ASYMPTOTIC BEHAVIOR FOR MINIMIZERS OF A GINZBURG-LANDAU-TYPE FUNCTIONAL IN HIGHER DIMENSIONS ASSOCIATED WITH n–HARMONIC MAPS

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Abstract. We describe the behavior as $\varepsilon \to 0$ of minimizers for a Ginzburg-Landau functional

$$E_\varepsilon(u;\Omega) = \int_\Omega \big[\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\big]\,dx$$

in the space $H_g^{1,n}(\Omega;\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ and the boundary data $g : \partial\Omega \to S^{n-1}$ has a nonzero topological degree. Some recent results of Bethuel, Brezis and Hélein, and of Struwe on the two-dimensional problem, are extended to higher-dimensional cases. New proofs for their results are also presented in this paper.

1. Introduction. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega \cong S^{n-1}$, and let g be a smooth function, $g : \partial\Omega \to S^{n-1}$. We may associate with g a topological degree d. Let

$$H_g^{1,p}(\Omega;\mathbb{R}^n) = \{u \in H^{1,p}(\Omega,\mathbb{R}^n) : u|_{\partial\Omega} = g\}.$$

Let us consider for $\varepsilon > 0$ the Ginzburg-Landau-type functional

$$E_\varepsilon(u;\Omega) = \int_\Omega \big[\frac{|\nabla u|^p}{p} + \frac{1}{4\varepsilon^p}(1-|u|^2)^2\big]\,dx \tag{1.1}$$

where $1 < p \le n$.

The functional E_ε is related to models introduced by Ginzburg and Landau in [13] for the study of phase transitions. For the scalar-value case, numerous mathematically interesting results have been obtained by many authors (see [9], [15], [23], [22] and [24]).

In the vector-value case (i.e., $n \ge 2$), it is well known that $H_g^{1,p}(\Omega;\mathbb{R}^n)$ is nonempty and for $\varepsilon > 0$ the functional E_ε achieves its minimizer in $H_g^{1,p}(\Omega;\mathbb{R}^n)$ by a function u_ε; i.e.,

$$\nu(\varepsilon) := E_\varepsilon(u_\varepsilon;\Omega) = \min_{u \in H_g^{1,p}(\Omega;\mathbb{R}^n)} E_\varepsilon(u;\Omega). \tag{1.2}$$

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Define

$$H_g^{1,p}(\Omega; S^{n-1}) := \{u \in H_g^{1,p}(\Omega; \mathbb{R}^n) : |u| = 1 \text{ a.e. on } \Omega\}.$$

If $p < n$, it can be easily proven that u_ε converges strongly to a p-harmonic map since $H_g^{1,p}(\Omega, S^{n-1})$ is always nonempty. For $p = n \geq 2$, $H_g^{1,n}(\Omega, S^{n-1})$ is empty if the degree $d \neq 0$. The value $\nu(\varepsilon)$ may go to infinity as $\varepsilon \to 0$. The first part of the energy $E_\varepsilon(u; \Omega)$, $\int_\Omega \frac{1}{n} |\nabla u|^n \, dx$, is conformal invariant allowing change of variable x, so it is interesting to study asymptotic behavior of minimizers u_ε of E_ε in $H_g^{1,n}$ for the case $p = n$. For $p = n = 2$, Bethuel, Brezis and Hélein (see [1], [2], [3] and [4]) first proved many beautiful results about asymptotic behavior for minimizers of E_ε. One of the main results in [4] is the following:

Theorem ([4]). *Let $n = p = 2$ and u_ε be a minimizer of the minimizing problem* (1.2). *If Ω is star-shaped, there is a subsequence $\{u_{\varepsilon_k}\}$ which converges uniformly on a compact set of $\Omega \backslash \Sigma$ to a harmonic map with values in S^1 and the singular set Σ is exactly $|d|$ points in Ω.*

An extension to non-star-shaped domains of the above work was obtained by Struwe (see [25] and [26]).

In this paper we consider the Ginzburg-Landau functional in the case of $p = n \geq 2$.

A function $u(x) \in H_g^{1,n}(\Omega, \mathbb{R}^n)$ is said to be a critical point of the Ginzburg-Landau functional (1.1) if $u(x) \in H_g^{1,n}(\Omega, \mathbb{R}^n)$ is a weak solution to the following Euler-Lagrange equation:

$$-\nabla \cdot (|\nabla u|^{n-2} \nabla u) = \frac{1}{\varepsilon^n} u(1 - |u|^2) \text{ in } \Omega, \tag{1.3}$$

$$u|_{\partial\Omega} = g. \tag{1.4}$$

One special case of interest is $\Omega = B$ and $g = x$ on ∂B where B is the unit ball in $\mathbb{R}^n$. For each $\varepsilon > 0$, we can find a symmetric solution to equations (1.3)–(1.4) of the form $u_\varepsilon = f_\varepsilon(r) \frac{x}{|x|}$.

Theorem 1.1. *Assume that $n \geq 2$. Let $\Omega = B$ and let $g = x$ be the boundary data. For each $\varepsilon > 0$, there exists a symmetric u_ε to the Ginzburg-Landau equation* (1.3) *with* (1.4). *For this sequence of critical points u_ε, there exists a subsequence (u_{ε_k}) such that as $\varepsilon_k \to 0$*

$$u_{\varepsilon_k} \rightharpoonup \frac{x}{|x|} \text{ in } H^{1,n}_{\text{loc}}(B \backslash \{0\}, \mathbb{R}^n).$$

Theorem 1.1 is proved directly by using the Pohozaev identity (see Lemma 2.3).

A map $u : \Omega \to S^{n-1}$ is called an n-harmonic map if $u \in H^{1,n}(\Omega, S^{n-1})$ satisfies

$$\nabla \cdot (|\nabla u|^{n-2} \nabla u) + |\nabla u|^n u = 0 \tag{1.5}$$

in the distribution sense.

For a general case, we give a partial answer to the problem posed by Bethuel, Brezis and Hélein in their book (see Problem 17 in [4]) in the following:

Theorem 1.2. *Let $d \neq 0$ be the degree of the boundary data g. For each $\varepsilon > 0$, there exists a minimizer u_ε for E_ε. For this sequence of minimizers u_ε, there exists a subsequence (u_{ε_k}) and finite points x_l, $l = 1, \dots J$, such that as $\varepsilon_k \to 0$*

$$u_{\varepsilon_k} \rightharpoonup u \quad in \quad H^{1,n}_{\text{loc}}(\Omega \backslash \{x_1, \dots, x_J\}, \mathbb{R}^n),$$

where u is an n-harmonic map with values in S^{n-1}. Moreover, u_{ε_k} converges to u weakly in $H^{1,q}$ for $q < n$.

For the proof of Theorem 1.2, we modify Bethuel, Brezis and Hélein's main ideas in [4] and the Struwe's ideas in [25]. For $n = 2$, Bethuel, Brezis and Hélein in [2] showed the estimate $|\nabla u_\varepsilon| \leq \frac{C}{\varepsilon}$ holds, where C is a constant independent of ε. It seems that their proof can not be applied to the case $n \geq 3$. To overcome this difficulty, we first regularize the functional (1.1) by following an idea of Uhlenbeck in [27] (also see [12]) and rescale the minimization problem (1.2) as in [25] to establish Theorem 2.2. The proof of Theorem 2.2 relies on the fact that for $x_0 \in \bar{\Omega}$ and for some $\rho > 0$ we have

$$\int_{B_{\rho\varepsilon}(x_0) \cap \Omega} |\nabla u_\varepsilon|^n \, dx \leq C,$$

where C is a uniform constant for ε. Based on a Bochner-type inequality, a local bounded theorem (see Theorems 8.17 of Gilbarg and Trudinger's book, [14]) and the reverse Hölder inequality (see [11, Theorem 3.9, page 159], or [19]), we obtain an interior estimate for $|\nabla u_\varepsilon|$ (see (i) of Theorem 2.2). Using the reverse Hölder inequality (see [21]) and Sobolev imbedding theorem, we get $|u_\varepsilon| \geq \frac{1}{2}$ near the boundary $\partial\Omega$ (see (ii) of Theorem 2.2).

Another difficult step (see Theorem 3.10) in the proof of Theorem 1.2 is to show that there exists a finite collection of points x_k for $k = 1, \dots, J$ such that for any $\sigma > 0$

$$E(u_\varepsilon; \Omega \backslash \cup B_\sigma(x_k)) \leq C(\sigma) \tag{1.6}$$

where $C(\sigma)$ is a constant independent of ε. For $n = 2$, this result was first proven by Bethuel, Brezis and Hélein, with a simplified proof given by Struwe in [25]. But their proofs rely heavily on the following result of Brezis, F. Merle and Rivière in [6].

Theorem ([6]). *Assume $\varepsilon \leq R_0 \leq R \leq L$. Let $x_0 \in \Omega$ and denote*

$$A_{R,R_0} = B_R(x_0) \backslash B_{R_0}(x_0) \cap \Omega$$

and let $u \in H^{1,2}(A_{R,R_0}, \mathbb{R}^2)$ be a function satisfying $\frac{1}{2} \leq |u| \leq 1$ in $A_!R, R_0$. Assume that there exists a constant K such that

$$\frac{1}{\varepsilon^2} \int_\Omega (1 - |u|^2)^2 \, dx \leq K. \tag{1.7}$$

Then there exists a constant $C(K,d)$

$$\int_{A_{R,R_0}} |\nabla u|^2 \geq \pi |d| \ln \frac{R}{R_0} - C(K,d)$$

where d *is the degree of* u *on each* $\partial B_r(x_0)$, $R_0 \leq r \leq R$.

The condition (1.7) in the above theorem can be replaced in [20] by the following weaker assumption; i.e., there exists a constant K such that

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)^2\,dx \leq K(|\ln\varepsilon|+1) \quad\text{and}\quad \frac{1}{\varepsilon^2}\int_{B_{\varepsilon^{1/2}}(x_0)} (1-|u|^2)^2\,dx \leq K. \tag{1.8}$$

The assumption (1.8) is applied in [25]. However, all proofs about the Brezis, F. Merle and Rivière's theorem in [6], [25] and [20] are based on two-dimensional complex analysis and seem not to apply in the case $n \geq 3$. We prove Brezis-Merle-Rivière's theorem by a new approach which is easily extended to higher-dimensional cases (see Theorem 3.9). Roughly speaking, combining a result of Brezis, Coron and Leib in [5, Theorem 8.2] with the reverse Hölder inequality due to [16, Section 6] and [10] for minimizing a functional among maps from a domain into S^{n-1}, we set up a new minimization problem of a functional over maps from S^{n-1} into S^{n-1} with the topological degree d. Then we compare a minimizer of this new minimization problem with u_ε to prove Theorem 3.9. The estimate (1.6) is finally proven using an idea of Struwe in [25]. Other proofs of Theorem 1.2 are extended from [25] to higher-dimensional cases.

Remark 1.3. The number J of the singular points x_k in Theorem 1.2 is exactly $|d|$ following [4]. If $n = 2$, Theorem 1.2 holds for any minimizer u_ε of the functional (1.1) by our proofs.

Related results for p-harmonic maps have been obtained by Hardt and Lin in [18] for $n = 2$, and by Chen and Hardt in [7] for $n \geq 2$.

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2. Some lemmas and the proof of Theorem 1.1.

Lemma 2.1. *There exists a constant* $C_1 = C_1(\Omega, g)$ *such that for* $0 < \varepsilon \leq 1$,

$$\nu(\varepsilon) \leq |d| \frac{(n-1)^{\frac{n}{2}}}{n} |S^{n-1}| |\ln\varepsilon| + C_1, \tag{2.1}$$

where $|S^{n-1}|$ *denotes the area of the unit sphere* S^{n-1} *in* $\mathbb{R}^n$.

Proof. Without loss of generality, we may assume that $d > 0$. We can follow the steps in [25] by deleting d balls. Let x_i $(i = 1, \ldots, d)$ be d different points inside Ω such that

$$B_\rho(x_i) \cap B_\rho(x_j) = \emptyset \quad \text{for} \quad i \neq j,$$

where ρ is small enough. We then introduce Dirichlet boundary conditions

$$g_i(x) = \frac{x - x_i}{|x - x_i|} \quad \text{on} \quad \partial B_\rho(x_i)$$

to obtain a new domain $\tilde{\Omega} = \Omega \backslash \cup_{i=1}^d B_\rho(x_i)$. Choose u_0 be a function from $\tilde{\Omega}$ into S^{n-1} with $u_0 = g$ on $\partial\Omega$ and $u_0 = g_i$ on each $\partial B_\rho(x_0)$ and

$$\int_{\tilde{\Omega}} |\nabla u_0|^n \leq C.$$

As in [25], we can thus reduce to the case $\Omega = B = B_1^n(0)$ and $g(x) = x$. Set

$$u_\varepsilon(x) = f_\varepsilon(x)\frac{x}{|x|}$$

where $f_\varepsilon(x) \cong \tanh(\frac{r}{\sqrt{2}\varepsilon})$. Since $\nabla u_\varepsilon(x) = \nabla f_\varepsilon(r) \cdot \frac{x}{|x|} + f_\varepsilon(x)\nabla\frac{x}{|x|}$, we have

$$|\nabla u_\varepsilon(x)|^2 = |\frac{\partial}{\partial r} f_\varepsilon(r)|^2 + |f_\varepsilon(r)|^2 |\nabla \frac{x}{|x|}|^2 = \frac{1}{2}\frac{(1-f)^2}{\varepsilon^2} + \frac{(n-1)f^2}{r^2}$$

by a simple calculation.

For $a > 0$ and $b > 0$, we have

$$\sum_{i=1}^{n-1} a^i b^{n-i} \leq C(a^{n-1}b + b^n).$$

Then using this inequality we obtain

$$\begin{aligned}
E_\varepsilon(u_\varepsilon) &= \frac{1}{n}\int_\Omega |\nabla u|^n \, dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-|u|^2)^2 \, dx \\
&\leq \int_\Omega \frac{1}{n}\Big[\frac{(n-1)f^2}{r^2} + \frac{(1-f^2)^2}{2\varepsilon^2}\Big]^{\frac{n}{2}} dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-f^2)^2 \, dx \\
&\leq \int_\Omega \frac{1}{n}\Big(\frac{(n-1)^{1/2}|f|}{r} + \frac{|1-f^2|}{\sqrt{2}\varepsilon}\Big)^n dx + \frac{1}{4\varepsilon^n}\int_\Omega (1-f^2)^2 \, dx \\
&\leq \int_\Omega \frac{(n-1)^{n/2}}{n}\frac{|f|^n}{r^n} \, dx + \sum_{1=1}^{n-1} C\int_\Omega \frac{1}{n}\Big(\frac{(n-1)^{1/2}|f|}{r}\Big)^i\Big(\frac{|1-f^2|}{\sqrt{2}\varepsilon}\Big)^{n-i} \\
&\quad + \frac{C}{4\varepsilon^n}\int_\Omega (1-f^2)^2 \, dx \\
&\leq \frac{(n-1)^{n/2}}{n}|S^{n-1}|\int_0^1 \frac{|f|^n}{r} \, dr + C\int_0^1 \Big(\frac{|1-f^2|}{\sqrt{2}\varepsilon}\Big)^n r^{n-1} \, dr \\
&\quad + C\int_0^1 \Big(\frac{(n-1)^{1/2}|f|}{r}\Big)^{n-1}\frac{|1-f^2|}{\sqrt{2}\varepsilon} r^{n-1} \, dr + \frac{C}{\varepsilon^n}\int_0^1 (1-f^2)^2 r^{n-1} \, dr,
\end{aligned}$$

where C is a constant. By changing the variable $s=\frac{r}{\sqrt{2}\varepsilon}$ we have

$$\begin{aligned}&\int_0^1\big[\big(\frac{|1-f^2|}{\sqrt{2}\varepsilon}\big)^n+\big(\frac{|f|}{r}\big)^{n-1}\frac{|1-f^2|}{\sqrt{2}\varepsilon}\big]\,r^{n-1}\,dr+\frac{C}{\varepsilon^n}\int_0^1(1-f^2)^2\,r^{n-1}dr\\&\le\int_0^\infty\big[\big(\frac{|1-|\tanh(s)|^2|}{\sqrt{2}}\big)^n s^{n-1}+|\tanh(s)|^{n-1}\big(\frac{|1-\tanh^2(s)|}{\sqrt{2}}\big)\big]\,ds\\&\quad+C\int_0^\infty(1-|\tanh(s)|^2)^2\,s^{n-1}ds<+\infty\end{aligned}$$

and

$$\int_0^1\frac{|f|^n}{r}\,dr\le\int_1^{\frac{1}{\sqrt{2}\varepsilon}}\frac{|\tanh(s)|^n}{s}\,ds+\int_0^1\frac{|\tanh(s)|^n}{s}\,ds\le|\ln\varepsilon|+C,$$

where C is a constant. Therefore Lemma 2.1 is proved. □

Let u_ε be a minimizer of the functional E_ε. We do not know whether the minimizer u_ε is regular. However we find a new minimizer which can be approximated by a sequence of smooth maps. Following Uhlenbeck's idea in [27] (see also [12]), we regularize the minimization problem (1.2) by minimizing the functionals:

$$I_\varepsilon^\eta(v;\Omega_\varepsilon)=\int_\Omega\big[\frac{(|\nabla v|^2+\eta\varepsilon^{-2})^{\frac{n}{2}}}{n}+\frac{1}{4\varepsilon^n}(1-|v|^2)^2\big]\,dx$$

over all functions $v\in H_g^{1,n}(\Omega;\mathbb{R}^n)$ where $\eta>0$ is a small constant. Let u_ε^η be the minimizer. Hence u_ε^η is also a smooth solution of the following equation:

$$-\nabla\cdot\big[(|\nabla u|^2+\eta\varepsilon^{-2})^{\frac{n-2}{2}}\nabla u\big]=\frac{1}{\varepsilon^n}u(1-|u|^2)\text{ in }\Omega.\tag{2.2}$$

Since $I_\varepsilon^\eta(u_\varepsilon^\eta;\Omega)\le I_\varepsilon^\eta(u_\varepsilon;\Omega)$, $u_\varepsilon^\eta\rightharpoonup\bar u_\varepsilon$ in $H_g^{1,n}(\Omega;\mathbb{R}^n)$ as $\eta\to0$. By the weakly low semicontinuity of I_ε^η, we have

$$\lim_{\eta\to0}I_\varepsilon^\eta(u_\varepsilon^\eta;\Omega)=E_\varepsilon(\bar u_\varepsilon,\Omega)=\min_{v\in H_g^{1,n}(\Omega;\mathbb{R}^n)}E_\varepsilon(v;\Omega).$$

Therefore $u^\eta\to\bar u_\varepsilon$ strongly in $H_g^{1,n}(\Omega;\mathbb{R}^n)$ and $\bar u_\varepsilon$ is a new minimizer of E_ε. Moreover, repeating Uhlenbeck's proofs, we may show $\bar u_\varepsilon\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, although this result is not needed here.

Denote for $\rho>0$

$$\Omega^{(\rho\varepsilon)}:=\{x\in\Omega:\ \mathrm{dist}\,(x,\partial\Omega)\ge\rho\varepsilon\}.$$

Theorem 2.2. *Any critical point $u\in H_g^{1,n}(\Omega;\mathbb{R}^n)$ of E_ε satisfies the estimate $|u|\le1$ almost everywhere on Ω. For each ε, there exists a minimizer u_ε of the functional E_ε such that u_ε can be approximated in $H_g^{1,n}$ by a sequence of minimizers u_ε^η of the functional I_ε^η. Then there exist constants ρ and $C_2=C_2(\Omega,g,\rho)$ such that*

$$\overline{\lim_{\eta\to0}}\,|\nabla u_\varepsilon^\eta|\le C_2(\Omega,g,\rho)\varepsilon^{-1}\quad\textit{almost everywhere on}\ \ \Omega^{(\rho\varepsilon)}.\tag{i}$$

Moreover there exists a $\delta > 0$ *such that*

$$|u_\varepsilon| \geq \frac{1}{2} \quad on \quad \Omega\backslash\Omega^{(\delta\varepsilon)}. \tag{ii}$$

Proof. Choose $\Phi = u - \frac{u}{|u|}\min\{1, |u|\}$ as a test function in equation (1.3) and define $\Omega_+ = \{x \in \Omega : |u(x)| > 1 \text{ a.e. on } \Omega\}$. Then we have

$$\nabla\Phi = \begin{cases} 0, & \text{a.e. } x \in \Omega\backslash\Omega_+ \\ \nabla u - [\frac{\nabla u}{|u|} - \frac{u(u\cdot\nabla u)}{|u|^3}], & \text{a.e. } x \in \Omega_+. \end{cases}$$

This implies that

$$\int_{\Omega_+} |\nabla u|^n(1 - \frac{1}{|u|})\,dx + \int_{\Omega_+} |\nabla u|^{n-2}\frac{|u\cdot\nabla u|^2}{|u|^3}\,dx \\ + \frac{1}{\varepsilon^n}\int_{\Omega_+}(1-|u|^2)|u|(1-|u|)\,dx = 0,$$

so $\text{meas}(\Omega_+) = 0$. Hence $|u| \leq 1$ almost everywhere as claimed.

Moreover, rescaling equation (1.3) by $\tilde{u}(x) = u(\varepsilon x)$, we have

$$-\nabla\cdot(|\nabla\tilde{u}|^{n-1}\nabla\tilde{u}) = \tilde{u}(1-|\tilde{u}|^2) \quad \text{in} \quad \Omega_\varepsilon := \Omega/\varepsilon. \tag{2.3}$$

We regularize the solution to equation (2.3) by minimizing the rescaled functional

$$\tilde{I}^\eta(v;\Omega_\varepsilon) = \int_{\Omega_\varepsilon}\Big[\frac{(|\nabla v|^2+\eta)^{\frac{n}{2}}}{n} + \frac{1}{4}(1-|v|^2)^2\Big]\,dx$$

over all functions $v \in H^{1,n}_{\tilde{g}}(\Omega_\varepsilon)$ where $\tilde{g}(x) = g(\varepsilon x)$ and $\eta > 0$ is a small constant. Let u^η be the minimizer. Hence u^η is a smooth solution of the following equation:

$$-\nabla\cdot\big[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\nabla u\big] = u(1-|u|^2) \quad \text{in} \quad \Omega_\varepsilon. \tag{2.4}$$

Choosing $\Phi = u^\eta - \frac{u^\eta}{|u^\eta|}\min\{1, |u^\eta|\}$ as a test function in equation (2.4), we obtain that $|u^\eta| \leq 1$ a.e. on Ω_ε. $u^\eta \to \tilde{u}_\varepsilon$ strongly in $H^{1,n}_{\tilde{g}}(\Omega_\varepsilon;\mathbb{R}^n)$ as $\eta \to 0$ where $\tilde{u}$ is a minimizer of E_ε. For simplicity we denote u^η by u. Denote $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{ik} = \frac{\partial^2}{\partial x_i \partial x_k}$. By equation (2.4) we have

$$\begin{aligned} &\partial_i\{(|\nabla u|^2+\eta)^{\frac{n-2}{2}}[\delta_{ij} + \frac{(n-2)u^\alpha_{x_i}u^\alpha_{x_j}}{|\nabla u|^2+\eta}]\partial_{kj}u^\beta\partial_k u^\beta\} \\ &= \partial_i\{(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial_{ki}u^\beta\partial_k u^\beta + \partial_k[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}]\partial_i u^\beta\partial_k u^\beta\} \\ &= \partial_i\{\partial_k[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial_i u^\beta]\partial_k u^\beta\} \\ &= \partial_i\{\partial_{x_k}[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial u^\beta]\}\partial_k u^\beta + \partial_k[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial_i u^\beta]\partial_{ki}u^\beta \\ &= \partial_k[(|u|^2-1)u^\beta]\partial_k u^\beta + \partial_k[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial_i u^\beta]\partial_{ki}u^\beta \end{aligned}$$

Define $a_{ij}=\delta_{ij}+\frac{(p-2)u^\alpha_{x_i}u^\alpha_{x_j}}{|\nabla u|^2+\eta}, a^{\alpha\beta}_{ij}=\delta_{ij}\delta_{\alpha\beta}+\frac{(p-2)u^\alpha_{x_i}u^\beta_{x_j}}{|\nabla u|^2+\eta}$. Applying the above identity and setting $V=(|\nabla u|^2+\eta)^{\frac{n}{2}}+1$, we have

$$\begin{aligned}LV:&=(a_{ij}V_{x_j})_{x_i}=\partial_i[n(|\nabla u|^2+\eta)^{\frac{n-2}{2}}a_{ij}\partial_k j u^\beta\partial_u^\beta]\\&=n\partial_k[(|u|^2-1)u^\beta]\partial_k u^\beta+\partial_k[(|\nabla u|^2+\eta)^{\frac{n-2}{2}}\partial_i u^\beta]\partial_{ki}u^\beta\\&=n(|u|^2-1)|\nabla u^\beta|^2+2n|u\cdot\nabla u|^2+n(|\nabla u|^2+\eta)^{\frac{n-2}{2}}a^{\alpha\beta}_{ij}\partial_{ki}u^\alpha\partial_{kj}u^\beta\\&\geq n(|u|^2-1)|\nabla u|^2\geq -c(n)V,\end{aligned}\tag{2.5}$$

where $c(n)$ is an absolute constant. (2.5) is a so-called Bochner-type inequality.

Note that u^η is a minimizer of $\tilde{I}^\eta$ with $|u^\eta|\leq 1$. Consider a new functional

$$\mathbb{F}(u,\Omega_\varepsilon)=\int_{\Omega_\varepsilon}f(x,u,\nabla u)\,dx=\int_{\Omega_\varepsilon}\big[\frac{(\eta+|\nabla u|^2)^{\frac{n}{2}}}{n}+\frac{1}{4}\min\{(1-|u|^2)^2,1\}\big]\,dx.$$

Then u_η is also a minimizer of $\mathbb{F}$.

Let x_0 be an interior point of Ω_ε; i.e., $B_{4\rho}(x_0)\subset\Omega_\varepsilon$ for some $\rho>0$. Using the standard L^p-estimate of the functional $\mathbb{F}$ (see [11, Theorem 3.1, page 159] and [19]), there exist constants $\delta>0$ and $C(\rho)>0$ (independent of η) such that

$$\Big(\int_{B_{2\rho}(x_0)}|\nabla u^\eta|^{n+\delta}\,dx\Big)^{\frac{1}{n+\delta}}\leq C(\rho)\Big(\int_{B_{3\rho}(x_0)}|\nabla u^\eta|^n\,dx\Big)^{\frac{1}{n}}+C(\rho),\tag{2.6}$$

where $C(\rho)$ is a uniform constant for $\eta<1$. Using (2.5), (2.6) and Theorem 8.17 in [14] we have

$$\sup_{B_\rho(x_0)}|\nabla u|^n\leq C\Big(\int_{B_{2\rho}(x_0)}|\nabla u|^{n+\delta}\Big)^{\frac{n}{n+\delta}}\leq C\int_{B_{2\rho}(x_0)}|\nabla u|^n\,dx+C(\Omega,g).$$

Then letting $\eta\to 0$, it implies

$$\overline{\lim_{\eta\to 0}}\,|\nabla u^\eta|^n\leq C\int_{B_{2\rho}(x_0)}|\nabla\tilde{u}_\varepsilon|^n\,dx+C(\Omega,g).\tag{2.7}$$

Now consider the boundary case. For $x_0\in\partial\Omega_\varepsilon$, we know that u^η is C^1-continuous at x_0. By the standard method in [14], for each ε there exists a transformation (g^ε_{ij}) from $\Omega_\varepsilon\cap B_\rho(x_0)$ to the domain $B^+_\rho(0):=B_\rho(0)\cap\mathbb{R}^n_+$. We claim that these transformations are uniform for ε. Set $x_1=\varepsilon x_0\in\partial\Omega$. After a translation $Y_\varepsilon(\tilde{x})=\tilde{x}-\frac{1}{\varepsilon}x_1+x_1$, we have $Y_\varepsilon(\partial\Omega_\varepsilon\cap B_\rho(x_0))\cap(\partial\Omega\cap B_\rho(x_1))=x_1$. Let P_T be the tangent plane of both $\partial\Omega$ and $Y_\varepsilon(\partial\Omega_\varepsilon)$ at x_1. We know that $Y_\varepsilon(\partial\Omega_\varepsilon)$ locally lies between P_T and $\partial\Omega$ in a neighborhood of x_1; i.e., $\partial\Omega_\varepsilon\cap B_\rho(x_0)$ is flatter than $\partial\Omega\cap B_\rho(x_1)$. Thus this proves our claim. Then there exists a constant c independent of ε such that

$$c^{-1}|\xi|^2\leq\sum_{i,j}g^\varepsilon_{ij}\xi_i\xi_j\leq c|\xi|^2,\quad\text{for}\ \ \xi\in\mathbb{R}^n.$$

Therefore we only need to consider the norm of the gradient of the map u

$$|\nabla u|^2 = \sum_{i,j} g^{\varepsilon}_{ij} D_i u D_j u$$

on the domain $B^+_\rho(0)$ instead of $B_\rho(x_0)\cap\Omega_\varepsilon$. Note that the boundary data g is a smooth function. Then from the argument of Jost and Meier in [21], we know that the reverse Hölder inequality also holds at the boundary point x_0; i.e., there exists a constant $\delta > 0$ and $C(\rho)$ such that

$$\Big(\int_{B_{2\rho}(x_0)\cap\Omega_\varepsilon} |\nabla u^\eta|^{n+\delta}\, dx\Big)^{\frac{1}{n+\delta}} \le C(\rho)\Big(\int_{B_{3\rho}(x_0)\cap\Omega_\varepsilon} |\nabla u^\eta|^{n}\, dx\Big)^{\frac{1}{n}} + C(\rho). \qquad (2.6b)$$

Let x_0 be a point in $\bar\Omega_\varepsilon$. For $3\rho \le t < s \le 4\rho$, let ϕ be a smooth function in Ω_ε such that

$$\phi = \begin{cases} 0, & \text{for } x \in \Omega_\varepsilon \backslash B_s(x_0) \\ 1, & \text{for } x \in B_t(x_0) \end{cases}$$

with $|\phi| \le 1$, and $|\nabla\phi| \le \frac{C}{s-t}$. Let u_0 be a given smooth vector-value function in $\bar\Omega$ with $u_0|_{\partial\Omega} = g$ with $\int_\Omega |\nabla u_0|^n\, dx \le C$. Setting $\tilde u_0(x) = u_0(\varepsilon x)$, we have

$$\int_{\Omega_\varepsilon} |\nabla \tilde u_0|^n dx \le C.$$

Choosing $(\tilde u - \tilde u_0)\phi$ as a test function in equation (2.3), we obtain

$$\int_{B_{4\rho}(x_0)\cap\Omega_\varepsilon} |\nabla\tilde u|^{n-2}\nabla\tilde u\nabla(\tilde u - \tilde u_0)\phi\, dx + \int_{B_{4\rho}(x_0)\cap\Omega_\varepsilon} |\nabla\tilde u|^{n-2}\nabla\tilde u\cdot(\tilde u - \tilde u_0)\nabla\phi\, dx$$
$$= \int_{B_{4\rho}(x_0)\cap\Omega_\varepsilon} (1 - |\tilde u|^2)\tilde u\cdot(\tilde u - \tilde u_0)\phi\, dx.$$

Denote

$$h(t) = \int_{B_t(x_0)\cap\Omega_\varepsilon} |\nabla\tilde u|^n.$$

Noting that $|u| \le 1$ and using the standard “filling hole” technique in [11], we obtain

$$h(t) \le \theta h(s) + C\Big(\frac{1}{(s-t)^n} + 1\Big),$$

where $\theta < 1$ is a constant. Then from [11, Lemma 3.1, pages 161–162] or [19, Lemma 2.2] we have

$$h(3\rho) = \int_{B_{3\rho}(x_0)\cap\Omega_\varepsilon} |\nabla\tilde u|^n\, dx \le C\Big(1 + \frac{1}{\rho^n}\Big) = C(\theta,\rho,g), \qquad (2.8)$$

where $C(\theta, \rho, g)$ is a constant independent of ε.

Let x_0 be a interior point of Ω. Combining (2.7) with (2.8), we have

$$\overline{\lim_{\eta\to 0}} |\nabla u^\eta| \le C(\Omega, g) \quad \text{on } B_\rho(x_0).$$

Rescaling $\tilde{u}_\varepsilon$ back to u_ε, (i) is proved.

Let x_0 be a boundary point; i.e., $x_0 \in \partial\Omega_\varepsilon$. By (2.6b), (2.8) and applying the Sobolev imbedding theorem, there exists an $\alpha > 0$ such that for $|x_1 - x_2| \le 2\rho$ there holds

$$\begin{aligned} |\tilde{u}(x_1) - \tilde{u}(x_2)| &= \lim_{\eta\to 0} |\tilde{u}^\eta(x_1) - \tilde{u}^\eta(x_2)| \le C \lim_{\eta\to 0} \|\tilde{u}\|_{H^{1,n+\delta}} |x_1 - x_2|^\alpha \\ &\le C \lim_{\eta\to 0} \|\tilde{u}\|_{H^{1,n}} |x_1 - x_2|^\alpha \le \tilde{C}|x_1 - x_2|^\alpha. \end{aligned}$$

Since $|\tilde{u}| = 1$ on $\partial\Omega_\varepsilon$, there exists $\delta_1 > 0$ such that for $|x - x_0| \le \delta_1$ there holds

$$|\tilde{u}_\varepsilon(x_1) - \tilde{u}_\varepsilon(x_0)| \le \tilde{C}\delta_1^\alpha.$$

Choosing δ_1 small enough, we obtain

$$|\tilde{u}(x)| \ge \frac{1}{2} \quad \text{in} \;\; \Omega_\varepsilon \backslash \Omega_\varepsilon^{(\delta_1)},$$

where $\Omega_\varepsilon^{(\delta_1)} := \{x \in \Omega_\varepsilon : \text{ dist } (x, \partial\Omega_\varepsilon) \ge \delta_1\}$. Rescaling back to u_ε, (ii) is obtained. This proves Theorem 2.2 $\Box$

In the next lemma, we assume that for each ε, the minimizer u_ε can be approximated by minimizers u_ε^η in $H_g^{1,n}(\Omega; \mathbb{R}^n)$.

For $\rho > 0$ let

$$\begin{aligned} f^{(\delta)}(\rho) &:= f(x_0, \rho, B_\rho \cap \Omega^{(\delta\varepsilon)}) = \overline{\lim_{\eta\to 0}}\, \rho \int_{\partial B_\rho(x_0) \cap \Omega^{(\delta\varepsilon)}} \Big[\frac{|\nabla u_\varepsilon^\eta|^n}{n} + \frac{(1 - |u_\varepsilon^\eta|^2)^2}{4\varepsilon^n}\Big]\, d\tau, \\ f(\rho, \eta) &:= f(x_0, u^\eta, \rho, B_\rho \cap \Omega) = \rho \int_{\partial B_\rho(x_0) \cap \Omega} \Big[\frac{|\nabla u_\varepsilon^\eta|^n}{n} + \frac{(1 - |u_\varepsilon^\eta|^2)^2}{4\varepsilon^n}\Big]\, d\tau \end{aligned}$$

with $d\tau$ denoting the area element on ∂B_ρ.

The following lemma is related to the Courant Lemma, as in [25].

Lemma 2.3. (i) *For $0 < \varepsilon \le e^{-1}$ there exists a constant C_3 such that*

$$\overline{\lim_{\eta\to 0}} \inf_{\varepsilon^{1/2} \le \rho \le \varepsilon^{1/4}} f(\rho, \eta) \le 4\frac{E_\varepsilon\big(u_\varepsilon, \Omega \cap B_{\varepsilon^{1/4}}(x_0)\big)}{|\ln \varepsilon|} \le C_3$$

and

$$\overline{\lim_{\eta\to 0}} \inf_{5\varepsilon^{1/4} \le \rho \le 5\varepsilon^{1/8}} f(\rho, \eta) \le 2C_3.$$

(ii) *There are constants* γ *and* $\varepsilon_0 = \varepsilon_0(\Omega, g) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$\inf_{B_{\rho(x_0)} \cap \Omega^{(\delta)}} |u_\varepsilon| \geq \frac{1}{2}$$

whenever $\varepsilon^{1/2} \leq \rho \leq \varepsilon^{1/4}$ *and* $f^{(\delta)}(\rho) \leq \gamma$.

Proof. (i) As in [25], we have

$$\overline{\lim_{\eta\to 0}} \inf_{\varepsilon^{1/2} \leq \rho \leq \varepsilon^{1/4}} f(\rho, \eta) \leq 4 \overline{\lim_{\eta\to 0}} \frac{\int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} f(\rho, \eta) \frac{d\rho}{\rho}}{|\ln \varepsilon|} \leq 4 \frac{\overline{\lim}_{\eta\to 0} E_\varepsilon(u_\varepsilon^\eta; \Omega \cap B_{\varepsilon^{1/4}}(x_0))}{|\ln \varepsilon|} \leq C.$$

The second inequality is also proved as in [25].

(ii) Choose $\varepsilon_1 = \varepsilon_1(\Omega) > 0$ such that for $0 < \rho < \varepsilon_1^{1/4}$ the domain $D = \Omega^{(\delta\varepsilon)} \cap B_\rho(x_0)$ is strongly star-shaped; i.e., $r_0 \cdot x \geq \frac{1}{4}\rho$ for $x \in \partial D$ where r_0 denotes the outer unit normal. Let $\tau = (\tau^1, \ldots, \tau^{n-1})$ denote a smooth basis of tangent vector fields along ∂D. Let u_ε^η be a smooth solution to equation (2.2). We drop ε and η for u_ε^η. By equation (2.2) we have the following Pohozaev identity:

$$\begin{aligned}
&\sum_{i,j=1}^n [\,(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} u_{x_j} x_i u_{x_i}]_{x_j} \\
&= \sum_{i,j=1}^n [(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} u_{x_j}]_{x_j} x_i u_{x_i} + \sum_{i,j=1}^n (|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} x_i u_{x_j} u_{x_i x_j} \\
&\quad + (|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} |\nabla u|^2 \\
&= \frac{1}{\varepsilon^n}(|u|^2 - 1)u \sum_{i=1}^n x_i u_{x_i} + \sum_{i=1}^n \frac{1}{n}\big[x_i\big((|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n}{2}}\big)_{x_i} + (|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n}{2}}\big] \\
&\quad - \eta\varepsilon^{-2}(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} \\
&= \frac{1}{4\varepsilon^n} \sum_{i=1}^n \big[(|u|^2 - 1)^2 x_i\big]_{x_i} - \frac{n}{4\varepsilon^n}(1 - |u|^2)^2 + \sum_{i=1}^n \frac{1}{n}\big(x_i(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n}{2}}\big)_{x_i} \\
&\quad - \eta\varepsilon^{-2}(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}}.
\end{aligned}$$

Integrating both sides of the above equality gives

$$\begin{aligned}
&\int_{\partial D} \partial_{r_0} u(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} x \cdot \nabla u \, d\tau + \frac{n}{4\varepsilon^n}\int_D (1 - |u|^2)^2 \, dx \\
&= \int_{\partial D} \big[\frac{(1 - |u|^2)^2}{4\varepsilon^n} + \frac{(|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n}{2}}}{n}\big] r_o \cdot x \, d\tau - \eta\varepsilon^{-2}\int_D (|\nabla u|^2 + \eta\varepsilon^{-2})^{\frac{n-2}{2}} \, dx.
\end{aligned}$$

Note

$$\partial_{r_0} u |\nabla u|^{n-2}(x \cdot \nabla u) \geq r_0 \cdot x |\partial_{r_0} u|^n - |\partial_{r_0} u| \rho |\nabla_\tau u| |\nabla u|^{n-2} \geq \frac{r_0 \cdot x}{n} |\partial_{r_0} u|^n - C\rho |\nabla_\tau u|^n.$$

Letting $\eta \to 0$, we have

$$\frac{1}{\varepsilon^n}\int_D (1-|u_\varepsilon|^2)^2\,dx = \frac{1}{\varepsilon^n}\lim_{\eta\to 0}\int_D (1-|u_\varepsilon^\eta|^2)^2\,dx$$
$$\le C\rho\,\overline{\lim_{\eta\to 0}}\int_{\partial D}\big[\frac{|\nabla_\tau u_\varepsilon^\eta|^n}{n}+\frac{(1-|u_\varepsilon^\eta|^2)^2}{4\varepsilon^n}\big]d\tau \le Cf^{(\delta)}(\rho)\le C_4\gamma. \tag{2.9}$$

If $|u_\varepsilon(x_1)| < \frac{1}{2}$ for some $x_1 \in D$, by Theorem 2.2 we have

$$|u_\varepsilon(y)| \le \frac{3}{4} \quad \text{for} \quad |x_1 - y| < \frac{\varepsilon}{4C_2}.$$

Hence

$$\int_D \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^n}\,dx \ge C_5 > 0. \tag{2.10}$$

Choosing ε_1 and γ small enough gives Lemma 2.3. $\square$

Now consider a special case $\Omega = B$ and $g = x$ on $\partial\Omega$. We define a symmetric class $\mathcal{X}$ in $H_g^{1,n}$; i.e., a function $u(x) \in H_g^{1,n}$ belongs to the symmetric class $\mathcal{X}$ if there exists a function $f(r) : [0,1] \to \mathbb{R}$ such that the functional $u(x)$ has the form of $u(x) = f(r)\frac{x}{|x|}$, where $r = |x|$.

For a function $u(x) \in \mathcal{X}$, by a simple calculation, we have

$$|\nabla u|^2 = |\nabla f(r)|^2 + 2\nabla f(r)\cdot\nabla\frac{x}{|x|} + f^2(r)|\nabla\frac{x}{|x|}|^2 = f_r^2(r) + f^2(r)\frac{n-1}{r^2}.$$

For a function $u(x) \in \mathcal{X}$, define an energy for the corresponding $f(r)$ by

$$E_\varepsilon^{(S)}(f(r)) := E_\varepsilon(u(x);B) = \int_B \big[\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\big]\,dx$$
$$= |S^{n-1}|\int_0^1 \big[\frac{1}{n}(f_r^2(r) + \frac{n-1}{r^2}f^2(r))^{\frac{n}{2}} + \frac{1}{4\varepsilon^n}(1-f^2(r))^2\big]\,r^{n-1}\,dr.$$

A function $f(r)$ belongs to the space $\mathcal{H}_1^{1,n}[0,1]$ if and only if $f(r)$ satisfies

$$\int_0^1 (f_r^n + f^n)\,r^{n-1}\,dr < +\infty \quad \text{and} \quad f(1) = 1.$$

Consider the minimization problem

$$\min_{f\in\mathcal{H}_1^{1,n}[0,1]} E_\varepsilon^{(S)}(f(r)). \tag{2.11}$$

Since $E_\varepsilon^{(S)}(f(r))$ is weak low semicontinuous on $\mathcal{H}_1^{1,n}[0,1]$, the functional $E_\varepsilon^{(S)}(f)$ achieves its minimizer in $\mathcal{H}_1^{1,n}[0,1]$ by a function $f_\varepsilon(r)$. But we do not know whether the minimum $f_\varepsilon(r)$ is regular. Following the idea of Uhlenbeck in [27] again, we regularize the minimization problem (2.11) by minimizing the functional

$$I^\eta(f(r)) = |S^{n-1}| \int_0^1 \big[\frac{1}{n}(f_r^2(r) + \frac{n-1}{r^2} f^2(r) + \eta)^{\frac{n}{2}} + \frac{1}{4\varepsilon^n}(1 - f^2(r))^2\big]\, r^{n-1}\, dr$$

over all functions $f(r) \in \mathcal{H}_1^{1,n}[0,1]$ where $\eta > 0$ is a small constant. Let $f^\eta(r)$ be the minimizer of I^η in $\mathcal{H}_1^{1,n}[0,1]$. Hence $u^\eta(x) = f^\eta(|x|)\frac{x}{|x|}$ is a smooth solution of the following equation:

$$-\nabla \cdot \big[(|\nabla u|^2 + \eta)^{\frac{n-2}{2}} \nabla u\big] = \frac{1}{\varepsilon^n} u(1 - |u|^2) \quad \text{in} \quad B. \tag{2.12}$$

Let f_ε be a minimizer of E_ε in $\mathcal{H}_1^{1,n}[0,1]$. Since $I^\eta(f^\eta) \le I^\eta(f_\varepsilon)$, $f^\eta \to \tilde{f}_\varepsilon$ weakly in $\mathcal{H}_1^{1,n}[0,1]$. Since I^η is weak low semicontinuous, we have

$$\lim_{\eta \to 0} I^\eta(f^\eta(r)) = E_\varepsilon^{(S)}(f_\varepsilon(r)).$$

Define $\tilde{u}_\varepsilon = \tilde{f}_\varepsilon(|x|)\frac{x}{|x|}$. Then $u^\eta \to \tilde{u}_\varepsilon$ strongly in $H_g^{1,n}$. Letting $\eta \to 0$ in equation (2.12), $\tilde{u}_\varepsilon$ is a critical point of $E_\varepsilon(u;\Omega)$. The corresponding $\tilde{f}_\varepsilon(r)$ is a minimizer of $E_\varepsilon^{(S)}(f(r))$ in $\mathcal{H}_1^{1,n}[0,1]$.

Lemma 2.4. *Let $u_\varepsilon = f_\varepsilon(r)\frac{x}{|x|}$ be a critical point of E_ε which is regularized by solutions of equation* (2.12). *Then we have:*

(i) *There exists a constant C_6 independent of ε such that*

$$\int_0^1 \frac{(1 - f_\varepsilon^2(r))^2}{\varepsilon^n}\, r^{n-1}\, dr \le C_6.$$

(ii) *For each ρ, $0 < \rho < 1$, there exist two constants C_7 and C_8 independent of ρ and ε such that*

$$\int_\rho^1 \big|\partial_r f_\varepsilon(r)\big|^n r^{n-1}\, dr \le C_7 |\ln \rho| + C_8.$$

Proof. At first, we suppose that the critical point $u_\varepsilon = f_\varepsilon(|x|)\frac{x}{|x|}$ is smooth. Let $D = B_r(0)$ and r_0 denote the outer unit normal of D. Let $\sigma = (\sigma^1, \dots, \sigma^{n-1})$ denote a smooth basis of tangent vector fields along ∂D. Using the Pohozaev identity as in the proof of Lemma 2.3, we have

$$\int_{\partial D} r_0 \cdot x |\partial_r u|^n\, d\sigma + \frac{1}{\varepsilon^n}\int_D (1 - |u|^2)^2\, dx \le Cr \int_{\partial D} \big[\frac{|\partial_\sigma u|^n}{n} + \frac{(1 - |u|^2)^2}{4\varepsilon^n}\big]\, d\sigma. \tag{2.13}$$

Letting $u_\varepsilon(x) = f_\varepsilon(|x|)\frac{x}{|x|}$ and setting $D = B$ in (2.13), we get

$$\frac{1}{\varepsilon^n}\int_B (1-|u_\varepsilon|^2)^2\,dx \le C\int_{\partial B}\frac{|\partial_\sigma g|^n}{n}\,d\sigma \le C_6.$$

This proves (i).

Setting $D = B_r$ in (2.13), we obtain

$$r^{n-1}|\partial_r f_\varepsilon(r)|^n \le C\int_{\partial B_r}\big[\frac{|\partial_\sigma u_\varepsilon|^n}{n} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^n}\big]\,d\sigma \le \frac{C}{r} + \int_{\partial B_r}\frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^n}\,d\sigma.$$

Integrating the above inequality and using (i) gives

$$\int_\rho^1 |\partial_r f_\varepsilon(r)|^n r^{n-1}\,dr \le C\ln\frac{1}{\rho} + \int_B \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^n}\,dx \le C_7|\ln\rho| + C_8.$$

This proves (ii). If the critical point $u_\varepsilon(x)$ is not smooth, we repeat the above proofs using solutions $u^\eta = f^\eta\frac{x}{|x|}$ of equation (2.12) instead of u_ε. The conclusion of Lemma 2.4 follows by letting $\eta \to 0$. □

Proof of Theorem 1.1. Applying Lemma 2.4, there exists a constant C such that

$$E_\varepsilon(u_\varepsilon; B\backslash B_\rho(0)) \le C$$

for each $\rho > 0$. Thus $u_\varepsilon \rightharpoonup \frac{x}{|x|}$ in $H^{1,2}_{\text{loc}}(B\backslash B_\rho(0);\mathbb{R}^n)$. □

3. Proof of Theorem 1.2. We again consider the general domain Ω and boundary data g, and assume that u_ε is a minimizer of E_ε such that u_ε is approximated by u_ε^η and u_ε^η is a minimizer of the functional I_ε^η.

For $0 < \varepsilon < \varepsilon_0$ and minimizers u_ε of E_ε, consider the set

$$\Sigma_\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| < \tfrac{1}{2}\} = \{x \in \Omega^{(\delta\varepsilon)} : |u_\varepsilon(x)| < \tfrac{1}{2}\}$$

and its cover $(B_{\varepsilon^{\frac14}}(x))_{x\in\Sigma_\varepsilon}$. For $x \in \Sigma_\varepsilon$ let $\varepsilon^{1/2} < \rho(x) < \varepsilon^{1/4}$ be determined as in Lemma 2.3 such that

$$\frac{4E_\varepsilon(u_\varepsilon; \Omega^{(\delta\varepsilon)} \cap B_{\varepsilon^{1/4}}(x))}{|\ln\varepsilon|} \ge f^{(\delta)}(\rho(x), x, \varepsilon, u_\varepsilon) \ge \gamma.$$

By Vitali's covering lemma there exists a finite collection of disjoint balls $B_i = B_{\varepsilon^{1/4}}(x_i)$, $x_i \in \Sigma_\varepsilon$, $1 \le i \le I = I(u_\varepsilon)$ such that

$$(\Omega\bigcap\bigcup_{x\in\Sigma_\varepsilon} B_{\varepsilon^{1/4}}) \subset \bigcup_i B_{5\varepsilon^{1/4}}(x_i).$$

Moreover, we obtain the uniform bound

$$I \le \sum_i \frac{4E_\varepsilon(u_\varepsilon; \Omega\cap B_{\varepsilon^{1/4}}(x_0))}{|\ln\varepsilon|} \le \frac{4E_\varepsilon(u_\varepsilon;\Omega)}{|\ln\varepsilon|} \le C_3\gamma^{-1} := I_0 \tag{3.1}$$

on the number of "bad" balls B_i.

For $x_0 \in \Omega$, there exist constants $\rho_0^\eta \in [5\varepsilon^{1/4}, 5\varepsilon^{1/8}]$ such that

$$\overline{\lim_{\eta\to0}}\, f(\rho_0^\eta, x_0, \varepsilon, u_\varepsilon^\eta) = \overline{\lim_{\eta\to0}} \inf_{5\varepsilon^{1/4}\le\rho\le5\varepsilon^{1/8}} f(\rho, x_0, \varepsilon, u_\varepsilon^\eta) < 2C_3$$

and let $D = \Omega\cap B_{5\varepsilon^{1/4}}(x_0)$. Repeating the same proof in Lemma 2.3 (ii), we have

Lemma 3.1. *There exists a constant* $C_9 = C_9(\Omega, g) > 0$ *such that*

$$\frac{1}{\varepsilon^n}\int_D (1-|u_\varepsilon|^2)^2\,dx \le C_9$$

uniformly in $0 < \varepsilon < \varepsilon_0$ *for* $1 \le i \le I$.

Combining Theorem 2.2 with Lemma 3.1 we have from [25]

Lemma 3.2. *There exists a number* $J_0 = J_0(\Omega, g) \in \mathbb{N}$ *such that for any disjoint collection of balls* $B_{\varepsilon/5}(x_j)$, $x_j \in \Omega$, $1 \le j \le J$ *with* $|u_\varepsilon(x_j)| < \frac{1}{2}$, *we have* $J \le J_0$.

Theorem 8.2 of [5] gives

Lemma 3.3. *Let* $\phi : S^{n-1} \to S^{n-1}$ *be a* C^0*-map with* $\deg\phi = d$. *Then*

$$\int_{S^{n-1}} |\nabla_\tau \phi|^{n-1}\,dx \ge |d|(n-1)^{\frac{n-1}{2}}|S^{n-1}|,$$

where $|S^{n-1}|$ *denotes the area of* S^{n-1}.

Lemma 3.4. *Assume that* $\varepsilon \le R_0 < R \le L$ *where* L *is a constant. Let* $\phi(r,\tau) : S^{n-1}\times[R_0,R] \to S^{n-1}$ *be a* C^0*-map. For each fixed* r, $R_0 \le r \le R$, *the degree of the map* $\phi(r,\cdot)$ *is* d. *Then we have*

$$\int_{R_0}^{R} \Big(\int_{S^{n-1}} |\nabla_\tau\phi|^{n-\frac12} d\tau\Big)^{\frac{n}{n-\frac12}} r^{-1}dr \ge |d|^{\frac{n}{n-1}}(n-1)^{\frac{n}{2}}|S^{n-1}|^{\frac{2n}{2n-1}} \ln\frac{R}{R_0}.$$

Proof. By Hölder's inequality, we have

$$\int_{S^{n-1}} |\nabla_\tau\phi|^{n-1}\,d\tau \le \Big(\int_{S^{n-1}} |\nabla_\tau\phi|^{n-\frac12}\Big)^{\frac{n-1}{n-\frac12}}|S^{n-1}|^{\frac{1}{2n-1}}.$$

By Lemma 3.1, we have

$$\begin{aligned}\Big(\int_{S^{n-1}} |\nabla_\tau\phi|^{n-\frac12}\,d\tau\Big)^{\frac{n}{n-\frac12}} &\ge \Big(\int_{S^{n-1}} |\nabla_\tau\phi|^{n-1}\,d\tau\Big)^{\frac{n}{n-1}}|S^{n-1}|^{-\frac{n}{n-1}(\frac{1}{2n-1})}\\ &\ge \big(|d|(n-1)^{\frac{n-1}{2}}\big)^{\frac{n}{n-1}}|S^{n-1}|^{\frac{2n}{2n-1}}.\end{aligned}$$

The desired result is proved. □

Lemma 3.5. *Assume that* $\varepsilon \le R_0 < R \le L$. *Suppose that* $u : B_R(x_0)\backslash B_{R_0}(x_0) \to \mathbb{R}^n$ *with* $\frac12 \le |u| \le 1$ *and* $u \in H^{1,n}(B_R(x_0)\backslash B_{R_0}(x_0), \mathbb{R}^n)$. *Assume that there exists a constant* K *such that*

$$\frac{1}{\varepsilon^n}\int_{B_R(x_0)} (1-|u|^2)^2\,dx \le K(|\ln\varepsilon| + 1)$$

and

$$\frac{1}{\varepsilon^n}\int_{B_{\varepsilon^{1/2}}(x_0)}(1-|u|^2)^2\,dx \le K.$$

Then for any α *with* $0<\alpha<1$, *there exists a constant* $C(\alpha,K)$ (*independent of* ε) *such that*

$$\int_{R_0}^{R}\big[\int_{S^{n-1}}(1-|u|^2)^2\,d\tau\big]^{\alpha}r^{-1}\,dr \le C(\alpha,K).$$

Proof. Without loss of generality, we assume that $\varepsilon \le R_0 \le \varepsilon^{1/2} < R \le L$.

Choose p and q such that $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$ with $\frac{1}{p}+\frac{1}{q}=1$. By the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{R_0}^{R}\big(\int_{S^{n-1}}(1-|u|^2)^2 d\tau\big)^{\alpha}r^{-1}\,dr\\
&=\int_{\varepsilon^{1/2}}^{R}\big(\int_{S^{n-1}}(1-|u|^2)^2 r^{n-1}d\tau\big)^{\alpha}r^{-\alpha(n-1)-1}\,dr\\
&\quad+\int_{R_0}^{\varepsilon^{1/2}}\big(\int_{S^{n-1}}(1-|u|^2)^2 r^{n-1}d\tau\big)^{\alpha}r^{-\alpha(n-1)-1}\,dr\\
&\le\big[\int_{\varepsilon^{1/2}}^{R}\int_{S^{n-1}}(1-|u|^2)^2\,d\tau\,r^{n-1}\,dr\big]^{\frac{1}{p}}\big[\int_{\varepsilon^{1/2}}^{R}\int_{S^{n-1}}r^{-\frac{\alpha(n-1)-1}{1-\alpha}}d\tau\ dr\big]^{\frac{1}{q}}\\
&\quad+\big[\int_{R_0}^{\varepsilon^{1/2}}\int_{S^{n-1}}(1-|u|^2)^2\,d\tau\,R^{n-1}\,dr\big]^{\frac{1}{p}}\big[\int_{R_0}^{\varepsilon^{1/2}}\int_{S^{n-1}}r^{-\frac{\alpha(n-1)-1}{1-\alpha}}d\tau\,dr\big]^{\frac{1}{q}}\\
&\le\big[\frac{1}{\varepsilon^n}\int_{B_R(x_0)}(1-|u|^2)^2\,d\tau r^{n-1}\,dr\big]^{\frac{1}{p}}\big[\varepsilon^{\frac{nq}{p}}|S^{n-1}|\int_{\varepsilon^{1/2}}^{R}r^{-nq+n-1}d\tau\,dr\big]^{\frac{1}{q}}\\
&\quad+\big[\frac{1}{\varepsilon^n}\int_{B_{\varepsilon^{1/2}}(x_0)}(1-|u|^2)^2\,d\tau r^{n-1}\,dr\big]^{\frac{1}{p}}\big[\varepsilon^{\frac{nq}{p}}|S^{n-1}|\int_{R_0}^{\varepsilon^{1/2}}r^{-nq+n-1}\,dr\big]^{\frac{1}{q}}\\
&\le\big[K(|\ln\varepsilon|+1)\big]^{\frac{1}{p}}|S^{n-1}|^{\frac{1}{q}}\varepsilon^{\frac{n}{p}}\frac{1}{(nq-n)^{1/q}}\big[\varepsilon^{-\frac{(nq-n)}{2}}\big]^{1/q}\\
&\quad+K^{\frac{1}{p}}|S^{n-1}|^{\frac{1}{q}}\frac{1}{(nq-n)^{1/q}}\big[\varepsilon^{\frac{nq}{p}}\varepsilon^{n(1-q)}\big]^{\frac{1}{q}}\\
&=K^{1/p}|S^{n-1}|^{1/q}(nq-n)^{-\frac{1}{q}}\big[(|\ln\varepsilon|+1)^{1/p}\varepsilon^{\frac{n}{2p}}+1\big]\le C
\end{aligned}$$

for $\varepsilon\le\varepsilon_0$. □

Lemma 3.6 (Reverse Hölder inequality)**.** *Consider the functional*

$$\mathbb{A}(u,\Omega)=\int_{\Omega}A(x,u,\nabla u)\,dx,$$

where A is a measurable function satisfying the uniform growth condition:

$$\lambda^{-1}|z|^{n-\frac{1}{2}} - \mu \leq A(x,y,z) \leq \lambda |z|^{n-\frac{1}{2}} + \mu.$$

Let v be a minimizer for the functional $\mathbb{A}(u)$ in $H^{1,n-\frac{1}{2}}(\Omega; S^{n-1})$. Then for every $B_r(a) \subset \Omega$, there exists a $\beta > 0$ such that

$$\Big(\int_{B_{\frac{r}{2}}(a)} |\nabla v|^{(1+\beta)(n-\frac{1}{2})}\,dx\Big)^{\frac{1}{1+\beta}} \leq C(r)\Big(\int_{B_r(a)} |\nabla v|^{n-\frac{1}{2}}\,dx + 1\Big),$$

where $C(r)$ is a constant depending on r.

For the proof of Lemma 3.6, we refer to see Section 6 of [16], pages 314–317. The idea comes from Giaquinta's book, [11].

Assume that $u(x) = u(r\frac{x}{|x|}) = u(r,\tau)$ with $\frac{1}{2} \leq |u| \leq 1$. Denote

$$\mathbb{A}_r(\phi, S^{n-1}) = \int_{S^{n-1}} |u|^{n-1/2} |\nabla_\tau \phi|^{n-1/2}\,d\tau$$

and $V_d = \{\phi \in H^{1,n-\frac{1}{2}} \cap C^0(S^{n-1}, S^{n-1}) : \deg\phi = d\}$.

Lemma 3.7. *There exists a map $\phi_0 \in V_d$ such that*

$$\int_{S^{n-1}} |u|^{n-\frac{1}{2}} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\,d\tau = \min_{\phi\in V_d} \int_{S^{n-1}} |u|^{n-\frac{1}{2}} |\nabla_\tau \phi|^{n-\frac{1}{2}}\,d\tau.$$

Moreover, there exists $\beta > 0$ such that

$$\Big(\int_{S^{n-1}} |\nabla_\tau \phi_0|^{(1+\beta)(n-\frac{1}{2})}\,d\tau\Big)^{\frac{1}{1+\beta}} \leq C \int_{S^{n-1}} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\,d\tau,$$

where C is a constant.

Proof. The proof of existence is due to [7]. Let ϕ_k be a minimizing sequence in V_d. Then $\phi_k \rightharpoonup \phi_0$ in $H^{1,n-\frac{1}{2}}(S^{n-1}, S^{n-1})$. Moreover by the Sobolev imbedding theorem, ϕ_k converges uniformly to ϕ_0 in $C^{0,\gamma}$ for $\gamma \in (0,1)$.

Let τ_1 and τ_2 be two points on S^{n-1}. Let $|\tau_1 - \tau_2|_{S^{n-1}}$ be the distance between τ_1 and τ_2 on S^{n-1}. Let τ_0 be a point on S^{n-1} and denote

$$\tilde{B}^{n-1}_\rho(\tau_0) = \{\tau \in S^{n-1} : |\tau - \tau_0|_{S^{n-1}} \leq \rho\}.$$

Since ϕ_0 is Hölder continuous on S^{n-1}, there exists a $\rho > 0$ such that if $|\tau_1 - \tau_2|_{S^{n-1}}$ for τ_1 and τ_2 on S^{n-1} , then

$$\big|\phi_0(\tau_1) - \phi_0(\tau_2)\big|_{S^{n-1}} \leq \frac{1}{2}.$$

For $\tau_0 \in S^{n-1}$, denote

$$\mathbb{A}(\phi, \tilde{B}_\rho^{n-1}(\tau_0)) = \int_{\tilde{B}_\rho^{n-1}(\tau_0)} |u|^{n-\frac{1}{2}} |\nabla_\tau \phi|^{n-\frac{1}{2}}\, d\tau.$$

Let $\psi : \tilde{B}_\rho^{n-1}(\tau_0) \to S^{n-1}$ with $\psi|_{\partial \tilde{B}_\rho^{n-1}(\tau_0)} = \phi_0|_{\partial \tilde{B}_\rho^{n-1}(\tau_0)}$ and

$$\left|\psi(\tilde{B}_\rho^{n-1}(\tau_0)) - \phi_0(\tau_0)\right|_{S^{n-1}} \leq \frac{3}{4}.$$

Let

$$\tilde{\phi} = \begin{cases} \phi_0, & \text{for } \tau \in S^{n-1} \backslash \tilde{B}_\rho^{n-1}(x_0) \\ \psi, & \text{for } \tau \in \tilde{B}_\rho^{n-1}(x_0). \end{cases}$$

Then $\deg \tilde{\phi} = \deg \phi_0 = d$. Since ϕ_0 is minimizer of $\mathbb{A}(\phi, S^{n-1})$ on V_d,

$$\mathbb{A}(\phi_0, \tilde{B}_\rho^{n-1}(\tau_0)) \leq \mathbb{A}(\psi, \tilde{B}_\rho^{n-1}(\tau_0)).$$

Thus ϕ_0 is a local minimizer of $\mathbb{A}$ in $H^{1,n-\frac{1}{2}}(\tilde{B}_\rho^{n-1}(\tau_0); S^{n-1})$ with an obstacle $\mu = \{y \in S^{n-1} : |y - \phi_0(\tau_0)|_{S^{n-1}} \leq \frac{3}{4}\}$. Similarly to Lemma 3.6 (see [10]), we have the following reverse Hölder inequality:

$$\Big(\int_{\tilde{B}_{\frac{\rho}{2}}^{n-1}(y_0)} |\nabla_\tau \phi_0|^{(1+\beta)(n-\frac{1}{2})} d\tau\Big)^{\frac{1}{1+\beta}} \leq C \int_{\tilde{B}_\rho^{n-1}(y_0)} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\, d\tau + C.$$

Since S^{n-1} is a compact manifold without boundary, this proves Lemma 3.7. □

Lemma 3.8. *Let a, b be two constants with $a > 0$, $a + b \geq 0$. Then we have*

$$(a+b)^{\frac{1}{2n-1}} \geq a^{\frac{1}{2n-1}} - |b|^{\frac{1}{2n-1}}, \tag{3.2}$$

$$(a+b)^{\frac{2n}{2n-1}} \geq a^{\frac{2n}{2n-1}} - \sum_{i=0}^{2n-1} C_{2n}^i |a|^{\frac{i}{2n-1}} |b|^{\frac{2n-i}{2n-1}} \tag{3.3}$$

where $C_{2n}^0 = 1$ and $C_{2n}^i = \frac{2n(2n-1)\cdots(2n-i+1)}{i!}$ for $i = 1, \ldots, 2n-1$.

Proof. Since

$$[|b|^{\frac{1}{2n-1}} + (a+b)^{\frac{1}{2n-1}}]^{2n-1} \geq |b| + (a+b) \geq a,$$

then the first inequality (3.2) is proved. Note

$$(a+b)^{\frac{2n}{2n-1}} = \big[(a+b)^{2n}\big]^{\frac{1}{2n-1}} = \big[a^{2n} + \sum_{i=0}^{2n-1} C_{2n}^i a^i b^{2n-i}\big]^{\frac{1}{2n-1}}.$$

Then from the inequality (3.2) we have

$$(a+b)^{\frac{2n}{2n-1}} \geq a^{\frac{2n}{2n-1}} - \big|\sum_{i=0}^{2n-1} C_{2n}^i a^i b^{2n-i}\big|^{\frac{1}{2n-1}} \geq a^{\frac{2n}{2n-1}} - \sum_{i=0}^{2n-1} C_{2n}^i a^{\frac{i}{2n-1}} |b|^{\frac{2n-i}{2n-1}}.$$

(3.3) is proved. □

Theorem 3.9. *Let* $A_{R,R_0} = (B_R(x_0)\backslash B_{R_0}(x_0)) \cap \Omega$ *with* $\varepsilon \le R_0 < R \le L$. *Assume that* $u \in H^{1,n}_g(\Omega;\mathbb{R}^n)$ *and* $\frac{1}{2} \le |u| \le 1$ *on* A_{R,R_0}. *Assume that there exists a constant* K *such that*

$$\frac{1}{\varepsilon^n}\int_{A_{R,R_0}}(1-|u|^2)^2\,dx \le K(|\ln\varepsilon|+1) \quad \textit{and} \quad \frac{1}{\varepsilon^n}\int_{B_{\varepsilon^{1/2}}(x_0)}(1-|u|^2)^2\,dx \le K.$$

Then for $\varepsilon \le \varepsilon_0$ *there holds*

$$\int_{A_{R,R_0}}|\nabla u|^n\,dx \ge |d|^{\frac{n}{n-1}}(n-1)^{\frac{n}{2}}|S^{n-1}|\ln\frac{R}{R_0} - C(K,d,g),$$

where $C(K,d,g)$ *is a constant* (*independent of* ε) *and* d *is the degree of* u *on each* $\partial(B_r(x_0)\cap\Omega)$, $R_0 \le r \le R$.

Proof. As in [25] or in [4], we assume that

$$A_{R,R_0} = B_R(x_0)\backslash B_{R_0}(x_0) \subset \Omega.$$

Denote

$$\phi(r,\tau) := \frac{u(x)}{|u(x)|} = \frac{u(r\frac{x}{|x|})}{|u(r\frac{x}{|x|})|} = \frac{u(r\tau)}{|u(r\tau)|}, \quad r = |x|, \tau = \frac{x}{|x|}.$$

Then $\phi(r,\tau) : S^{n-1} \to S^{n-1}$ with $\deg\phi(r,\cdot) = d$ for each r with $R_0 \le r \le R$.

Since $\frac{1}{2} \le |u| \le 1$ on A_{R,R_0}, we have

$$|\nabla u|^2 = |\nabla|u||^2 + |u|^2\big|\nabla\frac{u}{|u|}\big|^2.$$

Then

$$|\nabla u|^2 \ge |u|^2 r^{-2}|\nabla_\tau\phi(r,\cdot)|^2.$$

By the Hölder inequality, we obtain

$$\int_{S^{n-1}}|u|^{n-\frac{1}{2}}|\nabla_\tau\phi|^{n-\frac{1}{2}}\,d\tau \le |S^{n-1}|^{\frac{1}{2n}}\Big(\int_{S^{n-1}}|u|^n|\nabla_\tau\phi|^n\Big)^{\frac{n-\frac{1}{2}}{n}}.$$

Therefore

$$\begin{aligned}\int_{A_{R,R_0}}|\nabla u|^n\,dx &\ge \int_{R_0}^R\int_{S^{n-1}}|u|^n|\nabla_\tau\phi|^n d\tau\, r^{-1}\,dr \\ &\ge \int_{R_0}^R|S^{n-1}|^{-\frac{1}{2n-1}}\Big(\int_{S^{n-1}}|u|^{n-\frac{1}{2}}|\nabla_\tau\phi|^{n-\frac{1}{2}}\Big)^{\frac{n}{n-\frac{1}{2}}} r^{-1}\,dr.\end{aligned}$$

Let $\phi_0 = \phi_0(r)$ be a minimizer of $\mathbb{A}_r$ on V_d. Set

$$a = \int_{S^{n-1}}|\nabla_\tau\phi_0|^{n-\frac{1}{2}}\,d\tau, \quad b = \int_{S^{n-1}}(1-|u|^{n-\frac{1}{2}})|\nabla_\tau\phi_0|^{n-\frac{1}{2}}\,d\tau.$$

By Lemma 3.7, we have

$$\begin{aligned}
&\int_{A_{R,R_0}} |\nabla u|^n\,dx \geq \int_{R_0}^{R} |S^{n-1}|^{-\frac{1}{2n-1}} \Big(\int_{S^{n-1}} |u|^{n-\frac{1}{2}} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\Big)^{\frac{n}{n-\frac{1}{2}}} r^{-1}\,dr \\
&= \int_{R_0}^{R} |S^{n-1}|^{-\frac{1}{2n-1}} (a+b)^{\frac{n}{n-\frac{1}{2}}} r^{-1}\,dr \\
&\geq \int_{R_0}^{R} |S^{n-1}|^{-\frac{1}{2n-1}} a^{\frac{n}{n-\frac{1}{2}}} r^{-1}\,dr - \sum_{i=0}^{2n-1} C_{2n}^{i} \int_{R_0}^{R} |S^{n-1}|^{-\frac{1}{2n-1}} a^{\frac{i}{2n-1}} b^{\frac{2n-i}{2n-1}} r^{-1}\,dr \\
&:= I_1 - I_2.
\end{aligned}$$

By Lemma 3.4, we have

$$I_1 \geq |d|^{\frac{n}{n-1}} (n-1)^{\frac{n}{2}} |S^{n-1}| \ln \frac{R}{R_0}.$$

Since $\frac{1}{2} \leq |u| \leq 1$, then $1 - |u|^{n-\frac{1}{2}} \leq C(1-|u|^2)$. By Lemma 3.7 and the Hölder inequality

$$\begin{aligned}
b &\leq \Big(\int_{S^{n-1}} (1-|u|^2)^{\frac{1+\beta}{\beta}}\,dx\Big)^{\frac{\beta}{1+\beta}} \Big(\int_{S^{n-1}} |\nabla_\tau \phi_0|^{(1+\beta)(n-\frac{1}{2})}\,d\tau\Big)^{\frac{1}{1+\beta}} \\
&\leq C\Big(\int_{S^{n-1}} (1-|u|^2)^2\,dx\Big)^{\frac{\beta}{1+\beta}} \int_{S^{n-1}} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\,d\tau.
\end{aligned}$$

There exists a constant C such that

$$a = \int_{S^{n-1}} |\nabla_\tau \phi_0|^{n-\frac{1}{2}}\,d\tau \leq 2^{n-\frac{1}{2}} \min_{\phi \in V_d} \mathbb{A}(\phi, S^{n-1}) \leq C.$$

By Lemma 3.5 $I_2 \leq C(K,d)$. This proves Theorem 3.9. □

Now consider the cover $(B_{\varepsilon/5}(x))_{x\in\Sigma_\varepsilon}$ of Σ_ε. Again by Vitali's covering lemma we can find a disjoint collection of balls $B_{\varepsilon/5}(x_j)$, $x_j \in \Sigma_\varepsilon$, $1 \leq j \leq J$ such that

$$\Sigma_\varepsilon \subset \bigcup_j B_\varepsilon(x_j).$$

By Lemma 3.2 we have $J \leq J_0$ independent of ε.

For each $\varepsilon > 0$ and any corresponding minimizer u_ε we fix this choice of (x_j). Given $\sigma > 0$ we denote

$$\Omega^\sigma = \Omega^\sigma_\varepsilon = \Omega \backslash \bigcup_j B_\sigma(x_j).$$

Set $G^\sigma_\varepsilon = \bigcup_{j=1}^J B_\sigma(x_j) \backslash \bigcup_{j=1}^J B_\varepsilon(x_j)$.

Theorem 3.10. *There exists a constant $C_{11} > 0$ such that for any $\sigma > 0$*

$$E_\varepsilon(u_\varepsilon; \Omega^\sigma) \le (n-1)^{n/2}|S^{n-1}||d||\ln\sigma| + C_{11}$$

uniformly in $0 < \varepsilon < \varepsilon_0$.

Proof. We give the proof following that in [25]. Fix a point x_j, $j \in \{1, \dots, J\}$. We suppose $x_j = 0$. For $R < R_1$ denote by $d_{j,R}$ the topological degree of the map $u : \partial(\Omega \cap B_R(0)) \cong S^{n-1} \to S^{n-1}$. Let R_ε^σ denote the set of all numbers $R \in [\varepsilon, \sigma]$ such that $\partial B_R(x_j) \bigcap B_\varepsilon(x_{j'}) = \emptyset$ for all $j \ne j'$ and such that for some collection $J_R \subset \{1, \dots, J\}$, satisfying $J_R \subset J_{R'}$ if $R' \le R$, the family $\{B_R(x_j)\}_{j \in J_R}$ is disjoint and

$$\bigcup_{j \in J} B_\varepsilon(x_j) \subset \bigcup_{j \in J_{R'}} B_{R'}(x_j) \subset \bigcup_{j \in J_R} B_R(x_j), \quad \text{if } R' \le R.$$

Note that R_ε^σ is the union of closed intervals $[R_0^{(l)}, R^{(l)}]$, $1 \le l \le L$, whose right endpoints correspond to a number $R = R^{(l)}$ such that

$$\partial B_R(x_j) \bigcap \overline{B_R(x_{j'})} \ne \emptyset$$

for some pair $j \ne j' \in J_R$ and whose left endpoints correspond to a number $R_0^{(l)}$ such that $\overline{B_{R^{(l-1)}}(x_{j'})} \setminus \bigcup_{j \in J_0} B_{R_0^{(l)}}(x_j) \ne \emptyset$ for $j' \notin J_{R_0^{(l)}}$. $J_R = J^{(l)}$ is a constant for $R \in [R_0^{(l)}, R^{(l)}]$ and $J^{(l+1)} \subset J^{(l)}$, $J^{(l+1)} \ne J^{(l)}$. Thus $L \le J \le J_0 = L_0(\Omega, g)$, independently of ε. Moreover, there exists a constant $M = M(\Omega, g) > 0$ such that

$$R_0^{(1)} \le M\varepsilon, \quad R^{(L)} \ge \frac{\sigma}{M} \quad \text{and } R_0^{(l+1)} \le MR^{(l)} \tag{3.4}$$

for all $l = 1, \dots, L-1$. Finally, observe that for all $R \in R_\varepsilon^\sigma$ and $J \in J_R$

$$|d| = |\sum_{j \in J_R} d_{i,R}| \le \sum_{j \in J_R} |d_{i,R}|. \tag{3.5}$$

Applying (3.4), (3.5), Lemma 2.1, Lemma 3.1 and Theorem 3.9 we have

$$\begin{aligned}
\int_{G_\varepsilon^\sigma} |\nabla u_\varepsilon|^n \, dx &\ge \sum_{l=1}^{L} \sum_{j \in J^{(l)}} \int_{A_{R^{(l)}, R_0^{(l)}}(x_j)} |\nabla u_\varepsilon|^n \, dx \\
&\ge \sum_l \sum_j |S^{n-1}|(n-1)^{n/2}|d_{j,R^{(l)}}| \ln(R^{(l)}/R_0^{(l)}) - C(K, g) \\
&\ge |S^{n-1}|(n-1)^{n/2}|d| \sum_l (\ln R^{(l)} - \ln R_0^{(l)}) - C(K, g) \\
&\ge |S^{n-1}|(n-1)^{n/2}|d| \ln(\frac{\sigma}{\varepsilon}) - C(K, g, \Omega).
\end{aligned} \tag{3.6}$$

This proves Theorem 3.10. □

From the Theorem 2.1 of [8], we have

Lemma 3.11. *Let Ω be an open domain in $\mathbb{R}^n$. For each $k \in \mathbb{N}$, let $u^k \in H^{1,p}(\Omega, \mathbb{R}^n)$ be a solution to the following equation:*

$$-\nabla \cdot (|\nabla u_k|^{p-2} \nabla u_k) = F_k,$$

where $p \geq 2$. If $\int_\Omega |\nabla u_k|^p \, dx \leq C$, $|u_k| \leq 1$ and $\|F_k\|_{L^1(\Omega)} \leq C$, then $u_k \rightharpoonup u$ weakly in $H^{1,p}(\Omega, \mathbb{R}^n)$ and $\nabla u_k \to \nabla u$ strongly in $L^q_{loc}(\Omega, \mathbb{R}^n)$ for all $q < p$.

Proof of Theorem 1.2. Consider any subsequence of minimizers $u_k = u_{\varepsilon_k}$ where $\varepsilon_k \to 0$ as $k \to \infty$. Let $(x_{j,k})$, $1 \leq j \leq J_k$, denote the corresponding centers of the "bad" balls. Note that $J_k \leq J_0$. Passing to a subsequence, if necessary, we assume that $J_k = J$ independent of k and $x_{j,k} \to x_j$ as $k \to \infty$ for each $j = 1, \ldots, J$.

For $\sigma > 0$ let $\Omega^\sigma = \Omega \backslash \cup_j B_\sigma(x_j)$. For any $\sigma > 0$ and $k \leq k_0(\sigma)$, by Theorem 3.10 we have

$$\frac{1}{2} \int_{\Omega^\sigma} |\nabla u_k|^2 \, dx \leq E_{\varepsilon_k}(u_k, \Omega^\sigma) \leq C_{12} |\ln \sigma| + C_{13}.$$

Choosing $\sigma = \sigma_k \to 0$ and passing to a further subsequence, we obtain that $u_k \to u$ weakly locally in $H^{1,n}_{\text{loc}}(\Omega \backslash \{x_1, \ldots, x_J\}; \mathbb{R}^n)$. Since u_k minimizes E_k, we have

$$-\nabla \cdot (|\nabla u_k|^{n-2} \nabla u_k) = \frac{1}{\varepsilon_k} (1 - |u_k|^2) u_k.$$

Let $\mathbb{K}_1$ be a compact subdomain of $\Omega \backslash \{x_1, \ldots, x_J\}$. There exists another compact subdomain $\mathbb{K}_2$ such that $\mathbb{K}_1 \subset\subset \mathbb{K}_2 \subset\subset \Omega \backslash \{x_1, \ldots, x_J\}$. Choose σ small enough such that $\mathbb{K}_2 \subset \Omega^\sigma$. Using $u\phi$ as a test function, we have

$$\frac{1}{(\varepsilon_k)^n} \int_\Omega (1 - |u_k|^2) |u_k|^2 \phi \, dx = \int_\Omega |\nabla u_k|^n \phi \, dx + \int_\Omega |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla \phi \, dx,$$

where ϕ is a smooth function on Ω, $\phi \equiv 1$ on $\mathbb{K}_1$ and $\phi \equiv 0$ outside $\Omega \backslash \mathbb{K}_2$. By Theorem 3.10 we have

$$\frac{1}{(\varepsilon_k)^n} \int_{\mathbb{K}_1} (1 - |u_k|^2) |u_k|^2 \, dx \leq C. \tag{3.7}$$

Since $|u_k| \geq \frac{1}{2}$ on $\mathbb{K}_1$, we have

$$\frac{1}{(\varepsilon_k)^n} \int_{\mathbb{K}_1} (1 - |u_k|^2) \, dx \leq C, \tag{3.8}$$

where C is a constant independent of ε_k. Setting $F_k = \frac{1}{\varepsilon_k^n} (1 - |u_k|^2) u_k$ and $p = n$ in Lemma 3.11, we have

$$|\nabla u_k|^{n-2} \nabla u_k \rightharpoonup |\nabla u|^{n-2} \nabla u.$$

Therefore,

$$-\nabla \cdot (|\nabla u|^{n-2} \nabla u) \wedge u = - \lim_{k \to \infty} \nabla \cdot (|\nabla u_k|^{n-2} \nabla u_k) \wedge u_k = 0.$$

Hence u is a weak n-harmonic map in $\mathbb{K}_1$ (see [8]). Since $\mathbb{K}_1$ is any compact subdomain of $\Omega\backslash\{x_1, \dots, x_J\}$, u_{ε_k} converges to u weakly in $H^{1,n}_{\text{loc}}(\Omega\backslash\{x_1, \dots, x_J\}; \mathbb{R}^n)$. Moreover, $u_{\varepsilon_k} \rightharpoonup u$ in $H^{1,p}(\Omega; \mathbb{R}^n)$ for $p < n$ following [25]. □

Added in proof. In a recent paper "Degenerate elliptic systems and applications to Ginzburg-Landau type equations, Part one," Z.C. Han and Y.Y. Li have independently obtained that Theorem 1.2 holds for any sequence of minimizers by a different method.

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