quad v_\varepsilon \in R(\mathcal{A}^*). \tag{3.16}$$

This yields

$$\begin{aligned}&\langle\eta_\varepsilon(t)+(\mathcal{A}^{-1}u_\varepsilon)(t),v_\varepsilon(t)-\varepsilon(\mathcal{A}^{-1}u_\varepsilon)(t)-\varepsilon\eta_\varepsilon(t)-w_1(t)\rangle_0\\&+\langle\xi_\varepsilon(t)+(\mathcal{A}^*)^{-1}v_\varepsilon(t),u_\varepsilon(t)-\varepsilon((\mathcal{A}^*)^{-1}v_\varepsilon)(t)-\varepsilon\xi_\varepsilon(t)-w_2(t)\rangle_0\\&\geq G(v_\varepsilon(t)-\varepsilon(\mathcal{A}^{-1}u_\varepsilon)(t)-\varepsilon\eta_\varepsilon(t)-f(t),u_\varepsilon(t)-\varepsilon(\mathcal{A}^*)^{-1}v_\varepsilon(t)\\&\quad-\varepsilon\xi_\varepsilon(t)-g(t))-G(w_1(t)-f(t),w_2(t)-g(t)),\quad \text{a.e.}\quad t\in(0,T)\end{aligned}\tag{3.17}$$

for all $w_1\in N(\mathcal{A}),\ w_2\in N(\mathcal{A}^*)$.

On the other hand, by (3.7) we have

$$G(v,u)\geq(4\omega)^{-1}(|u|^2+|v|^2)+C_4,\quad\forall\,(u,v)\in X\times X\tag{3.18}$$

whilst by (3.10) we see that

$$G(w_1-f,w_2-g)\leq\langle w_1-f_2,p\rangle_0+\langle w_2-g_2,q\rangle_0-\gamma_1|p|-\gamma_2|g|+C_3,\quad\text{a.e.}\ \ t\in(0,T)$$

for all $p\in N(\mathcal{A}),\ q\in N(\mathcal{A}^*)$.

By virtue of (3.12), (3.13) the latter yields

$$G(w_1-f,w_2-g)\leq C_5,\quad\text{a.e.}\quad t\in(0,T)\tag{3.19}$$

for all $(w_1,w_2)\in N(\mathcal{A})\times N(\mathcal{A}^*)$, $|w_1(t)|=|w_2(t)|=\lambda$, for almost every $t\in(0,T)$ and λ sufficiently small.

By (3.8), (3.17), (3.18), (3.19) we get the estimate

$$\begin{aligned}&\int_0^T(|u_\varepsilon(t)|^2+|v_\varepsilon(t)|^2+\varepsilon|\eta_\varepsilon(t)|^2+\varepsilon|\xi_\varepsilon(t)|^2)\,dt\\&+\int_0^T(|\eta_\varepsilon(t)|+|\xi_\varepsilon(t)|)\,dt\leq C_6,\quad\forall\,\varepsilon>0.\end{aligned}\tag{3.20}$$

Since by virtue of (3.4) and Hypothesis (jj) $N(\mathcal{A})$ and $N(\mathcal{A}^*)$ are finite dimensional it follows by (3.20) that $\{\eta_\varepsilon\}$, $\{\xi_\varepsilon\}$ are compact in $C([0,T];X)$ whilst $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ are weakly compact in $L^2([0,T];X)$. Hence on a subsequence, again denoted ε, we have

$$\begin{array}{lll}u_\varepsilon\longrightarrow u^*, & v_\varepsilon\longrightarrow v^* & \text{weakly in } L^2([0,T];X)\\ \eta_\varepsilon\longrightarrow\eta^*, & \xi_\varepsilon\longrightarrow\xi^* & \text{strongly in } C([0,T];X)\\ \mathcal{A}^{-1}u_\varepsilon\longrightarrow\mathcal{A}^{-1}u^*, & (\mathcal{A}^*)^{-1}v_\varepsilon\longrightarrow(\mathcal{A}^*)^{-1}v^* & \text{strongly in } L^2([0,T];X)\end{array}$$

because $\mathcal{A}^{-1}$ and $(\mathcal{A}^*)^{-1}$ are compact.

Thus letting ε tend to zero in (3.15) we get

$$(\eta^*+\mathcal{A}^{-1}u^*,\xi^*+(\mathcal{A}^*)^{-1}v^*)\in\partial G(v^*-f,u^*-g),\quad\text{a.e.}\quad t\in(0,T).$$

Hence $y^* = \mathcal{A}^{-1}u^* + \eta^*$ and $p^* = (\mathcal{A}^*)^{-1}v^* + \xi^*$ satisfy system (1.2) as desired.

Now we shall consider the particular case

$$X = L^2(\Omega), \quad H(y,p) = \int_\Omega h(y(x), p(x))\,dx \quad \forall\, y, p \in L^2(\Omega), \tag{3.21}$$

where $h : R \times R \to R$ is a continuous convex function satisfying the conditions

$$\gamma_1|y| + \gamma_2|p| + C_7 \le h(y,p) \le \omega(y^2 + p^2) + C_8, \quad \forall\, y, p \in R, \tag{3.22}$$

where $\gamma_1, \gamma_2, \omega > 0$. Here Ω is an open and bounded subset of R^n. In this case the Hamiltonian system (1.2) has the following form:

$$\begin{aligned} & y_t + Ay \in \partial_p h(y,p) + g \quad \text{in} \quad Q = \Omega \times (0,T) \\ & p_t - A^*p \in -\partial_y h(y,p) - f \\ & y(x,0) = y(x,T), \quad p(x,0) = p(x,T) \quad \forall\, x \in \Omega \end{aligned} \tag{3.23}$$

or, more precisely,

$$\begin{aligned} & (\mathcal{A}y)(t) \in \partial_p h(y(x,t), p(x,t)) + g(x,t), \quad \text{a.e.} \quad (x,t) \in Q \\ & (\mathcal{A}^*p)(t) \in \partial_y h(y(x,t), p(x,t)) + f(x,t), \quad \text{a.e.} \quad (x,t) \in Q \end{aligned} \tag{3.24}$$

where $\partial h = (\partial_y h, \partial_p) : R^2 \to R^2$ is the subdifferential of h.

Theorem 2 is not directly applicable since the left-hand side of inequality (3.22) is weaker than condition (3.10). We have however the following result.

Theorem 3. *Assume that Hypotheses* (jj), (3.22) *hold and that $f, g \in L^\infty(Q)$ are such that*

$$\|(I - \widetilde{P})f\|_{L^\infty(Q)} < \gamma_1, \quad \|(I-P)g\|_{L^\infty(Q)} < \gamma_2. \tag{3.25}$$

Then system (3.23) *has at least one weak solution*

$$(y,p) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega)).$$

Proof. We set $G_0 = h^*$. Then by (3.22) we see that

$$G_0(u,v) \ge (4\omega)^{-1}(v^2 + u^2) + C_9, \quad \forall\, v, u \in R \tag{3.26}$$

and

$$\begin{aligned} G_0(w_1(x,t) - f(x,t), w_2(x,t) - g(x,t)) &\le (w_1(x,t) - f(x,t))p \\ &+ (w_2(x,t) - g(x,t))q - \gamma_1|p| - \gamma_2|q|, \quad \forall\, p, q \in R, \quad \text{a.e.} \quad (x,t) \in Q \end{aligned} \tag{3.27}$$

for $w_1 \in N(\mathcal{A}),\ w_2 \in N(\mathcal{A}^*),\ \|w_1\|_{L^\infty(Q)} = \|w_2\|_{L^\infty(Q)} = \lambda$.

Recalling that

$$G(v,u) = \int_\Omega G_0(v(x), u(x))\,dx \quad \forall\, v, u \in L^2(\Omega)$$

it follows by (3.17), (3.26), (3.27) that

$$\begin{aligned} & \int_Q (u_\varepsilon^2(x,t) + v_\varepsilon^2(x,t) + \varepsilon\eta_\varepsilon^2(x,t) + \varepsilon\xi_\varepsilon^2(x,t))\,dx\,dt \\ & + \int_Q (|\eta_\varepsilon(x,t)| + |\xi_\varepsilon(x,t)|)\,dx\,dt \le C_{10}, \quad \forall\, \varepsilon > 0. \end{aligned}$$

From now on the proof is identical with that of Theorem 2.

Lemma 1. *Let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be defined by*

$$\begin{aligned} Ay &= -\Delta y + b(x) \cdot \nabla y + c(x)y, \\ D(A) &= H_0^1(\Omega) \cap H^2(\Omega), \end{aligned} \tag{3.28}$$

where $b \in W^{1,\infty}(\Omega; R^n)$, $c \in L^\infty(\Omega)$ and Ω is an open bounded subset of R^n with smooth boundary $\partial\Omega$.

Then system (3.23) *has the form*

$$\begin{aligned} & y_t - \Delta y + b\nabla y + cy \in \beta(y,p) + g && \text{in } Q \\ & p_t + \Delta p + \operatorname{div}(bp) - cp \in -\alpha(y,p) - f && \text{in } Q \\ & y = p = 0 && \text{in } \partial\Omega \times (0,T) \\ & y(x,0) = y(x,T), \quad p(x,0) = p(x,T) && \text{in } \Omega \end{aligned} \tag{3.29}$$

where $(\alpha(y,p), \beta(y,p)) = \partial h(y,p)$, $\forall\, (y,p) \in R \times R$ satisfy the conditions

$$|\alpha(y,p)| + |\beta(y,p)| \le 2\omega(|y| + |p|) + C, \quad \forall\, y,p \in R \tag{3.30}$$

$$[0,\gamma_1] \times [0,\gamma_2] \subset \operatorname{int} R(F) \tag{3.31}$$

$F(y,p) = (\alpha(y,p), \beta(y,p))$.

Then assumption (3.22) holds and so by Theorem 3 we have

Corollary 2. *Assume that $\partial h = (\alpha,\beta)$ satisfy conditions* (3.30), (3.31) *and*

$$\|f\|_{L^\infty(Q)} < \gamma_1, \quad \|g\|_{L^\infty(Q)} < \gamma_2. \tag{3.32}$$

Then for ω sufficiently small, system (3.29) *has at least one weak solution $(y,p) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$.*

By regularity theory for parabolic boundary problems it follows that (y,p) is a strong solution to (3.29); i.e.,

$$y, p \in W^{1,2}([\delta_1, \delta_2]; L^2(\Omega)) \cap L^2(\delta_1, \delta_2; H_0^1(\Omega) \cap H^2(\Omega))$$

for all $0 < \delta_1 < \delta_2 < T$.

Consider the elliptic system

$$\begin{aligned} -\Delta y + b\nabla y + cy &\in \beta(y,p) + g && \text{in } \Omega \\ \Delta p + \operatorname{div}(bp) - cp &\in -\alpha(y,p) - f && \text{in } \Omega \end{aligned} \tag{3.33}$$

where b, c are as above, $(\alpha,\beta) = \partial h$ satisfy (3.30), (3.31) and

$$\|f\|_{L^\infty(\Omega)} < \gamma_1, \quad \|g\|_{L^\infty(\Omega)} < \gamma_2. \tag{3.34}$$

We may write (3.33) in the form (1.1) or (3.24) in the space $\mathcal{X} = L^2(\Omega)$ where $\mathcal{A} = A$ is given by (3.28). It is easily seen that $R(A)$ is closed in $L^2(\Omega)$. Then, by the same proof, it follows that Theorem 3 remains true in the present case and so we have (see Corollary 2)

Corollary 3. *Under assumptions* (3.30), (3.31), (3.34) *if* $\|A^{-1}\|_{L(R(A),L^2(\Omega))} < 2\omega$, *then problem* (3.33) *has at least one solution* $(y,p) \in (H_0^1(\Omega) \cap H^2(\Omega))^2$.

Remark 2. There is a substantial body of work concerning elliptic systems of the form (3.33) in the resonant case but we omit the references and refer to [6] for some recommended results.

4. Second-order periodic Hamiltonian systems. We shall study here the periodic problem

$$\begin{aligned} &y''(t) + Ay(t) \in \partial_p H(y(t),p(t)) + g(t), \quad t \in (0,T) \\ &p''(t) + Ap(t) \in \partial_y H(y(t),p(t)) + f(t), \\ &y^{(k)}(0) = y^{(k)}(T), \quad p^{(k)}(0) = p^{(k)}(T), \quad k = 0,1, \end{aligned} \tag{4.1}$$

where $H : X \times X \to R$ is a continuous convex function, and

(kk) $A : D(A) \subset X \to X$ *is a self-adjoint positively defined operator,* $D(A^{1/2})$ *is compact in* X *and the eigenvalues* λ_n^2 *to* A *satisfy the condition*

$$\inf\{|\lambda_n - |\mu_m|| : \lambda_n \neq |\mu_m|\} > 0, \quad \mu_m = 2m\pi/T, \ m \in Z. \tag{4.2}$$

Let $\mathcal{W} : D(\mathcal{W}) \subset L^2([0,T];X) \to L^2([0,T];X)$ be the linear operator defined by

$$\mathcal{W}y = f \tag{4.3}$$

if and only if

$$\int_0^T (y(t), \varphi''(t) + A\varphi(t))\,dt = \int_0^T (f(t),\varphi(t))\,dt \tag{4.4}$$

$\forall\, \varphi \in Y_1 = \{\varphi \in C^2([0,T];H) \cap C([0,T];D(A)),\ \varphi^{(k)}(0) = \varphi^{(k)}(T),\ k = 0,1\}$. We set $\mathcal{X} = L^2([0,T];X)$.

Lemma 1. *Under assumption* (kk) *the operator* $\mathcal{W}$ *is self-adjoint and has closed range* $R(\mathcal{W})$ *in* $\mathcal{X}$. *Moreover,* $\mathcal{W}^{-1}$ *is continuous and compact from* $R(\mathcal{W})$ *to* $\mathcal{X}$ *and*

$$\|\mathcal{W}^{-1}\|_{L(R(\mathcal{W}),\mathcal{X})} \leq \rho^{-1} \tag{4.5}$$

where

$$\rho = \inf\{|\lambda_n^2 - |\mu_m|^2| : \lambda_n \neq |\mu_m|\}. \tag{4.6}$$

Proof. It is readily seen that $\mathcal{W}$ is linear and self-adjoint. If $f = \mathcal{W}y$ we have

$$y = \sum_{|\mu_m| \neq \lambda_n} y_{nm}\, e^{i\mu_m t}\, \varphi_n \tag{4.7}$$

where φ_n are the eigenfunctions of A and

$$y_{nm} = (\lambda_n^2 - \mu_m^2)^{-1} f_{nm}. \tag{4.8}$$

Here f_{nm} are the Fourier coefficients of f with respect to the complete system

$$\{e^{i\mu_m t}\varphi_n\} \subset L^2([0,T];X) = \mathcal{X}.$$

Then by (4.6) and the Parseval equality we see that

$$\|y\|_{\mathcal{X}} \leq \rho^{-1}\|f\|_{\mathcal{X}} \tag{4.9}$$

which implies that $R(\mathcal{W})$ is closed. Hence

$$\mathcal{X} = R(\mathcal{W}) \oplus N(\mathcal{W}) \tag{4.10}$$

and $\mathcal{W}^{-1}$ is continuous from $R(\mathcal{W})$ to $\mathcal{X}$. Moreover, by (4.7), (4.8) we have

$$\|y'\|_{\mathcal{X}} + \|A^{1/2}y\|_{\mathcal{X}} \leq C_1\|f\|_{\mathcal{X}}, \quad \forall\, f \in R(\mathcal{W}). \tag{4.11}$$

Since $D(A^{1/2})$ is compactly imbedded in H the latter implies that $\mathcal{W}^{-1}$ is compact in $R(\mathcal{W})$ as claimed.

By a weak solution to system (4.1) we mean a pair

$$(y,p) \in L^2([0,T];X) \times L^2([0,T];X)$$

which satisfies this system in the weak sense (4.4); i.e.,

$$\begin{aligned} \mathcal{W}y &\in \partial_p H(y,p) + g, \quad \text{a.e.} \quad t \in (0,T) \\ \mathcal{W}p &\in \partial_y H(y,p) + f, \quad \text{a.e.} \quad t \in (0,T). \end{aligned} \tag{4.12}$$

Applying Theorem 1 where $\mathcal{A} = \mathcal{W}$ and Lemma 1 we get

Corollary 4. *Assume that the convex continuous function* $H : X \times X \to R$ *satisfies conditions* (3.7), (3.9) *and* A *satisfies Hypothesis* (kk). *Assume further that*

$$\rho > 2\omega. \tag{4.13}$$

Then for all $f, g \in L^2([0,T];X)$ *satisfying assumption* (iv) *system* (4.1) *has at least one weak solution* $(y,p) \in L^2([0,T];X) \times L^2([0,T];X)$.

We have also

Theorem 4. *If* $N(\mathcal{W})$ *is finite dimensional and* $f, g \in C([0,T];X)$ *then in Corollary* 4 *assumptions* (3.9), (i.v) *can be weakened to* (3.10), (3.12), (3.13) *and the solution* (y,p) *is* $W^{1,2}([0,T];X) \cap L^2(0,T;D(A^{1/2}))^2$.

We omit the proof which is exactly the same as that of Theorem 2. As regards the regularity of weak solution (y,p), it follows by estimate (4.11).

In particular, Hypothesis (kk) is satisfied on $X = L^2(0,\pi)$ by the operator

$$Ay = -\frac{d^2y}{dx^2}, \quad D(A) = H_0^1(0,\pi) \cap H^2(0,\pi),$$

if T is a rational multiple of π. In this case

$$\rho = \inf\left\{ \left|n^2 - \left(\frac{2m\pi}{T}\right)^2\right| : n \neq \left|\frac{2m\pi}{T}\right| \right\} \tag{4.14}$$

and Theorems 3 and 4 applied to

$$H(y,p) = \int_0^\pi h(y(x),p(x))\,dx, \quad \forall\, y,p \in L^2(0,\pi) \tag{4.15}$$

(see (3.19), (3.20)) lead to an existence result for the system

$$\begin{aligned} &y_{tt} - y_{xx} \in \partial_p h(y,p) + g && \text{in } \; Q = (0,\pi)\times(0,T)\\ &p_{tt} - p_{xx} \in \partial_p h(y,p) + f && \text{in } \; Q\\ &y = p = 0 && \text{in } \; x = 0,\pi\\ &y(x,0) = y(x,T), \quad p(x,0) = p(x,T) && \text{in } \; (0,\pi). \end{aligned} \tag{4.16}$$

To be more specific we shall consider the particular case where

$$h(y,p) = j(y) + \psi(p), \quad \forall\, y,p \in R;$$

i.e.,

$$\begin{aligned} &y_{tt} - y_{xx} = b(p) + g && \text{in } \; Q\\ &p_{tt} - p_{xx} = a(y) + f && \\ &y = p = 0 && \text{in } \; x = 0,\pi\\ &y(x,0) = y(x,T), \quad p(x,0) = p(x,T) && \text{in } \; (0,\pi) \end{aligned} \tag{4.17}$$

where $a : R \to R$ and $b : R \to R$ are continuous increasing functions (or, more generally, maximal monotone graphs) which satisfy the following condition:

$$|a(r)|,\ |b(r)| \leq \gamma|r| + C_2, \quad \forall\, r \in R. \tag{4.18}$$

Assume also that $f, g \in L^\infty(Q)$ and

$$b(-\infty) + \delta \leq -g_2(x,t) \leq b(+\infty) - \delta, \qquad \text{a.e. } \; (x,t) \in Q \tag{4.19}$$

$$a(-\infty) + \delta \leq -f_2(x,t) \leq a(+\infty) - \delta, \qquad \text{a.e. } \; (x,t) \in Q \tag{4.20}$$

where $\delta > 0$ and $f_2 = (I-P)f$, $g_2 = (I-P)g$ and $P : L^2(Q) \to R(\mathcal{W})$ is the projection operator on $R(\mathcal{W})$.

Theorem 5. *Assume that conditions* (4.18), (4.19), (4.20) *hold and that* T *is a rational multiple of* π. *If* $2\rho > \gamma$ *then system* (4.17) *has at least one weak solution* $(y,p) \in L^2(Q) \times L^2(Q)$.

Proof. By Corollary 4, for each $\varepsilon > 0$ the system

$$\begin{aligned} \mathcal{W}y_\varepsilon &= b(p_\varepsilon) + \varepsilon p_\varepsilon + g \\ \mathcal{W}p_\varepsilon &= a(y_\varepsilon) + \varepsilon py\varepsilon + f \end{aligned} \tag{4.21}$$

has at least one solution $(y_\varepsilon, p_\varepsilon) \in L^2(Q) \times L^2(Q)$. We set $y_\varepsilon = y_\varepsilon^1 + y_\varepsilon^2$, $p_\varepsilon = p_\varepsilon^1 + p_\varepsilon^2$ where $y_\varepsilon^1, y_\varepsilon^2 \in R(\mathcal{W})$; $y_\varepsilon^2, p_\varepsilon^2 \in N(\mathcal{W})$.

By (4.21) we have

$$\begin{aligned} \int_Q ((b(p_\varepsilon)+g)p_\varepsilon + \varepsilon|p_\varepsilon|^2)\,dx\,dt + \int_Q ((a(y_\varepsilon)+f)y_\varepsilon + \varepsilon|y_\varepsilon|^2)\,dx\,dt \\ \leq (2\rho)^{-1}(\|\mathcal{W}y_\varepsilon\|^2_{L^2(Q)} + \|\mathcal{W}p_\varepsilon\|^2_{L^2(Q)}). \end{aligned} \tag{4.22}$$

Since by (4.18)

$$\begin{aligned} (b(p_\varepsilon)+g)p_\varepsilon &\geq (\gamma^{-1}-\lambda)|b(p_\varepsilon)|^2 + gp_\varepsilon + C_\lambda, \quad \text{a.e.} \quad (x,t) \in Q \\ (a(y_\varepsilon)+f)y_\varepsilon &\geq (\gamma^{-1}-\lambda)|a(y_\varepsilon)|^2 + fy_\varepsilon + C_\lambda, \quad \text{a.e.} \quad (x,t) \in Q \end{aligned}$$

for all $\lambda > 0$, it follows by (4.22) that

$$\begin{aligned} &(\gamma^{-1}-\lambda)\int_Q (|b(p_\varepsilon)|^2 + |a(y_\varepsilon)|^2)\,dx\,dt + \varepsilon\int_Q ((y_\varepsilon)^2 + (p_\varepsilon)^2)\,dx\,dt \\ &\leq (2\rho)^{-1}\int_Q ((b(p_\varepsilon)+\varepsilon p_\varepsilon + g)^2 + (a(y_\varepsilon)+\varepsilon y_\varepsilon + f)^2)\,dx\,dt \\ &\qquad + \int_Q (gp_\varepsilon + fy_\varepsilon)\,dx\,dt + C_\lambda, \quad \forall\,\lambda > 0. \end{aligned}$$

Since $\gamma < 2\rho$ the latter implies that

$$\begin{aligned} \|b(p_\varepsilon)\|^2_{L^2(Q)} + \|a(p_\varepsilon)\|^2_{L^2(Q)} + \varepsilon\big(\|y_\varepsilon\|^2_{L^2(Q)} + \|p_\varepsilon\|^2_{L^2(Q)}\big) + \|\mathcal{W}y_\varepsilon\|^2_{L^2(Q)} \\ + \|\mathcal{W}p_\varepsilon\|^2_{L^2(Q)} \leq C_3\Big(1 + \int_Q (gp_\varepsilon + fy_\varepsilon)\,dx\,dt\Big), \quad \forall\,\varepsilon > 0. \end{aligned} \tag{4.23}$$

This yields

$$\|y_\varepsilon^1\|^2_{L^\infty(Q)} + \|p_\varepsilon^1\|^2_{L^\infty(Q)} \leq C_4(1 + \|y_\varepsilon^2\|_{L^1(Q)} + \|p_\varepsilon^2\|_{L^1(Q)}), \quad \forall\,\varepsilon > 0 \tag{4.24}$$

because under our assumptions $\mathcal{W}^{-1}$ is bounded from $R(\mathcal{W})$ to $L^\infty(Q)$ (see e.g. [8]).

In order to conclude the proof, i.e., to pass to the limit in (4.21), we need the following estimate:

$$\|y_\varepsilon^2\|_{L^2(Q)} + \|p_\varepsilon^2\|_{L^2(Q)} \le C_5, \quad \forall\, \varepsilon > 0. \tag{4.25}$$

We note first that

$$\|p_\varepsilon^2\|_{L^1(Q)} + \|y_\varepsilon^2\|_{L^1(Q)} \le C_6, \quad \forall\, \varepsilon > 0. \tag{4.26}$$

Indeed, we have

$$\begin{aligned}(b(p_\varepsilon) + g_2)p_\varepsilon &= j^*(b(p_\varepsilon)) + j(p_\varepsilon) + g_2 p_\varepsilon \\ &\ge q p_\varepsilon - j^*(q) + g_2 p_\varepsilon + j^*(b(p_\varepsilon)), \quad \text{a.e.} \quad (x,t) \in Q,\ \forall\, q \in R.\end{aligned}$$

By (4.19), this yields

$$\begin{aligned}(b(p_\varepsilon) + g_2)p_\varepsilon &\ge \frac{\delta}{2}|p_\varepsilon| - j^*\big(\frac{\delta}{2}\operatorname{sgn} p_\varepsilon - g_2\big) + j^*(b(p_\varepsilon)) \\ &\ge 2^{-1}\delta|p_\varepsilon| + j^*(b(p_\varepsilon)) + C_7, \quad \text{a.e.} \quad (x,t) \in Q\end{aligned}$$

and, by (4.21), (4.24), we get as above (j^* is the conjugate of j)

$$\int_Q |p_\varepsilon^2(x,t)|\, dx\, dt \le C_8, \quad \forall\, \varepsilon > 0$$

and, similarly,

$$\int_Q |y_\varepsilon^2(x,t)|\, dx\, dt \le C_9, \quad \forall\, \varepsilon > 0.$$

Since $b(p_\varepsilon) + \varepsilon p_\varepsilon + g$ and $a(p_\varepsilon) + \varepsilon y_\varepsilon + f$ belong to $R(\mathcal{W})$ whilst

$$R(\mathcal{W}) = \{\psi \in L^2(Q) : \textstyle\int_0^\pi (\psi(x, t-x) - \psi(x, t+x))\, dx = 0, \quad \text{a.e. } t \in (0,\tau)\}$$

($\tau = 2\pi/a = T/b$ where $T = 2\pi b/a$), arguing as in [4] we conclude by (4.19), (4.20), (4.24) and (4.26) that $\{y_\varepsilon^2\}$ and $\{p_\varepsilon^2\}$ are bounded in $L^\infty(Q)$ as claimed.

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