Advances in Differential Equations Volume 1, Number 4, July 1996, pp. 635 – 674

UNIFORM STABILIZATION OF SPHERICAL SHELLS BY BOUNDARY DISSIPATION*

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(Submitted by: G. Da Prato)

Abstract. We consider an established model of a thin, shallow spherical shell. Under homogeneous boundary conditions, the "energy" remains constant in time (conservative problem). We then introduce suitable dissipative feedback controls on the boundary (forces, shears, moments) and show that

(i) the resulting closed loop feedback problem generates a s.c. semigroup of contractions on a natural function space;
(ii) the corresponding "energy" (norm of the semigroup solutions) decays exponentially in the uniform topology.

As a consequence of the above uniform stabilization result, we obtain—via a result of [15]—a corresponding exact controllability result by explicit boundary controls, which improves upon the recent result of [4, 5]. Energy (multipliers) methods are used, along with semigroup methods. In the process of absorbing lower-order terms to obtain the final energy inequality, we establish a unique continuation result for the system of strongly coupled equations describing the dynamics of the shell, which is of interest in its own right. To this end, Carleman estimates are used after a suitable change of variable.

1. Introduction; Statement of main results. The model. We consider a thin elastic spherical shell whose reference configuration in spherical coordinates (r, θ, ϕ) is the region: $r \in [R-h, R+h],\ \theta \in [0, \theta_0],\ \phi \in [0, 2\pi)$, where the half-thickness h of the shell, the middle surface ray R, and the opening angle $\theta_0 < \pi$ are given. A spherical shell has two characteristic parameters, *thinness* and *shallowness* defined respectively by: $\eta = h/R\sin\theta_0$; and $\beta = (R - R\cos\theta_0)/R\sin\theta_0$. We restrict our interest to a *thin* and *shallow* spherical shell, so that for some fixed $\ell > 0$, we take $\ell = R\sin\theta_0$ and suppose $\eta = h/\ell << 1$ and $\beta << 1$. This latter condition implies θ_0 sufficiently small. Starting from the Koiter linear shell model as in [4], [5], [6], [12], we arrive at the shallow shell approximation by introducing the coordinate $\rho = R\theta$ and by replacing $\cot\theta$ by $\frac{1}{\theta}$. Then, the axially symmetric vibrations for the meridianal and radial middle surface displacements (u, w) can be written in the following form on $(0, T) \times (0, \rho_0)$:

$$u_{tt} + \frac{e}{R}v_{tt} - L(u) - \frac{e}{R}L(v) + \frac{(1+\nu)}{R}w' = 0; \tag{1.1a}$$

$$w_{tt} - \frac{e}{\rho}[v_{tt}\rho]' + \frac{e}{\rho}[L(v)\rho]' - \frac{(1+\nu)}{\rho R}(u\rho)' + \frac{2(1+\nu)}{R^2}w = 0; \tag{1.1b}$$

Received for publication June 1995.
*Research of I.L. and R.T. partially supported by the National Science Foundation under Grant N.S.F. DMS 9204338.

AMS Subject Classifications: 35, 93.

$$v \equiv \frac{u}{R} + w';\ L(u) \equiv u'' + \frac{u'}{\rho} - \frac{u}{\rho^2};\ \rho_0 = R\theta_0;\ e \equiv \frac{h^2}{3};\quad 0 < \nu < 1; \tag{1.1c}$$

$\nu =$ Poisson's ratio, where the prime symbol $'$ denotes differentiation with respect to ρ, along with the initial conditions

$$u(0,\cdot) = u_0,\quad u_t(0,\cdot) = u_1;\quad w(0,\cdot) = w_0,\quad w_t(0,\cdot) = w_1, \tag{1.1d}$$

and the following homogeneous boundary conditions at $\rho = 0$:

$$u = w' = L(v) = 0, \qquad \rho = 0,\quad t > 0, \tag{1.1e}$$

and the following dissipative boundary conditions at $\rho = \rho_0$:

$$u' - \frac{(1+\nu)}{R} w + \nu \frac{u}{\rho_0} = -u_t - u; \tag{1.1f}$$

$$ev' = -v_t; \qquad \rho = \rho_0, \quad t > 0. \tag{1.1g}$$

$$eL(v) - ev_{tt} = w_t. \tag{1.1h}$$

By using the auxiliary variable $v = \frac{u}{R} + w'$ into equations (1.1a-b), we readily see that the u-equation (1.1a) is a generalized wave equation with a high, *third order* coupling in the radial displacement w, containing terms such as w''' and w'_{tt}; similarly, the w-equation (1.1b) is a generalized Kirchoff equation with a high, *third order* coupling in the meridianal displacement u containing terms such as u''', u'_{tt}. Well-posedness of system (1.1) will be asserted in Theorem 1.1 below.

With system (1.1) we associate the energy functional

$$E(t) = E_k(t) + E_p(t), \tag{1.2}$$

$$2E_k(t) \equiv \int_0^{\rho_0} [u_t^2 + w_t^2 + ev_t^2]\rho \, d\rho, \tag{1.3}$$

$$\begin{aligned} 2E_p(t) \equiv e \int_0^{\rho_0} \Big[(v')^2\rho + \frac{v^2}{\rho}\Big] \, d\rho + (1-\nu) \int_0^{\rho_0} \Big[\Big(u' - \frac{w}{R}\Big)^2 \rho + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)^2 \rho\Big] \, d\rho \\ + \nu \int_0^{\rho_0} \Big[\Big(u' - \frac{w}{R}\Big)\sqrt{\rho} + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)\sqrt{\rho}\Big]^2 \, d\rho + u^2(\rho_0)\rho_0. \end{aligned} \tag{1.4}$$

The main goal of this paper is to establish the following result in Theorem 1.4: that the dissipation taking place at the boundary of the shell in (1.1f–g–h)—via forces, shears, and moments applied to the edge—causes the energy (1.2) of the well-posed system (1.1) to decay exponentially to zero as $t \to \infty$. Preliminary to the study of the asymptotic behavior of the dissipative system (1.1) is the need to analyze well-posedness and regularity of its solutions, in order to justify the calculations in the energy method applied to (1.1). The required regularity result, in terms of weighted Sobolev spaces, is given in Theorem 1.2, while a well-posedness result of system (1.1) in the sense of semigroups, is given by Theorem 1.1.

Function spaces. Well-posedness, regularity, and stability results pertaining to the shell model (1.1) require the introduction of the following weighted spaces:
(i)

$$L_2^\rho(0,\rho_0) = \{u : u\sqrt{\rho} \in L_2(0,\rho_0)\}, \text{ with norm } \|u\|_{L_2^\rho} = \{\int_0^{\rho_0} u^2\rho\, d\rho\}^{\frac{1}{2}}; \tag{1.5}$$

(ii)

$$\mathcal{U}_\rho^1(0,\rho_0) = \{u : \frac{u}{\sqrt{\rho}},\ u'\sqrt{\rho} \in L_2(0,\rho_0),\ u(0) = 0\}, \tag{1.6a}$$

with norm

$$\|u\|_{\mathcal{U}_\rho^1} = \{\int_0^{\rho_0} [\frac{u^2}{\rho} + (u')^2\rho]\, d\rho\}^{\frac{1}{2}}, \tag{1.6b}$$

where we note that $\frac{u}{\sqrt{\rho}},\ u'\sqrt{\rho} \in L_2(0,\rho_0)$, hence $uu' \in L_1(0,\rho_0)$, makes the function

$$u^2(\rho) = u^2(\rho_0) + \int_{\rho_0}^{\rho} \frac{d}{dr}(u^2)dr = u^2(\rho_0) + 2\int_{\rho_0}^{\rho} uu'dr \tag{1.6c}$$

absolutely continuous, so that the condition $u(0)$ in (1.6a) is well defined;
(iii)

$$\mathcal{W}_\rho^2(0,\rho_0) = \{w : w\sqrt{\rho} \in L_2(0,\rho_0),\ w' \in \mathcal{U}_\rho^1(0,\rho_0)\} \tag{1.7a}$$

with norm

$$\|w\|_{\mathcal{W}_\rho^2} = \{\int_0^{\rho_0} w^2\rho\, d\rho + \|w'\|_{\mathcal{U}_\rho^1}^2\}^{\frac{1}{2}} = \{\int_0^{\rho_0} \big[w^2\rho + \frac{(w')^2}{\rho} + (w'')^2\rho\big]\, d\rho\}^{\frac{1}{2}}, \tag{1.7b}$$

where $w\sqrt{\rho},\ \frac{w'}{\sqrt{\rho}} \in L_2(0,\rho_0)$, as in the definition of $\mathcal{W}_\rho^2(0,\rho_0)$, hence $ww' \in L_1(0,\rho_0)$, makes—as in (1.6c)—$w^2(\rho)$ absolutely continuous;
(iv)

$$\mathcal{V}_\rho^1(0,\rho_0) = \{(u,w) \in L_2^\rho(0,\rho_0) \times L_2^\rho(0,\rho_0) : \tag{1.8a}$$
$$v = \frac{u}{R} + w' \in L_2^\rho(0,\rho_0) \text{ or, equivalently, } w' \in L_2^\rho(0,\rho_0)\}$$

with norm

$$\|(u,w)\|_{\mathcal{V}_\rho^1} = \{\|u\|_{L_2^\rho}^2 + \|w\|_{L_2^\rho}^2 + e\|v\|_{L_2^\rho}^2\}^{\frac{1}{2}}; \tag{1.8b}$$

(v) with $\epsilon > 0$ arbitrary, as well as for $\epsilon = 0$,

$$\mathcal{H}_{\epsilon,\rho}^2(0,\rho_0) = \{u \in \mathcal{U}_\rho^1 : \frac{u}{\rho^{\frac{3}{2}-\epsilon}},\ \frac{u'}{\rho^{\frac{1}{2}-\epsilon}},\ u''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0)\}, \tag{1.9a}$$

with norm

$$\|u\|_{\mathcal{H}_{\epsilon,\rho}^2} = \{\int_0^{\rho_0} [\frac{u^2}{\rho^{3-2\epsilon}} + \frac{(u')^2}{\rho^{1-2\epsilon}} + (u'')^2\rho^{1+2\epsilon}]\, d\rho\}^{\frac{1}{2}}. \tag{1.9b}$$

Note: Henceforth, we shall often drop specification of the interval $(0, \rho_0)$ in the notation of the above spaces.

Equivalence of energy $E(u, w)$ in (1.2) with the $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \times \mathcal{V}^1_\rho$-norm. We shall establish, in Proposition 2.1 below, the following topological equivalence between the "potential energy space" (where $E_p(u, w)$ in (1.2) is well defined), and the weighted space $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$: there exist constants $0 < c < C < \infty$ such that

$$c\big[\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\big] \leq E_p(u, w) \leq C\big[\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\big]; \tag{1.10}$$

$(u, w) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$, while from (1.3) and (1.8b) the "kinetic energy" satisfies

$$E_k(u, w) = \|(u_t, w_t)\|^2_{\mathcal{V}^1_\rho}, \tag{1.11}$$

whereby (1.10) and (1.11) show that the correct space for well-posedness and energy decay of problem (1.1) is the space $[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho$ for $[[u, w], [u_t, w_t]]$:

$$c\|\{[u, w], [u_t, w_t]\}\|^2_{[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho} \leq E(u, w) \leq C\|\{[u, w], [u_t, w_t]\}\|^2_{[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho}. \tag{1.12}$$

Statement of main results. Because of space restrictions, the well-posedness issue is treated in detail in [8], with an essential, brief summary being provided in Sections 3 and 4 below, to the extent strictly needed in the present paper. The main results are as follows.

Theorem 1.1 ([8]) (well-posedness of (1.1))**.** *Let the initial data satisfy*

$$u_0 \in \mathcal{U}^1_\rho;\ \ w_0 \in \mathcal{W}^2_\rho;\ \ (u_1, w_1) \in \mathcal{V}^1_\rho. \tag{1.13}$$

Then, with reference to problem (1.1), *there exists a unique solution in the following sense*:

$$\text{the map: } \{u_0, w_0, u_1, w_1\} \to \{u(t), w(t), u_t(t), w_t(t)\}$$

defines a s.c. uniformly bounded semigroup $e^{\mathcal{A}_s t}$ on the space

$$\mathcal{E} \equiv \{[u, w], [u_1, w_1] : u \in \mathcal{U}^1_\rho;\ \ w \in \mathcal{W}^2_\rho;\ \ [u_1, w_1] \in \mathcal{V}^1_\rho\} \tag{1.14a}$$

$$= [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho; \tag{1.14b}$$

$$\|\{u, w, u_1, w_1\}\|^2_{\mathcal{E}} = \|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho} + \|[u_1, w_1]\|^2_{\mathcal{V}^1_\rho}. \tag{1.14c}$$

The semigroup is actually a contraction on the space $\mathcal{E}$, topologized, however, by the norm, defined in (3.3), (3.4) *below, of the space*

$$\mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}) \tag{1.15}$$

equivalent to the norm (1.14c), *where A and M are strictly positive self-adjoint operators defined in Section 3 below, and analyzed in detail in* [8].

Theorem 1.2 ([8]) (Regularity). *With reference to* (1.14), *let* $\mathcal{A}_s : \mathcal{E} \supset \mathcal{D}(\mathcal{A}_s) \to \mathcal{E}$ *be the infinitesimal generator of the s.c. semigroup of contractions describing the solutions of the shell problem* (1.1), *whose domain* $\mathcal{D}(\mathcal{A}_s)$ *is studied in* (4.2.1) *below. Let the initial conditions satisfy* $[u_0, w_0, u_1, w_1] \in \mathcal{D}(\mathcal{A}_s)$*; hence, according to* (4.2.1),

$$[u_0, w_0] \in \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 : u_0 \in \mathcal{H}_{\epsilon,\rho}^2,\ v_0 = \frac{u_0}{R} + w_0' \in \mathcal{H}_{\epsilon,\rho}^2;\ [u_1, w_1] \in \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2. \tag{1.16}$$

Then, the semigroup solution of problem (1.1) *guaranteed by Theorem* 1.1 *satisfies the following regularity properties:*

$$\{u(t), w(t), u_t(t), w_t(t)\} \in C([0,\infty]; \mathcal{D}(\mathcal{A}_s)); \tag{1.17}$$

i.e., explicitly via (4.2.1) *below,*

$$\big\{u, v = \frac{u}{R} + w'\big\} \in C\big([0,\infty]; \mathcal{H}_{\epsilon,\rho}^2 \times \mathcal{H}_{\epsilon,\rho}^2\big), \quad \forall\ \epsilon > 0, \tag{1.18}$$

$$\{u_t, w_t\} \in C\big([0,\infty]; \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2\big), \tag{1.19}$$

$$\{u_{tt}, w_{tt}\} \in C\big([0,\infty]; \mathcal{V}_\rho^1\big). \tag{1.20}$$

Theorem 1.3 (Dissipativity). *Let the initial data* $\{u_0, w_0, u_1, w_1\} \in \mathcal{E}$ *be as defined by* (1.14). *Then*

(i) *the following energy identity holds true for the solutions of* (1.1) *guaranteed by Theorem* 1.1 :

$$E(t) + \rho_0 \int_0^t \big\{u_t^2(t,\rho_0) + v_t^2(t,\rho_0) + w_t^2(t,\rho_0)\big\}\, dt = E(0), \tag{1.21}$$

so that the energy $E(t)$ *is nonincreasing.*

(ii) *Moreover, the following "trace" regularity holds true for the solution of* (1.1), *where* $v = \frac{u}{R} + w'$ *as in* (1.1c)*: There is a constant* $C > 0$ *such that*

$$\int_0^\infty [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + v_t^2(t,\rho_0)]\, dt \le C \|\{u_0, w_0, u_1, w_1\}\|_{\mathcal{E}}^2. \tag{1.22}$$

(iii) *In particular, if the B.C. at* $\rho = \rho_0$ *are likewise homogeneous*

$$u' - \frac{(1+\nu)}{R} w + \nu \frac{u}{\rho_0} \equiv -u; \tag{1.23a}$$

$$v' \equiv 0; \qquad \rho = \rho_0, \quad t > 0, \tag{1.23b}$$

$$L(v) - v_{tt} \equiv 0, \tag{1.23c}$$

where $v = \frac{u}{R} + w'$ *as in* (1.1c), *then the corresponding problem* (1.1a–b–c–d–e), (1.23a–b–c) *is energy preserving:* $E(t) \equiv E(0)$.

That the (dissipative) B.C. (1.1f–g–h) make the energy actually decay exponentially is the content of the main result of the present paper.

Theorem 1.4. (Stabilization)**.** *Let the inital data* $\{u_0, w_0, u_1, w_1\} \in \mathcal{E}$ *be as defined in* (1.14). *Then, the following energy decay holds true for the solutions of problem* (1.1), *guaranteed by Theorem* 1.1 : *there exist positive constants* C, a, *such that*

$$E(t) \leq Ce^{-at}E(0), \qquad t \geq 0. \tag{1.24a}$$

Equivalently, the s.c. contraction semigroup $e^{\mathcal{A}_s t}$ *on* $\mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}})$ *guaranteed by Theorem 1.1 is uniformly stable here*:

$$\left\|e^{\mathcal{A}_s t}\right\|_{\mathcal{L}(\mathbf{E})} \leq Ce^{-at}, \qquad t \geq 0. \tag{1.24b}$$

Thus, if we consider the (open loop) mixed problem, which consists of equations (1.1a–b), the initial conditions (1.1d), the homogeneous B.C. (1.1e) at $\rho = 0$, and the following nonhomogeneous B.C.,

$$u' - \frac{(1+\nu)}{R}w + \nu\frac{u}{\rho_0} = g_1; \tag{1.25a}$$

$$ev' = g_2; \qquad \text{at } \rho = \rho_0, \quad t > 0, \tag{1.25b}$$

$$eL(v) - ev_{tt} = g_3, \tag{1.25c}$$

then we can say that such a problem is: (i) uniformly stabilizable (for positive times) on the space $\mathcal{E}$, equivalently on the space $\mathbf{E}$, by means of the feedback controls $g_1 = -u_t - u$; $g_2 = -v_t$; $g_3 = w_t$, by virtue of Theorem 1.4; (ii) such feedback controls g_i lie all in $L_2(0, \infty)$, by virtue of Theorem 1.3, equation (1.22); (iii) for $g_1 = g_2 = g_3 = 0$, the system is norm-preserving on $\mathbf{E}$, indeed describes a unitary group on $\mathbf{E}$, $t \in \mathbf{R}$, by virtue of Theorem 3.1(i). As a consequence, a well-known result ([15]) applies, and yields a corresponding exact controllability result on $\mathcal{E}$ (or $\mathbf{E}$) with constructive $L_2(0, T)$-controls g_1, g_2, g_3, for T sufficiently large.

Corollary 1.5. *With reference to the open loop mixed problem* (1.1a–b), (1.25a–b–c), *given* T *sufficiently large and given initial conditions* (1.1d): $\{[u_0, w_0], [u_1, w_1]\} \in [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho$, *there exist* (*constructively,* [15]) *boundary controls* $\{g_1, g_2, g_3\} \in [L_2(0, T)]^3$, *such that the corresponding solution* $\{u(t), w(t)\}$ *guaranteed by Theorem* 1.1 *satisfies* $u(T, \cdot) = w(T, \cdot) = u_t(T, \cdot) = w_t(T, \cdot) = 0$.

Remark 1.1. The methods of this paper can be adapted to show in a direct way (i.e., without passing through stabilization) the exact controllability result in Corollary 1.5, with an estimate on the required time T. To this end, one works with the simpler homogeneous problem $g_1 = g_2 = g_3 \equiv 0$, rather than with the feedback problem $g_1 = -u_t - u$; $g_2 = -v_t$; $g_3 = w_t$.

Remark 1.2. Corollary 1.5 improves upon the exact controllability result presented in [4, 5] in a few ways: (i) the boundary controls g_1, g_2, g_3 are asserted in Corollary 1.5 to be constructive and in $L_2(0, T)$. In contrast, no regularity of the steering controls is provided in [4, 5]; (ii) the analysis of [4, 5], unlike the one of the present paper, requires the assumption that the radius R, or the opening angle θ_0, be suitably small (beyond the requirements of the model); (iii) the model in [4, 5] is simplified over our model (1.1), in that [4, 5] does not contain the term v_{tt} in our equations (1.1a–b) (i.e., the operator M in the subsequent model (3.2) is the identity).

Literature. Problems related to boundary stabilization of wave-like and plate-like equations have received considerable attention in recent years, and we can only refer to [7, 13], [2], [10], and references therein. However, in the case of the dynamic equations of shells, very little is apparently known, particularly in the context of control/stabilization theory. Indeed, in these latter two areas, the only paper known to the authors is [4], which deals only with exact controllability, as already referenced, and compared with the results of the present paper, in Remark 1.2 above. Generally, direct adaptation of the techniques developed in the context of wave-like or plate-like equations are not fully adequate when dealing with the coupled equations of shells with coupling taking place through high-order terms (e.g., the wave equation (1.1a) in u contains coupling terms w'_{tt} and w''' and, similarly, the Kirchoff equation (1.1b) in w contains coupling terms u''' and u'_{tt}), and with space-variable coefficients. A specific case in point is the issue of unique continuation, which is intrinsic to the problems of uniform stabilization and exact controllability, in that it permits the absorption of lower-order terms (see Sections 8 through 10). Unique continuation is generally poorly understood for coupled systems. Indeed, one of the main contributions of the present paper is a new unique continuation result across the boundary, which is shown for the present shell model using Carleman estimates; see Theorem 8.1. It is this result that, in studying either uniform stabilization or else exact controllability, permits one to avoid the assumptions in [4, 5] that R, or θ_0, be suitably small.

Properties of function spaces. We shall collect here, for easy reference, properties pertaining to the above spaces to be invoked in the sequel.

(p.1) $u \in \mathcal{U}_\rho^1 \to u$ is absolutely continuous.
(p.2) $w \in \mathcal{W}_\rho^2 \to w$ and w' are absolutely continuous.
(p.3) $u \in \mathcal{U}_\rho^1(0, \rho_0) \to u \in H^1(\epsilon, \rho_0),\ \epsilon > 0$, and $u\sqrt{\rho} \in H^1(0, \rho_0)$; thus

$$\|u\|^2_{H^1(\epsilon,\rho_0)} \le \max\{\rho_0, \frac{1}{\epsilon}\}\|u\|^2_{\mathcal{U}_\rho^1(0,\rho_0)}. \tag{1.26}$$

(p.4) $(u, w) \in \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \to (u, w) \in \mathcal{V}_\rho^1$.
(p.5) We have $\|w'\|^2_{\mathcal{U}_\rho^1} \le \|w\|^2_{\mathcal{W}_\rho^2}$, so that with $v = \frac{u}{R} + w'$,

$$\|v\|_{\mathcal{U}_\rho^1} \le C\{\|u\|_{\mathcal{U}_\rho^1} + \|w\|_{\mathcal{W}_\rho^2}\};\ \text{and}\ c\big[\|u\|^2_{\mathcal{U}_\rho^1} + \|w\|^2_{\mathcal{W}_\rho^2}\big] \le \|u\|^2_{\mathcal{U}_\rho^1} + \|v\|^2_{\mathcal{U}_\rho^1} + \alpha^2\|w\|^2_{L_2^\rho}. \tag{1.27}$$

(p.6) For $\epsilon \ge 0$, recalling (1.9), we have

$$u \in \mathcal{H}^2_{\epsilon,\rho}(0, \rho_0) \to \big(u'\rho^{\frac{1}{2}-\epsilon}\big),\ \text{and}\ \big(\frac{u}{\rho^{\frac{1}{2}-\epsilon}}\big) \in H^1(0, \rho_0) \subset C[0, \rho_0], \tag{1.28}$$

so that, *a fortiori* $u(0),\ u(\rho_0), u'(\rho_0)$ are well defined.

Indeed, (p.1) was observed below (1.6c), while (p.2) for w was observed below (1.7b), and (p.2) for w' follows from (p.1). As to (p.3), we have $(u\sqrt{\rho})' = u'\sqrt{\rho} + \frac{1}{2}\rho^{-\frac{1}{2}}u \in L_2(0, \rho_0)$, and

$$\|u\|^2_{H^1(\epsilon,\rho_0)} = \int_\epsilon^{\rho_0} u^2\, d\rho + \int_\epsilon^{\rho_0} (u')^2\, d\rho \le \rho_0 \int_\epsilon^{\rho_0} \frac{u^2}{\rho}\, d\rho + \frac{1}{\epsilon}\int_\epsilon^{\rho_0} (u')^2 \rho\, d\rho,$$

and (1.26) follows via (1.6b). Properties (p.4) and (p.5) follow from the definitions (1.6), (1.7), (1.8), via (p.1) and (p.2).

Finally, we prove (p.6) = (1.28). Indeed, $u \in \mathcal{H}^2_{\epsilon,\rho}$ with $\epsilon \ge 0$ means, by (1.9), that both

$$u'\rho^{\frac{1}{2}+\epsilon} \in L_2(0, \rho_0) \quad \text{and} \quad \big(u'\rho^{\frac{1}{2}+\epsilon}\big)' = u''\rho^{\frac{1}{2}+\epsilon} + \big(\frac{1}{2} + \epsilon\big)\frac{u'}{\rho^{\frac{1}{2}-\epsilon}} \in L_2(0, \rho_0),$$

so that $u'\rho^{\frac{1}{2}+\epsilon} \in H^1(0, \rho_0)$, as well as

$$\frac{u}{\rho^{\frac{1}{2}-\epsilon}} \in L_2(0, \rho_0), \quad \text{and} \quad \big(\frac{u}{\rho^{\frac{1}{2}-\epsilon}}\big)' = \frac{u'}{\rho^{\frac{1}{2}-\epsilon}} - \big(\frac{1}{2} - \epsilon\big)\frac{u}{\rho^{\frac{3}{2}+\epsilon}} \in L_2(0, \rho_0),$$

so that $\frac{u}{\rho^{\frac{1}{2}-\epsilon}} \in H^1(0, \rho_0)$; then (1.28) is proved.

Two additional (compactness) properties will be seen in Section 8, Appendix A, the second being a consequence of (p.6) = (1.28).

2. Preliminary results. In this section we collect a few preliminary results to be invoked in the sequel. We begin with the proof of the equivalence of the energy E_p given by (1.10), hence of E given by (1.12). To this end, given u, w from (1.1a–b) and recalling $v = \frac{u}{R} + w'$ from (1.1c), we define (with $0 < \nu < 1$):

$$\begin{aligned} 2\hat{E}_p(u, w) \equiv e \int_0^{\rho_0} \big[(v')^2\rho + \frac{v^2}{\rho}\big]\, d\rho + u^2(\rho_0)\rho_0 \\ + (1 - \nu)\int_0^{\rho_0} \big[\big(u' - \frac{w}{R}\big)^2\rho + \big(\frac{u}{\rho} - \frac{w}{R}\big)^2\rho\big]\, d\rho. \end{aligned} \tag{2.1}$$

By comparing (2.1) with (1.4), we readily see that

$$\hat{E}_p(u, w) \le E_p(u, w) \le \text{ const } \hat{E}_p(u, w), \tag{2.2}$$

and thus the energy $\hat{E}_p(u, w)$ is topologically equivalent to the energy $E_p(u, w)$, so that they may be interchanged. The following result, already announced in (1.10), provides a topological relation between the energy space and the weighted space $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$.

Proposition 2.1. *$E_p(u, w)$, equivalently $\hat{E}_p(u, w)$ by* (2.2)*, is bounded and coercive on the space $\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$ so that* (1.10) *holds true: there exist constants $0 < c < C < \infty$ such that for $(u, w) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$,*

$$c\big[\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\big] \le E_p(u, w) \sim \hat{E}_p(u, w) \le C\big[\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\big], \tag{2.3a}$$

$$\|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho} = \Big\{\int_0^{\rho_0} \big[\frac{u^2}{\rho} + (u')^2\rho\big]\, d\rho + \int_0^{\rho_0} \big[w^2\rho + \frac{(w')^2}{\rho} + (w'')^2\rho\big]\, d\rho\Big\}^{\frac{1}{2}}, \quad (2.3b)$$

via (1.6b), (1.7b), *so that by* (1.2) *and* (1.11), *we obtain* (1.12) :

$$E(u,w) \sim \|\{[u,w],[u_t,w_t]\}\|^2_{[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho]\times \mathcal{V}^1_\rho} \equiv \|\{[u,w],[u_t,w_t]\}\|^2_{\mathcal{E}}. \quad (2.4)$$

Proof. That $\hat{E}_p(u,w)$ is well defined and bounded for $(u,w) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$ follows from (1.6), (1.7), (2.1). Next, let $u \in \mathcal{U}^1_\rho$ in (1.6), so that

$$u'\sqrt{\rho} - \frac{u}{\sqrt{\rho}} \equiv f(\rho) \in L_2(0,\rho_0). \quad (2.5)$$

We now multiply (2.5) by $\frac{u}{\sqrt{\rho}}$ and integrate over $[0,\rho_0]$. We then obtain after using $2u'u = \frac{d}{d\rho}(u^2)$ and $u(0)=0$,

$$\int_0^{\rho_0} \big[u'u - \frac{u^2}{\rho}\big]\, d\rho = \int_0^{\rho_0} f\frac{u}{\sqrt{\rho}}\, d\rho = \tfrac{1}{2}u^2(\rho_0) - \int_0^{\rho_0} \frac{u^2}{\rho}\, d\rho, \quad \text{hence}$$

$$\int_0^{\rho_0} \frac{u^2}{\rho}\, d\rho \le \tfrac{1}{2}u^2(\rho_0) + \int_0^{\rho_0} |f|\frac{|u|}{\sqrt{\rho}}\, d\rho \le \tfrac{1}{2}\big[u^2(\rho_0) + \int_0^{\rho_0} \big(f^2 + \frac{u^2}{\rho}\big)\, d\rho\big], \quad (2.6)$$

and collecting the first and last terms in (2.6),

$$\int_0^{\rho_0} \frac{u^2}{\rho}\, d\rho \le u^2(\rho_0) + \int_0^{\rho_0} f^2\, d\rho. \quad (2.7)$$

Next, we estimate from (2.5), after adding and subtracting, $w\sqrt{\rho}/R$, with $w \in \mathcal{W}^2_\rho$,

$$\begin{aligned}\int_0^{\rho_0} f^2\, d\rho &= \int_0^{\rho_0} \big(u'\sqrt{\rho} - \frac{u}{\sqrt{\rho}}\big)^2\, d\rho \\ &\le 2\int_0^{\rho_0} \big(u'\sqrt{\rho} - \frac{w}{R}\sqrt{\rho}\big)^2\, d\rho + 2\int_0^{\rho_0} \big(\frac{u}{\sqrt{\rho}} - \frac{w\sqrt{\rho}}{R}\big)^2\, d\rho \quad (2.8)\\ &\le 2E_p(t), \quad (2.9)\end{aligned}$$

where in the last step we have recalled the second integral term in the definition of $E_p(u,w)$ in (1.4). Substituting (2.9) in (2.7) yields, after recalling the last term in (1.4):

$$\int_0^{\rho_0} \frac{u^2}{\rho}\, d\rho \le \frac{1}{\rho_0}u^2(\rho_0)\rho_0 + 2E_p(t) \le \big(\frac{1}{\rho_0} + 1\big)2E_p(t). \quad (2.10)$$

Recalling (2.8)–(2.10), we obtain

$$\int_0^{\rho_0} (u')^2\rho\, d\rho = 2\int_0^{\rho_0} \big(u'\sqrt{\rho} - \frac{u}{\sqrt{\rho}}\big)^2\, d\rho + 2\int_0^{\rho_0} \frac{u^2}{\rho}\, d\rho \le \text{ const } E_p(t). \quad (2.11)$$

Then, (2.10) and (2.11) together say, by (1.6b), that

$$E_p(t) \geq c_1 \|u\|^2_{\mathcal{U}^1_\rho}, \qquad c_1 > 0, \tag{2.12}$$

as desired. Moreover, adding and subtracting we obtain

$$\begin{aligned}\int_0^{\rho_0} w^2 \rho \, d\rho &= R^2 \int_0^{\rho_0} \Big[\frac{w}{R} - \frac{u}{\rho} + \frac{u}{\rho}\Big]^2 \rho \, d\rho \\ &\leq 2R^2 \int_0^{\rho_0} \Big(\frac{w}{R} - \frac{u}{\rho}\Big)^2 \rho \, d\rho + 2R^2 \int_0^{\rho_0} \frac{u^2}{\rho} \, d\rho \leq \text{ const } E_p(t),\end{aligned} \tag{2.13}$$

where in the last step we have invoked the inequality from the second term on the right-hand side of (2.8) to (2.9), as well as (2.10). Furthermore, still after adding and subtracting,

$$\int_0^{\rho_0} \frac{(w')^2}{\rho} \, d\rho \leq 2 \int_0^{\rho_0} \Big(\frac{u}{R} + w'\Big)^2 \frac{1}{\rho} \, d\rho + \frac{2}{R^2} \int_0^{\rho_0} \frac{u^2}{\rho} \, d\rho \leq \text{ const } E_p(t), \tag{2.14}$$

where in the last step we have invoked the first integral term on the right-hand side of (1.4) with $v = \frac{u}{R} + w'$, as well as (2.10). Finally, again after adding and subtracting,

$$\int_0^{\rho_0} (w'')^2 \rho \, d\rho \leq 2 \int_0^{\rho_0} \Big(\frac{u'}{R} + w''\Big)^2 \rho \, d\rho + \frac{2}{R^2} \int_0^{\rho_0} (u')^2 \rho \, d\rho \leq \text{ const } E_p(t), \tag{2.15}$$

where in the last step we have invoked the first integral term on the right-hand side of (1.4) with $v' = \frac{u'}{R} + w''$, as well as (2.11). Then, (2.13)–(2.15) together say, by (1.7c), that

$$E_p(t) \geq c_2 \|w\|^2_{\mathcal{W}^2_\rho}, \qquad c_2 > 0. \tag{2.16}$$

Finally, combining (2.12) with (2.16) yields the lower bound (coercivity) in (2.3), while the upper bound (boundedness) for $(u, w) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho$ via (1.6), (1.7), (2.1), was already observed at the outset.

Proposition 2.2. *With reference to the operator L in* (1.1c), *we have for $u, \phi \in \mathcal{U}^1_\rho$ and suitably smooth as needed*:

$$\begin{aligned}\int_0^{\rho_0} (Lu)\phi\rho \, d\rho &= u'(\rho_0)\phi(\rho_0)\rho_0 - \int_0^{\rho_0} u'\phi'\rho \, d\rho - \int_0^{\rho_0} \frac{u\phi}{\rho} \, d\rho \\ &= u'(\rho_0)\phi(\rho_0)\rho_0 - (u, \phi)_{\mathcal{U}^1_\rho}\end{aligned} \tag{2.17}$$

$$\int_0^{\rho_0} (L\phi)\phi\rho \, d\rho = \phi'(\rho_0)\phi(\rho_0)\rho_0 - \|\phi\|^2_{\mathcal{U}^1_\rho} \tag{2.18}$$

$$\int_0^{\rho_0} (L\phi)\phi'\rho^2 \, d\rho = \frac{1}{2}\big[(\phi'(\rho_0))^2\rho_0^2 - \phi^2(\rho_0)\big]. \tag{2.19}$$

Proof. Straightforward integration by parts yields (2.17)—which specializes then to (2.18) for $u = \phi$—as well as (2.19), recalling in this latter case that $\phi(0) = 0$ by (1.6a) for $\phi \in \mathcal{U}^1_\rho$.

3. Semigroup formulation of problem (1.1). Well-posedness Theorem 1.1. The proof of Theorem 1.1 (well-posedness) is given in [8], along with the results contained in the present section. Briefly, we employ a two-step procedure: (i) we shall first replace the original problem (1.1) with the simplified version (3.1) below, by dropping some lower-order terms, and then establish semigroup generation for (3.1) (Theorem 3.1); (ii) next, by a perturbation argument ([14]), we use (i) for (3.1) to assert semigroup generation of the original model (1.1). The simplified version of problem (1.1) on $(0, T] \times (0, \rho_0)$ to be analyzed here is

$$u_{tt} + \frac{e}{R} v_{tt} - L(u) - \frac{e}{R} L(v) = 0; \tag{3.1a}$$

$$w_{tt} - \frac{e}{\rho}[v_{tt}\rho]' + \frac{e}{\rho}[L(v)\rho]' + \alpha^2 w = 0; \tag{3.1b}$$

$$\alpha^2 = \frac{2(1+\nu)}{R^2};\ v = \frac{u}{R} + w';\ Lu = u'' + \frac{u'}{\rho} - \frac{u}{\rho^2}, \tag{3.1c}$$

along with the following homogeneous boundary conditions at $\rho = 0$,

$$u = v = Lv = 0, \qquad \rho = 0,\ t > 0, \tag{3.1d}$$

and the following nonhomogeneous boundary conditions at $\rho = \rho_0$,

$$u' + u = -u_t; \tag{3.1e}$$

$$ev' = -v_t; \qquad \rho = \rho_0, \quad t > 0. \tag{3.1f}$$

$$eL(v) - ev_{tt} = w_t, \tag{3.1g}$$

The closed-loop mixed problem (3.1) may be written abstractly as

$$M \begin{bmatrix} u_{tt} \\ w_{tt} \end{bmatrix} + A \begin{bmatrix} u \\ w \end{bmatrix} + BB^* \begin{bmatrix} u_t \\ w_t \end{bmatrix} = 0, \tag{3.2}$$

with operators M, A, B identified explicitly in [8]. More precisely, M is continuous (bounded) $\mathcal{V}_\rho^1 \to [\mathcal{V}_\rho^1]'$, self-adjoint and strictly positive on $L_2^\rho \times L_2^\rho$, and coercive on $\mathcal{V}_\rho^1$, so that

$$\mathcal{D}\big(M^{\frac{1}{2}}\big) \equiv \mathcal{V}_\rho^1 \text{ (equivalent norms).} \tag{3.3}$$

Similarly, A is continuous (bounded) $\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \to [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2]'$; self-adjoint and strictly positive on $L_2^\rho \times L_2^\rho$, and coercive on $\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2$, so that

$$\mathcal{D}\big(A^{\frac{1}{2}}\big) \equiv \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \text{ (equivalent norms).} \tag{3.4}$$

Finally, B is continuous (bounded) from $R^3 \to [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2]'$, and B^* is continuous (bounded) from $\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2$ onto $R_{\rho_0}^3$ (= the space R^3 topologized with weight ρ_0):

$$B \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} -g_1\delta_{\rho_0} - \frac{g_2}{R}\delta_{\rho_0} \\ \\ \frac{\rho_0}{\rho} g_2 \delta'_{\rho_0} + g_3\delta_{\rho_0} \end{bmatrix} : \mathbf{R}^3 \to [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2]' \equiv \big[\mathcal{D}(A^{\frac{1}{2}})\big]', \tag{3.5a}$$

where δ_{ρ_0} is the Dirac mass concentrated at ρ_0, and δ'_{ρ_0} its derivative, and

$$B^* \begin{bmatrix} \bar{u} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} -\bar{u}(\rho_0) \\ -\bar{v}(\rho_0) \\ \bar{w}(\rho_0) \end{bmatrix} : \mathcal{D}(A^{\frac{1}{2}}) \equiv \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \to \mathbf{R}^3_{\rho_0}. \tag{3.5b}$$

The first-order model corresponding to (3.1) is then

$$\frac{d}{dt} \begin{bmatrix} u \\ w \\ u_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}A & -M^{-1}BB^* \end{bmatrix} \begin{bmatrix} u \\ w \\ u_t \\ w_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} u \\ w \\ u_t \\ w_t \end{bmatrix}; \tag{3.6}$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -M^{-1}A & -M^{-1}BB^* \end{bmatrix} : \mathbf{E} \supset \mathcal{D}(\mathcal{A}) \to \mathbf{E} \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}); \tag{3.7}$$

with domain $\mathcal{D}(\mathcal{A})$ identified in [8]. A key description of $\mathcal{D}(\mathcal{A})$—critical for our subsequent development—is given below in (4.2.1). A direct application of the Lumer-Phillips theorem then yields part (iii) of the following result.

Theorem 3.1 ([8])**.** (i) *The operator* $\mathcal{A}_0$ *obtained from* $\mathcal{A}$ *in* (3.7) *by taking* $B = 0$ *is skew-adjoint on* $\mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}}) = \mathbf{E}$*, and thus it generates a unitary group* $e^{\mathcal{A}_0 t}$ *on* $\mathbf{E}$.

(ii) *The operator* $\mathcal{A}$ *in* (3.7) *is dissipative*

$$Re(\mathcal{A}x, x)_{\mathbf{E}} \equiv -\|B^* x_2\|^2_{L_2^\rho \times L_2^\rho} \leq 0, \ \ x = [x_1, x_2] \in \mathcal{D}(\mathcal{A}); \ \ i.e., \ x_2 \in \mathcal{D}(A^{\frac{1}{2}}). \tag{3.8}$$

(iii) $\mathcal{A}$ *is, in fact, maximal dissipative on* $\mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}})$*. Thus,* $\mathcal{A}$ *generates a s.c. contraction semigroup* $e^{\mathcal{A}t}$ *on* $\mathbf{E}$ *(which becomes uniformly bounded on* $[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho)$:

$$[u(t), w(t), u_t(t), w_t(t)] = e^{\mathcal{A}t}[u_0, w_0, u_1, w_1] \tag{3.9}$$

for the solution $\{u, w\}$ *of problem* (3.1a–g).

4. Description of $\mathcal{D}(A)$ and of $\mathcal{D}(\mathcal{A})$.

In order to study the regularity of solutions to problem (3.1), hence of solutions to problem (1.1) and obtain Theorem 1.2, it is critical to characterize more explicitly the domain $\mathcal{D}(A)$ of the operator A in (3.1), as well as the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$ in (3.7).

4.1. Description of $\mathcal{D}(A)$.

Proposition 4.1.1 ([8])**.** *With reference to the operator* A*:* $L_2^\rho \times L_2^\rho \supset \mathcal{D}(A) \to L_2^\rho \times L_2^\rho$ *in* (3.1)*, and recalling the spaces in* (1.6), (1.7), (1.9)*, we have*

$$\mathcal{D}(A) \subset \left\{ \begin{array}{l} (u, w) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho : \ u \in \mathcal{H}^2_{\epsilon,\rho}, \ v = \dfrac{u}{R} + w' \in \mathcal{H}^2_{\epsilon,\rho}; \ i.e., \\ u\rho^{-\frac{3}{2}+\epsilon}; \ u'\rho^{-\frac{1}{2}+\epsilon}; \ u''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0); \\ v\rho^{-\frac{3}{2}+\epsilon}; \ v'\rho^{-\frac{1}{2}+\epsilon}; \ v''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0); \\ u = v = Lv = 0 \ \ at \ \rho = 0; \ u' + u = 0, \ v' = Lv = 0 \ \ at \ \rho = \rho_0, \end{array} \right\} \tag{4.1.1}$$

where $\epsilon > 0$ *is arbitrarily small* (*however,* ϵ *cannot be taken equal to zero*).

The proof of Proposition 4.1.1 hinges crucially on the following lemma. To state it, we first recall the expression Lu from (1.1c), and define the operator $\mathcal{L}$ by:

$$\mathcal{L}: \ L_2^\rho \supset \mathcal{D}(\mathcal{L}) \to L_2^\rho; \ \mathcal{L}u = Lu = u'' + \frac{u'}{\rho} - \frac{u}{\rho^2}; \tag{4.1.2a}$$

$$\mathcal{D}(\mathcal{L}) = \big\{ u \in \mathcal{U}_\rho^1 \text{ (hence } u(0) = 0) : Lu \in L_2^\rho; \ |u'(\rho_0)| < \infty \big\}. \tag{4.1.2b}$$

Lemma 4.1.2 ([8])**.** *With reference to* (4.1.2), *and with* $\epsilon > 0$ *arbitrarily small, we have, recalling* (1.9),

$$\mathcal{H}_{0,\rho}^2 \subset \mathcal{D}(\mathcal{L}) \subset \mathcal{H}_{\epsilon,\rho}^2 \equiv \big\{ u \in \mathcal{U}_\rho^1 : \ u\rho^{-\frac{3}{2}+\epsilon}, \ u'\rho^{-\frac{1}{2}+\epsilon}, \ u''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0) \big\}, \tag{4.1.3}$$

where ϵ *cannot be taken equal to zero.*

4.2. Description of $\mathcal{D}(\mathcal{A})$.

Proposition 4.2.1 ([8])**.** *With reference to the operator* $\mathcal{A} : \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \times \mathcal{V}_\rho^1 \supset \mathcal{D}(\mathcal{A}) \to \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \times \mathcal{V}_\rho^1$ *in* (3.7), *where we recall that* $\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 \equiv \mathcal{D}(A^{\frac{1}{2}})$ *and* $\mathcal{V}_\rho^1 \equiv \mathcal{D}(M^{\frac{1}{2}})$ (*norm equivalence*) *from* (3.4) *and* (3.3), *we have*

$$\begin{bmatrix} u_0 \\ w_0 \\ u_1 \\ w_1 \end{bmatrix} \in \mathcal{D}(\mathcal{A}) \to \left\{ \begin{array}{l} \text{(i)} \quad (u_1, w_1) \in \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2; \\ \text{(ii)} \quad (u_0, w_0) \in \mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2 : \\ u_0 \in \mathcal{H}_{\epsilon,\rho}^2, \ v_0 = \dfrac{u_0}{R} + w_0' \in \mathcal{H}_{\epsilon,\rho}^2; \ \textit{i.e.,} \\ u_0\rho^{-\frac{3}{2}+\epsilon}; \ u_0'\rho^{-\frac{1}{2}+\epsilon}; u_0''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0); \\ v_0\rho^{-\frac{3}{2}+\epsilon}; \ v_0'\rho^{-\frac{1}{2}+\epsilon}; v_0''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0); \\ u_0 = v_0 = Lv_0 = 0 \ \textit{at} \ \rho = 0 \\ u_0' + u_0 = -u_1, \ ev_0' = -v_1; \ eLv_0 = w_1 \ \textit{at} \ \rho = \rho_0. \end{array} \right\} \tag{4.2.1}$$

Finally, if $\mathcal{A}_s$ (s for shell) is the generator in Theorem 1.1 corresponding to the original, full model (1.1), then $\mathcal{D}(\mathcal{A}_s) = \mathcal{D}(\mathcal{A})$. Moreover, to prove Theorem 1.2 (regularity) one then uses Theorem 3.1 and (4.2.1). Details are given in [8].

5. Proof of regularity Theorem 1.2.

Step 1. To prove the regularity Theorem 1.2 for the solutions of the original system (1.1), it suffices to establish the same regularity result for the corresponding simplified system (3.1), for then the former follows from the latter via perturbation as in Section 3.

Step 2.

Theorem 5.1. *With reference to problem* (3.1), *let the initial condition* $[u_0, w_0, u_1, w_1] \in \mathcal{D}(\mathcal{A}) \subset \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \times \mathcal{V}^1_\rho$, *see* (1.6)–(1.8), *so that, by Proposition* 4.2.1, *we then have*

$$\begin{cases} u_0 \in \mathcal{H}^2_{\epsilon,\rho};\ v_0 = \frac{u_0}{R} + w_0' \in \mathcal{H}^2_{\epsilon,\rho};\ (u_1, w_1) \in \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho \\ \text{and the B.C. (4.2.9) and (4.2.10) are satisfied.} \end{cases} \tag{5.1}$$

Then, the corresponding semigroup solution of (3.1), *guaranteed by Theorem* 3.1, *possesses the following regularity properties*:

$$[u(t), w(t), u_t(t), w_t(t)] = e^{\mathcal{A}t}[u_0, w_0, u_1, w_1] \in C(0, \infty; \mathcal{D}(\mathcal{A})), \ \ or \tag{5.2}$$

$$[[u_t(t), w_t(t)], [u_{tt}(t), w_{tt}(t)]] = e^{\mathcal{A}t}\mathcal{A}[u_0, w_0, u_1, w_1] \in C\big(0, \infty; \big[\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho\big] \times \mathcal{V}^1_\rho\big); \tag{5.3}$$

hence, explicitly via Proposition 4.2.1, *equation* (4.2.1),

$$(u_t(t), w_t(t)) \in C\big(0, \infty; \mathcal{U}^1_\rho \times \mathcal{W}^2_\rho\big) \tag{5.4}$$

$$(u_{tt}(t), w_{tt}(t)) \in C\big(0, \infty; \mathcal{V}^1_\rho\big) \tag{5.5}$$

$$\Big(u(t), v(t) = \frac{u(t)}{R} + w'(t)\Big) \in C\big(0, \infty; \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{H}^2_{\epsilon,\rho}\big), \quad \epsilon > 0. \tag{5.6}$$

Proof. The proof is an immediate consequence of the semigroup result of well-posedness Theorem 3.1 for problem (3.1), combined with the description of $\mathcal{D}(\mathcal{A})$ given by (4.2.1) of Proposition 4.2.1.

6. Proof of dissipativity Theorem 1.3. Let $\{u, w\}$ be a regular solution of problem (1.1), in the sense that it satisfies the regularity properties (1.18), (1.19), (1.20) of Theorem 1.2 for smooth initial data $[u_0, w_0, u_1, w_1] \in \mathcal{D}(\mathcal{A})$, as described there. We shall derive the dissipativity identity (1.21) first for such regular solutions for which all the subsequent calculations of this section are justified, and then extend its validity by density to all semigroup solutions guaranteed by Theorem 1.1. Thus, with $\{u, w\}$ a regular solution, we multiply the u-equation (1.1a) by ρu_t, the w-equation (1.1b) by ρw_t and integrate over $(0, \rho_0)$.

Step 1 (kinetic energy).

$$\int_0^{\rho_0} (u_{tt}u_t + w_{tt}w_t)\rho\, d\rho = \tfrac{1}{2}\int_0^{\rho_0} \frac{d}{dt}(u_t^2 + w_t^2)\rho\, d\rho; \tag{6.1}$$

$$\begin{aligned} &\frac{e}{R}\int_0^{\rho_0} v_{tt}u_t\,\rho\, d\rho - e\int_0^{\rho_0} \frac{[v_{tt}\rho]'}{\rho} w_t\,\rho\, d\rho \\ &\qquad = e\int_0^{\rho_0} v_{tt}\frac{u_t}{R}\,\rho\, d\rho + e\int_0^{\rho_0} v_{tt}w_t'\,\rho\, d\rho - e[v_{tt}\rho w_t]_{\rho=\rho_0} \\ \text{(by (1.1c))} &\qquad = e\int_0^{\rho_0} v_{tt}v_t\rho\, d\rho - e[v_{tt}\rho w_t]_{\rho=\rho_0} = \frac{e}{2}\int_0^{\rho_0}\frac{d}{dt}(v_t^2)\rho\, d\rho - e[v_{tt}\rho w_t]_{\rho=\rho_0}. \end{aligned} \tag{6.2}$$

Thus, summing up (6.1) and (6.2),

$$\begin{aligned} &\int_0^{\rho_0} \big(u_{tt} + \frac{e}{R} v_{tt}\big) u_t \rho \, d\rho + \int_0^{\rho_0} \big(w_{tt} - e\frac{[v_{tt}\rho]'}{\rho}\big) w_t \rho \, d\rho \\ \text{(by (1.3))} \ &= \frac{1}{2}\frac{d}{dt} E_k(t) - e v_{tt}(\rho_0) w_t(\rho_0)\rho_0. \end{aligned} \tag{6.3}$$

Step 2 (principal part of operator involving L). From (2.17) we obtain

$$-\int_0^{\rho_0} (Lu) u_t \rho \, d\rho = -u'(\rho_0) u_t(\rho_0)\rho_0 + \int_0^{\rho_0} \big[u' u_t' \rho + \frac{u u_t}{\rho}\big] \, d\rho \tag{6.4}$$

$$= -u'(\rho_0) u_t(\rho_0)\rho_0 + \tfrac{1}{2}\frac{d}{dt}\int_0^{\rho_0} \big[(u')^2 \rho + \frac{u^2}{\rho}\big] \, d\rho = -u'(\rho_0) u_t(\rho_0)\rho_0 + \tfrac{1}{2}\frac{d}{dt}\big(\|u\|^2_{\mathcal{U}^1_\rho}\big).$$

Integrating by parts,

$$\begin{aligned} &-\frac{e}{R}\int_0^{\rho_0} (Lv) u_t \rho \, d\rho + e\int_0^{\rho_0} \frac{[(Lv)\rho]'}{\rho} w_t \rho \, d\rho \\ &= -e\int_0^{\rho_0} (Lv)\frac{u_t}{R}\rho \, d\rho - e\int_0^{\rho_0} (Lv) w_t' \rho \, d\rho + e(Lv)(\rho_0) w_t(\rho_0)\rho_0 \\ \text{(by (1.1c))} \ &= -e\int_0^{\rho_0} (Lv) v_t \rho \, d\rho + e(Lv)(\rho_0) w_t(\rho_0)\rho_0 \\ \text{(by (6.4))} \ &= \frac{e}{2}\ \frac{d}{dt}\big(\|v\|^2_{\mathcal{U}^1_\rho}\big) - e v'(\rho_0) v_t(\rho_0)\rho_0 + e(Lv)(\rho_0) w_t(\rho_0)\rho_0 \qquad (6.5) \\ &= \frac{e}{2}\ \frac{d}{dt}\big(\|v\|^2_{\mathcal{U}^1_\rho}\big) + v_t^2(\rho_0)\rho_0 + w_t^2(\rho_0)\rho_0 + e v_{tt}(\rho_0) w_t(\rho_0)\rho_0, \qquad (6.6) \end{aligned}$$

after using the B.C. (1.1g) and (1.1h) in the last two terms, respectively, of (6.5).

Step 3. Summing up (6.3), (6.4), and (6.6) results, after cancellation of $e v_{tt}(\rho_0) w_t(\rho_0)\rho_0$, in

$$\begin{aligned} &\int_0^{\rho_0} \big(u_{tt} + \frac{e}{R} v_{tt}\big) u_t \rho \, d\rho + \int_0^{\rho_0} \big(w_{tt} - e\frac{[v_{tt}\rho]'}{\rho}\big) w_t \rho \, d\rho \\ &\quad - \int_0^{\rho_0} \big[Lu + \frac{e}{R}(Lv)\big] u_t \rho \, d\rho + e\int_0^{\rho_0} \frac{[(Lv)\rho]'}{\rho} w_t \rho \, d\rho \\ &\quad = \tfrac{1}{2}\ \frac{d}{dt} E_k(t) + \frac{1}{2}\ \frac{d}{dt}\big(\|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho}\big) + v_t^2(\rho_0)\rho_0 + w_t^2(\rho_0)\rho_0 \\ &\quad + \big[u(\rho_0) u_t(\rho_0)\rho_0 + u_t^2(\rho_0)\rho_0 + \nu u(\rho_0) u_t(\rho_0) - \frac{(1+\nu)}{R} w(\rho_0) u_t(\rho_0)\rho_0\big], \end{aligned} \tag{6.7}$$

where the term in the square bracket [ ] is $-u'(\rho_0) u_t(\rho_0)\rho_0$ after use of the B.C. (1.1f).

Step 4 (lower-order terms). We handle the last three (lower-order) terms in (1.1a–b), by integration by parts:

$$\begin{aligned}
&\frac{(1+\nu)}{R}\Big\{\int_0^{\rho_0} w' u_t \rho\, d\rho - \int_0^{\rho_0}\Big[\frac{(u\rho)'}{\rho} - \frac{2}{R}w\Big] w_t\rho\, d\rho\Big\}\\
&= \frac{(1+\nu)}{R}\Big\{w(\rho_0)u_t(\rho_0)\rho_0 - \int_0^{\rho_0}[w(u_t\rho)' + (u\rho)'w_t]\, d\rho + \frac{2}{R}\int_0^{\rho_0}\tfrac{1}{2}\frac{d}{dt}(w^2)\rho\, d\rho\Big\}\\
&= \frac{(1+\nu)}{R}\Big\{w(\rho_0)u_t(\rho_0)\rho_0 - \frac{d}{dt}\int_0^{\rho_0} w(u\rho)'\, d\rho + \frac{d}{dt}\int_0^{\rho_0}\frac{w^2}{R}\rho\, d\rho\Big\}. \qquad (6.8)
\end{aligned}$$

Step 5. Thus, the energy method of multiplying the u-equation (1.1a) by ρu_t and the w-equation (1.1b) by ρw_t and summing up the resulting expressions obtained after integrating by parts in $[0,\rho_0]$ amounts to setting to zero the sum of the right-hand sides of (6.7) and (6.8). In so doing, cancellation of the term $\frac{(1+\nu)}{R}w(\rho_0)u_t(\rho_0)\rho_0$ takes place, and we obtain

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}E_k(t) + \tfrac{1}{2}\frac{d}{dt}\big(e\|v\|^2_{\mathcal{U}^1_\rho}\big) + \tfrac{1}{2}\frac{d}{dt}\Big(\|u\|^2_{\mathcal{U}^1_\rho} + \frac{2(1+\nu)}{R^2}\|w\|^2_{L^\rho_2}\Big)\\
&\quad - \tfrac{1}{2}\frac{d}{dt}\frac{2(1+\nu)}{R}\int_0^{\rho_0} w[u'\rho + u]\, d\rho + [u_t^2(t,\rho_0) + v_t^2(t,\rho_0) + w_t^2(t,\rho_0)]\rho_0\\
&\quad + \tfrac{1}{2}\frac{d}{dt}(u^2(t,\rho_0))\rho_0 + \tfrac{1}{2}\frac{d}{dt}(\nu u^2(t,\rho_0)) \equiv 0. \qquad (6.9)
\end{aligned}$$

Step 6. We claim that (6.9) can be rewritten, equivalently, as

$$\tfrac{1}{2}\frac{d}{dt}E_k(t) + \tfrac{1}{2}\frac{d}{dt}E_p(t) + \rho_0[u_t^2(t,\rho_0) + v_t^2(t,\rho_0) + w_t^2(t,\rho_0)] \equiv 0, \qquad (6.10)$$

which, by recalling E_p in (1.4) and comparing (6.10) with (6.9), requires verifying that

$$\begin{aligned}
&(1-\nu)\int_0^{\rho_0}\Big[\Big(u' - \frac{w}{R}\Big)^2\rho + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)^2\rho\Big]\, d\rho\\
&\quad + \nu\int_0^{\rho_0}\Big[\Big(u' - \frac{w}{R}\Big)\sqrt{\rho} + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)\sqrt{\rho}\Big]^2 d\rho\\
&\equiv \|u\|^2_{\mathcal{U}^1_\rho} + \frac{2(1+\nu)}{R^2}\|w\|^2_{L^\rho_2} - \frac{2(1+\nu)}{R}\int_0^{\rho_0}(wu'\rho + wu)\, d\rho + \nu u^2(t,\rho_0) \qquad (6.11)\\
&= \int_0^{\rho_0}\Big[\frac{u^2}{\rho} + (u')^2\rho + 2(1+\nu)\frac{w^2}{R^2}\rho - 2\frac{w}{R}u'\rho - 2\frac{w}{R}u - 2\rho\nu\frac{w}{R}u' - 2\nu\frac{w}{R}u\Big]\, d\rho \qquad (6.12)\\
&\quad + \nu u^2(t,\rho_0).
\end{aligned}$$

Indeed, we verify that

$$\begin{aligned}
&(1-\nu)\Big[\Big(u' - \frac{w}{R}\Big)^2\rho + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)^2\rho\Big] + \nu\Big[\Big(u' - \frac{w}{R}\Big)\sqrt{\rho} + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)\sqrt{\rho}\Big]^2\\
&= \Big(u' - \frac{w}{R}\Big)^2\rho + \Big(\frac{u}{\rho} - \frac{w}{R}\Big)^2\rho + 2\nu\rho\Big(u' - \frac{w}{R}\Big)\Big(\frac{u}{\rho} - \frac{w}{R}\Big) \qquad (6.13)\\
&= (u')^2\rho + 2\frac{w^2}{R^2}\rho + \frac{u^2}{\rho} - 2u'\frac{w}{R}\rho - 2u\frac{w}{R} + 2\nu uu' - 2\rho\nu u'\frac{w}{R} - 2\nu u\frac{w}{R} + 2\rho\nu\frac{w^2}{R^2},
\end{aligned}$$

and by comparing the right-hand sides of (6.12) and (6.13), we see that all we need to check now is that

$$2\nu \int_0^{\rho_0} uu'\, d\rho = \nu u^2(t,\rho_0), \tag{6.14}$$

which is, indeed, true by recalling the B.C. (1.1e) for u:

$$2\nu \int_0^{\rho_0} uu'\, d\rho = \nu \int_0^{\rho_0} (u^2)'\, d\rho = \nu[u^2(t,\rho_0)]$$

and (6.14) follows. Thus (6.10) has been verified. Integrating in t yields identity (1.21) first for smooth initial data $[u_0, w_0, u_1, w_1] \in \mathcal{D}(\mathcal{A})$ as in Theorem 1.2, then for all $[[u_0, w_0], [u_1, w_1]] \in [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho$ by extension, and Theorem 1.3 is proved.

Proof of stabilization Theorem 1.4: Preliminary identity and *a priori* **estimates.** In this section we begin the proof of Theorem 1.4. We shall establish, by energy or multiplier methods, a basic identity from which we shall derive a preliminary energy estimate [(7.33) below], which is the desired estimate modulo lower-order terms. These will then be absorbed in the forthcoming Section 9. We begin with:

Theorem 7.1. (i) *With reference to problem* (1.1), *whose solution is guaranteed by Theorem* 1.1, *the following identity holds true*:

$$\tfrac{1}{2}\int_0^T \big[\|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho} + \|u_t\|^2_{L_2^\rho} + 3\|w_t\|^2_{L_2^\rho} + e\|v_t\|^2_{L_2^\rho}\big]\, dt + (E_nT)_0^T + IT + BT \equiv 0, \tag{7.1}$$

where the end terms $(E_nT)_0^T$, *the interior terms* IT, *and the boundary terms* BT *are defined by*

$$(E_nT)_0^T = \int_0^{\rho_0} \big\{u_t\big[u'\rho^2 + \frac{u\rho}{2}\big]\big\}_0^T\, d\rho + \int_0^{\rho_0} \big\{w_t\big[w'\rho^2 - \frac{w\rho}{2}\big]\big\}_0^T\, d\rho$$
$$+ e\int_0^{\rho_0} \big\{v_t\big[v'\rho^2 + \frac{v\rho}{2}\big]\big\}_0^T\, d\rho; \tag{7.2}$$

$$IT = \frac{(1+\nu)}{R}\int_0^T \int_0^{\rho_0} \big[wu'\rho + wu - \frac{3}{R}w^2\rho\big]\, d\rho\, dt; \tag{7.3}$$

$$BT = -\tfrac{1}{2}\int_0^T \big[u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)\big]\rho_0^2\, dt \tag{7.4}$$
$$- \tfrac{1}{2}\int_0^T \big[(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0\big]\, dt$$
$$+ e\int_0^T \big\{w_t(t,\rho_0)\big[w'(t,\rho_0)\rho_0^2 - \frac{w(t,\rho_0)\rho_0}{2}\big]\big\}\, dt$$
$$- \tfrac{1}{2e}\int_0^T \big[(v_t(t,\rho_0))^2\rho_0^2 + v_t(t,\rho_0)v(t,\rho_0)\rho_0\big]\, dt$$
$$+ \tfrac{1}{2}\int_0^T \big[u^2(t,\rho_0) + ev^2(t,\rho_0)\big]\, dt$$
$$- \frac{(1+\nu)}{2R}\int_0^T u(t,\rho_0)w(t,\rho_0)\rho_0\, dt + \frac{(1+\nu)}{R^2}\int_0^T w^2(t,\rho_0)\rho_0^2\, dt.$$

(ii) *As to* (7.2), *we have that there exists a constant* $c > 0$, *independent of* T, *such that*

$$\left|(E_nT)_0^T\right| \le c[E(0) + E(T)], \tag{7.5}$$

while, as to (7.3), *we have, given* $\epsilon > 0$, *that there exists* $c_\epsilon > 0$ *such that*

$$|IT| \le \epsilon \int_0^T \|u\|^2_{\mathcal{U}^1_\rho} + c_\epsilon \int_0^T \int_0^{\rho_0} w^2 \rho \, d\rho \, dt. \tag{7.6}$$

(iii) *The following a priori energy estimate holds true*:

$$\int_0^T \left\{ \left(\tfrac{1}{2} - \epsilon\right) \|u\|^2_{\mathcal{U}^1_\rho} + \frac{e}{2}\|v\|^2_{\mathcal{U}^1_\rho} + \tfrac{1}{2}\|u_t\|^2_{L_2^\rho} + \tfrac{3}{2}\|w_t\|^2_{L_2^\rho} + \tfrac{e}{2}\|v_t\|^2_{L_2^\rho} \right\} dt$$
$$\le c\{E(0) + E(T)\} + |BT| + c_\epsilon \int_0^T \int_0^{\rho_0} w^2 \rho \, d\rho \, dt. \tag{7.7}$$

Remark 7.1. In the proof below, the multipliers used for the wave equation (1.1a) in u and the beam-equation (1.1b) in w with "high" B.C. (of second and third order, involving w'' and w''') are inspired by the successful experience of the past several years on wave-like and plate-like equations (e.g., [9] and [8]), with the correction of a factor ρ, since the underlying spaces have a weight.

Proof. (i) We start with regular solutions, in the sense of Theorem 1.2, satisfying properties (1.18), (1.19), (1.20), and multiply the u-equation (1.1a) by $\left[u'\rho^2 + \frac{u\rho}{2}\right]$, and the w-equation (1.1b) by $\left[w'\rho^2 - \frac{w\rho}{2}\right]$, and integrate by parts over $(0,T) \times (0,\rho_0)$. We shall obtain identity (7.1) first for regular solutions, for which the subsequent computations are justified, and then extend by density to all semigroup solutions.

Step 1 (Principal part involving L)**.** By applying identities (2.17) and (2.18) we obtain at once

$$-\int_0^{\rho_0} (Lu)\left[u'\rho^2 + \frac{u\rho}{2}\right] d\rho = \tfrac{1}{2}\left[\|u\|^2_{\mathcal{U}^1_\rho} - u'(\rho_0)u(\rho_0)\rho_0\right] + \tfrac{1}{2}[u^2(\rho_0) - (u'(\rho_0))^2\rho_0^2]. \tag{7.8}$$

Moreover, by integrating by parts on the second integral we obtain

$$-\frac{e}{R}\int_0^{\rho_0} (Lv)\left[u'\rho^2 + \frac{u\rho}{2}\right] d\rho + e\int_0^{\rho_0} \frac{[(Lv)\rho]'}{\rho}\left[w'\rho - \frac{w}{2}\right]\rho \, d\rho$$
$$= -e\int_0^{\rho_0} (Lv)\left[\frac{u'}{R}\rho^2 + \frac{u}{R}\,\frac{\rho}{2}\right] d\rho - e\int_0^{\rho_0} (Lv)\left[w''\rho^2 + w'\frac{\rho}{2}\right] d\rho$$
$$+ e(Lv)(\rho_0)w'(\rho_0)\rho_0^2 - \frac{e}{2}(Lv)(\rho_0)w(\rho_0)\rho_0 \tag{7.9}$$

$$\text{(by (1.1c))} = -e\int_0^{\rho_0} (Lv)\left[v'\rho^2 + v\tfrac{\rho}{2}\right] d\rho + e(Lv)(\rho_0)w'(\rho_0)\rho_0^2 - \tfrac{e}{2}(Lv)(\rho_0)w(\rho_0)\rho_0$$

$$[\text{by (7.8) with } u \text{ replaced by } v] \; = \tfrac{e}{2}\left[\|v\|^2_{\mathcal{U}^1_\rho} - v'(\rho_0)v(\rho_0)\rho_0\right] \tag{7.10}$$
$$+ \tfrac{e}{2}[v^2(\rho_0) - (v'(\rho_0))^2\rho_0^2] + e(Lv)(\rho_0)w'(\rho_0)\rho_0^2 - \tfrac{e}{2}(Lv)(\rho_0)w(\rho_0)\rho_0.$$

Thus, summing up (7.8) and (7.10) and integrating over $(0,T)$ yields

$$\begin{aligned}
&-\int_0^T\int_0^{\rho_0}(Lu)\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt-\frac{e}{R}\int_0^T\int_0^{\rho_0}(Lv)\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt\\
&=\tfrac12\int_0^T\big[\|u\|^2_{\mathcal{U}^1_\rho}+e\|v\|^2_{\mathcal{U}^1_\rho}\big]\,dt\\
&\quad+\tfrac12\int_0^T\big[-u'(t,\rho_0)u(t,\rho_0)\rho_0-(u'(t,\rho_0))^2\rho_0^2+u^2(t,\rho_0)\big]\,dt\\
&\quad+\tfrac{e}{2}\int_0^T\big[-v'(t,\rho_0)v(t,\rho_0)\rho_0-(v'(t,\rho_0))^2\rho_0^2+v^2(t,\rho_0)\big]\,dt\\
&\quad+e\int_0^T(Lv)(t,\rho_0)\big[w'(t,\rho_0)\rho_0^2-\tfrac12 w(t,\rho_0)\rho_0\big]\,dt.
\end{aligned}\tag{7.11}$$

Step 2 (lower-order terms)**.** Expanding the products, we compute

$$\begin{aligned}
&\frac{(1+\nu)}{R}\Big\{\int_0^{\rho_0}w'\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho-\int_0^{\rho_0}\frac{(u\rho)'}{\rho}\big[w'\rho^2-\frac{w\rho}{2}\big]\,d\rho\\
&\qquad+\frac{2}{R}\int_0^{\rho_0}w\big[w'\rho^2-\frac{w\rho}{2}\big]\,d\rho\Big\}\\
&=\frac{(1+\nu)}{R}\Big\{\int_0^{\rho_0}\big[\frac{u'w\rho}{2}+\frac{uw}{2}-\frac{uw'\rho}{2}\big]\,d\rho+\frac{2}{R}\int_0^{\rho_0}\tfrac12(w^2)'\rho^2\,d\rho-\frac{1}{R}\int_0^{\rho_0}w^2\rho\,d\rho\Big\}
\end{aligned}\tag{7.12}$$

(integrating by parts $-uw'\rho/2$)

$$=\frac{(1+\nu)}{R}\Big\{\int_0^{\rho_0}\big[u'w\rho+uw-\frac{3}{R}w^2\rho\big]\,d\rho-\frac12 u(\rho_0)w(\rho_0)\rho_0+\frac{1}{R}w^2(\rho_0)\rho_0^2\Big\}.$$

Thus, integrating (7.12) over $(0,T)$ yields

$$\begin{aligned}
&\frac{(1+\nu)}{R}\Big\{\int_0^T\int_0^{\rho_0}w'\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt-\int_0^T\int_0^{\rho_0}\frac{(u\rho)'}{\rho}\big[w'\rho^2-\frac{w\rho}{2}\big]\,d\rho\,dt\\
&\qquad+\frac{2}{R}\int_0^T\int_0^{\rho_0}w\big[w'\rho^2-\frac{w\rho}{2}\big]\,d\rho\,dt\Big\}\\
&=\frac{(1+\nu)}{R}\Big\{\int_0^T\int_0^{\rho_0}\big[u'w\rho+uw\big]\,d\rho\,dt-\frac{3}{R}\int_0^T\int_0^{\rho_0}w^2\rho\,d\rho\,dt\\
&\qquad-\frac12\int_0^T\big[u(t,\rho_0)w(t,\rho_0)\big]\rho_0\,dt+\frac{1}{R}\int_0^T w^2(t,\rho_0)\rho_0^2\,dt\Big\}.
\end{aligned}\tag{7.13}$$

Step 3 (terms involving second-time derivatives)**.** Integrating by parts in t,

$$\int_0^T u_{tt}\big[u'\rho^2+\frac{u\rho}{2}\big]\,dt+\int_0^T w_{tt}\big[w'\rho^2-\frac{w\rho}{2}\big]\,dt$$

$$= \big(u_t\big[u'\rho^2+\frac{u\rho}{2}\big]\big)_0^T+\big(w_t\big[w'\rho^2-\frac{w\rho}{2}\big]\big)_0^T-\tfrac12\int_0^T(u_t^2)'\rho^2\,dt \\ -\tfrac12\int_0^T u_t^2\rho\,d\rho-\tfrac12\int_0^T(w_t^2)'\rho^2\,dt+\tfrac12\int_0^T w_t^2\rho\,d\rho. \tag{7.14a}$$

Thus, integrating (7.14a) over $(0,\rho_0)$ yields

$$\int_0^T\int_0^{\rho_0}u_{tt}\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt+\int_0^T\int_0^{\rho_0}w_{tt}\big[w'\rho^2-\frac{w\rho}{2}\big]\,dt \tag{7.14b}$$
$$=\int_0^{\rho_0}\big(u_t\big[u'\rho^2+\frac{u\rho}{2}\big]\big)_0^T\,d\rho+\int_0^{\rho_0}\big(w_t\big[w'\rho^2-\frac{w\rho}{2}\big]\big)_0^T\,d\rho-\tfrac12\int_0^T u_t^2(t,\rho_0)\rho_0^2\,dt \\ +\tfrac12\int_0^T\int_0^{\rho_0}u_t^2\rho\,d\rho\,dt-\tfrac12\int_0^T w_t^2(t,\rho_0)\rho_0^2\,dt+\tfrac32\int_0^T\int_0^{\rho_0}w_t^2\rho\,d\rho\,dt.$$

Next, integrating by parts the second integral on the right-hand side,

$$\int_0^T\int_0^{\rho_0}\frac{e}{R}v_{tt}\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt-e\int_0^T\int_0^{\rho_0}\frac{[v_{tt}\rho]'}{\rho}\big[w'\rho^2-\frac{w}{2}\big]\rho\,d\rho\,dt$$
$$=\int_0^T\int_0^{\rho_0}ev_{tt}\big[\frac{u'}{R}\rho^2+\frac{u}{R}\,\frac{\rho}{2}\big]\,d\rho\,dt$$
$$-e\int_0^T v_{tt}(\rho_0)\rho_0\big[w'(\rho_0)\rho_0-\frac{w(\rho_0)}{2}\big]\,dt+e\int_0^T\int_0^{\rho_0}v_{tt}\rho\big[w''\rho+\frac{w'}{2}\big]\,d\rho\,dt$$
$$(\text{by (1.1c)})=e\int_0^{\rho_0}\int_0^T v_{tt}\big[v'\rho^2+\frac{v\rho}{2}\big]\,dt\;d\rho-e\int_0^T v_{tt}(\rho_0)\rho_0\big[w'(\rho_0)\rho_0-\frac{w(\rho_0)}{2}\big]\,dt$$
$$=e\int_0^{\rho_0}(v_tv'\rho^2)_0^T\,d\rho-\frac{e}{2}\int_0^T\int_0^{\rho_0}(v_t^2)'\rho^2\,d\rho\,dt \tag{7.15}$$
$$+\frac{e}{2}\int_0^{\rho_0}(v_tv\rho)_0^T\,d\rho-\frac{e}{2}\int_0^T\int_0^{\rho_0}v_t^2\rho\,d\rho\,dt-e\int_0^T v_{tt}(\rho_0)\big[w'(\rho_0)\rho_0^2-w(\rho_0)\frac{\rho_0}{2}\big]\,dt.$$

Finally, integrating in ρ the second integral on the right-hand side of (7.16) yields

$$\int_0^T\int_0^{\rho_0}\frac{e}{R}v_{tt}\big[u'\rho^2+\frac{u\rho}{2}\big]\,d\rho\,dt-e\int_0^T\int_0^{\rho_0}\frac{[v_{tt}\rho]'}{\rho}\big[w'\rho-\frac{w}{2}\big]\rho\,d\rho\,dt \tag{7.16}$$
$$=e\int_0^{\rho_0}(v_tv'\rho^2)_0^T\,d\rho-\frac{e}{2}\int_0^T v_t^2(\rho_0)\rho_0^2\,dt+\frac{e}{2}\int_0^T\int_0^{\rho_0}v_t^2\rho\,d\rho\,dt \\ +\frac{e}{2}\int_0^{\rho_0}(v_tv\rho)_0^T\,d\rho-e\int_0^T v_{tt}(\rho_0)\big[w'(\rho_0)\rho_0^2-w(\rho_0)\frac{\rho_0}{2}\big]\,dt.$$

Step 4. Thus, the energy methods of multiplying the u-equation (1.1a) by $\big[u'\rho^2+\frac{u\rho}{2}\big]$, the w-equation (1.1b) by $\big[w'\rho^2-\frac{w\rho}{2}\big]$, then integrating over $(0,T)\times(0,\rho_0)$, and summing up the resulting expressions amounts now to summing the right-hand sides of (7.11), (7.13), (7.14a), (7.14b), (7.15), and (7.16), and setting equal to zero their sum. We thus obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_0^T \big[\|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho} + \|u_t\|^2_{L^\rho_2} + 3\|w_t\|^2_{L^\rho_2} + e\|v_t\|^2_{L^\rho_2}\big]\,dt \\
&+ \int_0^{\rho_0} \big(u_t\big[u'\rho^2 + \frac{u\rho}{2}\big]\big)_0^T\,d\rho + \int_0^{\rho_0} \big(w_t\big[w'\rho^2 - \frac{w}{2}\rho\big]\big)_0^T\,d\rho \\
&+ e\int_0^{\rho_0} \big(v_t\big[v'\rho^2 + \frac{v\rho}{2}\big]\big)_0^T\,d\rho + \frac{(1+\nu)}{R}\int_0^T\int_0^{\rho_0}\big[wu'\rho + wu - \frac{3}{R}w^2\rho\big]\,d\rho\,dt \\
&+ \Big\{ - \tfrac{1}{2}\int_0^T \big[u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)\big]\rho_0^2\,dt \\
&- \tfrac{1}{2}\int_0^T \big[(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0\big]\,dt \\
&+ e\int_0^T \big[(Lv)(t,\rho_0)\big(w'(t,\rho_0)\rho_0^2 - \frac{1}{2}w(t,\rho_0)\rho_0\big)\big]\,dt \\
&- \tfrac{e}{2}\int_0^T \big[(v'(t,\rho_0))^2\rho_0^2 + v'(t,\rho_0)v(t,\rho_0)\rho_0\big]\,dt + \tfrac{1}{2}\int_0^T \big[u^2(t,\rho_0) + ev^2(t,\rho_0)\big]\,dt \\
&- \frac{(1+\nu)}{2R}\int_0^T u(t,\rho_0)w(t,\rho_0)\rho_0\,dt + \frac{(1+\nu)}{R^2}\int_0^T w^2(t,\rho_0)\rho_0^2\,dt \\
&- e\int_0^T v_{tt}(t,\rho_0)\big[w'(t,\rho_0)\rho_0^2 - w(t,\rho_0)\frac{\rho_0}{2}\big]\,dt\Big\} \equiv 0. \qquad (7.17)
\end{aligned}$$

In view of the definitions (7.2), (7.3), (7.4) of $(E_nT)_0^T$, IT, and BT, we see that equation (7.17) is precisely (7.1), once we take into account the B.C. of (1.1) as follows in the last parentheses $\{\quad\}$: (i) we first use the B.C. (1.1h), $e(Lv) - ev_{tt} = w_t$ at $\rho = \rho_0$, to combine the third integral term with the last integral term $\{\ \}$ to produce the third integral term in (7.4); (ii) next we use the B.C. (1.1g), $ev' = -v_t$ at $\rho = \rho_0$, in the fourth integral term in $\{\ \}$. This way, $\{\ \}$ becomes the expression (7.4) for BT, as desired.

(ii) Returning to (7.2), we estimate (by (1.6b))

$$\begin{aligned}
\Big|\int_0^{\rho_0} \big(u_t\big[u'\rho^2 + \frac{u\rho}{2}\big]\big)\,d\rho\Big| &= \Big|\int_0^{\rho_0} \big[(u_t\sqrt{\rho})(u'\sqrt{\rho})\rho + (u_t\sqrt{\rho})\frac{u}{\sqrt{\rho}}\frac{\rho}{2}\big]\,d\rho\Big| \\
&\le c\int_0^{\rho_0}\big[u_t^2\rho + (u')^2\rho + \frac{u^2}{\rho}\big]\,d\rho = c\Big\{\int_0^{\rho_0} u_t^2\rho\,d\rho + \|u\|^2_{\mathcal{U}^1_\rho}\Big\} \qquad (7.18)
\end{aligned}$$

Similarly, by (1.7b), writing now $w_tw'\rho^2 = (w_t\sqrt{\rho})(w'\sqrt{\rho})\rho^2$ and similarly for $v_tv'\rho^2$:

$$\Big|\int_0^{\rho_0} \big(w_t\big[w'\rho^2 - \frac{w\rho}{2}\big]\big)\,d\rho\Big| \le c\Big\{\int_0^{\rho_0} w_t^2\rho\,d\rho + \|w\|^2_{\mathcal{W}^2_\rho}\Big\} \qquad (7.19)$$

$$\begin{aligned}
\Big|\int_0^{\rho_0} e\big(v_t\big[v'\rho^2 + \frac{v\rho}{2}\big]\big)\,d\rho\Big| &\le c\Big\{e\int_0^{\rho_0} v_t^2\rho\,d\rho + \|v\|^2_{\mathcal{U}^1_\rho}\Big\} \qquad (7.20) \\
\text{(by (1.12))} \ &= c\Big\{e\int_0^{\rho_0} v_t^2\rho\,d\rho + \|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\Big\}.
\end{aligned}$$

Thus, combining (7.18)–(7.20) and recalling the definition (7.2), we obtain

$$|(E_nT)_0^T| \leq c\Big\{\int_0^{\rho_0} (u_t^2\rho + w_t^2\rho + ev_t^2\rho)\,d\rho + \|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\Big\}_{t=0} \tag{7.21}$$

$$+ c\Big\{\int_0^{\rho_0} (u_t^2\rho + w_t^2\rho + ev_t^2\rho)\,d\rho + \|u\|^2_{\mathcal{U}^1_\rho} + \|w\|^2_{\mathcal{W}^2_\rho}\Big\}_{t=T} \leq c[E(0) + E(T)],$$

recalling, in the last step, half of the equivalence in Proposition 2.1, along with (1.2), (1.3), (1.4). Thus, estimate (7.5) is proved.

Returning now to (7.3), we estimate similarly

$$\begin{aligned} |IT| &= \Big|\frac{(1+\nu)}{R}\int_0^T\int_0^{\rho_0}\Big[(u'\sqrt{\rho})(w\sqrt{\rho}) + \Big(\frac{u}{\sqrt{\rho}}\Big)(w\sqrt{\rho}) - \frac{3}{R}w^2\rho\Big]\,d\rho\,dt\Big| \\ &\leq \epsilon\int_0^T\int_0^{\rho_0}\Big[(u')^2\rho + \frac{u^2}{\rho}\Big]\,d\rho\,dt + c_\epsilon\int_0^T\int_0^{\rho_0} w^2\rho\,d\rho\,dt \\ &= \epsilon\int_0^T \|u\|^2_{\mathcal{U}^1_\rho}\,dt + c_\epsilon\int_0^T\int_0^{\rho_0} w^2\rho\,d\rho\,dt, \end{aligned} \tag{7.22}$$

and estimate (7.6) is proved.

(iii) We return to identity (7.1), where we use estimates (7.5), (7.6), thereby obtaining (7.7).

We next use also the B.C. (1.1f) in (7.4) and estimate the boundary term BT.

Proposition 7.2. *With reference to the term BT in (7.4) we have that there exists a constant $c > 0$ and, for any $\epsilon_0 > 0$, there exists a constant $c_{\epsilon_0} > 0$ such that*

$$\begin{aligned} |BT| &\leq \epsilon_0\int_0^T E(t)\,dt + c_{\epsilon_0}\int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)]\,dt \\ &\quad + c\Big\{\int_0^T [u^2(t,\rho_0) + w^2(t,\rho_0)]\rho_0^2\,dt + \int_0^T\int_0^{\rho_0} w^2\rho\,d\rho\,dt\Big\}. \end{aligned} \tag{7.23}$$

Proof. We return to (7.4). In its second integral term, we use the B.C. (1.1f) and thus express u' in terms of u_t and u at $\rho = \rho_0$; hence,

$$\int_0^T [(u'(t,\rho_0))^2\rho_0^2 + u'(t,\rho_0)u(t,\rho_0)\rho_0]\,dt \leq c\int_0^T [u_t^2(t,\rho_0) + u^2(t,\rho_0) + w^2(t,\rho_0)]\,dt. \tag{7.24}$$

Next, we recall that $w' \in \mathcal{U}^1_\rho$ (by definition (1.7) of $w \in \mathcal{W}^2_\rho$), whereby $w' \in H^1(I_{\rho_0})$, $I_{\rho_0} = (\rho_0/2, \rho_0)$, by property (p.3) in (1.26), and trace theory gives

$$\begin{aligned} \int_0^T |w'(t,\rho_0)|^2\,dt &\leq C\int_0^T \|w'(t,\cdot)\|^2_{H^1(I_{\rho_0})}\,dt \leq C\int_0^T \|w(t,\cdot)\|^2_{H^2(I_{\rho_0})} \\ &\leq C\int_0^T \|w(t,\cdot)\|^2_{\mathcal{W}^2_\rho(I_{\rho_0})}\,dt \leq \text{ const }\int_0^T E_p(t)\,dt, \end{aligned} \tag{7.25}$$

where, in the last step, we have recalled one way (coercivity) of Proposition 2.1, equation (2.3). Then (7.24) and (7.25), along with $v = \frac{u}{R} + w'$, allow us to obtain from (7.4),

$$|BT| \leq \epsilon_0 \int_0^T E(t)\,dt + C_{\epsilon_0} \int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + e v_t^2(t,\rho_0)]\,dt$$
$$+ C \int_0^T [u^2(t,\rho_0) + w^2(t,\rho_0)]\,dt + \frac{e}{2} \int_0^T v^2(t,\rho_0)\,dt. \tag{7.26}$$

However, a finer analysis than (7.25) is needed to handle the last integral term of (7.26). By $v = \frac{u}{R} + w'$, it suffices to have

$$\int_0^T |w'(t,\rho_0)|^2\,dt \leq \epsilon_0' \int_0^T \|w(t,\cdot)\|^2_{\mathcal{W}^2_\rho(I_{\rho_0})} + C_{\epsilon_0'} \int_0^T \int_{\frac{\rho_0}{2}}^{\rho_0} w^2 \rho\,d\rho\,dt \tag{7.27}$$

for any $\epsilon_0' > 0$, whereby then (7.27), used in the last integral term of (7.26) with $v^2(t,\rho_0) \leq C\{u^2(t,\rho_0) + (w'(t,\rho_0))^2\}$ does yield (7.23) as desired. It remains to establish (7.27). To do this we use a finer trace theory than in (7.25) and obtain, with $\epsilon > 0$:

$$\int_0^T |w'(t,\rho_0)|^2\,dt \leq C \int_0^T \|w'(t,\cdot)\|^2_{H^{\frac{1}{2}+\epsilon}(I_{\rho_0})}\,dt \leq C \int_0^T \|w(t,\cdot)\|^2_{H^{\frac{3}{2}+\epsilon}(I_{\rho_0})}\,dt, \tag{7.28}$$

and we next omit specification of the interval $I_{\rho_0} = \left(\frac{\rho_0}{2}, \rho_0\right)$. If $z \in H^{\frac{3}{2}+\epsilon} = [H^2, L_2]_\theta$, with $2(1-\theta) = \frac{3}{2} + \epsilon$, then the moment (or interpolation) inequality yields

$$\|z\|^2_{H^{\frac{3}{2}+\epsilon}} \leq C\|z\|^{2(1-\theta)}_{H^2} \|z\|^{2\theta}_{L_2} = Cab. \tag{7.29}$$

Next, we use on the right-hand side of (7.29) the Young inequality, e.g., [7, page 105],

$$ab \leq \frac{1}{\alpha} a^\alpha + \frac{1}{\beta} b^\beta, \qquad \alpha > 1,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \tag{7.30a}$$

$$a = \epsilon_0^{(1-\theta)} \|z\|^{2(1-\theta)}_{H^2}; \quad b = \epsilon_0^{(\theta-1)} \|z\|^{2\theta}_{L_2}; \quad 2(1-\theta) = \frac{3}{2} + \epsilon; \tag{7.30b}$$

$$\alpha(1-\theta) = 1,\ \beta\theta = 1;\ \frac{1}{\alpha} = (1-\theta) = \frac{3}{4} + \frac{\epsilon}{2};\ \frac{1}{\beta} = \theta = \frac{1}{4} - \frac{\epsilon}{2}, \tag{7.30c}$$

to obtain

$$\|z\|^2_{H^{\frac{3}{2}+\epsilon}} \leq C\big\{\big(\frac{3}{4} + \frac{\epsilon}{2}\big)\epsilon_0 \|z\|^2_{H^2} + \big(\frac{1}{4} - \frac{\epsilon}{2}\big)\epsilon_0^{\frac{\theta-1}{\theta}} \|z\|^2_{L_2}\big\}. \tag{7.31}$$

Finally, using (7.31) with $z = w(t,\cdot)$ in (7.28) yields

$$\int_0^T |w'(t,\rho_0)|^2\,dt \leq C\big\{\big(\tfrac{3}{4} + \tfrac{\epsilon}{2}\big)\epsilon_0 \int_0^T \|w(t,\cdot)\|^2_{H^2(I_{\rho_0})}\,dt$$
$$+ \big(\tfrac{1}{4} - \tfrac{\epsilon}{2}\big)\epsilon_0^{\frac{\theta-1}{\theta}} \frac{2}{\rho_0} \int_0^T \int_{\rho_0/2}^{\rho_0} w^2 \rho\,d\rho\big\}, \tag{7.32}$$

which proves (7.27), as desired, with $\epsilon > 0$ fixed and $\epsilon_0 > 0$ arbitrarily small.

As a corollary of Theorem 7.1 and Proposition 7.2, we obtain the preliminary *a priori* stabilization estimate.

Theorem 7.3. *With reference to the solutions of problem* (1.1), *guaranteed by Theorem* 1.1, *the following a priori stabilization estimate holds true: there exists* $T_0 > 0$ *such that, for all* $T > T_0$, *there is a constant* $c_T > 0$ *such that*

$$E(0) + E(T) + \int_0^T E(t)\,dt \leq c_T\Big\{\int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)]dt + \ell.o.t.\Big\}; \tag{7.33}$$

$$|\ell.o.t.| \leq C\Big\{\int_0^T [u^2(t,\rho_0) + w^2(t,\rho_0)]\,dt + \int_0^T\int_0^{\rho_0} w^2\rho\,d\rho\,dt\Big\}, \tag{7.34}$$

where c_T *may be taken as* $c_T = \frac{1}{\mu_T}$, $\mu_T = \min\big\{\frac{1}{2}T - c_1, 1\big\} > 0$, *and* T_0 *as* $\frac{1}{2}T_0 > c_1$, *where* c_1 *is the constant, independent of* T, *identified below in* (7.36).

Proof. (i) First we return to identity (7.1) in which we make use of estimates (7.5) for $(E_nT)_0^T$; (7.6) for IT; and (7.23) for BT. This way we obtain, preliminarily, the following estimate, for all T:

$$\int_0^T E(t)\,dt \leq c[E(0)+E(T)]+C\Big\{\int_0^T [u_t^2(t,\rho_0)+w_t^2(t,\rho_0)+ev_t^2(t,\rho_0)]\,dt+\ell.o.t.\Big\}. \tag{7.35}$$

(ii) Next, invoking the dissipativity identity (1.21), we express $E(0)$ in terms of $E(T)$ and the boundary terms on the right-hand side of (7.35), while we use $E(T) \leq E(t)$, $0 \leq t \leq T$, on its left-hand side, thus obtaining

$$\begin{aligned} \tfrac{1}{2}TE(T) + \tfrac{1}{2}\int_0^T E(t)\,dt &\leq \tfrac{1}{2}\int_0^T E(t)\,dt + \tfrac{1}{2}\int_0^T E(t)\,dt \\ &\leq c_1E(T) + c\Big\{\int_0^T [u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)]\,dt + \ell.o.t.\Big\}. \end{aligned} \tag{7.36}$$

Moving $E(T)$ to the left-hand side and writing here $\big(\frac{1}{2}T - c_1\big)\big[\frac{1}{2}E(T) + \frac{1}{2}E(T)\big]$, where one $E(T)$ is replaced by $E(0)$ and boundary terms via (1.21), we obtain (7.33) from (7.36).

Our next task is to absorb the $\ell.o.t.$ via a compactness/uniqueness argument.

8. A unique continuation result. Key to the absorption of the $\ell.o.t.$ from the preliminary estimate (7.33) is the following unique continuation result for the shell equations (1.1a–b), which is of interest in itself.

Theorem 8.1. *Let* $\{u,w\} \in \mathcal{D}'((0,T)\times(0,\rho_0))$ *be a solution of equations* (1.1a), (1.1b), *here rewritten for convenience*

$$u_{tt} + \frac{e}{R}v_{tt} - Lu - \frac{e}{R}Lv + \frac{(1+\nu)}{R}w' = 0; \tag{8.1a}$$

$$w_{tt} - \frac{e}{\rho}[v_{tt}\rho]' + \frac{e}{\rho}[(Lv)\rho]' - \frac{(1+\nu)}{R}\,\frac{1}{\rho}[u\rho]' + \frac{2(1+\nu)}{R^2}w = 0, \tag{8.1b}$$

as well as of the following overdetermined B.C.

$$u = w' = Lv = 0 \qquad \text{at } \rho = 0,\ 0 < t < T; \tag{8.1c}$$

$$u = u' = w = w' = w'' = w''' = 0 \qquad \text{at } \rho = \rho_0,\ 0 < t < T, \tag{8.1d}$$

where $T > T_0,\ T_0$ *as in Theorem* 7.3. *Then, in fact,*

$$u \equiv w \equiv 0 \qquad \text{in } [0,T] \times [0,\rho_0]. \tag{8.2}$$

Proof. Step 1. For $T > T_0$, we define the space (recall Theorem 1.1 and (1.4)),

$$\begin{aligned} \mathcal{X}_T = \big\{ &\{[u,w],[u_t,w_t]\} \in C([0,T];[\mathcal{U}^1_\rho(0,\rho_0)\times\mathcal{W}^2_\rho(0,\rho_0)]\times\mathcal{V}^1_\rho(0,\rho_0)) \\ &\equiv C([0,T];\mathcal{E})\ \{u,w\} \text{ solution of the overdetermined problem (8.1a–d)}\big\}, \end{aligned} \tag{8.3}$$

which is a linear subspace of $C([0,T];\mathcal{E})$. We shall show

Proposition 8.2. *The vector space* $\mathcal{X}_T,\ T > T_0$, *defined in* (8.3) *is finite dimensional.*

Proof. We shall establish that the unit ball of $\mathcal{X}_T$ is compact. To this end, we follow the strategy of [10, Appendix], [2], which, however, requires now Carleman estimates to handle the resulting system (8.20) below. Let $\{u,w\} \in \mathcal{X}_T$, so that, in particular, for $t = 0$, the initial conditions satisfy$\{[u_0,w_0],[u_1,w_1]\} \in \mathcal{E} \equiv [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho$, and then $E(0) < \infty$ recalling Proposition 2.1, equation (2.4). Then

$$\{\bar{u} = u_t; \bar{w} = w_t\} \text{ satisfies the overdetermined problem (8.1a–d)}, \tag{8.4a}$$

as well, with corresponding boundary traces equal to zero:

$$\bar{u} = \bar{w}' = L\bar{v} = 0 \ \text{ at } \rho = 0; \quad \bar{u} = \bar{u}' = \bar{w} = \bar{w}' = \bar{w}'' = \bar{w}''' = 0 \ \text{ at } \rho = \rho_0. \tag{8.4b}$$

Let $\bar{E}(t) = E(\bar{u}(t),\bar{w}(t))$ be the energy of the solution $\{\bar{u},\bar{w}\}$ defined by (1.2)–(1.4), via (8.4). We can apply estimate (7.33) of Theorem 7.3 to $\bar{E}(t)$ where now the integral term on boundary values at $\rho = \rho_0$ on the right-hand side of (7.33) vanishes by (8.4b), and we thus obtain

$$\bar{E}(0) + \bar{E}(T) + \int_0^T \bar{E}(t)\,dt \le c_T\,\overline{\ell.o.t.}, \tag{8.5}$$

with $\overline{\ell.o.t.}$ estimated from (7.34) with u, w replaced by $\bar{u} = u_t,\ \bar{w} = w_t$, so that, invoking (8.3),

$$\begin{aligned} |\overline{\ell.o.t.}| &\le c\Big\{\int_0^T [\bar{u}^2(t,\rho_0) + \bar{w}^2(t,\rho_0)]\,dt + \int_0^T\int_0^{\rho_0} w_t^2\rho\,d\rho\,dt\Big\} \\ \text{(by (8.3))}\ &\le c_T\|\{[u_0,w_0],[u_1,w_1]\}\|^2_{[\mathcal{U}^1_\rho\times\mathcal{W}^2_\rho]\times\mathcal{V}^1_\rho} \sim c_T E(0) < \infty, \end{aligned} \tag{8.6}$$

where in the last step we have also recalled (1.12), (8.3). Thus, (8.6) inserted into (8.5) yields

$$\bar{E}(0) + \bar{E}(T) + \int_0^T \bar{E}(t)\,dt \le c_T E(0) < \infty, \tag{8.7}$$

which, in particular, implies for the initial condition $\bar{u}_0 = u_1$, $\bar{w}_0 = w_1$, $\bar{u}_1 = u_{tt}|_{t=0}$, $\bar{w}_1 = w_{tt}|_{t=0}$ that

$$\big\{[u_1, w_1], [u_{tt}|_{t=0}, w_{tt}|_{t=0}]\big\} \in \mathcal{E} \equiv [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho, \tag{8.8}$$

so that by the well-posedness Theorem 2.1 we obtain from (8.8),

$$\big\{[u_t, w_t], [u_{tt}, w_{tt}]\big\} \in C([0,T]; [\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho] \times \mathcal{V}^1_\rho) = C([0,T]; \mathcal{E}), \tag{8.9a}$$

a fortiori via (1.8),

$$u_{tt},\ w_{tt},\ v_{tt} = \frac{u_{tt}}{R} + w'_{tt} \in C([0,T]; L^\rho_2(0,\rho_0)), \tag{8.9b}$$

a "time regularity improvement" over (8.3). We now use equations (8.1a–b) to obtain a corresponding "space regularity improvement" over (8.3).

Lemma 8.3. *Let* $\{[u,w],[u_t,w_t]\} \in C([0,T];\mathcal{E})$ *be a solution of problem* (8.1a–d), *i.e., a solution in* $\mathcal{X}_T$ *satisfying, moreover, the additional regularity property* (8.9*a*), *hence* (8.9*b*). *Then, in fact, for* $\epsilon > 0$

$$\big\{u, v = \frac{u}{R} + w'\big\} \in C([0,T]; \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{H}^2_{\epsilon,\rho}). \tag{8.10}$$

Proof. We use equation (8.1a) = (1.1a) rewritten here as

$$L\big(u + \frac{e}{R}v\big) = u_{tt} + \frac{e}{R}v_{tt} + \frac{(1+\nu)}{R}w' \in C([0,T]; L^\rho_2(0,\rho_0)) \tag{8.11}$$

for $u + \frac{e}{R}v \in C([0,T];\mathcal{U}^1_\rho)$, where the regularity on the right-hand side follows from (8.9b) and $w \in C([0,T];\mathcal{W}^2_\rho)$; i.e., $w' \in C([0,T];\mathcal{U}^1_\rho)$ by (1.7a), $\mathcal{U}^1_\rho \subset L^\rho_2$. Moreover, the B.C. $u(0) = v(0) = u(\rho_0) = v(\rho_0) = u'(\rho_0) = v'(\rho_0) = 0$—which follow from (8.1c–d)—*a fortiori* guarantee that we can apply Lemma 4.1.2 on (8.11) and obtain

$$u + \frac{e}{R}v \in C([0,T]; \mathcal{H}^2_{\epsilon,\rho}). \tag{8.12}$$

Next, we multiply by ρ equation (8.1b) = (1.1b) and integrate in ρ from zero to ρ, using $(Lv)(0) = u(0) = v_{tt}(0) = 0$, to get

$$e[(Lv)\rho] = e[v_{tt}\rho] - \int_0^\rho \xi w_{tt} d\xi + \frac{(1+\nu)}{R}[u\rho] - \frac{2(1+\nu)}{R}\int_0^\rho \xi w\, d\xi. \tag{8.13}$$

Recalling the assumed regularity on $\{u,w\} \in C([0,T];\mathcal{U}^1_\rho \times \mathcal{W}^2_\rho)$ and v_{tt}, w_{tt} in (8.9b), we obtain from (8.13) and (1.7),

$$Lv \in C([0,T]; L^\rho_2(0,\rho_0)),\ v \in C([0,T];\mathcal{U}^1_\rho), \tag{8.14}$$

which, along with the B.C. $v(0) = v(\rho_0) = v'(\rho_0) = 0$ from (8.1c–d), *a fortiori* guarantee that we can apply Lemma 4.1.2 on (8.14) and obtain

$$v \in C([0,T]; \mathcal{H}^2_{\epsilon,\rho}), \qquad \epsilon > 0. \tag{8.15}$$

Thus, (8.15) and (8.12) yield (8.10), as desired.

Continuing with the proof of Proposition 8.2, we see that, at this stage, for $\{u, w, u_t, w_t\} \in \mathcal{X}_T$, we have obtained the following regularity results, expressed in terms of $\left\{u, v = \frac{u}{R} + w'\right\}$ (where $w \in \mathcal{W}^2_\rho$ implies $w' \in \mathcal{U}^1_\rho$ by (1.7a)):

$$\{u, v, u_t, v_t\} \in C([0,T]; \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{U}^1_\rho \times \mathcal{U}^1_\rho) \tag{8.16}$$

from (8.10) and (8.9a), and moreover,

$$\{u_t, v_t, u_{tt}, v_{tt}\} \in C([0,T]; \mathcal{U}^1_\rho \times \mathcal{U}^1_\rho \times \mathcal{L}^\rho_2 \times \mathcal{L}^\rho_2) \tag{8.17}$$

by (8.9b). But

$$\text{the injections } \mathcal{H}^2_{\epsilon,\rho} \hookrightarrow \mathcal{U}^1_\rho \text{ and } \mathcal{U}^1_\rho \hookrightarrow L^\rho_2 \text{ are compact} \tag{8.18}$$

(see Appendix A, at the end of this section), and thus by (8.16)–(8.18), we can invoke [17] and conclude that

$$\begin{aligned} \text{the injection } \{u, v, u_t, v_t\} &\in C\big([0,T]; \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{H}^2_{\epsilon,\rho} \times \mathcal{U}^1_\rho \times \mathcal{U}^1_\rho\big) \\ &\hookrightarrow C\big([0,T]; \mathcal{U}^1_\rho \times \mathcal{U}^1_\rho \times L^\rho_2 \times L^\rho_2\big) \text{ is compact,} \end{aligned} \tag{8.19}$$

which, in terms of $\{u, w, u_t, w_t\}$, means (recall property (p.5) = (1.27)) that the unit ball of $\mathcal{X}_T$ is compact, as desired. Proposition 8.2 is proved.

Remark 8.1. The above argument can be repeated over and over leading to C^∞-regularity in time and space for the elements of $\mathcal{X}_T$.

Step 2. Since $\mathcal{X}_T$ is a finite-dimensional, invariant subspace under $\frac{d^2}{dt^2}$, it follows that $\frac{d^2}{dt^2}\colon \mathcal{X}_T \to \mathcal{X}_T$ can be represented by a (finite-dimensional) matrix. We shall show that $\mathcal{X}_T = \{0\}$, the trivial space, and so uniqueness as in (8.2) follows via (8.3). Indeed, for the sake of contradiction, let dim $\mathcal{X}_T > 0$, so that then the matrix representing $\frac{d^2}{dt^2}$ must possess an eigenvalue, denoted by λ, and a (nonzero) eigensolution $\{u, w, u_t, w_t\} \in \mathcal{X}_T$. Then such solution satisfies the following eigenvalue problem corresponding to (8.1):

$$-Lu - \frac{e}{R}Lv + \frac{(1+\nu)}{R}w' + \lambda u + \frac{e}{R}\lambda v = 0; \tag{8.20a}$$

$$\frac{e}{\rho}[(Lv)\rho]' - \frac{(1+\nu)}{R}\frac{1}{\rho}(u\rho)' + \frac{2(1+\nu)}{R^2}w + \lambda w - \frac{e}{\rho}\lambda(v\rho)' \equiv 0; \tag{8.20b}$$

$$u = w' = Lv = 0 \qquad \text{at } \rho = 0,\ 0 < t < T; \tag{8.20c}$$

$$u = u' = w = w' = w'' = w''' = 0 \qquad \text{at } \rho = \rho_0,\ 0 < t < T, \tag{8.20d}$$

which consists of two coupled elliptic equations, for t fixed, with solution components in $C^1([0,T]\times(0,\rho_0))$ [in fact, in $C^\infty((0,T)\times(0,\rho_0))$].

Step 3. We shall now follow the strategy presented in the Appendix B in the case of a dynamic problem which is slightly simplified over (8.1), which is corresponding to (8.20) in that equation (8.20a) contains also the lower-order term $\frac{(1+\nu)}{R}w'$. As a result of the presence of this term, however, it will not be possible any longer to decouple equations (8.20a) and (8.20b), by virtue of the new variable z,

$$z = u + \frac{e}{R}v = \big(1+\frac{e}{R^2}\big)u + \frac{e}{R}w'; \quad \text{or} \quad u = cz - \frac{ec}{R}w', \; c = \frac{R^2}{R^2+e}, \tag{8.21}$$

as is the case in Appendix B. The resulting equations, (8.22a) and (8.22b) below, appear then mildly coupled, and we shall then apply Carleman estimates to each of them for t fixed, but arbitrary, to deduce uniqueness of the system; i.e., the desired conclusion (8.2). This contradicts the existence of an eigensolution for problem (8.20), so that $\mathcal{X}_T$ is the trivial space, as desired.

Lemma 8.4. *The transformation z in* (8.21) *changes the original problem* (8.20) *into the new problem*

$$Lz - \lambda z = \frac{(1+\nu)}{R}w'; \tag{8.22a}$$

$$\frac{R^2e}{R^2+e}[(L-\lambda)w'\rho]' + \frac{2e}{R^2+e}(1+\nu)(w'\rho)' + \big[\lambda + \frac{2(1+\nu)}{R^2}\big]w\rho \equiv \frac{R(1+\nu)}{R^2+e}(z\rho)', \tag{8.22b}$$

which is a coupled system of elliptic equations, for t fixed, second order in z and fourth order in w, along with the overdetermined B.C.,

$$z \equiv w' = 0, \qquad \text{at } \rho = 0,\; 0<t<T, \tag{8.22c}$$

$$z \equiv z' = 0; \quad w = w' = w'' = w''' = 0, \qquad \text{at } \rho = \rho_0,\; 0<t<T. \tag{8.22d}$$

Proof. Inserting z defined by (8.21) into (8.20a) yields (8.22a), a coupled version of the uncoupled equation (8.4a) in Appendix B. On the other hand, (8.20a) multiplied first by ρ and then differentiated in ρ, also yields the identity

$$\frac{e}{R}[(Lv-\lambda v)\rho]' = [(\lambda u - Lu)\rho]' + \frac{(1+\nu)}{R}(w'\rho)', \tag{8.23}$$

a variation of equation (B.6) in Appendix B, which inserted into equation (8.20b), rewritten now for convenience as

$$\frac{e}{\rho}[(Lv-\lambda v)\rho]' + \lambda w + \frac{2(1+\nu)}{R^2}w - \frac{(1+\nu)}{R}\;\frac{1}{\rho}(u\rho)' \equiv 0, \tag{8.24}$$

yields

$$\frac{R}{\rho}[(\lambda u - Lu)\rho]' + \frac{(1+\nu)}{\rho}(w'\rho)' + \lambda w + \frac{2(1+\nu)}{R^2}w - \frac{(1+\nu)}{R}\ \frac{1}{\rho}(u\rho)' \equiv 0. \tag{8.25}$$

Next, we insert $u = cz - \frac{ec}{R}w'$ from (8.21) into (8.25) twice to get

$$\frac{Rc}{\rho}[(\lambda z - Lz)\rho]' + \frac{ec}{\rho}[(L-\lambda)w'\rho]' + \frac{(1+\nu)}{\rho}(w'\rho)'\big(1 + \frac{ec}{R^2}\big) \\ + \lambda w + \frac{2(1+\nu)}{R^2}w = \frac{(1+\nu)c}{R\rho}(z\rho)'. \tag{8.26}$$

Finally, we use (8.22a) in the first term of (8.26), and recall the constant c from (8.21) to obtain (8.22b). The B.C. (8.22c–d) follow from (8.20c–d) via (8.21).

Step 4. Proposition 8.5 (Basic estimate)**.** *With reference to problem* (8.22a–b–c–d), *the following estimate holds true for t fixed: there exists a constant $C > 0$ such that*

$$\sum_{|\alpha|+k\le 1}(1+\tau)^{2\delta(2-|\alpha|-k)}\tau^{2k}\int_{\psi(\rho)\ge -y_0} e^{2\tau\psi(\rho)}|D^\alpha z|^2\,d\rho \\ + \sum_{|\alpha|+k\le 3}(1+\tau)^{2\delta(4-|\alpha|-k)}\tau^{2k}\int_{\psi(\rho)\ge -y_0} e^{2\tau\psi(\rho)}|D^\alpha w|^2\,d\rho \\ \le C\int_{-2y_0\le\psi(\rho)\le -y_0} e^{2\tau\psi(\rho)}\big[\sum_{\alpha=0}^{1}|D^\alpha z|^2 + \sum_{\alpha=0}^{3}|D^\alpha w|^2\big]\,d\rho, \tag{8.27}$$

where

$$\begin{cases} 0<\delta\le 1;\ \tau \ge \ \textit{some}\ \tau_1 > 0;\ \psi(\rho) = \rho - \rho_0 - c(\rho_0-\rho)^2, \\ c\ \textit{small positive constant;}\ y_0\ \textit{small positive number.} \end{cases} \tag{8.28}$$

Proof. Let t be fixed but arbitrary in $(0,T)$. By virtue of the homogeneous B.C. (8.22d), we can extend by zero the weak solutions across the boundary $\rho = \rho_0$, so that the extensions

$$z_{\text{ext}} = \begin{cases} z & \text{in } (0,\rho_0], \\ 0 & \rho > \rho_0; \end{cases} \qquad w_{\text{ext}} = \begin{cases} w & \text{in } (0,\rho_0], \\ 0 & \rho > \rho_0; \end{cases}$$

satisfy the weak form of (8.22a–b). Henceforth, such extensions will be simply denoted by z and w, respectively. We shall then apply the Carleman estimates to such extensions z and w ([19, equation (1.1), page 359]), with t fixed. To this end, we let $\phi(y) \in C^\infty(\mathbf{R})$, $\phi(y) \equiv 1$ for $y \ge -y_0$ and $\phi(y) \equiv 0$ for $y \le -2y_0$, y_0 a small positive constant. Furthermore, we let

$$p_2(\rho, D) = \text{symbol associated with second-order elliptic operator } [L-\lambda] \text{ in (8.22a)} \tag{8.29}$$

$$p_4(\rho, D) = \text{symbol associated with fourth-order elliptic operator on the left-hand side of (8.22b).} \tag{8.30}$$

Then, the Carleman estimates ([19, equation (1.1), page 359]) for equations (8.22a) and (8.22b) are as follows, $\forall\ \tau \geq$ some positive $\tau_0 > 0$:

$$\sum_{|\alpha|+k\leq 1}(1+\tau)^{2\delta(2-|\alpha|-k)}\tau^{2k}\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|D^\alpha\phi(\psi)z|^2\,d\rho \leq c\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|p_2(\rho,D)\phi(\psi)z|^2\,d\rho; \tag{8.31}$$

$$\sum_{|\alpha|+k\leq 3}(1+\tau)^{2\delta(4-|\alpha|-k)}\tau^{2k}\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|D^\alpha\phi(\psi)w|^2\,d\rho \leq c\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|p_4(\rho,D)\phi(\psi)w|^2\,d\rho. \tag{8.32}$$

We note that

$$D^\alpha\phi(\psi)z = \phi(\psi)D^\alpha z + [D^\alpha,\phi(\psi)]z; \tag{8.33}$$

$$D^\alpha\phi(\psi)w = \phi(\psi)D^\alpha w + [D^\alpha,\phi(\psi)]w; \tag{8.34}$$

where the commutators in (8.33), (8.34) are differential operators of at most zero order and of at most second order, respectively, supported in $-2y_0 \leq \psi \leq -y_0$, by the definition of ϕ. Similarly,

$$p_2(\rho,D)\phi(\psi)z = \phi(\psi)p_2(\rho,D)z + [p_2(\cdot,D),\phi(\psi)]z, = \phi(\psi)\big[\frac{(1+\nu)}{R}w'\big] + [p_2(\cdot,D),\phi(\psi)]z, \tag{8.35}$$

recalling the right-hand side of (8.22a) via (8.29), as well as

$$p_4(\rho,D)\phi(\psi)w = \phi(\psi)p_4(\rho,D)w + [p_4(\cdot,D),\phi(\psi)]w, = \phi(\psi)\big[\frac{R(1+\nu)}{R^2+e}(z\rho)'\big] + [p_4(\cdot,D),\phi(\psi)]w, \tag{8.36}$$

recalling the right-hand side of (8.22b) via (8.30), where

$$[p_2(\cdot,D),\phi(\psi)] \in S^1(\mathbf{R}); \qquad [p_4(\cdot,D),\phi(\psi)] \in S^3(\mathbf{R}), \tag{8.37}$$

both homogeneous symbols in (8.37) being supported in $-2y_0 \leq \psi \leq -y_0$. From (8.33),

$$\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|D^\alpha\phi(\psi)z|^2\,d\rho \geq \int_{\psi\geq -y_0}e^{2\tau\psi(\rho)}|\phi(\psi)D^\alpha z + [D^\alpha,\phi(\psi)]z|^2\,d\rho = \int_{\psi\geq -y_0}e^{2\tau\psi(\rho)}|D^\alpha z|^2\,d\rho, \tag{8.38}$$

since, over $\psi \geq -y_0$, $[D^\alpha,\phi(\psi)]$ vanishes and $\phi(\psi)=1$. Similarly,

$$\int_{-\infty}^{\infty}e^{2\tau\psi(\rho)}|D^\alpha\phi(\psi)w|^2\,d\rho \geq \int_{\psi\geq -y_0}e^{2\tau\psi(\rho)}|D^\alpha w|^2\,d\rho. \tag{8.39}$$

Next, we use (8.38) and (8.39) on the left-hand side of (8.31) and (8.32), respectively; while on the right-hand sides we use (8.35) and (8.36), respectively, and we finally sum up the resulting expressions. We thus obtain, recalling the symbols in (8.37) and where they are supported:

$$\begin{aligned}&\sum_{|\alpha|+k\leq 1}(1+\tau)^{2\delta(2-|\alpha|-k)}\tau^{2k}\int_{\psi\geq -y_0} e^{2\tau\psi(\rho)}|D^\alpha z|^2\,d\rho\\&\quad+\sum_{|\alpha|+k\leq 3}(1+\tau)^{2\delta(4-|\alpha|-k)}\tau^{2k}\int_{\psi\geq -y_0} e^{2\tau\psi(\rho)}|D^\alpha w|^2\,d\rho\\&\leq c\Big\{\int_{\psi\geq -y_0} e^{2\tau\psi(\rho)}|w'|^2\,d\rho+\int_{\psi\geq -y_0} e^{2\tau\psi(\rho)}|z'|^2\,d\rho\\&\quad+\int_{-2y_0\leq\psi\leq -y_0} e^{2\tau\psi(\rho)}\big[|z|^2+|z'|^2+\sum_{\alpha=0}^{3}|D^\alpha w|^2\big]\,d\rho\Big\}.\end{aligned}\tag{8.40}$$

Finally, by taking τ sufficiently large, say $\tau \geq$ some $\tau_1 > 0$, on the left-hand side of (8.40), we can absorb the integral terms on $\psi \geq -y_0$, for $|w'|^2$ and $|z'|^2$ on the right-hand side of (8.40) by the corresponding terms on the left-hand side, and thus obtain (8.27) from (8.40).

Step 5. Corollary 8.6. (i) *With reference to problem* (8.22a–b–c–d), *the following estimate holds true for t fixed: there exists a constant $c > 0$ such that*

$$\begin{aligned}&\sum_{|\alpha|+k\leq 1}(1+\tau)^{2\delta(2-|\alpha|-k)}\tau^{2k}\int_{\psi\geq -y_0}|D^\alpha z|^2\,d\rho\\&\quad+\sum_{|\alpha|+k\leq 3}(1+\tau)^{2\delta(4-|\alpha|-k)}\tau^{2k}\int_{\psi\geq -y_0}|D^\alpha w|^2\,d\rho\\&\leq C\int_{-2y_0\leq\psi\leq -y_0}\big[\sum_{\alpha=0}^{1}|D^\alpha z|^2+\sum_{\alpha=0}^{3}|D^\alpha w|^2\big]\,d\rho=C_{(z,w)},\end{aligned}\tag{8.41}$$

with the notation of (8.28), *and where $C_{(z,w)}$ is a constant depending on the solution $\{z, w\}$, with t fixed.*

(ii) *Consequently,*

$$z\equiv 0,\quad w\equiv 0\quad \text{on}\quad [0,\rho_0]\ \text{at each}\ t,\tag{8.42}$$

for the solution of problem (8.22a–d).

Proof. (i) On the left-hand side of (8.27) we use $\psi \geq -y_0$ hence $e^{2\tau\psi} \geq e^{-2\tau y_0}$, and on the right-hand side of (8.27), we majorize by $\psi \leq -y_0$, hence obtaining $e^{2\tau\psi} \leq e^{-2\tau y_0}$, so that we take out of the integrals and then cancel $e^{-2\tau y_0}$ from both sides, obtaining (8.41), where $C_{(z,w)}$ is a constant depending on the solution $\{z; w\}$ for t fixed in $(0, T)$.

(ii) It suffices to take $|\alpha| = k = 0$ in (8.41), thus obtaining

$$\int_{\psi\geq -y_0}[|z|^2+|w|^2]\,d\rho\leq\frac{1}{(1+\tau)^{4\delta}}\to 0\quad\text{as}\quad\tau\to\infty,\tag{8.43}$$

and thus

$$z \equiv 0;\ w \equiv 0 \quad \text{on } \psi \geq -y_0. \tag{8.44}$$

Recalling the definition of $\psi(\rho)$ in (8.28), we see that $\psi \geq -y_0$ implies

$$0 \leq \rho_0 - \rho \leq \frac{-1 + \sqrt{1 + 4cy_0}}{2c} < \rho_0,$$

by taking y_0 sufficiently small; in particular, $0 < \rho \leq \rho_0$; i.e., the above unique continuation argument holds true on the semi-open interval $(0, \rho_0]$ due to a possible singularity of the coefficients of the elliptic operator at $\rho = 0$, which would prevent use of the Carleman estimate. On the other hand, $z \equiv 0$, $w \equiv 0$ for $0 < \rho \leq \rho_0$, t fixed, along with the B.C. $z = w' = 0$ at $\rho = 0$ in (8.22c), and $\{z, w\} \in C^1[0, \rho_0]$, implies (8.42), as desired.

Step 6. We can now finally complete the proof of Theorem 8.1: If $\{u, w\}$ solves problem (8.1a–b–c–d), then $\{z, w\}$, z as in (8.21) solves problem (8.22a–b–c–d), and hence $z \equiv w \equiv 0$, by Corollary 8.6(ii). Then, in view of (8.21), we have $u \equiv 0$ as well, and Theorem 8.1 is proved.

Appendix 8A: Two compactness results. In this appendix we establish the two compactness results needed in (8.18).

Proposition 8A.1. *With reference to the spaces* $L_2^\rho = L_2^\rho(0, \rho_0)$ *and* $\mathcal{U}_\rho^1 = \mathcal{U}_\rho^1(0, \rho_0)$ *defined in* (1.5) *and* (1.6), *we have*

$$\textit{the injection } \mathcal{U}_\rho^1 \hookrightarrow L_2^\rho \textit{ is compact.} \tag{8A.1}$$

Proposition 8A.2. *With reference to the space* $\mathcal{H}_{\epsilon,\rho}^2 = \mathcal{H}_{\epsilon,\rho}^2(0, \rho_0)$ *defined in* (1.7), *we have*

$$\textit{the injection } \mathcal{H}_{\epsilon,\rho}^2 \hookrightarrow \mathcal{U}_\rho^1 \textit{ is compact.} \tag{8A.2}$$

Proof of Proposition 8A.1. Let $u \in \mathcal{U}_\rho^1(0, \rho_0)$ so that by (1.6), $\frac{u}{\sqrt{\rho}}$, $u'\sqrt{\rho} \in L_2(0, \rho_0)$. Then, (i) $u\sqrt{\rho} \in L_2(0, \rho_0)$, and (ii) $(u\sqrt{\rho})' = u'\sqrt{\rho} + \frac{1}{2}\ \frac{u}{\sqrt{\rho}} \in L_2(0, \rho_0)$; i.e., $u\sqrt{\rho} \in H^1(0, \rho_0)$ as in property (p.3). But the injection $H^1(0, \rho_0) \hookrightarrow L_2(0, \rho_0)$ is compact. Thus, $\{u_n\}$ a bounded sequence in $\mathcal{U}_\rho^1$, implies that we can extract a subsequence $\{u_n\sqrt{\rho}\}$ convergent in $L_2(0, \rho_0)$,i.e., a subsequence $\{u_n\}$ convergent in $L_2^\rho(0, \rho_0)$.

Proof of Proposition 8A.2. (i) We first convey the idea by taking $\epsilon = 0$. If $u \in \mathcal{H}_{0,\rho}^2$, then, by property (p.6) = (1.28) with $\epsilon = 0$, we have

$$u'\sqrt{\rho},\ \frac{u}{\sqrt{\rho}} \in H^1(0, \rho_0),\ \text{with injection } \hookrightarrow L_2(0, \rho_0) \text{ compact.} \tag{8A.3}$$

Thus, if $\{u_n\}$ is a bounded sequence in $\mathcal{H}_{0,\rho}^2$, then by (8A.3), we can extract a subsequence $\{u_{n_k}\}$ such that $\{u'_{n_k}\sqrt{\rho}\}$ and $\{\frac{u_{n_k}}{\sqrt{\rho}}\}$ are convergent in $L_2(0, \rho_0)$; i.e., $\{u_{n_k}\}$ is convergent in $\mathcal{U}_\rho^1(0, \rho_0)$, as desired.

(ii) If $\epsilon > 0$, whereby now $u\rho^{-\frac{3}{2}+\epsilon}$, $u'\rho^{-\frac{1}{2}+\epsilon}$, $u''\rho^{\frac{1}{2}+\epsilon} \in L_2(0,\rho_0)$ by (1.9), then (8A.3) is replaced by $u'\rho^{\frac{1}{2}+\epsilon} \in H^1(0,\rho_0)$ and $\frac{u}{\rho^{\frac{1}{2}-\epsilon}} \in H^1(0,\rho_0)$ respectively, i.e., by

$$u'\rho^{\frac{1}{2}} \in H^{1-\epsilon'}(0,\rho) \quad \text{and} \quad \frac{u}{\rho^{\frac{1}{2}}} \in H^{1-\epsilon'}(0,\rho_0), \tag{8A.4}$$

respectively, for some $\epsilon' > 0$, $1-\epsilon' > 0$, with the injection $H^{1-\epsilon'}(0,\rho_0) \hookrightarrow L_2(0,\rho_0)$ still compact. Thus, given the sequence $\{u_n\}$ bounded in $\mathcal{H}^2_{\epsilon,\rho}$, then we can extract a convergent subsequence $\{u_{n_k}\}$ in $\mathcal{U}^1_\rho$, as before.

Appendix 8B: A uniqueness result for a slightly simplified model, obtained by the Holmgren-John theorem. In this Appendix B, we consider a slightly simplified problem over problem (8.1), namely, on $(0,T)\times(0,\rho_0)$:

$$u_{tt} + \frac{e}{R}v_{tt} - Lu - \frac{e}{R}Lv = 0; \tag{8B.1a}$$

$$w_{tt} - \frac{e}{\rho}[v_{tt}\rho]' + \frac{e}{\rho}[(Lv)\rho]' - \frac{(1+\nu)}{R}\frac{1}{\rho}[u\rho]' + \frac{2(1+\nu)}{R^2}w = 0; \tag{8B.1b}$$

$$u = w' = Lv = 0, \qquad \rho = 0,\ 0 < t < T; \tag{8B.1c}$$

$$u = u' = w = w' = w'' = w''' = 0, \qquad \rho = \rho_0,\ 0 < t < T. \tag{8B.1d}$$

Notice that problem (8B.1) differs from problem (8.1) *only in that equation (8B.1a) does not contain the lower-order term w'*. Problem (8B.1) displays a symmetry which allows decoupling of the two equations and hence will allow us to obtain a unique continuation theorem, by simply appealing, ultimately, to the Holmgren-John theorem ([16], [19]), without resorting to the more sophisticated Carleman estimates as in Step 3 of the proof of Theorem 8.1 for the original problem (8.1).

Theorem B.1. *Let $\{u,w\} \in \mathcal{D}'((0,T)\times(0,\rho_0))$ be a solution of the over-determined problem* (B.1), *where T is sufficiently large. Then, in fact,*

$$u \equiv w \equiv 0 \qquad \text{in } [0,T]\times[0,\rho_0]. \tag{8B.2}$$

Proof. (Sketch) Step 1. We introduce a new variable z,

$$z = u + \frac{e}{R}v = \big(1 + \frac{e}{R^2}\big)u + \frac{e}{R}w', \tag{8B.3}$$

in terms of which the first equation (8B.1a) becomes

$$z_{tt} - Lz = 0, \qquad (0,T)\times(0,\rho_0); \tag{8B.4a}$$

$$z \equiv 0 \text{ at } \rho = 0;\ z = z' = 0 \text{ at } \rho = \rho_0,\ 0 < t < T, \tag{8B.4b}$$

after recalling the B.C. (8B.1c–d) in (8B.3). Problem (8B.4) is a generalized wave equation with analytic coefficients in $(0, \rho_0)$, to which we can apply the Holmgren-John uniqueness theorem for T sufficiently large and obtain

$$z \equiv 0 \text{ or } u \equiv -kw', \quad \text{in } (0,T) \times (0, \rho_0), \tag{8B.5}$$

by (8B.3), with k a positive constant $= (e/R)/(1 + (e/R^2))$.

Step 2. On the other hand, equation (8B.1a), multiplied by ρ, also yields the identity

$$[(u_{tt} - Lu)\rho]' = \frac{e}{R}[(Lv - v_{tt})\rho]',$$

which inserted in equation (8B.1b) results in

$$w_{tt} + \frac{R}{\rho}[(u_{tt} - Lu)\rho]' - \frac{(1+\nu)}{R}\,\frac{1}{\rho}(u\rho)' + \frac{2(1+\nu)}{R^2}w = 0. \tag{8B.6}$$

Using $u = -kw'$ from (8B.5) in (8B.6) finally yields

$$w_{tt} - \frac{kR}{\rho}[(w'_{tt} - L(w'))\rho]' + \frac{(1+\nu)}{R}\,\frac{k}{\rho}(w'\rho)' + \frac{2(1+\nu)}{R^2}w = 0. \tag{8B.7}$$

rewritten as

$$\begin{aligned}\{1 - kR\frac{d^2}{d\rho^2} - \frac{kR}{\rho}\frac{d}{d\rho}\}w_{tt} + kR[L(w')]' + \frac{kR}{\rho}L(w') + \frac{(1+\nu)}{R}\frac{k}{\rho}[w'\rho]' \\ + \frac{2(1+\nu)}{R^2}w \equiv 0,\end{aligned} \tag{8B.8}$$

along with the B.C.

$$w' = Lw = 0 \text{ at } \rho = 0;\ w = w' = w'' = w''' = 0 \text{ at } \rho = \rho_0; \tag{8B.9}$$

after recalling (8B.5) in $Lv = 0$ at $\rho = 0$. The above equation (8B.8) is a generalized Kirchhoff equation. In fact, consider the operator $\left(-i\frac{d}{d\rho}\right)$ on $L_2(0, \rho_0)$ [or $-\frac{i}{\rho}\ \frac{d}{d\rho}$ on $L_2^\rho(0, \rho_0)$] with domain given by all absolutely continuous functions f on $(0, \rho_0)$ with first derivative f' in $L_2(0, \rho_0)$ which satisfy the B.C. $f(0) = f(\rho_0) = 0$ [as w_{tt} from (8B.9)]. Such an operator admits a self-adjoint extension ([18, page 384]), so that the operator in the brackets { } is still boundedly invertible on $L_2^\rho(0, \rho_0)$. The Homgren-John theorem then applies to (8B.8), (8B.9) and yields (8B.2).

9. Absorption of $\ell.o.t.$ in (7.33) via a compactness/uniqueness argument. In this section we use, as usual (e.g., [11]), a compactness/uniqueness argument—which critically relies on the unique continuation Theorem 8.1—to absorb the lower-order terms $\ell.o.t.$ in (7.33) by the boundary terms at the right-hand side of estimate (7.33). The final stabilization estimate is given in (9.2) below.

Theorem 9.1. (i) *With reference to problem* (1.1), *whose solutions have been shown in Theorem* 7.3 *to obey estimate* (7.33) *for all* $T >$ *some* $T_0 > 0$, *we have*: *There exists a constant* $C_T > 0$, *such that*

$$[\ell.o.t.\text{in } (7.33)] \le C_T \int_0^T \left[u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)\right] dt. \tag{9.1}$$

(ii) (*Stabilization estimate*) *Consequently, estimate* (7.33) *is improved to read*: *there exists* $T_0 > 0$ *such that, for all* $T > T_0$, *there exists a constant* $C_T > 0$ *such that*

$$E(0) + E(T) + \int_0^T E(t)\,dt \le C_T \int_0^T \left[u_t^2(t,\rho_0) + w_t^2(t,\rho_0) + ev_t^2(t,\rho_0)\right] dt. \tag{9.2}$$

Proof. Step 1. For the sake of contradiction, let $\{[u_{0n}, w_{0n}], [u_{1n}, w_{1n}]\}$ be a sequence of initial data in $\mathcal{E} \equiv [\mathcal{U}_\rho^1 \times \mathcal{W}_\rho^2] \times \mathcal{V}_\rho^1$, whose corresponding solutions $\{u_n(t), w_n(t)\}$ of problem (1.1a–h) satisfy

$$(\ell.o.t.)_n \equiv 1, \text{ while } \int_0^T \left[\dot{u}_n(t,\rho_0) + \dot{w}_n^2(t,\rho_0) + e\dot{v}_n(t,\rho_0)\right] dt \to 0, \text{ as } n \to \infty, \tag{9.3}$$

where a dot will indicate time derivative, $\frac{d}{dt} = \dot{}$, when the subscript n is involved. Such solutions $\{u_n(t), w_n(t)\}$, with energy $E_n(t)$ given by (1.2)–(1.4), thus obey estimate (7.33) of Theorem 7.3, and hence, by virtue of (9.2),

$$E_n(0) + E_n(T) + \int_0^T E_n(t)\,dt \le C_T, \text{ uniformly in } n. \tag{9.4}$$

It follows from (9.4) that there is a subsequence such that

$$\{u_{0n}, w_{0n}, u_{1n}, w_{1n}\} \to \text{ some } \{u_0, w_0, u_1, w_1\} \text{ in } \mathcal{E} \text{ weakly,}$$

and if $\{u(t), w(t), u(t), w(t)\}$ is the solution of (1.1a–h) corresponding to the limit $\{u_0, w_0, u_1, w_1\}$, as initial conditions, then

$$\{u_n(t), w_n(t), \dot{u}_n(t), \dot{w}_n(t)\} \to \{u(t), w(t), u_t(t), w_t(t)\} \text{ in } L_\infty(0,T;\mathcal{E}) \text{ -weak star,}$$
$$\text{or in } L_2(0,T;\mathcal{E}) \text{ -weakly.} \tag{9.5}$$

Since

$$\text{the injection } \mathcal{W}_\rho^2(0,\rho_0) \hookrightarrow L_2^\rho(0,\rho_0) \text{ is compact,} \tag{9.6}$$

then Aubin's compactness lemma applies ([1]) by (9.5) and (9.6), and yields

$$w_n \to w, \text{ strongly in } L_2(0,T; L_2^\rho(0,\rho_0)). \tag{9.7}$$

Moreover, by trace theory with $s > \frac{1}{2}$, if $I_{\rho_0} = \left(\frac{\rho_0}{2}, \rho_0\right)$, then

$$\int_0^T [u_n^2(t,\rho_0) + w_n^2(t,\rho_0)]\, dt \le C_T \int_0^T \left[\|u_n\|^2_{H^s(I_{\rho_0})} + \|w_n\|^2_{H^s(I_{\rho_0})}\right] dt. \tag{9.8}$$

Moreover, with $\frac{1}{2} < s < 1$, and recalling (9.5), (p.3) = (1.26), (1.7),

$$\begin{cases} u_n \in L_2(0,T;\mathcal{U}^1_\rho(I_{\rho_0})); & \qquad w_n \in L_2(0,T;\mathcal{W}^2_\rho(I_{\rho_0})); \\ u_n' \in L_2(0,T;L_2^\rho(I_{\rho_0})); \quad \text{and} & \qquad w_n' \in L_2(0,T;L_2^\rho(I_{\rho_0})); \\ \mathcal{U}^1_\rho(I_{\rho_0}) \underset{\text{continuous}}{\hookrightarrow} H^1(I_{\rho_0}) \underset{\text{compact}}{\hookrightarrow} H^s(I_{\rho_0}) & \qquad \mathcal{W}^2_\rho(I_{\rho_0}) \underset{\text{continuous}}{\hookrightarrow} H^3(I_{\rho_0}) \underset{\text{compact}}{\hookrightarrow} H^s(I_{\rho_0}). \end{cases} \tag{9.9}$$

Then (9.9) implies, by Aubin's lemma ([1]),

$$u_n \to u \text{ and } w_n \to w \text{ strongly in } L_2(0,T;H^s(I_{\rho_0})). \tag{9.10}$$

Thus, we combine (9.7) with the result of (9.10) using (9.8), and recall the left-hand side of (9.3) in estimate (7.34) for the n^{th} solution

$$1 \equiv (\ell.o.t.)_n \le C\Big\{\int_0^T [u_n^2(t,\rho_0) + w^2(t,\rho_0)]\, dt + \int_0^T \int_0^{\rho_0} w_n^2 \rho \, d\rho \, dt\Big\} \tag{9.11}$$

to obtain

$$1 \le C \int_0^T [u^2(t,\rho_0) + w^2(t,\rho_0)]\, dt + \int_0^T \int_0^{\rho_0} w^2 \rho \, d\rho \, dt. \tag{9.12}$$

Step 2. We now use the assumed convergence on the right-hand side of (9.3), so that the limit $\{u(t), w(t)\}$ is a solution of equations (1.1a–b), along with the B.C. (1.1e)

$$u = w' = Lv = 0 \qquad \text{at } \rho = 0, \tag{9.13}$$

as well as the following B.C.'s at $\rho = \rho_0$:

$$u' - \frac{(1+\nu)}{R} w + \nu \frac{u}{\rho_0} = -u; \tag{9.14a}$$

$$v' = 0; \quad \text{at } \rho = \rho_0. \tag{9.14b}$$

$$Lv = 0; \tag{9.14c}$$

$$u_t = w_t = w_t' = w_t = 0. \tag{9.14d}$$

Next, we define

$$\bar{u} \equiv u_t; \ \ \bar{w} \equiv w_t \quad \text{so that } \bar{v} = \frac{\bar{u}}{R} + \bar{w}'. \tag{9.15}$$

Then

Lemma 9.2. *We have that* $\{\bar{u}, \bar{w}\}$ *in* (9.15) *satisfies equations (1.1a–b) as well as the following B.C.*

$$\bar{u} = \bar{w}' = L\bar{v} = 0 \qquad \textit{at } \rho = 0; \tag{9.16}$$

$$\bar{u} = \bar{u}' = \bar{w} = \bar{w}' = \bar{w}'' = \bar{w}''' = 0 \qquad \textit{at } \rho = \rho_0. \tag{9.17}$$

Proof. By differentiation in t on (1.1a–b), (9.13), (9.14), we see that $\{\bar{u}, \bar{w}\}$ in (9.15) continues to satisfy equations (1.1a–b) and also (9.16), as well as

$$\bar{u}' - \frac{(1+\nu)}{R}\bar{w} + \frac{\nu\bar{u}}{\rho_0} = -\bar{u}; \tag{9.18a}$$

$$\bar{v}' = 0; \qquad \text{at } \rho = \rho_0. \tag{9.18b}$$

$$L\bar{v} = 0; \tag{9.18c}$$

$$\bar{u} = \bar{w} = \bar{w}' = \bar{v} = 0. \tag{9.18d}$$

But (9.18d), used in (9.18a), yields then $\bar{u}' = 0$ at $\rho = \rho_0$, as desired. Also, (9.18b) and $\bar{v} = 0$ from (9.18d), used in (9.18c), yield $\bar{v}'' = 0$ at $\rho = \rho_0$ via (1.1c). Moreover, using (1.1a) in the form

$$L\bar{u} = \bar{u}_{tt} + \frac{e}{R}\bar{v}_{tt} - \frac{e}{R}L\bar{v} + \frac{(1+\nu)}{R}\bar{w}' = 0 \qquad \text{at } \rho = \rho_0 \tag{9.19}$$

yields $L\bar{u} = 0$ at $\rho = \rho_0$ via (9.18c–d). This latter result, combined with $\bar{u} = 0$ in (9.18d) and $\bar{u}' = 0$ at $\rho = \rho_0$ already shown, yields $\bar{u}'' = 0$ at $\rho = \rho_0$, via (1.1c). Thus, $\bar{u}'' = \bar{v}'' = 0$ at $\rho = \rho_0$ already shown yield $\bar{w}''' = 0$ at $\rho = \rho_0$, as desired. Thus, (9.17) is proved.

Step 3. Proposition 9.3. *With reference to* $\{\bar{u}, \bar{w}\}$ *in* (9.15) *which is a solution of* (1.1a–b), (9.16), (9.17), *we have*

$$\bar{u} \equiv \bar{w} \equiv 0 \qquad \textit{in } \ (0,T) \times (0, \rho_0). \tag{9.20}$$

Proof. We apply the uniqueness Theorem 8.1 to $\{\bar{u}, \bar{w}\}$.

Step 4. By virtue of (9.20) and (9.15), we have, at this stage, that $\{u, w\}$ is then a solution of problem (1.1) with time-derivative terms equal to zero, i.e., of the stationary problem

$$-Lu - \frac{e}{R}Lv + \frac{(1+\nu)}{R}w' = 0; \tag{9.21a}$$

$$\frac{e}{\rho}[L(v)\rho]' - \frac{(1+\nu)}{\rho R}(u\rho)' + \frac{2(1+\nu)}{R^2}w = 0; \tag{9.21b}$$

with boundary conditions

$$u = w' = Lv = 0 \qquad \text{at } \rho = 0; \tag{9.21c}$$

$$u' - \frac{(1+\nu)}{R}w + \frac{\nu u}{\rho_0} = -u; \tag{9.21d}$$

$$v' = 0; \qquad \text{at } \rho = \rho_0. \tag{9.21e}$$

$$Lv = 0. \tag{9.21f}$$

We then have, improving upon (9.20):

Proposition 9.4. *Let* $\{u, w\}$ *be a solution of* (9.21). *Then, in fact,*

$$u \equiv w \equiv 0 \quad \text{in } (0,T) \times (0, \rho_0). \tag{9.22}$$

Proof. We multiply (9.21a) by $u\rho$ and (9.21b) by $w\rho$ (as in Section 7), integrate over $(0, \rho_0)$ and sum up the resulting equations. As to the principal terms, we have

$$- \int_0^{\rho_0} (Lu) u\rho \, d\rho - \frac{e}{R} \int_0^{\rho_0} (Lv) u\rho \, d\rho + e \int_0^{\rho_0} \frac{1}{\rho} [L(v)\rho]' w\rho \, d\rho \tag{9.23}$$

$$= -u'(\rho_0) u(\rho_0)\rho_0 + \|u\|^2_{\mathcal{U}^1_\rho} - \frac{e}{R} v'(\rho_0) u(\rho_0)\rho_0$$

$$+ e\big(v, \frac{u}{R}\big)_{\mathcal{U}^1_\rho} + e(Lv)(\rho_0)\rho_0 - e \int_0^{\rho_0} (Lv) w' \rho \, d\rho$$

$$= \big[u(\rho_0) + \nu \frac{u(\rho_0)}{\rho_0} - \frac{(1+\nu)}{R} w(\rho_0)\big] u(\rho_0)\rho_0 + \|u\|^2_{\mathcal{U}^1_\rho} + e\|v\|^2_{\mathcal{U}^1_\rho}. \tag{9.24}$$

To obtain (9.24), we apply equation (2.17) of Proposition 2.2 to the first two integral terms as well as to the integral term in (9.23), which is obtained by integrating by parts the third integral term. Finally, we use the B.C. (9.21d–f). As to the lower-order terms' contributions, we have by integrating by parts:

$$\int_0^{\rho_0} w' u\rho \, d\rho - \int_0^{\rho_0} \frac{1}{\rho} (u\rho)' w\rho \, d\rho + \frac{2}{R} \int_0^{\rho_0} ww\rho \, d\rho$$

$$= 2 \int_0^{\rho_0} w' u\rho \, d\rho - u(\rho_0) w(\rho_0)\rho_0 + \frac{2}{R} \int_0^{\rho_0} w^2 \rho \, d\rho \tag{9.25}$$

$$= w(\rho_0) u(\rho_0)\rho_0 - 2 \int_0^{\rho_0} wu' \rho \, d\rho - 2 \int_0^{\rho_0} wu \, d\rho + \frac{2}{R} \int_0^{\rho_0} w^2 \rho \, d\rho.$$

Multiplying (9.25) by $\frac{(1+\nu)}{R}$ and adding up (9.24) yields, by virtue of (9.21a–b), after a cancellation of $\frac{(1+\nu)}{R} w(\rho_0) u(\rho_0)\rho_0$,

$$e\|v\|^2_{\mathcal{U}^1_\rho} + u^2(\rho_0)\rho_0 + \Big\{ \|u\|^2_{\mathcal{U}^1_\rho} + \frac{2(1+\nu)}{R^2} \|w\|^2_{L^\rho_2}$$

$$- \frac{2(1+\nu)}{R} \int_0^{\rho_0} (wu'\rho + wu) \, d\rho + \nu u^2(t, \rho_0) \Big\} = 0. \tag{9.26}$$

Thus, (9.26) is the result of multiplying (9.21a) by $u\rho$; (9.21b) by $w\rho$; integrating over $(0, \rho_0)$; and summing up the resulting equations. We next recall identity (6.11) for the term within bracket { } in (9.26). Thus, (9.26) is rewritten as

$$e\|v\|^2_{\mathcal{U}^1_\rho} + u^2(\rho_0)\rho_0 + (1-\nu) \int_0^{\rho_0} \big[\big(u' - \frac{w}{R}\big)^2 \rho + \big(\frac{u}{\rho} - \frac{w}{R}\big)^2 \rho \big] \, d\rho$$

$$+ \nu \int_0^{\rho_0} \big[\big(u' - \frac{w}{R}\big)\sqrt{\rho} + \big(\frac{u}{\rho} - \frac{w}{R}\big)\sqrt{\rho} \big]^2 \, d\rho \equiv 0, \tag{9.27}$$

or, recalling definition (2.1) for $\hat{E}_P(u, w)$,

$$2\hat{E}_P(u, w) = 0, \tag{9.28}$$

and hence, by the equivalence (2.3a) of Proposition 2.1, we conclude from (9.28) that $u \equiv w \equiv 0$, as desired in (9.22).

Step 5. Having shown $u \equiv w \equiv 0$ in $[0, T] \times [0, \rho_0]$, see (9.22), we then obtain a contradiction with (9.12). The proof of Theorem 9.1 is complete.

10. Completion of the proof of Theorem 1.4. The basic stabilization estimate (9.2) of Theorem 9.1 is

$$E(0) + E(T) + \int_0^T E(t)\, dt \leq C_T \int_0^T \left[u_t^2(t, \rho_0) + w_t^2(t, \rho_0) + ev_t^2(t, \rho_0)\right] dt, \tag{10.1}$$

T large enough, for the solutions of problem (1.1), while the dissipativity identity (1.21) of Theorem 1.3 is

$$\rho_0 \int_0^T \left[u_t^2(t, \rho_0) + w_t^2(t, \rho_0) + ev_t^2(t, \rho_0)\right] dt = E(0) - E(T). \tag{10.2}$$

Thus, inserting the integral term (10.2) in (10.1) yields

$$E(0) + E(T) + \int_0^T E(t)\, dt \leq C_T[E(0) - E(T)], \tag{10.3}$$

and *a fortiori*, dropping positive terms,

$$E(T) \leq \frac{C_T}{1 + C_T} E(0), \qquad T \text{ large enough.} \tag{10.4}$$

Since $C_T/(1 + C_T) < 1$, (10.4) is equivalent, in the notation of Theorem 1.1 or (1.24), with $e^{A_s t}$ the s.c. semigroup of contractions defining the solutions of problem (1.1) in the space $\mathbf{E} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(M^{\frac{1}{2}})$, to the condition

$$\|e^{A_s T}\|_{\mathcal{L}(\mathbf{E})} < 1 \quad \text{ for suitable } T > 0. \tag{10.5}$$

Thus, standard semigroup theory then yields to the desired uniform decay (1.24) for the energy, equivalently (1.25) for the semigroup $e^{A_s t}$. The proof of Theorem 1.4 is complete.

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