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POSITIVE PERIODIC SOLUTIONS FOR SEMILINEAR REACTION DIFFUSION SYSTEMS ON $\mathbb{R}^N$

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Abstract. The existence and nonexistence of positive time-periodic solutions of semilinear reaction diffusion systems of the type

$$\begin{cases} \partial_t u + \mathcal{A}_1 u = au - bg_1(u)u - h_1(u,v)u \\ \partial_t v + \mathcal{A}_2 v = dv - fg_2(v)v + h_2(u)v \end{cases} \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

is discussed. Here the coefficients a, b, d, f, and the coupling terms g_1, g_2, h_1, h_2 depend on space and time and satisfy suitable conditions. The differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are elliptic with space- and time-dependent coefficients. In particular, the time dependence is always assumed to be periodic with a fixed common period. A topological method, based on the Leray-Schauder degree, is used to obtain the existence of positive time-periodic solutions. Finally, we apply our results to various population models.

1. Introduction. This paper is devoted to the study of the following semilinear reaction diffusion system on $\mathbb{R}^N$:

$$\begin{cases} \partial_t u + \mathcal{A}_1 u = au - bg_1(u)u - h_1(u,v)u \\ \partial_t v + \mathcal{A}_2 v = dv - fg_2(v)v + h_2(u)v \end{cases} \text{in } \quad \mathbb{R}^N \times (0,\infty). \tag{1}$$

Here the coefficients a, b, d, f the nonlinearities g_1, g_2, h_1, h_2 and the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$ depend on space and time. In particular the time dependence is assumed to be periodic with a fixed period $T > 0$. The differential operators $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ are uniformly elliptic for each $t \in [0,T]$. Furthermore, the functions h_1, g_1, g_2 are nonnegative, whereas h_2 may change sign.

The main point of interest will be the existence of positive T-periodic solutions, whose components u and v are both not identically zero in $\mathbb{R}^N \times [0,T]$. We call such solutions T-periodic coexistence solutions or briefly coexistence solutions. Moreover, we will prove extinction results; i.e., we give conditions that guarantee that coexistence solutions do not exist and that solutions of the system (1) to positive initial values will converge to the trivial solution or to a periodic solution of the form $(u,0)$ or $(0,v)$ as time tends to ∞.

For bounded domains and special nonlinearities h_1 and h_2 coexistence and extinction problems were recently studied in numerous research articles (see e.g. [15], [21], [16],

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[10], [6]). The general strategy is the following: In a first step the existence of coexistence solutions is reduced to finding positive fixed points of the period map or time-T map associated with the system. In a second step, various techniques from nonlinear functional analysis are applied to this problem. For bounded domains a detailed description of this program is given in [15] for a competition model and a predator-prey model from populations dynamics. Moreover, in [21] a topological method is used to obtain a unified approach for different population models.

We will extend the results known for bounded domains and special nonlinearities to the problem in $\mathbb{R}^N$ and to a more general class of nonlinearities. For all the techniques mentioned above the fact that the period map is a compact self-map of the positive cone of an ordered Banach space is an instrumental property. For the problem in $\mathbb{R}^N$, however, this compactness is lost. Moreover, we will not impose the usual sign restrictions on $\partial_v h_1$ and $\partial_u h_2$, which make (1) quasimonotonous and allow the application of iteration techniques.

The loss of compactness described above, in particular impedes the adaptation of the bifurcation method used by Brown and Hess in [6], as well as the abstract theory of Dancer (cf. [9]), applied by López-Gómez in [21], to our problem. This forces us to look for a different approach: In order to find coexistence solutions of (1) we will show that the period map is a compact self-map of a certain invariant retract of the positive cone of an ordered Banach space. This allows one to define a fixed-point index related to the well-known Leray-Schauder degree. We then compute the local fixed-point index of the trivial fixed point $(0, 0)$ and of certain fixed points of the form $(u, 0)$ or $(0, v)$, so-called semitrivial fixed points. This is done by constructing a homotopy between the period map associated with (1) and the period map of a decoupled system. The index of fixed points of the period map, which is associated with the decoupled system, can then be computed. Finally, under certain conditions on the stability of the semitrivial solutions, a so called "index-counting" argument will yield the existence of a coexistence solution. The decoupling technique described turns out to be quite flexible allowing for rather general coupling terms h_1 and h_2.

To apply the decoupling technique we mentioned before, a detailed knowledge of the decoupled system is imperative. For our class of problems the decoupled system consists of two equations describing diffusive logistic growth on $\mathbb{R}^N$. This phenomenon is extensively studied in [18] and these results play an essential rôle in our analysis.

Our main coexistence result is Theorem 9.7 and our main extinction result is Theorem 10.1. In Section 11 we apply our main results to different population models from mathematical biology.

Finally, we remark that our results cover the corresponding elliptic problem and that our decoupling technique also applies to systems on bounded domains.

2. The abstract evolution problem. In this section we formulate equation (1) as an abstract evolution problem in the Banach lattice $X := C_0(\mathbb{R}^N)$, the space of continuous functions on $\mathbb{R}^N$ that vanish at infinity. The positive cone of pointwise nonnegative functions in X will be denoted by X^+. We note that the interior of X^+ is empty, and that the *quasi-interior* of X^+ is given by

$$qint(X^+) := \{u \in X^+ : u(x) > 0,\ x \in \mathbb{R}^N\}.$$

For $u, v \in X^+$ we write $u \gg v$ if $u - v \in qint(X^+)$, $u > v$ if $u - v \in X^+ \setminus \{0\}$ and $u \geq v$ if $u - v \in X^+$. Furthermore, we set $\dot{X}^+ := X^+ \setminus \{0\}$ and $\mathbf{X}^+ := X^+ \times X^+$. Throughout this paper we fix $T > 0$ and $\mu \in (0,1)$ and make the following assumptions:

For $k = 1, 2$ the operators $\mathcal{A}_k$ are second-order uniformly elliptic operators of the form

(A1) $$\mathcal{A}_k(x,t) = -\sum_{i,j=1}^{N} a_{ij}^k(x,t)\partial_i\partial_j + \sum_{i=1}^{N} a_i^k(x,t)\partial_i \ ,$$

where $a_{ij}^k \in BUC^{1+\mu,\frac{\mu}{2}}(\mathbb{R}^N \times \mathbb{R})$, $a_i \in BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^N \times \mathbb{R})$, T-periodic in the second variable for all $i, j \in \{1, \dots, N\}$.

(A2) $a, b, d, f \in BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^N \times \mathbb{R})$, T-periodic in the second variable.

(A3) For $R > 0$ we have $h_2, g_i \in BUC^{\mu,\frac{\mu}{2},2}(\mathbb{R}^N \times \mathbb{R} \times [0,R])$ and $h_1 \in BUC^{\mu,\frac{\mu}{2},2,2}(\mathbb{R}^N \times \mathbb{R} \times [0,R]^2)$, T-periodic in the second variable.

For $k = 1, 2$ and each $t \in [0,T]$ we denote the X-realization of $\mathcal{A}_k(t)$ by $A_k(t)$. It is shown in [27] that for each $t \in [0,T]$ the operator $-A_k(t)$ generates an analytic C_0-semigroup on X. Moreover, it is shown in [17] that also the $BUC(\mathbb{R}^N)$-realization $-A_k^{BUC}(t)$ generates an analytic C_0-semigroup on $BUC(\mathbb{R}^N)$. We note that in general the domains of $A_k(t)$ depend on t. As in [17] and [18] we restrict ourselves to time-independent domains; i.e., we assume

(A4) For $k = 1, 2$ the domains $D(A_k^{BUC}(t))$ and $D(A_k(t))$ are independent of $t \in \mathbb{R}$.

In particular assumption (A4) is satisfied whenever $N = 1$ or $\mathcal{A}_k(x,t)$ is of the form

$$\mathcal{A}_k(x,t) = a_k(t)\mathcal{A}_k(x), \quad k = 1, 2,$$

for T-periodic $a_k \in BUC^{\frac{\mu}{2}}(\mathbb{R})$. To treat system (1), for each $t \in [0,T]$ we set

$$\mathbf{A}(t) := \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{X} := X \times X \ ,$$

with domain $D(\mathbf{A}) := D(A_1) \times D(A_2)$. Of course, $-\mathbf{A}(t)$ generates a C_0-semigroup on $\mathbf{X}$ for each $t \in [0,T]$. Assumption (A4) allows us to use the classical theory of evolution equations due to Sobolevskii and Tanabe (cf. e.g. [2] or [17]). Hence, we infer the existence of a unique evolution operator

$$\mathbf{U} : \triangle_T \to \mathcal{L}(\mathbf{X}) \ ; \quad \triangle_T := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\ \} \ ,$$

associated with the family $\{\mathbf{A}(t) : t \in [0,T]\}$. Assumption (A4) also implies the existence of an evolution system in the BUC-setting. Although we will work in the C_0-setting, it will be necessary to use the evolution system on BUC in the proof of Proposition 7.2.

To deal with the nonlinearities in (1) we introduce the following superposition operator $\mathbf{F} : [0,T] \times \mathbf{X} \longrightarrow \mathbf{X}$:

$$\mathbf{F}(t,\mathbf{w})(x) := \begin{pmatrix} a(x,t)u(x,t) - b(x,t)g_1(x,t,u(x,t))u(x,t) - h_1(x,t,u(x,t),v(x,t))u(x,t) \\ \\ d(x,t)u(x,t) - f(x,t)g_2(x,t,v(x,t))v(x,t) + h_2(x,t,u(x,t))v(x,t) \end{pmatrix},$$

where $\mathbf{w} := (u, v)$. It is shown (e.g. in [11], Section 15) that the superposition operator F satisfies

$$F \in C^{\frac{\mu}{2},2}([0,T] \times \mathbf{X}, \mathbf{X}) \ . \tag{2}$$

For $\mathbf{w}_0 \in \mathbf{X}$ we consider the following initial value problem in $\mathbf{X}$:

$$\dot{\mathbf{w}} + \mathbf{A}(t)\mathbf{w} = \mathbf{F}(t, \mathbf{w}), \quad \mathbf{w}(0) = \mathbf{w}_0, \quad t \in (0, T]. \tag{3}$$

By a local solution of (3) we mean a function

$$\mathbf{w} \in C([0, \varepsilon), \mathbf{X}) \cap C^1((0, \varepsilon), \mathbf{X}) \quad \text{for some} \ \ \varepsilon > 0,$$

which satisfies (3) for $t \in (0, \varepsilon)$. We state the following existence and regularity result (see e.g. [11] Section 16).

Theorem 2.1. *The following assertions hold.*

a) *For each* $\mathbf{w}_0 \in \mathbf{X}$ *there exists a unique maximal solution* $\mathbf{w}(\cdot, \mathbf{w}_0)$ *of* (3)*. By* $J(\mathbf{w}_0)$ *we denote the maximal interval of existence.* $J(\mathbf{w}_0)$ *is either the interval* $[0, T]$ *or* $[0, t^+(\mathbf{w}_0))$*, where* $t^+(\mathbf{w}_0) \in [0, T]$ *is called the positive escape time of* $\mathbf{w}(\cdot, \mathbf{w}_0)$.

b) *If* $\mathbf{w}(\cdot, \mathbf{w}_0)$ *satisfies the a priori estimate*

$$\|\mathbf{w}(t, \mathbf{w}_0)\|_{\mathbf{X}} \leq C \ , \quad t \in [0, t^+(\mathbf{w}_0)),$$

then $J(\mathbf{w}_0) = [0, T]$*; i.e.,* $\mathbf{w}(\cdot, \mathbf{w}_0)$ *is a global solution of* (3).

c) *Each global solution of* (3) *is a classical solution of* (1)*; i.e.,*

$$u, v \in BUC(\mathbb{R}^N \times [0, T]) \cap BUC^{2+\mu, 1+\frac{\mu}{2}}(\mathbb{R}^N \times (\varepsilon, T)) \quad \textit{for } \varepsilon \in (0, T).$$

We conclude this section by introducing the shift map or period map, which will be of crucial importance.

Definition 2.2. On the subset of $\mathbf{X}$: $D(\mathbf{S}) := \{\mathbf{x} \in \mathbf{X} : t^+(\mathbf{x}) = T\}$ we define the period map, or time-T-map associated with (3) by

$$\mathbf{S} : D(\mathbf{S}) \longrightarrow \mathbf{X}, \quad \mathbf{S}(\mathbf{x}) := \mathbf{w}(T, \mathbf{x}).$$

Of course, there is a one-to-one correspondence between fixed points of $\mathbf{S}$ and T-periodic solutions of (3).

3. The stability of T-periodic solutions. In this section we introduce the stability concepts for T-periodic solutions of (3) that we will use in the subsequent sections. Moreover, we recall the principle of linearized stability. The *resolvent set* and the *spectrum* of a linear bounded operator T on a Banach space will be denoted by $\rho(T)$ and $\sigma(T)$, respectively. Furthermore, we write $r(T)$ and $r_{ess}(T)$ for the *spectral radius* and the *radius of the essential spectrum* (see [25]).

Definition 3.1. Let $\mathbf{v} \in C^1(\mathbb{R}, \mathbf{X})$ be a T-periodic solution of (3) and set $\mathbf{v}_0 := \mathbf{v}(0)$. Let $\mathcal{M}$ be a subset of $\mathbf{X}$. We introduce the following stability concepts:

a) $\mathbf{v}$ is (*Lyapunov-*) *stable with respect to* $\mathcal{M}$ if to every $\varepsilon > 0$ there exists a $\delta > 0$ such that $t^+(\mathbf{w}_0) = \infty$ and $\|\mathbf{w}(t; \mathbf{w}_0) - \mathbf{v}(t)\|_{\mathbf{X}} < \varepsilon$ for $t > 0$, whenever $\mathbf{w}_0 \in \mathbb{B}_{\mathbf{X}}(\mathbf{v}_0, \delta) \cap \mathcal{M}$.

b) $\mathbf{v}$ is *asymptotically stable with respect to* $\mathcal{M}$ if $\mathbf{v}$ is stable and there exists a $\delta > 0$ such that $\|\mathbf{w}(t; \mathbf{w}_0) - \mathbf{v}(t)\|_{\mathbf{X}} \to 0$ as $t \to \infty$, whenever $\mathbf{w}_0 \in \mathbb{B}_{\mathbf{X}}(\mathbf{v}_0, \delta) \cap \mathcal{M}$. If $\|\mathbf{w}(t; \mathbf{w}_0) - \mathbf{v}(t)\|_{\mathbf{X}} \to 0$ as $t \to \infty$ for $\mathbf{w}_0 \in \mathcal{M}$, then $\mathbf{v}$ is *globally* asymptotically stable with respect to $\mathcal{M}$.

c) $\mathbf{v}$ is *exponentially asymptotically stable with respect to* $\mathcal{M}$ if $\mathbf{v}$ is stable and there exist constants $\delta > 0$, $M > 0$ and $\omega > 0$, such that $\|\mathbf{w}(t; \mathbf{w}_0) - \mathbf{v}(t)\|_{\mathbf{X}} \leq M e^{-\omega t}$ for $t > 0$, whenever $\mathbf{w}_0 \in \mathbb{B}_{\mathbf{X}}(\mathbf{v}_0, \delta) \cap \mathcal{M}$.

d) Finally, $\mathbf{v}$ is *unstable with respect to* $\mathcal{M}$ if $\mathbf{v}$ is not stable with respect to $\mathcal{M}$.

By (2) the superposition operator F is differentiable with respect to the second variable. Thus $D_2F(t_0, \mathbf{v}_0) \in \mathcal{L}(\mathbf{X})$ for $(t_0, \mathbf{v}_0) \in [0, T] \times \mathbf{X}$ and we may consider the linearization of (3) at a T-periodic solution $\mathbf{v}$:

$$\dot{\mathbf{w}} + \mathbf{A}(t)\mathbf{w} = D_2\mathbf{F}(t, \mathbf{v}(t))\mathbf{w}, \quad t > 0. \tag{4}$$

With (4) is associated an evolution operator $\mathbf{W}(t, s)$ with its period map $\mathbf{K} := \mathbf{W}(T, 0)$. The following proposition follows easily from the differentiable dependence of the solution of (3) on the initial value. A complete proof can be found in [11], Proposition 22.1.

Definition 3.2. Let $\mathbf{v}$ be a T-periodic solution of (3). Then $\mathbf{v}(0)$ is a fixed point of the period map $\mathbf{S}$ of (3) and the derivative of $\mathbf{S}$ at $\mathbf{v}(0)$ is given by $\mathbf{K}$; i.e., $D\mathbf{S}(\mathbf{v}(0)) = \mathbf{K}$.

Definition 3.3. A T-periodic solution $\mathbf{v}$ of (3) is called *linearly stable* if $r(\mathbf{K}) < 1$, *neutrally stable* if $r(\mathbf{K}) = 1$, and *linearly unstable* if $r(\mathbf{K}) > 1$.

The proof of the next theorem can be found in [11, Theorems 22.2 and 22.3].

Theorem 3.4 (Principle of linearized stability)**.**

a) *Assume that* $\sigma(\mathbf{K})$ *lies within the open complex unit disk. Then the* T*-periodic solution* $\mathbf{v}$ *of* (3) *is exponentially stable with respect to* $\mathbf{X}$.

b) *Assume that* $\sigma(\mathbf{K}) \cap \{\mu \in \mathbb{C} : |\mu| > 1\}$ *is a nonempty spectral set of* $\mathbf{K}$. *Then the* T*-periodic solution* $\mathbf{v}$ *of* (3) *is unstable with respect to* $\mathbf{X}$.

Remark 3.5. If $\mathbf{K}$ is compact, then the instability of a T-periodic solution of (1) may thus be established by finding an eigenvalue of $\mathbf{K}$ with modulus strictly greater than 1, whereas in the noncompact case one has to make sure that such an eigenvalue is isolated in order to obtain a spectral set that will allow a spectral decomposition by means of the Dunford spectral calculus.

4. The periodic parabolic eigenvalue problem. Periodic parabolic eigenvalues play an important rôle in the stability analysis of periodic solutions of periodic parabolic equations. The notion of periodic parabolic eigenvalues was introduced by Lazer in [20] and then extensively studied by Beltramo and Hess in [4], [5] and [15]. The corresponding notions and results for the problem on $\mathbb{R}^N$ were studied in [12], [13] and [17]. In this section we introduce the notation and the results that will be relevant for us in the sequel. We study the following problem:

$$\begin{cases} \partial_t\varphi + \mathcal{A}(x,t)\varphi - m(x,t)\varphi = \mu\varphi & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ \varphi(\cdot,t) \in X^+ & \text{for } t \in \mathbb{R}, \\ \varphi \text{ is } T\text{-periodic in t.} \end{cases} \tag{5}$$

Here $\mathcal{A}$ and m satisfy the conditions specified in $(A1)$, $(A4)$ and $(A2)$, respectively. A real number μ is said to be an eigenvalue of (5) if there exists a nontrivial classical solution φ of (5). Eigenvalues admitting positive eigenfunctions are called *principal eigenvalues*. Problem (5) is closely related to the stability properties of the zero solution of the following linear equation:

$$\partial_t u + \mathcal{A}(x,t)u = m(x,t)u \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{6}$$

As in Section 2 we can reformulate (6) as an abstract evolution equation in X:

$$\dot{u} + A(t)u = M(t)u \quad \text{for } t > 0. \tag{7}$$

Here for each $t \in [0,T]$ we write $A(t)$ for the X-realization of $\mathcal{A}(t)$ and $M(t) \in \mathcal{L}(X)$ for the multiplication operator induced by $m(\cdot,t) \in BUC(\mathbb{R}^N)$.

Let $U_m : \triangle_T \to \mathcal{L}(X)$ denote the evolution operator associated with (7). Then we set

$$S_m := U_m(T,0) \in \mathcal{L}(X);$$

i.e., S_m is the period map associated with (7).

For any real-valued function f we denote by f^+ and f^- its positive and negative part. Following Arendt and Batty ([3]; see also [13]) we define the class $\mathcal{E}$ consisting of all $f \in BUC(\mathbb{R}^N)$ for which $\int_G f(x)\,dx = \infty$ for any subset $G \subset \mathbb{R}^N$ containing arbitrarily large balls. For the rest of this section we make the following additional assumption on m:

$$(A5) \qquad [t \mapsto m^+(t,\cdot)] \in C(\mathbb{R},X) \quad \text{and} \quad \int_0^T m^-(t,\cdot)\,dt \in \mathcal{E}.$$

For later reference we list some properties of the periodic parabolic eigenvalue problem in the next proposition. For proofs in the case that $\mathcal{A}(t) = -\triangle$ we refer to [13, Section 6]. By the results in [17] it is straightforward to see that the assertions remain true in the present generality.

Proposition 4.1. a) *There is a one-to-one correspondence between the positive real eigenvalues of S_m and the real eigenvalues of* (5)*, given by $[\lambda \mapsto -\frac{1}{T}\log\lambda]$. Moreover, if $\lambda > 0$ is an eigenvalue of S_m with eigenfunction $\varphi_0 \in X$, then $\varphi(t) := e^{\mu t}U(t,0)\varphi_0$ is an eigenfunction of* (5) *corresponding to the eigenvalue $\mu = -\frac{1}{T}\log\lambda$.*

b) *It is shown in* [13, Corollary 6.6] *and* [17, Theorem 5.1] *that assumption* $(A5)$ *forces*

$$r_{ess}(P_m) < 1. \tag{8}$$

We note that this is the only instance where we need $C^{1+\mu}$*-regularity in* $x \in \mathbb{R}^N$ *for the top-order coefficients of* $\mathcal{A}$*, stated in* $(A1)$*, rather than* C^{μ}*-regularity. This additional regularity permits one to write* $\mathcal{A}$ *in divergence form, as required for the application of Theorem* 5.1 *in* [17]*. As a consequence of* (8)*, the spectral theory of positive irreducible operators implies that* $r(S_m)$ *is an algebraically simple eigenvalue provided* $r(S_m) \geq 1$*. Hence* $r(S_m) \geq 1$ *implies that* $r(S_m)$ *is the only principal eigenvalue of* S_m*.*

c) *If* $r(S_m) < 1$*, then the zero solution of* (7) *is exponentially stable. If* $r(S_m) = 1$*, then it is stable but not exponentially stable. Finally, if* $r(S_m) > 1$*, then it is unstable.*

d) *Let* (m_n) *be a sequence in* $BUC(\mathbb{R}^N \times [0,T])$*, such that each element satisfies* $(A2)$*. For each* $n \in \mathbb{N}$ *let* $S_{m_n} \in \mathcal{L}(X)$ *be the period map associated with the linear equation*

$$\partial_t u + \mathcal{A}(t)u = m_n(x,t)u \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Then

$$m_n \to m \quad \text{in } BUC(\mathbb{R}^N \times [0,T]) \text{ for } n \to \infty,$$

implies

$$r(S_{m_n}) \to r(S_m) \quad \text{as } n \to \infty.$$

e) *Let* $m_1, m_2 \in BUC(\mathbb{R}^N \times [0,T])$ *satisfy* $(A2)$ *and* $(A5)$*. Then if*

$$m_1(x,t) \leq m_2(x,t) \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,T],$$

then $r(S_{m_1}) \leq r(S_{m_2})$*. If* $r(S_{m_1})$ *and* $r(S_{m_2})$ *are eigenvalues of* S_{m_1} *and* S_{m_2}*, respectively, and if* $m_1(x_0,t_0) < m_2(x_0,t_0)$ *for some* $(x_0,t_0) \in \mathbb{R}^N \times [0,T]$*, then* $r(S_{m_1}) < r(S_{m_2})$*.*

f) *Let* $w \in BUC(\mathbb{R}^N \times [0,T])$ *satisfy* $(A2)$ *and* $(A5)$*. Then if* φ *is a positive* T*-periodic function in* $BUC^{2,1}(\mathbb{R}^N \times [0,T])$ *satisfying*

$$\partial_t \varphi + \mathcal{A}(t)\varphi = w(x,t)\varphi \quad \text{in } \mathbb{R}^N \times [0,T],$$

then $r(S_w) = 1$*.*

5. Diffusive logistic growth. In [18] the T-periodic diffusive logistic equation

$$\begin{cases} \partial_t u + \mathcal{A}(x,t)u = m(x,t)u - p(x,t)g(x,t,u)u & \text{in } \mathbb{R}^N \times [0,T], \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{9}$$

is studied. Following [18] we assume that g satisfies

$(A6)$ For $R > 0$ we have $g \in BUC^{\mu,\frac{\mu}{2},2}(\mathbb{R}^N \times \mathbb{R} \times [0,R])$, T-periodic in the second variable. Moreover, g is nonnegative and satisfies $g(\cdot,\cdot,0) \equiv 0$. Finally, $\partial_3 g > 0$ on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^+$ and $\lim_{\xi\to\infty} g(x,t,\xi) = \infty$ uniformly in $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Moreover, $\mathcal{A}$ satisfies $(A1)$, $(A4)$ and m satisfies $(A2)$ and $(A5)$. Finally, we assume that p is nonnegative and satisfies $(A2)$. The following theorem will be the basis for our analysis of system (1) in the subsequent sections; it is a direct consequence of Theorem 4.3 and Proposition 5.2 in [18].

Theorem 5.1. *Assume that the support of m^+ is contained in the support of p. Then the following assertions hold.*

a) *Assume that the trivial solution of (9) is linearly or neutrally stable. Then it is globally asymptotically stable with respect to initial data in X^+.*

b) *Assume that the trivial solution of (9) is linearly unstable. Then there exists a unique positive T-periodic solution u^* of (9). Furthermore u^* is everywhere positive and is globally asymptotically stable with respect to initial data in $X^+ \setminus \{0\}$.*

Note that the linearization of (9) at the trivial solution is given by (6). Hence the linear stability and the linear instability of the trivial solution of (9) is characterized by $r(S_m) < 1$ and $r(S_m) > 1$, respectively.

The following technical result is the analogue of Lemma 39.1 in [15] for the logistic equation on $\mathbb{R}^N$. The proof follows the approach to problems on $\mathbb{R}^N$ presented in [19].

Proposition 5.2. *Let (m_n) be a sequence in $BUC(\mathbb{R}^N \times [0,T])$ such that each element m_n satisfies (A2) and (A5). Furthermore, assume that*

$$m_n \to m \quad \text{in } C^{\eta,\frac{\eta}{2}}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}), \quad \text{for some } \eta \in (0,1), \quad \text{as } n \to \infty .$$

We define the sequence $(u_n) \in C([0,T], X^+)$ by

$$u_n := u^*(m_n) := \begin{cases} 0 & \text{if } r(S_{m_n}) \le 1, \\ \text{unique positive } T\text{-periodic solution of } (9) & \\ \text{with the coefficient } m \text{ replaced by } m_n & \text{if } r(S_{m_n}) > 1. \end{cases}$$

Then

$$u_n \to u^*(m) \quad \text{in } C^{\eta,\frac{\eta}{2}}(\mathbb{R}^N \times [0,T]) \quad \text{as } n \to \infty.$$

Note that in case $r(S_{m_n}) > 1$, the existence and uniqueness of the T-periodic solution $u^*(m_n)$ is guaranteed by Theorem 5.1.

Proof. We set

$$H_n(u) := m_n u - pg(u)u, \quad \text{and} \quad H(u) := mu - pg(u)u$$

and

$$E_\eta := C^{\eta,\frac{\eta}{2}}(\mathbb{R}^N \times [0,T]).$$

Endowed with the topology of uniform $C^{\eta,\frac{\eta}{2}}$-convergence on compact subsets of $\mathbb{R}^N \times [0,T]$, the space E_η is a Fréchet space.

By Lemma 2.5 in [19] it follows that $H, H_n : E_\eta \to E_\eta$ are Lipschitz continuous, uniformly on bounded subsets of E_η. We proceed in three steps:

i) We show that (u_n) is a bounded sequence in $BUC^{2\eta,\eta}(\mathbb{R}^N \times [0,T])$ for some $\eta \in (0,1)$. In the following we make use of the interpolation spaces $Z_\alpha, \alpha \in (0,1)$, obtained by continuous interpolation between

$$Z_0 := BUC(\mathbb{R}^N) \quad \text{and} \quad Z_1 := D(A^{BUC}(t)).$$

Here Z_1 is independent of t, by $(A4)$, and is equipped with the graph norm. For $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$ the Banach spaces Z_α can be characterized. In fact, it is shown in e.g. [17] and [24] that up to equivalent norms Z_α is the little Hölder space $buc^{2\alpha}(\mathbb{R}^N)$. It is easily seen by the parabolic maximum principle that (u_n) is a bounded sequence in $BUC(\mathbb{R}^N \times [0,T])$. Thus the sequence $(u_n(0) := u_n(\cdot,0))$ is bounded in $BUC(\mathbb{R}^N)$. By making use of interpolation theory (cf. [19], Propositions 1.5 and 1.3 or [11], Lemma 16.7), we also obtain the boundedness of the sequence $(u_n(0))$ in any space $Z_\alpha, \alpha \in (0,1)$. The sequence (h_n), defined by

$$h_n := H_n(u_n), \quad h_n : [0,T] \to Z_0,$$

is bounded in $L_\infty([0,T], Z_0)$.

Each element of the sequence (u_n) can be represented by the variation-of-constants formula

$$u_n(t) = V(t,0)u_n(0) + \int_0^t V(t,\tau)H_n(u_n(\tau))\,d\tau, \quad t \in [0,T]. \tag{9}$$

Here $V(t,s)$ denotes the evolution system on Z_0 generated by the family $\{\mathcal{A}(t) : t \in [0,T]\}$. Since for any $0 \le \alpha < \beta \le 1$, the space Z_β is continuously embedded in Z_α and since

$$V(t,s) \in \mathcal{L}(Z_0, Z_1) \quad \text{for} \quad (t,s) \in \triangle_T \quad \text{with } t > s,$$

(cf. [11, Section 2]), we obtain

$$V(t,s) \in \mathcal{L}(Z_\alpha, Z_\beta) \quad \text{for each } \alpha, \beta \in [0,1] \text{ and } (t,s) \in \triangle_T \text{ with } t > s.$$

So we have denoted by the same symbol $V(t,s)$ the various operators obtained by restricting the domain of definition and the range of $V(t,s)$.

It is shown in [11, Corollary 5.6] that for $0 \le \alpha \le \beta < 1$

$$K \in \mathcal{L}(Z_\beta \times L_\infty([0,T], Z_0), C^{\beta-\alpha}([0,T], Z_\alpha))\ ,$$

where K is defined by

$$K(u_0, g)(t) := V(t,0)u_0 + \int_0^t V(t,\tau)g(\tau)\,d\tau\ .$$

Since by (10)

$$u_n(t) = K(u_n(0), h_n)(t), \quad t \in [0,T],$$

and since we know that $(u_n(0))$ is bounded in any space $Z_\beta, \beta \in (0,1)$, we obtain the boundedness of (u_n) in $C^{\beta-\alpha}([0,T], Z_\alpha)$, $0 \le \alpha \le \beta < 1$. But since for $0 < \eta < min\{\alpha, \beta - \alpha\}$ the space $C^{\beta-\alpha}([0,T], Z_\alpha)$ is continuously embedded in $BUC^{2\eta,\eta}(\mathbb{R}^N \times [0,T])$ we are done.

Now we discuss the two possible cases (ii): $r(S_m) > 1$ and (iii): $r(S_m) \le 1$.

ii) Proposition 4.1 d) implies that there exists a $n_0 \in \mathbb{N}$ such that $r(S_{m_n}) > 1$ for $n \ge n_0$. Consequently by Theorem 5.1

$$u_n(x,t) > 0 \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,T] \text{ and } n \ge n_0.$$

Since $u_n(\cdot,t) \in C_0(\mathbb{R}^N)$ for $t \in [0,T]$, and because $C^{2\eta,\eta}(\bar{\Omega} \times [0,T])$ is compactly embedded in $C^{\eta,\frac{\eta}{2}}(\bar{\Omega} \times [0,T])$ for any bounded $\Omega \subset \mathbb{R}^N$, we find a subsequence of (u_n), that we still denote by (u_n), such that

$$u_n \to \tilde{u} \quad \text{in} \quad E_\eta \quad \text{for } n \to \infty.$$

By continuity we obtain

$$H_n(u_n) \to H(\tilde{u}) \quad \text{in} \quad E_\eta \quad \text{as } n \to \infty \enspace .$$

As shown in [19], Lemma 2.5,

$$L : D(L) \subset E_\eta \to E_\eta, \quad Lu := \partial_t u + \mathcal{A}(x,t)u,$$

is a closed operator on E_η. Since $L(u_n) = H_n(u_n)$ for $n \in \mathbb{N}$, and by the closedness of L, it follows that $\tilde{u} \in D(L)$ and $L\tilde{u} = H(\tilde{u})$. But, by Theorem 5.1, this is only possible if either $\tilde{u} = 0$ or $\tilde{u} = u^*(m)$. Suppose that $\tilde{u} = 0$. Since $Lu_n = (m_n - pg(u_n))u_n$, we obtain

$$r(S_{m_n - pg(u_n)}) = 1 \quad \text{for} \quad n \geq n_0 \ ,$$

by Proposition 4.1 f). Moreover, since $u_n \to 0$ for $n \to \infty$, Proposition 4.1 d) implies

$$r(S_{m_n - pg(u_n)}) \to r(S_m) \quad \text{for } n \to \infty \ .$$

Thus $r(S_m) = 1$ and we have a contradiction to our assumption. So we obtain $\tilde{u} = u^*(m)$.

iii) Now we assume that $r(S_m) \leq 1$. From Theorem 5.1 we immediately obtain that $u^*(m) = 0$. As in part (ii) we may consider a convergent subsequence (u_n) such that $u_n \to \tilde{u}$ in E_η for $n \to \infty$. Again we conclude that $L\tilde{u} = H(\tilde{u})$. If $\tilde{u}$ is not zero, then by Theorem 5.1 we obtain $r(S_m) > 1$, a contradiction. Thus $\tilde{u} = 0 = u^*(m)$ and the proof is completed.

6. Global existence of positive solutions. In this section we will establish the existence of global classical solutions of (1) for initial values in the positive cone $\mathbf{X}^+$. Throughout the following sections we make the following assumptions.

(*A*7) b and f are nonnegative and a and d satisfy (*A*5). Moreover, the support of a^+ is contained in the support of b and the support of d^+ is contained in the support of f.

(*A*8) g_1 and g_2 satisfy (*A*6).

(*A*9) $h_1(x,t,\xi,\eta) \geq 0$ for $(x,t,\xi,\eta) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+$. For each $(x,t) \in \mathbb{R}^N \times [0,T]$, the positive part h_2^+ of h_2 is monotone increasing in ξ. Moreover, $\partial_u h_1(x,t,\xi,0) \geq 0$ for $(x,t,\xi) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^+$. Finally, $h_1(\cdot,\cdot,\cdot,0) = h_2(\cdot,\cdot,\cdot,0) \equiv 0$.

By Theorem 2.1 the question of global existence for solutions of (1) is reduced to the problem of finding a priori bounds in $\mathbf{X}$, i.e., L_∞-bounds. We will obtain such bounds using the parabolic maximum principle.

Theorem 6.1. *To each initial value* $\mathbf{w}_0 \in \mathbf{X}^+$ *the system* (1) *possesses a unique global classical solution* $\mathbf{w}(t, \mathbf{w}_0)$. *Moreover, it is positive; i.e.,* $\mathbf{w}(t, \mathbf{w}_0) \in \mathbf{X}^+$ *for* $t \in [0, T]$. *Finally, if* $\mathbf{w}_0 \in \dot{X}^+ \times \dot{X}^+$, *then* $\mathbf{w}(t, \mathbf{w}_0) \in qint\,(\mathbf{X}^+)$ *for* $t > 0$.

Proof. By Theorem 2.1 we obtain the existence of a unique maximal classical solution

$$\mathbf{w}(t, \mathbf{w}_0) = (u(t, u_0, v_0), v(t, u_0, v_0)) \quad \text{for } t \in [0, t^+(\mathbf{w}_0)) .$$

To show the positivity of the maximal solution, we note that the first component u satisfies

$$\begin{cases} \partial_t u + \mathcal{A}_1(t) u - a_0 u = 0 & \text{in } \mathbb{R}^N \times (0, t^+), \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $a_0 := a - bg_1(u) - h_1(u, v)$. Applying the parabolic maximum principle we see that $u(t, u_0, v_0) \in X^+$ for $t \in [0, t^+)$. It is clear that for the second component $v(t, u_0, v_0)$ we can proceed analogously.

To find a priori bounds observe that, by the nonnegativity of h_1, the first component u satisfies

$$\partial_t u + \mathcal{A}_1(t) u = au - bg_1(u)u - h_1(u, v)u \leq au - bg_1(u)u \ \text{ in } \mathbb{R}^N \times (0, t^+).$$

Therefore, if we denote the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_1(t) w = aw - bg_1(w)w & \text{in } \mathbb{R}^N \times (0, T), \\ w(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{11}$$

by $\tilde{u}$, then the parabolic maximum principle implies $u(x, t) \leq \tilde{u}(x, t)$ in $\mathbb{R}^N \times (0, t^+)$. Since $\tilde{u}$ exists globally by Theorem 5.1, we conclude that there exists a constant c_1 such that $u(x, t) \leq c_1$ in $\mathbb{R}^N \times (0, t^+)$. Thus we have found an L_∞-bound for the first component. We will now use this bound to estimate the second component.

The monotonicity of h_2^+ implies

$$h_2(x, t, u(x, t)) \leq h_2^+(x, t, u(x, t)) \leq h_2^+(x, t, c_1) \quad \text{in } \mathbb{R}^N \times (0, t^+),$$

thus

$$\partial_t v + \mathcal{A}_2(t) v = dv - fg_2(v)v + h_2(u)v \leq (d + h_2^+(c_1))v - g_2(v)v \quad \text{in } \mathbb{R}^N \times (0, t^+).$$

Denoting the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_2(t) w = (d + h_2^+(c_1))w - fg_2(w)w & \text{in } \mathbb{R}^N \times (0, T), \\ w(\cdot, 0) = v_0 & \text{in } \mathbb{R}^N, \end{cases} .$$

by $\tilde{v}$, we obtain $v(x, t) \leq \tilde{v}(x, t)$ in $\mathbb{R}^N \times (0, t^+)$. Again the global existence of $\tilde{v}$ implies the existence of a constant c_2 such that $v(x, t) \leq c_2$ in $\mathbb{R}^N \times (0, t^+)$. Hence we are done by Theorem 2.1. $\square$

By the previous theorem, the period map $\mathbf{S}$ associated with system (1) now satisfies $D(\mathbf{S}) \supset \mathbf{X}^+$ and $\mathbf{S}(\mathbf{X}^+) \subset \mathbf{X}^+$. In the following we will consider $\mathbf{S}$ as a positive mapping $\mathbf{S} : \mathbf{X}^+ \to \mathbf{X}^+$. Furthermore, as discussed in Section 3, the period map is differentiable and the principle of linearized stability can thus be applied. It will be useful to represent the period map $\mathbf{S}$ componentwise as

$$\mathbf{S}(u,v) = (p(u,v)\,,\, q(u,v)), \tag{12}$$

where p and q are mappings from $\mathbf{X}^+$ to X^+. For later reference we list some evident properties of p and q.

Remark 6.2. a) Since $(0,0)$ is a fixed point of $\mathbf{S}$, we conclude that $p(0,0) = 0$ and $q(0,0) = 0$.

b) For each $w \in X^+$ it holds that $p(0,w) = 0$ and $q(w,0) = 0$. This means that $\mathbf{S}$ leaves $X^+ \times \{0\}$ and $\{0\} \times X^+$ invariant.

c) $p(\cdot,0)$ is the period map associated with the logistic equation

$$\partial_t w + \mathcal{A}_1(t)w = aw - bg_1(w)w \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{13}$$

Analogously $q(0,\cdot)$ is the period map associated with the logistic equation

$$\partial_t w + \mathcal{A}_1(t)w = dw - fg_2(w)w \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{14}$$

7. Coexistence solutions. We start with a definition. a) By a *coexistence solution* of (1) we mean a classical T-periodic solution $\mathbf{w}(x,t) = (u(x,t), v(x,t))$ of (1) with $\mathbf{w}(\cdot,t) \in \mathbf{X}^+$, $t \in [0,T]$, and such that neither u nor v is identically zero.

b) By a *semitrivial solution* of (1) we mean a nonnegative classical T-periodic solution of (1) of the form $(u(x,t),0)$ or $(0,v(x,t))$, where neither u nor v is identically zero.

c) We also introduce the following notation. For $w \in BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^N \times \mathbb{R})$, T-periodic in the second variable and $i = 1,2$, we denote by $S_w^{(i)} \in \mathcal{L}(X)$ the period map associated with the linear equation

$$\partial_t \varphi + \mathcal{A}_i(x,t,\partial)\varphi = w(x,t)\varphi \quad \text{in } \mathbb{R}^N \times (0,T].$$

We remark that, according to the above definition, the period maps of the linearizations of the logistic equations (13) and (14) at the trivial solution are given by

$$S_a^{(1)} \quad \text{and} \quad S_d^{(2)}. \tag{15}$$

To find coexistence solutions it will be useful to study the existence of semitrivial solutions of (1). It is clear that in order to find semitrivial solutions we have to study the existence of positive T-periodic solutions of the logistic equations (13) and (14) obtained from our system (1) by setting $u = 0$ and $v = 0$, respectively.

Now, by Theorem 5.1, we know that the existence of positive T-periodic solutions for the logistic equation depends only on the stability of its trivial solution. We also know that, if a positive periodic solution exists, then it is unique, stable and attracts initial data in $X^+\backslash\{0\}$. Therefore, besides the trivial solution, the system (1) can only have two, one, or no semitrivial solutions at all. Moreover, we have a complete qualitative description of the long-time behaviour of orbits $(\mathbf{S}^n(\mathbf{w}_0))$ of the discrete dynamical system $(\mathbf{S},\mathbf{X}^+)$ starting in the invariant set $(X^+ \times \{0\}) \cup (\{0\} \times X^+)$.

The next proposition motivates, a posteriori, the choice of the space $\mathbf{X}$.

Proposition 7.2. *Any nonnegative classical T-periodic solution* $\mathbf{w}$ *of* (1) *satisfies* $\mathbf{w}(t) \in \mathbf{X}$ *for* $t \in \mathbb{R}$.

Proof. Let $\mathbf{w} := (u, v)$ be a nonnegative classical T-periodic solution of (1). Then u satisfies

$$\partial_t u + \mathcal{A}_1(x, t, \partial)u = m_1(x, t)u \quad \text{in } \mathbb{R}^N \times [0, T],$$

where, by $(A7)$, $(A8)$ and $(A9)$, the function $m_1 := a - bg_1(u) - h_1(u, v)$ satisfies assumption $(A5)$. It is shown in [18, Lemma 3.1] that this is sufficient for $u(t) \in X$ for $t \in [0, T]$. For the second component v we proceed similarly. We consider the function $m_2 := d - fg_2(v) + h_2(u)$. It remains to show that m_2 satisfies $(A5)$. First note that by $(A9)$ we have $h_2(\cdot, t, u(\cdot, t)) \in X$ for $t \in [0, T]$. Moreover, we know that $r_{ess}(S_d^{(2)}) < 1$. We infer by Proposition 6.5 in [13] that

$$r_{ess}(S_d^{(2)}) = r_{ess}(S_{d+h_2(u)}^{(2)}).$$

By Corollary 6.6 in [13] we obtain that $d + h_2(u)$ satisfies $(A5)$ and finally, by the nonnegativity of $fg_2(v)$, we conclude that m_2 satisfies $(A5)$.

8. The stability of semitrivial solutions. To obtain the existence of coexistence solutions it will be necessary to make assumptions on the stability of semitrivial solutions. Consider a semitrivial solution of (1) of the form $(u^*, 0)$ and the corresponding fixed point $(u_0^*, 0)$ of the period map $\mathbf{S}$. Linearizing system (1) at $(u^*, 0)$ we obtain

$$\begin{cases} \partial_t \varphi + \mathcal{A}_1(t)\varphi = c_{11}(x, t)\varphi + c_{12}(x, t)\psi \\ \partial_t \psi + \mathcal{A}_2(t)\psi = c_{22}(x, t)\psi \end{cases} \quad \text{in } \mathbb{R}^N \times [0,\mathrm{T}], \tag{16}$$

where

$$c_{11} := a - b\partial_u g_1(u^*)u^* - bg_1(u^*) - \partial_u h_1(u^*, 0)u^* \ ,$$
$$c_{12} := -\partial_v h_1(u^*, 0)u^* \ , \quad c_{22} := d + h_2(u^*) \ .$$

By the componentwise representation (12), the period map $D\mathbf{S}(u_0^*, 0)$ associated with (16) may be written as

$$D\mathbf{S}(u_0^*, 0) = \begin{pmatrix} D_1 p(u_0^*, 0) & D_2 p(u_0^*, 0) \\ 0 & D_2 q(u_0^*, 0) \end{pmatrix}.$$

Moreover, using the notation introduced in Definition 7.1, we see that

$$D_1 p(u_0^*, 0) = S_{c_{11}}^{(1)} \ \text{ and } \ D_2 q(u_0^*, 0) = S_{c_{22}}^{(2)} \ .$$

We state a simple stability criterion for semitrivial solutions of (1).

Theorem 8.1. *The following assertions hold.*

a) *A semitrivial solution* $(u^*, 0)$ *of* (1) *is exponentially stable if* $r(S^{(2)}_{d+h_2(u^*)}) < 1$ *and it is unstable if* $r(S^{(2)}_{d+h_2(u^*)}) > 1$.

b) *A semitrivial solution* $(0, v^*)$ *of* (1) *is exponentially stable if* $r(S^{(1)}_{a-h_1(0,v^*)}) < 1$ *and it is unstable if* $r(S^{(1)}_{a-h_1(0,v^*)}) > 1$.

Proof. We only consider a semitrivial solution of the form $(u^*, 0)$. The other case is analogous. In order to discuss the spectrum of $D\mathbf{S}(u_0^*, 0)$, we first have a closer look at the spectrum of $S^{(1)}_{c_{11}}$. Since u^* is a positive T-periodic solution of the logistic equation (13), it satisfies

$$\partial_t u^* + \mathcal{A}_1(t)u^* = (a - bg_1(u^*))u^* \quad \text{in } \mathbb{R}^N \times [0, T].$$

It follows by Proposition 4.1 f) that $r(S^{(1)}_{a-bg_1(u^*)}) = 1$. Our assumptions $(A8)$ and $(A9)$ imply

$$c_{11} \leq a - bg_1(u^*) \quad \text{in } \mathbb{R}^N \times [0, T] .$$

Thus the monotonicity of the spectral radius (Proposition 4.1 e)) implies $r(S^{(1)}_{c_{11}}) \leq 1$. Assume that $r(S^{(1)}_{c_{11}}) = 1$. Then 1 is an eigenvalue of $S^{(1)}_{c_{11}}$. Since $c_{11} < a - bg_1(u^*)$, by Proposition 4.1 e) we obtain $r(S^{(1)}_{c_{11}}) < r(S^{(1)}_{a-bg_1(u^*)}) = 1$. This contradiction shows that in fact $r(S^{(1)}_{c_{11}}) < 1$. Hence the set $\{\mu \in \mathbb{C} : |\mu| \geq 1\}$ is contained in the resolvent set $\rho(S^{(1)}_{c_{11}})$.

Of course, $\lambda \in \{\mu \in \mathbb{C} : |\mu| \geq 1\}$ is in the spectrum of $D\mathbf{S}(u_0^*, 0)$ if and only if λ is in the spectrum of $S^{(2)}_{d+h_2(u^*)}$. Therefore, if $r(S^{(2)}_{d+h_2(u^*)}) < 1$, then the spectrum of $D\mathbf{S}(u_0^*, 0)$ is contained in the open unit disc. By the principle of linearized stability 3.4 we obtain the exponential stability of $(u^*, 0)$.

If on the other hand $r(S^{(2)}_{d+h_2(u^*)}) > 1$, we infer from Proposition 4.1 b) that the set

$$\{\mu \in \mathbb{C} : |\mu| > 1\} \cap \sigma(S^{(2)}_{d+h_2(u^*)})$$

consists of isolated points and therefore it is a spectral set. Hence

$$\{\mu \in \mathbb{C} : |\mu| > 1\} \cap \sigma(D\mathbf{S}(u_0^*, 0))$$

is a nonempty spectral set. Therefore the instability of $(u^*, 0)$ follows from the principle of linearized stability.

9. Coexistence results. In this section we prove our main result: We show the existence of coexistence solutions of the system (1) for three different situations. We consider the following three different cases, characterized by assumptions on the existence of semitrivial solutions and their stability properties.

Case I :

$(A10)$ $r(S^{(1)}_a) > 1, \quad r(S^{(2)}_d) > 1, \quad r(S^{(1)}_{a-h_1(0,v^*)}) > 1, \quad r(S^{(2)}_{d+\gamma h_2(u^*)}) > 1, \quad \gamma \in [0, 1].$

These assumptions imply the existence and the instability of the two semitrivial solutions $(u^*, 0)$ and $(0, v^*)$.

Case II :

$$r(S_a^{(1)}) > 1, \quad r(S_d^{(2)}) > 1, \quad r(S^{(1)}_{a-h_1(0,v^*)}) < 1, \quad r(S^{(2)}_{d+h_2(u^*)}) < 1. \tag{A11}$$

These assumptions imply the existence and the stability of the two semitrivial solutions $(u^*, 0)$ and $(0, v^*)$.

Case III :

$$r(S_a^{(1)}) > 1, \quad r(S_d^{(2)}) \le 1, \quad r(S^{(2)}_{d+h_2(u^*)}) > 1. \tag{A12}$$

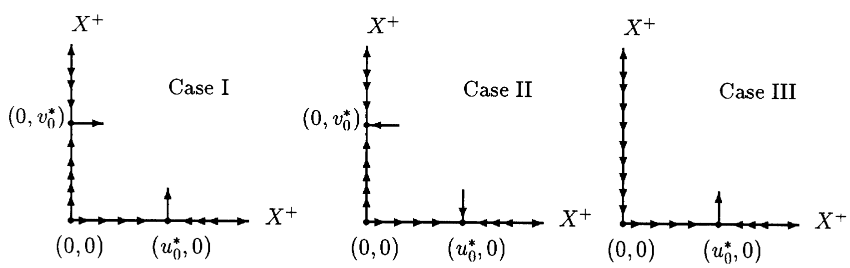
These assumptions imply the existence of an unstable semitrivial solution $(u^*, 0)$.

In the following we denote the fixed points corresponding to the semitrivial solutions $(u^*, 0)$ and $(0, v^*)$ by $(u_0^*, 0)$ and $(0, v_0^*)$, respectively. The following pictures describe the phase plane of the discrete dynamical system $(\mathbf{S}, \mathbf{X}^+)$ for the three different cases.

Since each of the assumptions $(A10)$, $(A11)$ and $(A12)$ implies that $r(S^{(2)}_{d+h_2^+(u^*)}) > 1$, by Theorem 5.1 the logistic equation

$$\partial_t w + \mathcal{A}_2(t) w = (d + h_2^+(u^*)) w - f g_2(w) w \quad \text{ in } \mathbb{R}^N \times (0, T],$$

possesses a unique positive T-periodic solution that we denote by $\hat{v}$. In the next proposition we use this solution $\hat{v}$ to construct an invariant and attractive subset of the positive cone $\mathbf{X}^+$.



Proposition 9.1. *The following assertions hold in each of the cases* I, II *and* III.

a) $\mathbf{S}$ *maps the set* $\mathcal{R} := \{(u, v) \in \mathbf{X}^+ : u \le u_0^*$ *and* $v \le \hat{v}_0\}$ *into itself and the restriction of* $\mathbf{S}$ *to* $\mathcal{R}$ *is a compact mapping.*

b) *Assume that there exist positive constants* c_0 *and* R_0 *such that*

$$d(x, t) \le -c_0 \quad \text{for } |x| \ge R_0 \quad \text{and} \quad t \in [0, T]\ ;$$

then $\mathcal{R}$ *attracts* $\mathbf{X}^+$*; i.e., for each* $\mathbf{w} \in \mathbf{X}^+$, $\mathbf{S}^n(\mathbf{w}) \to \mathcal{R}$ *in* $\mathbf{X}$ *as* $n \to \infty$. *Therefore* $\mathcal{R}$ *contains all fixed points of* $\mathbf{S}$.

Proof. a) To show the invariance of $\mathcal{R}$ let (u, v) be the solution of system (1) to an initial value (u_0, v_0) in $\mathcal{R}$. Since h_1 is nonnegative, the first component u satisfies

$$\begin{cases} \partial_t u + \mathcal{A}_1(t)u \le au - bg_1(u)u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(0) \le u_0^* & \text{in } \mathbb{R}^N. \end{cases}$$

The parabolic maximum principle then implies $u(x,t) \le u^*(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. By the monotonicity of $h_2^+(x,t,\cdot)$ we obtain the following inequalities for v:

$$\begin{cases} \partial_t v + \mathcal{A}_2(t)v \le (d + h_2^+(u^*))v - fg_2(v)v & \text{in } \mathbb{R}^N \times (0,\infty), \\ v(0) \le \hat{v}(0) & \text{in } \mathbb{R}^N. \end{cases}$$

This implies $v(x,t) \le \hat{v}(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. Thus the invariance of $\mathcal{R}$ is proved.

Finally, the compactness of the restriction of the period map $\mathbf{S}$ to the order interval $\mathcal{R}$ follows by Lemma 2.2 in [19].

b) Let now $(u_0, v_0) \in \mathbf{X}^+$ (without loss of generality $u_0 \neq 0$) and denote by (u, v) the solution of system (1) to the initial value (u_0, v_0). If w_1 is the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_1(t)w = aw - bg_1(w)w & \text{in } \mathbb{R}^N \times (0,\infty), \\ w(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

then the nonnegativity of h_1 and the parabolic maximum principle imply $u(x,t) \le w_1(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. By Theorem 5.1 $w_1(t) - u^*(t) \to 0$ in $\mathbf{X}$ as $t \to \infty$. Hence, by the monotonicity of h_2^+, to any $\varepsilon > 0$ we find a $t(\varepsilon)$ such that

$$h_2(x,t,u(x,t)) \le h_2^+(x,t,u^*(x,t) + \varepsilon) \quad \text{in } \mathbb{R}^N \times (t(\varepsilon),\infty).$$

For v this implies

$$\partial_t v + \mathcal{A}_2(t)v \le (d + h_2^+(u^* + \varepsilon))v - fg_2(v)v \quad \text{in } \mathbb{R}^N \times (t(\varepsilon),\infty).$$

Note that, by our assumption on d, the function $d + h_2^+(u^* + \varepsilon)$ satisfies the assumption $(A5)$ if ε is small enough. Hence the results of Section 5 can be applied to the logistic equation

$$\partial_t w + \mathcal{A}_2(t)w = (d + h_2^+(u^* + \varepsilon))w - fg_2(w)w \quad \text{in } \mathbb{R}^N \times [0,T].$$

Let w_2^ε be the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_2(t)w = (d + h_2^+(u^* + \varepsilon))w - fg_2(w)w & \text{in } \mathbb{R}^N \times (t(\varepsilon),\infty), \\ w(t(\varepsilon)) = v(t(\varepsilon)) & \text{in } \mathbb{R}^N. \end{cases}$$

Then, by the maximum principle, we obtain $v(x,t) \le w_2^\varepsilon(x,t)$ in $\mathbb{R}^N \times (t(\varepsilon),\infty)$. Since $r(S^{(2)}_{d+h_2^+(u^*)}) > 1$ and therefore $r(S^{(2)}_{d+h_2^+(u^*+\varepsilon)}) > 1$, by Theorem 5.1, there exists a unique positive T-periodic solution $\hat{v}_\varepsilon$ of the logistic equation

$$\partial_t w + \mathcal{A}_2(t)\triangle w = (d + h_2^+(u^* + \varepsilon))w - fg_2(w)w \quad \text{in } \mathbb{R}^N \times [0,T],$$

and by Proposition 5.2 and the monotone dependence of $\hat{v}_\varepsilon$ on ε

$$\hat{v}_\varepsilon \to \hat{v} \quad \text{in } C([0,T],X) \quad \text{as } \varepsilon \to 0.$$

Moreover, Theorem 5.1 implies that for each $\varepsilon > 0$

$$w_2^\varepsilon(t) - \hat{v}_\varepsilon(t) \to 0 \quad \text{in } X \text{ for } t \to \infty.$$

The two last assertions imply that for any $\delta > 0$ there exists an $\varepsilon > 0$ such that

$$|\hat{v}_\varepsilon(x,t) - \hat{v}(x,t)| < \tfrac{\delta}{2} \quad \text{in} \quad \mathbb{R}^N \times [0,T],$$

and we find a $t_0 \geq t(\varepsilon)$ such that

$$w_2^\varepsilon(x,t) \leq \hat{v}_\varepsilon(x,t) + \tfrac{\delta}{2} \quad \text{in} \quad \mathbb{R}^N \times (t_0,\infty).$$

Finally,

$$v(x,t) \leq w_2^\varepsilon(x,t) \leq \hat{v}(x,t) + \delta \quad \text{in} \quad \mathbb{R}^N \times (t_0,\infty).$$

Since this can be done for any $\delta > 0$ we conclude that $\mathbf{S}^n(u_0,v_0)$ converges to $\mathcal{R}$ as $n \to \infty$.

A decoupling technique. Since the invariant set $\mathcal{R}$ is a closed and convex subset of $\mathbf{X}$, by a theorem of Dugundji (cf. [14]), it is a retract of $\mathbf{X}$. Therefore, a fixed-point index for the period map $\mathbf{S}$ can be defined. This fixed-point index is intimately related to the well-known Leray-Schauder degree. For a discussion of the properties of the fixed-point index we refer to [1].

In the following we will compute the local fixed-point indices of the trivial and semitrivial fixed points of $\mathbf{S}$ appearing in the different cases I, II and III. By "index-counting," i.e., by using the additivity and the solution property of the fixed-point index, the existence of coexistence solutions will then immediately follow. To compute the local indices we construct a compact homotopy between the period map $\mathbf{S}$ and the period maps of certain decoupled systems. Then, by the homotopy invariance of the index, we will only have to study the period maps of these decoupled systems. We consider the two-parameter dependent systems

$$\begin{cases} \partial_t u + \mathcal{A}_1 u = au - bg_1(u)u - \gamma h_1(u,v)u \\ \partial_t v + \mathcal{A}_2 v = dv - fg_2(v)v + h_2(u)v \end{cases} \quad \text{in } \mathbb{R}^N \times [0,\mathrm{T}], \tag{17}$$

and

$$\begin{cases} \partial_t u + \mathcal{A}_1 u = au - bg_1(u)u - \delta h_1(u,v)u \\ \partial_t v + \mathcal{A}_2 v = dv - fg_2(v)v + \delta h_2(u)v \end{cases} \quad \text{in } \mathbb{R}^N \times [0,\mathrm{T}], \tag{18}$$

with parameters γ and δ in $[0,1]$.

Definition 9.2. a) For each γ and δ in $[0,1]$ we denote by $\mathbf{Q}_\gamma$ and $\mathbf{T}_\delta$ the period maps associated with (17) and (18), respectively.

b) For $\mathbf{x} \in \mathbf{X}^+$ and $r > 0$ we set $\mathbb{B}_{\mathcal{R}}(\mathbf{x},r) := \mathbb{B}_{\mathbf{X}}(\mathbf{x},r) \cap \mathcal{R}$.

In the next proposition we show that certain semitrivial fixed points exist for all values of the parameters γ and δ. Moreover, we show that those fixed points are isolated.

Proposition 9.3. a) *In case* I, *for each* $\delta \in [0,1]$ *the map* $\mathbf{T}_\delta$ *possesses the three fixed points* $(0,0)$, $(u_0^*,0)$ *and* $(0,v_0^*)$. *Moreover, these fixed points are isolated uniformly with respect to* $\delta \in [0,1]$. *More precisely, if* (u_0,v_0) *is one of the three fixed points, then we find an* $r_0 > 0$, *such that for each* $\delta \in [0,1]$ *it holds that* (u_0,v_0) *is the only fixed point of* $\mathbf{T}_\delta$ *in* $\mathbb{B}_{\mathcal{R}}((u_0,v_0),r_0)$.

b) *In cases* II *and* III, *for each* $\gamma \in [0,1]$ *the map* $\mathbf{Q}_\gamma$ *possesses the two fixed points* $(0,0)$ *and* $(u_0^*,0)$. *Moreover, the two fixed points are isolated uniformly with respect to* $\gamma \in [0,1]$.

Proof. a) We first consider the fixed point $(0,0)$. Assume on the contrary that there exist sequences (δ_n), $\delta_n \in [0,1]$, and (u_n^0,v_n^0) in $\mathcal{R}$ such that

$$\mathbf{T}_{\delta_n}(u_n^0,v_n^0) = (u_n^0,v_n^0) \quad \text{for} \quad n \in \mathbb{N},$$
$$(u_n^0,v_n^0) \to (0,0) \quad \text{in } \mathbf{X} \ \text{ for } n \to \infty.$$

Since there are no semitrivial solutions arbitrarily close to $(0,0)$, we can assume that the fixed points (u_n^0,v_n^0) are coexistence fixed points. We denote the T-periodic solution corresponding to the fixed point (u_n^0,v_n^0) by (u_n,v_n). Then, for $n \in \mathbb{N}$,

$$\partial_t u_n + \mathcal{A}_1(t)u_n = (a - bg_1(u_n) - \delta_n h_1(x,t,u_n,v_n))u_n \quad \text{in } \mathbb{R}^N \times [0,T].$$

By Proposition 4.1 f) and the positivity of u_n it follows that

$$r\big(S^{(1)}_{a-bg_1(u_n)-\delta_n h_1(u_n,v_n)}\big) = 1 \quad \text{for } n \in \mathbb{N}.$$

Moreover, since

$$a - bg_1(u_n) - \delta_n h_1(u_n,v_n) \to a \quad \text{in } BUC(\mathbb{R}^N \times [0,T]) \text{ for } n \to \infty,$$

we conclude by Proposition 4.1 d) that $r(S_a^{(1)}) = 1$. This is a contradiction to the assumptions of case I.

Next we consider the fixed point $(u_0^*,0)$. Assume that $(u_n^0,v_n^0) \to (u_0^*,0)$ in $\mathbf{X}$ for $n \to \infty$. Then, as above, we conclude that

$$\partial_t v_n + \mathcal{A}_2(t)v_n = (d - fg_2(v_n) + \delta_n h_2(u_n))v_n \quad \text{in } \mathbb{R}^N \times [0,T].$$

Hence $r(S^{(2)}_{d-fg_2(v_n)+\delta_n h_2(u_n)}) = 1$ for $n \in \mathbb{N}$. By possibly passing to a subsequence we may assume that $\delta_n \to \delta_0 \in [0,1]$ for $n \to \infty$. This implies

$$d - fg_2(v_n) + \delta_n h_2(u_n) \to d + \delta_0 h_2(u^*) \quad \text{in } BUC(\mathbb{R}^N \times [0,T]) \text{ as } n \to \infty.$$

Therefore $r(S^{(2)}_{d+\delta_0 h_2(u^*)}) = 1$, which is a contradiction to the assumptions of case I. For the fixed point $(0,v_0^*)$ one can proceed analogously.

b) The arguments are analogous to a). □

For the fixed points discussed in the above proposition a (local) fixed-point index is now well defined. In the next lemma we show that its value is independent of the parameter value.

Lemma 9.4. a) *In case* I *the fixed-point index* $i_{\mathcal{R}}(\mathbf{T}_\delta, \mathbf{w}_0)$ *is independent of* $\delta \in [0,1]$, *for each fixed point* $\mathbf{w}_0$ *in* $\{(0,0),(u_0^*,0),(0,v_0^*)\}$.

b) *In the cases* II *and* III *the fixed-point index* $i_{\mathcal{R}}(\mathbf{Q}_\gamma, \mathbf{w}_0)$ *is independent of* $\gamma \in [0,1]$, *for each fixed point* $\mathbf{w}_0$ *in* $\{(0,0),(u_0^*,0)\}$.

Proof. a) We consider the homotopy $h : [0,1] \times \mathcal{R} \to \mathcal{R}$, $h(\delta, \mathbf{x}) := \mathbf{T}_\delta(\mathbf{x})$. To show that h is compact it suffices to show that $h(\delta,\cdot) = \mathbf{T}_\delta(\cdot)$ is compact for each $\delta \in [0,1]$, which is clear by Proposition 9.1, and that h is continuous in δ uniformly with respect to $\mathbf{x} \in \mathcal{R}$. The last assertion follows by the results on parameter-dependent parabolic equations, as contained in [11, Chapter 18]. Thus h is a compact homotopy. By Lemma 9.3 to each fixed point in $\{(0,0),(u_0^*,0),(0,v_0^*)\}$ we find an open neighborhood U in $\mathcal{R}$ of the fixed point such that $h(\delta,\mathbf{x}) \neq \mathbf{x}$ for $(\delta,\mathbf{x}) \in [0,1] \times \partial U$. The assertion now follows from the homotopy invariance of the fixed-point index.

b) Here we consider the homotopy $h : [0,1] \times \mathcal{R} \to \mathcal{R}$, $h(\gamma,\mathbf{x}) := \mathbf{Q}_\gamma(\mathbf{x})$. The assertion now follows as in a). □

In the next lemma we determine all the fixed points of the period maps $\mathbf{T}_0$ and $\mathbf{Q}_0$.

Lemma 9.5. a) *In case* I, *the period map* $\mathbf{T}_0$ *possesses precisely the four fixed points* $\{(0,0),(u_0^*,0),(0,v_0^*),(u_0^*,v_0^*)\}$.

b) *In case* II, *the period map* $\mathbf{Q}_0$ *possesses precisely the three fixed points* $\{(0,0),(u_0^*,0),(0,v_0^*)\}$.

c) *In case* III, *the period map* $\mathbf{Q}_0$ *possesses precisely the three fixed points* $\{(0,0),(u_0^*,0),(u_0^*,\tilde{v}_0)\}$. *Here* $\tilde{v}$ *denotes the unique positive* T*-periodic solution of the logistic equation*

$$\partial_t w + \mathcal{A}_2(t)w = (d + h_2(u^*))w - fg_2(w)w \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Proof. a) For $\delta = 0$ the system (18) is decoupled. Therefore the period map $\mathbf{T}_0$ is given by

$$\mathbf{T}_0(u,v) = (p(u), q(v)),$$

where p and q are the period maps to the logistic equations (13) and (14), respectively. In case I, the fixed points of p are precisely 0 and u_0^*, while q has exactly the fixed points v_0^* and 0.

b) $\mathbf{Q}_0$ is of the form

$$\mathbf{Q}_0(u,v) = (p(u), q(u,v)).$$

As in a) the period map p has precisely the two fixed points 0 and u_0^*. Therefore, to find the fixed points of $\mathbf{Q}_0$, we just need to find the fixed points of $q(0,\cdot)$ and $q(u_0^*,\cdot)$. Note that $q(0,\cdot)$ is the period map associated with the logistic equation (14). Thus the fixed points of $q(0,\cdot)$ are precisely 0 and v_0^*. On the other hand, $q(u_0^*,\cdot)$ is the period map associated with the logistic equation

$$\partial_t v + \mathcal{A}_2(t)v = (d + h_2(u^*))v - fg_2(v)v \quad \text{in } \mathbb{R}^N \times [0,T].$$

By our assumption that $r(S^{(2)}_{d+h_2(u^*)}) < 1$, Theorem 5.1 implies that 0 is the only fixed point of $q(u_0^*,\cdot)$. This completes the proof of b).

c) In this case $\mathbf{Q}_0$ is of the same form as in b). The difference is that in case III we assume $r(S^{(2)}_{d+h_2(u^*)}) > 1$. Therefore, $q(u_0^*, \cdot)$ possesses precisely the two fixed points 0 and $\tilde{v}_0$. Furthermore, since we assume that $r(S_d^{(2)}) \leq 1$, the only fixed point of $q(0, \cdot)$ is 0. $\square$

After these preparations we pass to the computation of the fixed-point index of the trivial and semitrivial fixed points appearing in the cases I, II and III.

Concerning the proof of case b) of the following proposition, we note that we do not give a direct argument to show that the index of the semitrivial fixed points have index 1. But due to the fact that we know all fixed points of the relevant decoupled systems appearing, we obtain the result by using a suitable homotopy and the additivity of the index for each semitrivial fixed point separately. In that sense the proof of case b) is more subtle than the proofs of the cases a) and c).

Proposition 9.6. *The fixed-point index of the trivial and semitrivial fixed points of* $\mathbf{S}$ *has the following values:*

a) *case* I : $i_{\mathcal{R}}(\mathbf{S}, (0,0)) = 0, \quad i_{\mathcal{R}}(\mathbf{S}, (u_0^*, 0)) = 0, \quad i_{\mathcal{R}}(\mathbf{S}, (0, v_0^*)) = 0.$
b) *case* II : $i_{\mathcal{R}}(\mathbf{S}, (0,0)) = 0, \quad i_{\mathcal{R}}(\mathbf{S}, (u_0^*, 0)) = 1, \quad i_{\mathcal{R}}(\mathbf{S}, (0, v_0^*)) = 1.$
c) *case* III : $i_{\mathcal{R}}(\mathbf{S}, (0,0)) = 0, \quad i_{\mathcal{R}}(\mathbf{S}, (u_0^*, 0)) = 0.$

Proof. By Lemma 9.4, in case I it suffices to consider the period map $\mathbf{T}_0$, while in the cases II and III it is sufficient to consider the map $\mathbf{Q}_0$. We treat the three cases separately.

a) We first consider the fixed point $(0,0)$. We have to show that $i_{\mathcal{R}}(\mathbf{T}_0, (0,0)) = 0$. We define a compact homotopy h between $\mathbf{T}_0$ and the constant map $(u_0^*, 0)$ by

$$h : [0,1] \times \mathcal{R} \to \mathcal{R} \quad , \quad h(t, \mathbf{x}) = t\mathbf{T}_0(\mathbf{x}) + (1-t)(u_0^*, 0).$$

As mentioned in the proof of Lemma 9.5 a), the componentwise representation of $\mathbf{T}_0$ is of the form $\mathbf{T}_0(u,v) = (p(u), q(v))$. We show that the homotopy h is admissible, i.e., we show that there exists an $r_0 > 0$ such that

$$h(t, \mathbf{x}) \neq \mathbf{x} \quad \text{for} \quad (t, \mathbf{x}) \in [0,1] \times \partial\mathbb{B}_{\mathcal{R}}((0,0), r_0).$$

Suppose that there exists $(t_0, (u,v)) \in [0,1] \times \mathcal{R}$ such that $h(t_0, (u,v)) = (u,v)$. This means that

$$u = t_0 p(u) + (1-t_0)u_0^*, \quad v = t_0 q(v).$$

Since (u,v) is in $\mathcal{R}$ it holds that $u \leq u_0^*$ in X^+. Moreover, since p is the period map to the logistic equation (13), p is order preserving and we obtain $p(u) \leq p(u_0^*) = u_0^*$. This implies

$$u \geq t_0 p(u) + (1-t_0)p(u) = p(u) \quad \text{ in } X^+.$$

Iterating this relation we obtain $u \geq p^n(u)$, $n \in \mathbb{N}$. Next we distinguish the two cases:

i) $u = 0$: From the fact that $p(0) = 0$ and since $u_0^* > 0$ in X^+ it follows that $t_0 = 1$. This means that v satisfies $v = q(v)$. Thus v is a fixed point of the period map q associated with the logistic equation (14). Therefore, by our assumptions, v is either 0 or v_0^*. Hence in this case (u,v) is either $(0,0)$ or $(0,v_0^*)$. This implies that the positive number r_0 can be found.

ii) $u \neq 0$: Then, by Theorem 5.1,

$$p^n(u) \to u_0^* \quad \text{in } X \text{ for } n \to \infty.$$

This implies $u \geq u_0^*$ in X^+ and, therefore, $u = u_0^*$. Thus in this case (u,v) is of the form (u_0^*, v) and therefore has a positive distance from $(0,0)$, so that the required number r_0 exists. Now the homotopy invariance of the fixed-point index implies that $i(h(t,\cdot), \mathbb{B}_{\mathcal{R}}((0,0),r_0), \mathcal{R})$ is independent of $t \in [0,1]$. Thus, by the definition of the local index,

$$i_{\mathcal{R}}(\mathbf{T}_0,(0,0)) = i((u_0^*,0), \mathbb{B}_{\mathcal{R}}((0,0),r_0), \mathcal{R}).$$

But since the constant map $(u_0^*,0)$ has no fixed point in $\mathbb{B}_{\mathcal{R}}((0,0),r_0)$, by the solution property of the index we conclude that

$$i((u_0^*,0), \mathbb{B}_{\mathcal{R}}((0,0),r_0), \mathcal{R}) = 0.$$

Hence $i_{\mathcal{R}}(\mathbf{T}_0,(0,0)) = 0$.

Next we consider the fixed point $(u_0^*,0)$: Let h be the compact homotopy between $\mathbf{T}_0$ and the constant map (u_0^*,v_0^*) given by

$$h : [0,1] \times \mathcal{R} \to \mathcal{R}, \quad h(t,\mathbf{x}) = t\mathbf{T}_0(\mathbf{x}) + (1-t)(u_0^*,v_0^*).$$

Again we show that we can find an $r_0 > 0$ such that

$$h(t,\mathbf{x}) \neq \mathbf{x} \quad \text{for } (t,\mathbf{x}) \in [0,1] \times \partial\mathbb{B}_{\mathcal{R}}((u_0^*,0),r_0).$$

Suppose that there exists $(t_0,(u,v)) \in [0,1] \times \mathcal{R}$ such that $h(t_0,(u,v)) = (u,v)$, or in components,

$$u = t_0 p(u) + (1-t_0)u_0^*, \quad v = t_0 q(v) + (1-t_0)v_0^*.$$

We distinguish the two possible cases:

i) $u = 0$: As above this implies $q(v) = v$. Therefore v is either 0 or v_0^*. Thus in this case (u,v) is either $(0,0)$ or $(0,v_0^*)$. This implies that the number r_0 can be found.

ii) $u \neq 0$: Then, since $u_0^* = p(u_0^*) \geq p(u)$, it follows that

$$u = t_0 p(u) + (1-t_0)u_0^* \geq p(u).$$

Since p is order preserving we obtain $u \geq p^n(u)$, $n \in \mathbb{N}$. Moreover, by Theorem 5.1,

$$p^n(u) \to u_0^* \quad \text{in } X \text{ for } n \to \infty.$$

Therefore $u \geq u_0^*$, which implies that $u = u_0^*$. Thus in this case $(u,v) = (u_0^*,v)$, and v satisfies

$$v = t_0 q(v) + (1-t_0)v_0^*.$$

By considering the cases $v = 0$ and $v \neq 0$, as for u, we conclude that v either equals 0 or v_0^*. Therefore (u, v) is either $(u_0^*, 0)$ or (u_0^*, v_0^*) and, of course, in both cases the required $r_0 > 0$ exists.

By the homotopy invariance of the fixed-point index, $i(h(t, \cdot), \mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0), \mathcal{R})$ is independent of $t \in [0, 1]$. Thus, by the definition of the local index,

$$i_{\mathcal{R}}(\mathbf{T}_0, (u^*, 0)) = i((u_0^*, v_0^*), \mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0), \mathcal{R}).$$

But since the constant map (u_0^*, v_0^*) has no fixed point in $\mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0)$, by the solution property of the index, we conclude that

$$i((u_0^*, v_0^*), \mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0), \mathcal{R}) = 0.$$

Hence $i_{\mathcal{R}}(\mathbf{T}_0, (u_0^*, 0)) = 0$.

For the fixed point $(0, v_0^*)$ we can proceed exactly in the same way.

b) To compute the local index of the fixed point $(0, 0)$ we use the arguments of a) to find

$$i_{\mathcal{R}}(\mathbf{Q}_0, (0, 0)) = 0.$$

We pass to the fixed point $(u_0^*, 0)$: We have to show that $i_{\mathcal{R}}(\mathbf{Q}_0, (u_0^*, 0)) = 1$. In Lemma 9.5 b) we proved that $\mathbf{Q}_0$ possesses precisely the three fixed points $(0, 0)$, $(u_0^*, 0)$ and $(0, v_0^*)$. We have just shown that $i_{\mathcal{R}}(\mathbf{Q}_0, (0, 0)) = 0$. Now we will prove that $i_{\mathcal{R}}(\mathbf{Q}_0, (0, v_0^*)) = 0$. Then since $i(\mathbf{Q}_0, \mathcal{R}, \mathcal{R}) = 1$, the assertion follows by the additivity of the fixed-point index.

To compute the local index $i_{\mathcal{R}}(\mathbf{Q}_0, (0, v_0^*))$, we can proceed exactly in the same way as for the trivial fixed point $(0, 0)$; i.e., we consider again the compact homotopy h between $\mathbf{Q}_0$ and the constant map $(u_0^*, 0)$. Then we find an $r_0 > 0$ such that

$$h(t, \mathbf{x}) \neq \mathbf{x} \quad \text{for} \quad (t, \mathbf{x}) \in [0, 1] \times \partial\mathbb{B}_{\mathcal{R}}((0, v_0^*), r_0).$$

Thus $i_{\mathcal{R}}(\mathbf{Q}_0, (0, v_0^*)) = 0$, by the homotopy invariance and the solution property of the index.

Finally, to compute the index $i_{\mathcal{R}}(\mathbf{S}, (0, v_0^*))$, we interchange the rôles of u and v. More precisely, instead of considering the parameter-dependent system (17) we consider the parameter-dependent system

$$\begin{cases} \partial_t u + \mathcal{A}_1 u = au - bg_1(u)u - h_1(u, v)u \\ \partial_t v + \mathcal{A}_2 v = dv - fg_2(v)v + \beta h_2(u)v \end{cases} \quad \text{in} \quad \mathbb{R}^N \times [0,\mathrm{T}],$$

where $\beta \in [0, 1]$, together with the corresponding period maps $\mathbf{P}_\beta$. Then we can proceed as for the fixed point $(u_0^*, 0)$.

c) The local index of the fixed point $(0, 0)$ can again be computed as in part a). For the fixed point $(u_0^*, 0)$ we consider the compact homotopy

$$h : [0, 1] \times \mathcal{R} \to \mathcal{R}, \qquad h(t, \mathbf{x}) = t\mathbf{Q}_0(\mathbf{x}) + (1 - t)(u_0^*, \hat{v}_0).$$

Note that $\hat{v}_0$ was introduced to define the invariant set $\mathcal{R}$. We show that this homotopy is admissible; i.e., we show that there exists a $r_0 > 0$, such that

$$h(t, \mathbf{x}) \neq \mathbf{x} \qquad \text{for } (t, \mathbf{x}) \in [0,1] \times \partial\mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0).$$

Suppose that there exists $(t_0, (u, v)) \in [0,1] \times \mathcal{R}$ such that $h(t_0, (u, v)) = (u, v)$. Since $\mathbf{Q}_0$ is of the form $\mathbf{Q}_0(u, v) = (p(u), q(u, v))$, the first component u satisfies

$$u = t_0 p(u) + (1 - t_0) u_0^*.$$

We distinguish the two possible cases :

i) $u \neq 0$: Then as in the proof of a) it follows that $u = u_0^*$.

ii) $u = 0$: In this case (u, v) is of the form $(0, v)$ and therefore has a positive distance to $(u_0^*, 0)$.

Thus we can assume that $u = u_0^*$. The equation for v now reads

$$v = t_0 q(u_0^*, v) + (1 - t_0)\hat{v}_0. \tag{19}$$

Note that $q(u_0^*, \cdot)$ is the period map associated with the logistic equation

$$\partial_t w + \mathcal{A}_2(t) w = (d + h_2(u^*)) w - f g_2(w) w \quad \text{ in } \mathbb{R}^N \times (0, \infty). \tag{20}$$

Moreover, by our assumption $r(S^{(2)}_{d+h_2(u^*)}) > 1$, this logistic equation possesses the unique positive T-periodic solution $\tilde{v}$. Since $(u, v) \in \mathcal{R}$ we have that $v \leq \hat{v}_0$, and the monotonicity of $q(u_0^*, \cdot)$ implies

$$q(u_0^*, v) \leq q(u_0^*, \hat{v}_0) \quad \text{ in } \; X^+. \tag{21}$$

Let w_1 be the solution of the logistic equation (20) to the initial value $\hat{v}_0$. Then we obtain

$$\begin{cases} \partial_t w_1 + \mathcal{A}_2(t) w_1 \leq (d + h_2^+(u^*)) w_1 - f g_2(w_1) w_1 & \text{in } \mathbb{R}^N \times (0, \infty), \\ w_1(0) = \hat{v}_0 & \text{in } \mathbb{R}^N. \end{cases}$$

The parabolic maximum principle now implies $w_1(T) \leq \hat{v}(T)$ in X^+. Since, by definition, $w_1(T) = q(u_0^*, \hat{v}_0)$ and since $\hat{v}(T) = \hat{v}_0$, we obtain $q(u_0^*, \hat{v}_0) \leq \hat{v}_0$ in X^+. Applying this inequality together with (21) to equality (19), we obtain $v \geq q(u_0^*, v)$ in X^+. Thus, by the monotonicity of $q(u_0^*, \cdot)$, $v \geq q^n(u_0^*, v)$, $n \in \mathbb{N}$. It remains to distinguish the two possible cases

i) $v = 0$: In this case $(u, v) = (u_0^*, 0)$ and therefore the homotopy h is admissible.

ii) $v \neq 0$: In this case we know by Theorem 5.1 that $q^n(u_0^*, v) \to \tilde{v}_0$ in $\mathbf{x}$ for $n \to \infty$. Thus $v \geq \tilde{v}_0$ in $\mathbf{x}^+$. Since points of the form (u_0^*, v) with $v \geq \tilde{v}_0$ are bounded away from $(u_0^*, 0)$, the homotopy h is admissible in this case, too.

The homotopy invariance of the index now implies

$$i_{\mathcal{R}}(\mathbf{Q}_0, (u_0^*, 0)) = i((u_0^*, \hat{v}_0), \mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0), \mathcal{R}) = 0,$$

since the constant map $(u_0^*, \hat{v}_0)$ has no fixed point in $\mathbb{B}_{\mathcal{R}}((u_0^*, 0), r_0)$. $\square$

Our main result now follows immediately by the additivity of the fixed-point index.

Theorem 9.7. *In the cases* I, II *and* III *there exists a coexistence solution of the system* (1).

Proof. Since $i(\mathbf{S},\mathcal{R},\mathcal{R}) = 1$, Proposition 9.6 and the additivity of the fixed-point index imply that, besides the trivial and the semitrivial fixed points, there exists a further fixed point of $\mathbf{S}$ in $\mathcal{R}$. Since, by Theorem 5.1, there cannot exist other semitrivial fixed points, it is indeed a coexistence fixed point. □

In the next remark we show how further monotonicity assumptions on the nonlinearities h_1 and h_2, which guarantee that system (1) is not quasimonotonous and hence does not have a preserved order structure on $\mathbf{X}^+$, reflect themselves in the structure of the set of coexistence solutions.

Remark 9.8. If we additionally assume that h_1 does not depend on u and that $h_1(x,t,\cdot)$ and $h_2(x,t,\cdot)$ are monotone increasing for each (x,t) in $\mathbb{R}^N \times [0,T]$, then coexistence solutions of the system (1) cannot be ordered. More precisely, if (u_1,v_1) and (u_2,v_2) are two coexistence solutions with $u_1 \le u_2$ or $v_1 \le v_2$ in $\mathbb{R}^N \times [0,T]$, then they coincide.

Proof. Let (u_1,v_1) and (u_2,v_2) be two coexistence solutions of the system (1) and suppose that $v_1 \le v_2$ in $\mathbb{R}^N \times [0,T]$. We first show that this implies $u_2 \le u_1$ in $\mathbb{R}^N \times [0,T]$. By the monotonicity of $h_1(x,t,\cdot)$, the component u_1 satisfies

$$\partial_t u_1 + \mathcal{A}_1(t)u_1 \ge au_1 - bg_1(u_1)u_1 - h_1(v_2)u_1 \quad \text{in } \mathbb{R}^N \times [0,T].$$

Thus u_1 is a supersolution for the logistic equation

$$\partial_t w + \mathcal{A}_1(t)w = (a - h_1(v_2))w - bg_1(w)w \quad \text{in } \mathbb{R}^N \times [0,T]\,. \tag{22}$$

Therefore, if we denote the solution of (22) with the initial value $u_1(0)$ by w, then the maximum principle implies $w \le u_1$ in $\mathbb{R}^N \times [0,\infty)$. But since u_2 is the unique positive T-periodic solution of (22), it follows by Theorem 5.1 that

$$w(t) - u_2(t) \to 0 \quad \text{in} \ \ X \text{ for } t \to \infty.$$

Therefore, by the periodicity of u_1, we conclude that $u_2 \le u_1$. Similarly one shows that $u_2 \le u_1$ implies $v_2 \le v_1$.

10. Convergence to trivial or semitrivial solutions. In this section we will give conditions guaranteeing that coexistence solutions for the system (1) do not exist. In the cases we will consider, the solutions of (1) to positive initial values will converge either to the trivial solution or to a semitrivial solution. Results of this type are often called "extinction results."

Theorem 10.1. *Assume that there exist positive constants c_0 and R_0 such that*

$$d(x,t) \le -c_0 \quad \textit{for} \ \ |x| \ge R_0 \ \ \textit{and} \ \ t \in [0,T]\,; \tag{23}$$

then the following assertions hold.

a) *Assume that*

$$r(S_a^{(1)}) \le 1 \quad \textit{and} \quad r(S_d^{(2)}) \le 1.$$

Then extinction to the trivial solution $(0,0)$ *occurs; i.e., for each* $\mathbf{w} \in \mathbf{X}^+$,

$$\mathbf{S}^n(\mathbf{w}) \to (0,0) \quad \text{in } \mathbf{X} \text{ as } n \to \infty\ .$$

b) *Assume that*

$$r(S_a^{(1)}) > 1 \qquad \text{and} \qquad r(S^{(2)}_{d+h_2^+(u^*)}) \leq 1.$$

Then convergence to the semitrivial solution $(u^*, 0)$ *occurs. More precisely, for each* $\mathbf{w}$ *in* $\dot{X}^+ \times \dot{X}^+$,

$$\mathbf{S}^n(\mathbf{w}) \to (u_0^*, 0) \quad \text{in } \mathbf{X} \text{ as } n \to \infty\ .$$

Proof. a) Let (u_0, v_0) be in $\mathbf{X}^+$ and let (u, v) be the solution of the system (1) with the initial value (u_0, v_0). Then the positivity of h_1 implies

$$\begin{cases} \partial_t u + \mathcal{A}_1(t)u \leq au - bg_1(u)u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Denote by w_1 the solution of the logistic equation (13) with the initial value u_0. By the maximum principle we obtain $u(x,t) \leq w_1(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. Since, by Theorem 5.1, $w_1(t) \to 0$ in X for $t \to \infty$, the same is true for u. Therefore, by our assumptions on h_2, to every positive ε we find a $t(\varepsilon)$ such that $h_2(x,t,u(x,t)) \leq \varepsilon$ in $\mathbb{R}^N \times (t(\varepsilon), \infty)$. Hence for v we obtain

$$\partial_t v + \mathcal{A}_2(t)v \leq dv - fg_2(v)v + \varepsilon v \quad \text{in } \mathbb{R}^N \times (t(\varepsilon), \infty).$$

Note that, by our assumption on d, the function $d + \varepsilon$ satisfies $(A5)$ for ε sufficiently small. Let $\tilde{v}_\varepsilon$ be the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_2(t)w = (d+\varepsilon)w - fg_2(w)w & \text{in } \mathbb{R}^N \times (t(\varepsilon), \infty), \\ w(t(\varepsilon)) = v(t(\varepsilon)) & \text{in } \mathbb{R}^N. \end{cases}$$

Then, by the maximum principle,

$$v \leq \tilde{v}_\varepsilon \quad \text{in } \mathbb{R}^N \times (t(\varepsilon), \infty).$$

Moreover, by Theorem 5.1,

$$\tilde{v}_\varepsilon(t) - v_\varepsilon^*(t) \to 0 \quad \text{in } X \text{ for } t \to \infty,$$

where v_ε^* is the unique positive T-periodic solution of the logistic equation

$$\partial_t w + \mathcal{A}_2(t)w = (d+\varepsilon)w - fg_2(w)w \quad \text{in } \mathbb{R}^N \times [0,T]$$

if $r(S^{(2)}_{d+\varepsilon}) > 1$, and $v^*_\varepsilon := 0$ otherwise. Now, by Proposition 5.2 and our assumption $r(S^{(2)}_d) \le 1$, it follows that

$$v^*_\varepsilon \to 0 \quad \text{in } C^{\eta,\frac{\eta}{2}}(\mathbb{R}^N \times [0,T]) \;\text{ as } \varepsilon \to 0,$$

for any $\eta \in (0,1)$. By the monotone dependence of v^*_ε on ε and since $v^*_\varepsilon(\cdot,t) \in X$ for $t \in [0,T]$, we actually have the desired stronger assertion $v^*_\varepsilon \to 0$ in $C([0,T],X)$ as $\varepsilon \to 0$. This implies that $v(t) \to 0$ in X as $t \to \infty$.

b) Let now (u_0, v_0) be in $\dot{X}_+ \times \dot{X}^+$ and define w_1 as in a). Then, by the maximum principle, $u \le w_1$ in $\mathbb{R}^N \times (0,\infty)$, and by Theorem 5.1

$$w_1(t) - u^*(t) \to 0 \quad \text{in } X \text{ for } t \to \infty.$$

Thus, by the periodicity of u^*, it follows that for every positive ε there exists a $t(\varepsilon)$ such that

$$u \le u^* + \varepsilon \quad \text{in } \; \mathbb{R}^N \times (t(\varepsilon),\infty). \tag{24}$$

By the monotonicity of $h_2^+(x,t,\cdot)$, this implies

$$\partial_t v + \mathcal{A}_2(t)v \le (d + h_2^+(u^*+\varepsilon))v - fg_2(v)v \quad \text{in } \; \mathbb{R}^N \times (t(\varepsilon),\infty).$$

Again we point out that, by our assumption on d, the function $d + h_2^+(u^*+\varepsilon)$ satisfies $(A5)$. Let $\tilde{v}_\varepsilon$ be the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_2(t)w = (d + h_2^+(u^*+\varepsilon))w - fg_2(w)w & \text{in } \mathbb{R}^N \times (t(\varepsilon),\infty), \\ w(t(\varepsilon)) = v(t(\varepsilon)) & \text{in } \mathbb{R}^N. \end{cases}$$

By the maximum principle $v \le \tilde{v}_\varepsilon$ in $\mathbb{R}^N \times (t(\varepsilon),\infty)$. Moreover, by Theorem 5.1,

$$\tilde{v}_\varepsilon(t) - v^*_\varepsilon \to 0 \quad \text{in } X \text{ for } t \to \infty,$$

where v^*_ε is the unique positive T-periodic solution of the logistic equation

$$\partial_t w + \mathcal{A}_2(t)w = (d + h_2^+(u^*+\varepsilon))w - fg_2(w)w \quad \text{in } \mathbb{R}^N \times [0,T]$$

if $r(S^{(2)}_{d+e(u^*+\varepsilon)}) > 1$, and zero otherwise. Therefore, for any positive δ we find a $t_1(\delta)$ such that $v \le v^*_\varepsilon + \delta$ in $\mathbb{R}^N \times (t_1(\delta),\infty)$. But since we assume that $r(S^{(2)}_{d+h_2^+(u^*)}) \le 1$, Proposition 5.2 and the argument given in a) imply $v^*_\varepsilon \to 0$ in $C([0,T],X)$ as $\varepsilon \to 0$. Hence we obtain $v(t) \to 0$ in X for $t \to \infty$. Thus, by our assumptions on h_1, to every positive ε we find a $t_2(\varepsilon)$ such that

$$h_1(x,t,u(x,t),v(x,t)) \le \varepsilon \quad \text{in } \; \mathbb{R}^N \times (t_2(\varepsilon),\infty).$$

For u this implies

$$\partial_t u + \mathcal{A}_1(t)u \ge au - bg_1(u)u - \varepsilon u \quad \text{in } \; \mathbb{R}^N \times (t_2(\varepsilon),\infty).$$

Denote by $\tilde{u}_\varepsilon$ the solution of the initial value problem

$$\begin{cases} \partial_t w + \mathcal{A}_1(t)w = (a-\varepsilon)w - bg_1(w)w & \text{in } \mathbb{R}^N \times (t_2(\varepsilon),\infty), \\ w(t_2(\varepsilon)) = u(t_2(\varepsilon)) & \text{in } \mathbb{R}^N. \end{cases}$$

By the maximum principle we obtain $u \geq \tilde{u}_\varepsilon$ in $\mathbb{R}^N \times (t_2(\varepsilon),\infty)$. Theorem 5.1 implies

$$\tilde{u}_\varepsilon(t) - u^*_\varepsilon(t) \to 0 \quad \text{in } X \text{ for } t \to \infty,$$

where u^*_ε is the unique positive T-periodic solution of the logistic equation

$$\partial_t w + \mathcal{A}_1(t)w = (a-\varepsilon)w - bg_1(w)w \quad \text{in } \mathbb{R}^N \times [0,T]$$

if $r(S^{(1)}_{a-\varepsilon}) > 1$, and zero otherwise. Proposition 5.2 then implies

$$u^*_\varepsilon \to u^* \quad \text{in } C([0,T],X) \text{ as } \varepsilon \to 0.$$

This together with (24) implies

$$u(t) - u^*(t) \to 0 \quad \text{in } X \text{ as } t \to \infty.$$

11. Applications. We show how our results can be applied to different population models from mathematical biology. In the following u and v can be interpreted as the population densities of two interacting species. We will consider different types of interaction of the species.

In the following c and e are nonnegative functions in $BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^N \times [0,T])$, which are T-periodic in time.

A Competition Model. We consider the following system:

$$\begin{cases} \partial_t u + \mathcal{A}_1(x,t)u = a(x,t)u - b(x,t)u^2 - c(x,t)uv \\ \partial_t v + \mathcal{A}_2(x,t)v = d(x,t)v - f(x,t)v^2 - e(x,t)uv \end{cases} \quad \text{in } \mathbb{R}^N \times [0,T].$$

This is a model for the evolution of the population densities of two species whose interaction is competitive. This is system (1) in the special case

$$h_1(x,t,u,v) = c(x,t)v \quad \text{and} \quad h_2(x,t,u) = -e(x,t)u\,.$$

For this special system the assumptions $(A10)$ of case I read

$$r(S^{(1)}_a) > 1, \quad r(S^{(2)}_d) > 1, \quad r(S^{(1)}_{a-cv^*}) > 1, \quad r(S^{(2)}_{d-eu^*}) > 1.$$

The assumptions $(A11)$ of case II are

$$r(S^{(1)}_a) > 1, \quad r(S^{(2)}_d) > 1, \quad r(S^{(1)}_{a-cv^*}) < 1, \quad r(S^{(2)}_{d-eu^*}) < 1.$$

Under the above two sets of assumptions we can apply Theorem 9.7 to obtain the existence of a coexistence solution for the competition system. We point out that the assumptions of case III cannot be satisfied for the competition system, since the positivity of e, the assumption $r(S_d^{(2)}) \leq 1$ and the monotonicity of the spectral radius imply $r(S_{d-eu^*}^{(2)}) \leq 1$. In fact, if we assume that (23) holds, then by Theorem 10.1 the existence of the two semitrivial solutions $(u^*, 0)$ and $(0, v^*)$ is a necessary condition for the existence of coexistence solutions of the competition system.

A predator-prey model. We consider the system

$$\begin{cases} \partial_t u + \mathcal{A}_1(x,t)u = a(x,t)u - b(x,t)u^2 - c(x,t)uv \\ \partial_t v + \mathcal{A}_2(x,t)v = d(x,t)v - f(x,t)v^2 + e(x,t)uv \end{cases} \quad \text{in } \mathbb{R}^N \times [0,T].$$

This system models the interaction of a predator population v with its prey u. This is system (1) in the special case

$$h_1(x,t,u,v) = c(x,t)v \quad \text{and} \quad h_2(x,t,u) = e(x,t)u\ .$$

The assumptions of case I are

$$r(S_a^{(1)}) > 1, \quad r(S_d^{(2)}) > 1, \quad r(S_{a-cv^*}^{(1)}) > 1.$$

The assumptions of case III are

$$r(S_a^{(1)}) > 1, \quad r(S_d^{(2)}) \leq 1, \quad r(S_{d+eu^*}^{(2)}) > 1.$$

Here we point out that, by the nonnegativity of e, the assumption $r(S_{d+eu^*}^{(2)}) < 1$ of case II cannot be satisfied for the predator-prey system. We also remark that in case I the assumption $r(S_{d+\alpha eu^*}^{(2)}) > 1$, $\alpha \in [0,1]$, is always satisfied, since we assume that $r(S_d^{(2)}) > 1$ and since e is nonnegative.

The next theorem shows that for the predator-prey system case I and case III are the only situations where coexistence solutions occur.

Theorem 11.1. *The predator-prey system possesses a T-periodic coexistence solution if and only if the assumptions of case* I *or the assumptions of case* III *are satisfied. Moreover, coexistence solutions cannot be ordered. More precisely, if (u_1, v_1) and (u_2, v_2) are two coexistence solutions with $u_1 \leq u_2$ or $v_1 \leq v_2$ in $\mathbb{R}^N \times [0,T]$, then they coincide.*

Proof. In the cases I and III the existence of a coexistence solution follows by Theorem 9.7. So let (u, v) be a coexistence solution of the predator-prey system. Then Proposition 4.1 f) implies

$$r(S_{a-bu-cv}^{(1)}) = 1 \quad \text{and} \quad r(S_{d+eu-fv}^{(2)}) = 1.$$

From this, by Proposition 4.1 e), we conclude that $r(S_a^{(1)}) > 1$. Since $u \leq u^*$ in $\mathbb{R}^N \times [0,T]$, it follows, again by Proposition 4.1 e), that $r(S_{d+eu^*}^{(2)}) > 1$. Now if $r(S_d^{(2)}) \leq 1$, the conditions of case III are satisfied.

So assume that $r(S_d^{(2)}) > 1$. This implies the existence of the semitrivial solution $(0, v^*)$, and by the maximum principle we obtain $v^* \leq v$ in $\mathbb{R}^N \times [0, T]$. Thus, by Proposition 4.1 e),

$$r(S^{(1)}_{a-cv^*}) > r(S^{(1)}_{a-bu-cv}) = 1,$$

so that the conditions of case I are satisfied. The assertion that coexistence solutions cannot be ordered follows by Remark 9.8.

A predator-prey model with Holling-Tanner interaction. We consider the system

$$\begin{cases} \partial_t u + \mathcal{A}_1(x,t)u = a(x,t)u - b(x,t)u^2 - c(x,t)\frac{uv}{\kappa+u} \\ \partial_t v + \mathcal{A}_2(x,t)v = d(x,t)v - f(x,t)v^2 + e(x,t)\frac{uv}{\kappa+u} \end{cases} \quad \text{in } \mathbb{R}^N \times [0,T].$$

Here κ is a positive constant. This type of interaction models the saturation of the predator v in the presence of a high prey population u. This is system (1) in the special case

$$h_1(x,t,u,v) = c(x,t)\frac{v}{\kappa+u} \quad \text{and} \quad h_2(x,t,u) = e(x,t)\frac{u}{\kappa+u} \, .$$

For this special case the assumptions of case I are $r(S_a^{(1)}) > 1$, $r(S_d^{(2)}) > 1$, $r(S^{(1)}_{a-c\frac{v^*}{\kappa}}) > 1$. The assumptions of case III read $r(S_a^{(1)}) > 1$, $r(S_d^{(2)}) \leq 1$, $r(S^{(2)}_{d+e\frac{u^*}{\kappa+u^*}}) > 1$. Again the assumptions of case II cannot be satisfied for this system. By Theorem 9.7 we obtain the existence of a coexistence solution for the cases I and III. Moreover, we have the following result.

Theorem 11.2. *Assume that $r(S_d^{(2)}) \leq 1$ holds. Then the Holling-Tanner system admits a coexistence solution if and only if*

$$r(S_a^{(1)}) > 1 \quad \textit{and} \quad r(S^{(2)}_{d+e\frac{u^*}{\kappa+u^*}}) > 1$$

is satisfied.

Proof. The existence of a coexistence solution follows by Theorem 9.7. So let (u, v) be a coexistence solution of the Holling-Tanner system. Then, by Proposition 4.1 f), $r(S^{(1)}_{a-bu-\frac{cv}{\kappa+u}}) = 1$. Hence $r(S_a^{(1)}) > 1$. Since $u \leq u^*$ in $\mathbb{R}^N \times [0,T]$ we also have that $\frac{u}{\kappa+u} \leq \frac{u^*}{\kappa+u^*}$. Therefore, by Proposition 4.1 e) and f), we obtain

$$r(S^{(2)}_{d+e\frac{u^*}{\kappa+u^*}}) \geq r(S^{(2)}_{d+e\frac{u}{\kappa+u}}) > r(S^{(2)}_{d-fv+e\frac{u}{\kappa+u}}) = 1 \, .$$

Concluding remarks. To obtain explicit criteria for the estimation of the spectral radius we refer to Proposition 5.1 in [18] and to Section 17 in [15]. Together with estimates on $\|u^*\|_\infty$ and $\|v^*\|_\infty$, which could be derived using the parabolic maximum principle, it is indeed possible to obtain explicit conditions on the coefficients of the above systems in order to satisfy the estimates on the corresponding spectral radii.

Similar estimates are given in [15, Sections 33–40] for problems on bounded domains. Hence we refrain from giving the details.

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