Advances in Differential Equations Volume 1, Number 4, July 1996, pp. 547 – 578

MAXIMAL ATTRACTOR AND INERTIAL SETS FOR A CONSERVED PHASE FIELD MODEL

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(Submitted by: Roger Temam)

1. Introduction. In this paper we consider the following conserved phase field model proposed by Caginalp ([7]):

$$P_0 \quad \begin{cases} \tau\varphi_t = -\xi^2\Delta(\xi^2\Delta\varphi - g(\varphi) + 2u) & \text{in } \Omega\times\mathbb{R}^+, \\ u_t + \frac{\ell}{2}\varphi_t = K\Delta u & \text{in } \Omega\times\mathbb{R}^+, \\ \frac{\partial\varphi}{\partial n} = \frac{\partial\Delta\varphi}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times\mathbb{R}^+, \\ \varphi(x,0) = \varphi_0(x),\ u(x,0) = u_0(x) & x\in\Omega, \end{cases}$$

where Ω is an open bounded set in $\mathbb{R}^n$, $n = 1, 2, 3$, with smooth boundary $\partial\Omega$. Here $g = G'$, where G is a double well potential for which $g(s) = \frac{1}{2}(s^3 - s)$ and the unknown functions u and φ denote respectively the temperature and the order parameter or phase field. The dimensionless temperature u is scaled so that $u = 0$ corresponds to the standard planar equilibrium melting temperature and φ is scaled so that φ near 1 corresponds to the liquid phase and φ near -1 corresponds to the solid phase. The interface between liquid and solid described by the phase field model has finite width and contains all points where φ vanishes. In fact, Problem P_0 can be viewed as an approximating problem for the Stefan problem with surface tension ([6, 7]).

The positive constants ℓ and K represent the dimensionless latent heat and the diffusivity respectivly. The positive constants τ and ξ represent a relaxation time and a correlation length.

Problem P_0 can be viewed as a conserved version of the standard second-order phase field equations. In the second-order version the internal energy $e = \int_\Omega (u + \frac{\ell}{2}\varphi)$ is conserved. In the fourth-order version both the internal energy e and the "total mass" $M = \int_\Omega \varphi$ are conserved quantities. For the sake of generality we assume in this paper that the function g has the slightly more general form

$$g(s) = \sum_{j=0}^{2p-1} a_j s^j \quad \text{with} \quad a_{2p-1} > 0,\ p \geq 2 \ \text{ if } \ n = 1, 2 \ \text{ and } \ p = 2 \ \text{ if } \ n = 3.$$

Received for publication February 1995.
AMS Subject Classifications: 35G25, 35K22, 35K45.

While the formulation in P_0 is standard, it is possible to reformulate the problem in terms of the enthalpy $v = u + \frac{\ell}{2}\varphi$ and the order parameter φ, and the new problem that one obtains is easier to handle mathematically. It has the following form:

$$P_1 \quad \begin{cases} \tau\varphi_t = -\xi^2\Delta(\xi^2\Delta\varphi - g(\varphi) - \ell\varphi + 2v) & \text{in } \Omega\times\mathbb{R}^+ \quad (1.1)\\ v_t = K\Delta v - \frac{K\ell}{2}\Delta\varphi & \text{in } \Omega\times\mathbb{R}^+ \quad (1.2)\\ \frac{\partial\varphi}{\partial n} = \frac{\partial\Delta\varphi}{\partial n} = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \quad (1.3)\\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \quad (1.4)\\ \varphi(x,0) = \varphi_0(x),\ v(x,0) = v_0(x) & x\in\Omega, \quad (1.5) \end{cases}$$

where $v_0 = u_0 + \frac{\ell}{2}\varphi_0$.

For a mathematical study of Problem P_1, we introduce a Hilbert space H and associate to Problem P_1 a semigroup of operators $S(t)$ defined by $(\varphi(t), v(t)) = S(t)(\varphi_0, v_0)$ such that $(\varphi, v) = S(.)(\varphi_0, v_0) \in C(\mathbb{R}^+; H)$, and such that the mapping $(\varphi_0, v_0) \mapsto (\varphi(t), v(t))$ is a continuous operator from H into H. In analogy with the Cahn-Hilliard equation, which correspond to gradient flow in $(H^1(\Omega))'$ ([11]), it is possible to show that Problem P_1 corresponds to gradient flow in $(H^1(\Omega))' \times L^2(\Omega)$. Hence a natural space to work with appears to be: $H = (H^1(\Omega))' \times L^2(\Omega))$. Additionally, it will be convenient to define the space:

$$V = \{(\varphi, v) \in H^2(\Omega)\times H^1(\Omega); \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\}.$$

In what follows we suppose that $(\varphi_0, v_0) \in H$.

Furthermore, an essential remark for the study of Problem P_1 is that the space integrals of both φ and v are conserved in time, which we express in the form

$$\langle\varphi(t), 1\rangle_{(H^1)',H^1} = \langle\varphi_0, 1\rangle_{(H^1)',H^1} \qquad \text{for all } t > 0$$

and

$$\int_\Omega v(x,t)\,dx = \int_\Omega v_0(x)\,dx \qquad \text{for all } t > 0.$$

Hence it is convenient to introduce the function spaces

$$H_{\beta\gamma} = \{(\varphi, v) \in H;\ \frac{1}{|\Omega|}\langle\varphi, 1\rangle_{(H^1)',H^1} = \beta,\ \frac{1}{|\Omega|}\int_\Omega v = \gamma\}$$

$$\tilde{H}_\beta = \{\varphi \in L^2(\Omega);\ \frac{1}{|\Omega|}\int_\Omega \varphi(x) = \beta\}$$

$$\text{and} \quad \mathcal{H}_\alpha = \bigcup_{|\beta|,|\gamma|\le\alpha} H_{\beta\gamma}.$$

As already mentioned Problem P_1 is a gradient flow, and it has the following Lyapunov functional:

$$J(\varphi, v) = \int_\Omega \Big(\frac{\xi^4}{2}(\nabla\varphi)^2 + \xi^2 G(\varphi) + \frac{\ell\xi^2}{2}\varphi^2 - 2\xi^2 v\varphi + \frac{2\xi^2}{\ell}v^2\Big), \qquad (1.6)$$

where $G(s) = \int_0^s g(\tau)d\tau$. This property will be an essential tool in the proofs.

After a preliminary section where we recall two equivalent definitions of a scalar product in $(H^1(\Omega))'$, we show in Section 3 that there exists a unique solution $(\varphi(t), v(t)) = S(t)(\varphi_0, v_0)$ of Problem P such that $S(t)$ satisfies the usual semigroup properties and such that for each $t > 0$, $S(t)$ is a Lipschitz continuous operator from H into itself. In fact, as a consequence of these properties, one can also associate with Problem P_1 a semigroup in $(L^2(\Omega))^2$. In order to establish this fact we give a proof which is due in part to A. Debussche.

The semigroup $S(t)$ maps $\mathcal{H}_\alpha$ into itself for each $\alpha \geq 0$. In Section 4 we show that there exist absorbing sets in $\mathcal{H}_\alpha$ and in $(H^2(\Omega))^2 \cap \mathcal{H}_\alpha$. This implies that the semigroup possesses a maximal attractor $\mathcal{A}_\alpha$ in $\mathcal{H}_\alpha$, namely, a compact invariant set that attracts every bounded set of $\mathcal{H}_\alpha$. Moreover $\mathcal{A}_\alpha$ is connected.

In Section 5, we prove the existence of inertial sets, namely, compact sets which contain the attractor, which are positively invariant by the semigroup, which have a finite fractal dimension and which attract all solutions at an exponential rate so that we also obtain an upper bound for the fractal dimension of the attractor.

In Section 6 we prove that the solution becomes immediately very smooth, that is to say, that $(\varphi, v) \in (C^\infty(\bar{\Omega} \times (0,\infty)))^2$. In fact there exist bounded absorbing sets in $(C^m(\bar{\Omega}))^2$ for all $m \in \mathbb{N}^+$. This implies in particular that any function in the attractor belongs to $(C^\infty(\bar{\Omega}))^2$.

Our methods of proofs strongly rely on those of Temam ([14]), Nicolaenko, Scheurer and Temam ([13]), Eden, Foias, Nicolaenko and Temam ([9, 10]), and Eden, Milani and Nicolaenko ([8]). For similar studies of Caginalp's second-order phase field model, we refer to Brochet, Chen and Hilhorst ([2]) and Bates and Zheng ([4]).

In a forthcoming paper ([3]), we give similar results for a model for simultaneous order-disorder and phase separation which has been recently derived by Cahn and Novick-Cohen ([5]).

2. Preliminaries. Two alternative definitions for the inner product in $(H^1(\Omega))'$.

(i) We denote by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions and by w_k, $k = 1, \ldots,$ the corresponding eigenfunctions such that $|w_k|_{L^2(\Omega)} = 1$, $k = 1, \ldots.$ (Note that $w_1 = \frac{1}{\sqrt{|\Omega|}}$.) The $\{w_k\}$ are a complete orthonormal family in $L^2(\Omega)$ as well as a complete orthogonal family in $H^1(\Omega)$. Next we define the following scalar product in $(H^1(\Omega))'$:

$$(u, v)_{-1} = \langle u, w_1\rangle\langle v, w_1\rangle + \sum_{j=2}^{\infty} \frac{1}{\lambda_j}\langle u, w_j\rangle\langle v, w_j\rangle, \tag{2.1}$$

where the notation $\langle\cdot,\cdot\rangle$ is used to denote the duality product between $H^1(\Omega)$ and $(H^1(\Omega))'$. In what follows, we shall sometimes use the notation $(\cdot,\cdot)$ for the inner product in $L^2(\Omega)$.

We deduce from the density of $L^2(\Omega)$ in $(H^1(\Omega))'$ and from (2.1) that the $\{w_k\}$ are a complete orthogonal family in $(H^1(\Omega))'$. Note that we also have that for all

$\varphi \in (H^1(\Omega))'$

$$\varphi = (\varphi, w_1)_{-1} w_1 + \sum_{j=2}^{\infty} \lambda_j (\varphi, w_j)_{-1} w_j = \sum_{j=1}^{\infty} \langle \varphi, w_j \rangle w_j$$

and that

$$\langle w_i, w_j \rangle = \int_\Omega w_i w_j = \delta_{ij} \qquad i, j = 1, \dots .$$

(ii) For $u \in (H^1(\Omega))'$, we define

$$m(u) = \frac{1}{|\Omega|} \langle u, 1 \rangle$$

and

$$\bar{u} = u - m(u) \tag{2.2}$$

and give an alternative definition for the scalar product in $(H^1(\Omega))'$. For $u \in (H^1(\Omega))'$, let

$$\psi = Nu \tag{2.3}$$

be the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} -\Delta\psi = \bar{u} & \text{in the sense of distributions in } \Omega \\ \frac{\partial \psi}{\partial n} = 0 & \text{in the sense of distributions on } \partial\Omega \\ \int_\Omega \psi(x)\, dx = 0. \end{cases}$$

If $u, v \in (H^1(\Omega))'$ and if $\psi = Nu$ and $\chi = Nv$, then

$$(u, v)_{-1} = \frac{1}{|\Omega|} \langle u, 1 \rangle \langle v, 1 \rangle + \int_\Omega \nabla\psi \nabla\chi \tag{2.4}$$

and

$$\|u\|_{-1}^2 = |\Omega| (m(u))^2 + \int_\Omega |\nabla\psi|^2. \tag{2.5}$$

3. Existence of the semigroup. In this section we show the following well-posedness result:

Theorem 3.1. (i) *For any $(\varphi_0, v_0) \in H_{\beta\gamma}$, Problem P_1 has a unique solution (φ, v) which satisfies*

$$(\varphi, v) \in L^\infty(0, T; H_{\beta\gamma}) \cap L^2(0, T; (H^1(\Omega))^2), \quad \varphi \in L^{2p}(Q_T)$$

for all $T > 0$, where $Q_T := \Omega \times (0, T)$ and $(\varphi, v) \in C(\mathbb{R}^+; H_{\beta\gamma})$. If furthermore $\varphi_0 \in \tilde{H}_\beta$, then

$$\varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

(ii) *If* $(\varphi_0, v_0) \in V \cap H_{\beta\gamma}$*, then*

$$(\varphi, v) \in L^\infty(0,T;V) \cap L^2(0,T;H^4(\Omega)\times H^2(\Omega)), \quad (\varphi_t, v_t) \in L^2(Q_T).$$

(iii) *For any* $(\varphi_0, v_0) \in H_{\beta\gamma}$, $(\varphi, v) \in (C^\infty(\bar\Omega \times (0,\infty)))^2$*. Moreover, the mapping* $S(t) : (\varphi_0, v_0) \mapsto (\varphi(t), v(t))$ *is Hölder continuous with exponent* $\frac{1}{2}$ *on* H *for all* $t > 0$ *and* $\{S(t)\}_{t\geq 0}$ *is a semigroup on* $\mathcal{H}_\alpha$.

Finally the functional $J(\varphi(t), v(t))$ *defined in* (1.6) *decays along orbits so that it is a Lyapunov function for Problem* P_1.

Remark 3.2. The proof of Theorem 3.1 (iii) is postponed to Section 6.

Proof of Theorem 3.1. The proof relies on the Galerkin method. For each integer m we look for an approximate solution (φ_m, v_m) of the form

$$\varphi_m(t) = \frac{\beta}{\sqrt{|\Omega|}} w_1 + \sum_{i=2}^{m} \varphi_{im}(t) w_i, \quad v_m(t) = \frac{\gamma}{\sqrt{|\Omega|}} w_1 + \sum_{i=2}^{m} v_{im}(t) w_i$$

satisfying

$$\tau \int_\Omega \varphi_{mt} w_j + \xi^2 \int_\Omega (\xi^2 \Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)\Delta w_j = 0 \tag{3.1}$$

$$\int_\Omega v_{mt} w_j + K \int_\Omega \nabla v_m \nabla w_j = \frac{K\ell}{2} \int_\Omega \nabla\varphi_m \nabla w_j \tag{3.2}$$

for $j = 1, \dots, m$ and

$$\varphi_m(0) = \sum_{j=1}^{m} < \varphi_0, w_j > w_j, \quad v_m(0) = \sum_{j=1}^{m} (v_0, w_j) w_j. \tag{3.3}$$

Problem (3.1)–(3.3) is an initial value problem for a system of $2m$ ordinary differential equations, so that it has a unique solution (φ_m, v_m) on some interval $(0, T_m)$, $T_m > 0$; in fact the a priori estimates below show that $T_m = +\infty$.

We first recall some properties of the polynomial function g which will be useful in what follows.

(i) There exists a constant C_1 such that

$$g(s)s \geq \frac{3}{4} a_{2p-1} s^{2p} - C_1 \quad \text{for all } s \in \mathbb{R}. \tag{3.4}$$

(ii) For every $\varepsilon > 0$, there exists a constant $C_2 = C_2(\varepsilon)$ such that

$$|g(s)| \leq \varepsilon a_{2p-1} s^{2p} + C_2 \quad \text{for all } s \in \mathbb{R}. \tag{3.5}$$

(iii) There exists a positive constant C such that

$$g'(s) \geq -C \quad \text{for all } s \in \mathbb{R}. \tag{3.6}$$

We multiply the differential equations in (3.1) by $\frac{\varphi_{jm}(t)}{\lambda_j}$ for $j = 2, \dots, m$ and sum on j, setting

$$\bar{\varphi}_m(t) = \varphi_m(t) - \frac{1}{|\Omega|}\int_\Omega \varphi_m(t) = \sum_{j=2}^m \varphi_{jm}(t) w_j.$$

This gives

$$\tau \int_\Omega \varphi_{mt} \sum_{j=2}^m \frac{\varphi_{jm}(t)}{\lambda_j} w_j + \xi^2 \int_\Omega (\xi^2 \Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)\Delta(\sum_{j=2}^m \frac{\varphi_{jm}(t)}{\lambda_j} w_j) = 0,$$

so that

$$\tau \int_\Omega \varphi_{mt} N(\bar{\varphi}_m) - \xi^2 \int_\Omega (\xi^2 \Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)\bar{\varphi}_m = 0,$$

which in turn implies

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi_m\|_{-1}^2 + \xi^4 \int_\Omega (\nabla\varphi_m)^2 + \ell\xi^2 \int_\Omega \bar{\varphi}_m^2 = -\xi^2 \int_\Omega g(\varphi_m)\bar{\varphi}_m + 2\xi^2 \int_\Omega v_m \bar{\varphi}_m. \quad (3.7)$$

Then we use (3.4) and (3.5) to obtain

$$\begin{aligned} -\int_\Omega g(\varphi_m)\bar{\varphi}_m &= -\int_\Omega g(\varphi_m)\varphi_m + \int_\Omega g(\varphi_m) m(\varphi_m) \\ &\le -\frac{3}{4} a_{2p-1} \int_\Omega {\varphi_m}^{2p} + C + |m(\varphi_m)| \int_\Omega |g(\varphi_m)| \\ &\le -\frac{3}{4} a_{2p-1} \int_\Omega \varphi_m^{2p} + C + \varepsilon|\beta| a_{2p-1} \int_\Omega {\varphi_m}^{2p} \le -\frac{1}{2} a_{2p-1} \int_\Omega {\varphi_m}^{2p} + C, \end{aligned}$$

where we have chosen $\varepsilon = \frac{1}{4|\beta|}$. Furthermore we have that

$$\begin{aligned} 2\int_\Omega v_m \bar{\varphi}_m &= 2\int_\Omega v_m \varphi_m - 2\; m(\varphi_m)\int_\Omega v_m \\ &\le \varepsilon \int_\Omega {v_m}^2 + C_\varepsilon \int_\Omega {\varphi_m}^2 + 2|\beta||\gamma||\Omega| \le \varepsilon \int_\Omega {v_m}^2 + C_\varepsilon \delta \int_\Omega {\varphi_m}^{2p} + \tilde{C}_\varepsilon. \end{aligned}$$

Choosing $\delta < \frac{a_{2p-1}}{4C_\varepsilon}$, we get

$$2\int_\Omega v_m \bar{\varphi}_m \le \varepsilon \int_\Omega {v_m}^2 + \frac{a_{2p-1}}{4} \int_\Omega {\varphi_m}^{2p} + \tilde{C}_\varepsilon,$$

so that finally

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi_m\|_{-1}^2 + \xi^4 \int_\Omega (\nabla\varphi_m)^2 + \ell\xi^2 \int_\Omega \varphi_m^2 + \frac{\xi^2 a_{2p-1}}{4} \int_\Omega {\varphi_m}^{2p} \le \varepsilon\xi^2 \int_\Omega {v_m}^2 + C(\beta). \quad (3.8)$$

Next we multiply the differential equations in (3.2) by $v_{jm}(t)$ for $j = 1, \dots, m$ and sum on j to obtain

$$\int_\Omega v_{mt} v_m + K \int_\Omega (\nabla v_m)^2 = \frac{K\ell}{2} \int_\Omega \nabla\varphi_m \nabla v_m \le \frac{K}{2} \int_\Omega (\nabla v_m)^2 + \frac{K\ell^2}{8} \int_\Omega (\nabla\varphi_m)^2. \quad (3.9)$$

Adding (3.8) to the product of (3.9) by $\frac{4\xi^4}{K\ell^2}$ gives

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\|\varphi_m\|_{-1}^2 &+ \frac{2\xi^4}{K\ell^2}\frac{d}{dt}\int_\Omega {v_m}^2 + \frac{\xi^4}{2}\int_\Omega (\nabla\varphi_m)^2 + \frac{2\xi^4}{\ell^2}\int_\Omega (\nabla v_m)^2 \\ &+ \ell\xi^2 \int_\Omega \varphi_m^2 + \frac{\xi^2 a_{2p-1}}{4}\int_\Omega {\varphi_m}^{2p} \le \varepsilon\xi^2 \int_\Omega {v_m}^2 + C(\beta). \qquad (3.10)\end{aligned}$$

By Poincare's inequality we have that

$$\begin{aligned}\varepsilon \int_\Omega {v_m}^2 &= \varepsilon \int_\Omega (v_m - m(v_m))^2 + \varepsilon|\Omega|(m(v_m))^2 \\ &\le \varepsilon C(\Omega) \int_\Omega (\nabla v_m)^2 + \varepsilon|\Omega|(m(v_m))^2.\end{aligned}$$

Then we choose $\varepsilon = \frac{\xi^2}{2\ell^2 C(\Omega)}$ to obtain

$$\begin{aligned}\frac{d}{dt}\{\frac{\tau}{2}\|\varphi_m\|_{-1}^2 &+ \frac{2\xi^4}{K\ell^2}\int_\Omega {v_m}^2\} + \frac{\xi^4}{2}\int_\Omega (\nabla\varphi_m)^2 + \frac{\xi^4}{4\ell^2 C(\Omega)}\int_\Omega {v_m}^2 \\ &+ \frac{\xi^4}{4\ell^2}\int_\Omega (\nabla v_m)^2 + \ell\xi^2\int_\Omega \varphi_m^2 + \frac{\xi^2 a_{2p-1}}{4}\int_\Omega {\varphi_m}^{2p} \le C(\beta). \qquad (3.11)\end{aligned}$$

Thus there exists a constant C depending on β and γ and not on T such that

$$\|(\varphi_m, v_m)\|_{L^\infty(0,T;H_{\beta\gamma})} \le C(1 + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}),$$

$$\|(\varphi_m, v_m)\|_{L^2(0,T;(H^1(\Omega))^2)},\ \|\varphi_m\|_{L^{2p}(Q_T)} \le C(T + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}). \qquad (3.12)$$

We multiply the differential equations in (3.1) by φ_{jm} for $j = 1, \dots, m$ and sum on j to obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega \varphi_m^2 + \xi^4 \int_\Omega (\Delta\varphi_m)^2 + \ell\xi^2 \int_\Omega (\nabla\varphi_m)^2 = \xi^2 \int_\Omega g(\varphi_m)\Delta\varphi_m - 2\xi^2 \int_\Omega v_m \Delta\varphi_m. \qquad (3.13)$$

We have, using (3.6),

$$\begin{aligned}\xi^2 \int_\Omega g(\varphi_m)\Delta\varphi_m &= -\xi^2 \int_\Omega g'(\varphi_m)(\nabla\varphi_m)^2 \le C\xi^2 \int_\Omega (\nabla\varphi_m)^2 \\ &= -C\xi^2 \int_\Omega \varphi_m \Delta\varphi_m \le \frac{\xi^4}{4}\int_\Omega (\Delta\varphi_m)^2 + C^2 \int_\Omega \varphi_m^2\end{aligned}$$

and

$$-2\xi^2 \int_\Omega v_m \Delta\varphi_m \le 4\int_\Omega {v_m}^2 + \frac{\xi^4}{4}\int_\Omega (\Delta\varphi_m)^2$$

so that we deduce from (3.13) that

$$\begin{aligned}
&\|\varphi_m\|_{L^\infty(0,T;L^2(\Omega))} \le C(1+\|(\varphi_0,v_0)\|_{L^2(\Omega)^2}),\\
&\|\varphi_m\|_{L^2(0,T;H^2(\Omega))} \le C(T+\|(\varphi_0,v_0)\|_{L^2(\Omega)^2}).
\end{aligned} \tag{3.14}$$

Moreover it follows from (3.11) that $\int_t^{t+\delta}\int_\Omega \varphi_m^2 \le C$ so that applying the uniform Gronwall lemma to (3.13) we deduce that for all $\delta > 0$ there exists $C = C(\delta)$ such that

$$\|\varphi_m\|_{L^\infty(\delta,T;L^2(\Omega))}, \|\varphi_m\|_{L^2(\delta,T;H^2(\Omega))} \le C(T+\|(\varphi_0,v_0)\|_{H_{\beta\gamma}}). \tag{3.15}$$

Next we show the a priori estimates necessary to prove Theorem 3.1(ii). We consider the functional

$$J(\varphi_m,v_m) = \int_\Omega (\frac{\xi^4}{2}(\nabla\varphi_m)^2 + \xi^2 G(\varphi_m) + \frac{\ell\xi^2}{2}{\varphi_m}^2 - 2\xi^2 v_m\varphi_m + \frac{2\xi^2}{\ell}v_m^2)$$

and show that it is a Lyapunov functional for the problem (3.1)–(3.3). We have that

$$\begin{aligned}
\frac{d}{dt}J(\varphi_m,v_m) = \int_\Omega (&-\xi^4\Delta\varphi_m\varphi_{mt} + \xi^2 g(\varphi_m)\varphi_{mt} + \ell\xi^2\varphi_m\varphi_{mt}\\
&- 2\xi^2 v_m\varphi_{mt} - 2\xi^2\varphi_m v_{mt} + \frac{4\xi^2}{\ell}v_m v_{mt})
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}J(\varphi_m,v_m) = &\int_\Omega (-\xi^4\Delta\varphi_m + \xi^2 g(\varphi_m) + \ell\xi^2\varphi_m - 2\xi^2 v_m)\varphi_{mt}\\
&+ \int_\Omega (-2\xi^2\varphi_m + \frac{4\xi^2}{\ell}v_m)v_{mt}.
\end{aligned} \tag{3.16}$$

In order to compute the right-hand side of (3.16), we multiply the differential equations in (3.1) by $\frac{{\varphi'}_{jm}(t)}{\lambda_j}$, $j = 2, \dots, m$, and sum on j. This gives

$$\tau\int_\Omega \varphi_{mt}\sum_{j=2}^m \frac{{\varphi'}_{jm}(t)}{\lambda_j}w_j + \xi^2\int_\Omega (\xi^2\Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)\Delta(\sum_{j=2}^m \frac{{\varphi'}_{jm}(t)}{\lambda_j}w_j) = 0$$

and thus

$$\tau\int_\Omega \varphi_{mt}N(\sum_{j=2}^m {\varphi'}_{jm}(t)w_j) - \xi^2\int_\Omega (\xi^2\Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)(\sum_{j=2}^m {\varphi_{jm}}'(t)w_j) = 0;$$

that is,

$$\tau \int_\Omega \varphi_{m t} N(\bar{\varphi}_{mt}) - \xi^2 \int_\Omega (\xi^2 \Delta\varphi_m - g(\varphi_m) - \ell\varphi_m + 2v_m)\bar{\varphi}_{mt} = 0. \tag{3.17}$$

Next we deduce from (3.2) that

$$\int_\Omega v_{mt} v_m + K \int_\Omega (\nabla v_m)^2 = \frac{K\ell}{2} \int_\Omega \nabla\varphi_m \nabla v_m,$$

and

$$\int_\Omega v_{mt}\varphi_m + K \int_\Omega \nabla v_m \nabla\varphi_m = \frac{K\ell}{2} \int_\Omega (\nabla\varphi_m)^2,$$

so that

$$\begin{aligned}\int_\Omega (-2\xi^2\varphi_m + \frac{4\xi^2}{\ell} v_m) v_{mt} &= 4\xi^2 K \int_\Omega \nabla v_m \nabla\varphi_m - \xi^2 K\ell \int_\Omega (\nabla\varphi_m)^2 - \frac{4\xi^2 K}{\ell} \int_\Omega (\nabla v_m)^2 \\ &= -\xi^2 K \int_\Omega (\sqrt{\ell}\nabla\varphi_m - \frac{2}{\sqrt{\ell}} \nabla v_m)^2. \end{aligned} \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16) gives

$$\frac{d}{dt} J(\varphi_m, v_m) = -\tau \|\bar{\varphi}_{mt}\|_{-1}^2 - \xi^2 K \int_\Omega (\sqrt{\ell}\nabla\varphi_m - \frac{2}{\sqrt{\ell}} \nabla v_m)^2.$$

Thus J decreases along trajectories and hence

$$\|\varphi_m\|_{L^\infty(0,\infty;H^1(\Omega))} \le C(1 + \|(\varphi_0, v_0)\|_V). \tag{3.19}$$

Furthermore, since

$$\frac{d}{dt} J(\varphi_m, v_m) \le 0, \;\; J(\varphi_m, v_m) \ge -C,$$

and since by (3.11)

$$\int_t^{t+\delta} J(\varphi_m, v_m) \le C(1 + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}),$$

it follows from the uniform Gronwall lemma that

$$\|\varphi_m\|_{L^\infty(\delta,\infty;H^1(\Omega))} \le C(1 + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}). \tag{3.20}$$

Next we multiply the differential equations in (3.2) by $\lambda_j v_{jm}$, for $j = 1, \dots, m$, and sum on j to obtain

$$\frac{1}{2}\frac{d}{dt} \int_\Omega (\nabla v_m)^2 + K \int_\Omega (\Delta v_m)^2 = \frac{K\ell}{2} \int_\Omega \Delta\varphi_m \Delta v_m \le \frac{K}{2} \int_\Omega (\Delta v_m)^2 + \frac{K\ell^2}{8} \int_\Omega (\Delta\varphi_m)^2. \tag{3.21}$$

Adding (3.21) with $\frac{K\ell^2}{2\xi^4}\cdot$(3.13), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\nabla v_m)^2+\frac{\tau K\ell^2}{4\xi^4}\frac{d}{dt}\int_\Omega\varphi_m^2+\frac{K}{2}\int_\Omega(\Delta v_m)^2+\frac{K\ell^2}{4}\int_\Omega(\Delta\varphi_m)^2$$
$$\leq\frac{2K\ell^2}{\xi^4}\int_\Omega {v_m}^2+\frac{C^2K\ell^2}{2\xi^4}\int_\Omega\varphi_m^2. \tag{3.22}$$

Using (3.12), (3.14) and integrating in time we deduce that

$$\|v_m\|_{L^\infty(0,\infty;H^1(\Omega))}\leq C(1+\|(\varphi_0,v_0)\|_V),$$
$$\|v_m\|_{L^2(0,T;H^2(\Omega))}\leq C(T+\|(\varphi_0,v_0)\|_V), \tag{3.23}$$

and using (3.15) and the uniform Gronwall lemma gives

$$\|v_m\|_{L^\infty(\delta,\infty;H^1(\Omega))}\leq C(1+\|(\varphi_0,v_0)\|_{H_{\beta\gamma}}),$$
$$\|v_m\|_{L^2(\delta,T;H^2(\Omega))}\leq C(T+\|(\varphi_0,v_0)\|_{H_{\beta\gamma}}), \tag{3.24}$$

where the positive constant C depends on δ.

Next we multiply the differential equations in (3.1) by ${\lambda_j}^2\varphi_{jm}(t)$ for $j=1,\ldots,m$ and sum on j. This gives

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega(\Delta\varphi_m)^2+\xi^4\int_\Omega(\Delta^2\varphi_m)^2+\ell\xi^2\int_\Omega(\nabla\Delta\varphi_m)^2$$
$$=-2\xi^2\int_\Omega\Delta v_m\Delta^2\varphi_m+\xi^2\int_\Omega\Delta(g(\varphi_m))\Delta^2\varphi_m. \tag{3.25}$$

We now estimate the terms on the right-hand side of (3.25). We have that

$$2\xi^2\int_\Omega\Delta v_m\Delta^2\varphi_m\leq\frac{\xi^4}{8}\int_\Omega(\Delta^2\varphi_m)^2+8\int_\Omega(\Delta v_m)^2 \tag{3.26}$$

and

$$\xi^2\int_\Omega\Delta g(\varphi_m)\Delta^2\varphi_m\leq\frac{\xi^4}{4}\int_\Omega(\Delta^2\varphi_m)^2+\int_\Omega(\Delta g(\varphi_m))^2.$$

Furthermore, it follows from the inequality (1.40) in [13] that

$$\int_\Omega(\triangle g(\varphi_m))^2\leq C_1+\epsilon||\varphi_m||^2_{H^4(\Omega)},$$

where p is arbitrary if $n=1,2$ and $p=2$ if $n=3$. Since by Theorem 6.1, $||\cdot||_{D(A^2)}$ and $||\cdot||_{H^4(\Omega)}$ are equivalent norms on $D(A^2)$, and since

$$||\varphi_m||^2_{L^2(\Omega)}\leq\beta^2|\Omega|^{-1}+C\int_\Omega(\triangle^2\varphi_m)^2,$$

we obtain that

$$\int_\Omega (\triangle g(\varphi_m))^2 \le \frac{\xi^4}{2}\int_\Omega (\triangle^2 \phi_m)^2 + C(\beta) \tag{3.27}$$

where p is arbitrary if $n = 1, 2$ and $p = 2$ if $n = 3$.

Finally (3.25) implies that

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta\varphi_m)^2 + \frac{\xi^4}{8}\int_\Omega (\Delta^2\varphi_m)^2 \le 8\int_\Omega (\Delta v_m)^2 + C(\beta). \tag{3.28}$$

Adding (3.28), $\frac{4C(\beta)}{\xi^4}\cdot$ (3.13) and $\frac{32}{K}\cdot$ (3.22), we obtain

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta\varphi_m)^2 + C\frac{d}{dt}\int_\Omega \varphi_m^2 + \frac{16}{K}\frac{d}{dt}\int_\Omega (\nabla v_m)^2 + \frac{\xi^4}{8}\int_\Omega (\Delta^2\varphi_m)^2 \\ + C_1\int_\Omega (\Delta\varphi_m)^2 + 8\int_\Omega (\Delta v_m)^2 \le C_2(\int_\Omega {v_m}^2 + \int_\Omega \varphi_m^2) + C(\beta),\end{aligned} \tag{3.29}$$

which in turn implies that

$$\|(\varphi_m, v_m)\|_{L^\infty(0,T;V)} \le C(1 + \|(\varphi_0, v_0)\|_V), \tag{3.30}$$

$$\|(\varphi_m, v_m)\|_{L^2(0,T;H^4(\Omega)\times H^2(\Omega))} \le C(T + \|(\varphi_0, v_0)\|_V), \tag{3.31}$$

and that

$$\|(\varphi_m, v_m)\|_{L^\infty(\delta,T;V)\cap L^2(\delta,T;H^4(\Omega)\times H^2(\Omega))} \le C(\delta)(T + \|(\varphi_0, v_0)\|_H). \tag{3.32}$$

Multiplying (3.1) and (3.2) by $\varphi'_{jm}(t)$ and $v'_{jm}(t)$ respectively, we then sum on $j = 1, \dots, m$. Then using (3.27) and (3.31), we deduce that

$$\|(\varphi_{mt}, v_{mt})\|_{(L^2(Q_T))^2} \le C(T + \|(\varphi_0, v_0)\|_V). \tag{3.33}$$

Next we complete the proof of existence of a solution of Problem P_1. To begin with, we give the existence proof in the case that the initial condition $(\varphi_0, v_0) \in V \cap H_{\beta\gamma}$. From the uniformly bounded sequence of functions $\{(\varphi_m, v_m)\}_{m=1}^\infty$ we can extract a subsequence, which we denote again by (φ_m, v_m) such that as $m \to \infty$,

$$(\varphi_m, v_m) \rightharpoonup (\varphi, v) \;\text{ weakly in } L^2(0,T;H^4(\Omega)\times H^2(\Omega)) \;\text{ and weak}^* \text{ in } L^\infty(0,T;V),$$

$$(\varphi_m, v_m) \to (\varphi, v) \quad \text{strongly in } (L^2(Q_T))^2 \;\text{ and a.e. in } Q_T,$$

$$(\varphi_{mt}, v_{mt}) \rightharpoonup (\varphi_t, v_t) \quad \text{weakly in } (L^2(Q_T))^2,$$

so that in particular $(\varphi, v) \in C([0,T];(L^2(\Omega))^2)$. Let $\tilde V = \{v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$ and let $f \in L^2(0,T;\tilde V)$ with $f_t \in L^2(Q_T)$ and define

$$f_j(t) = \int_\Omega f(x,t) w_j(x)\, dx.$$

We multiply the differential equations in (3.1), (3.2) by $f_j(t)$, integrate on $[0,T]$ and let $m \to \infty$. After summing the equations for $j = 1, \dots, m$ and letting $m \to \infty$, one deduces that (φ, v) is a solution of Problem P_1.

Next we consider the case that $(\varphi_0, v_0) \in H_{\beta\gamma}$ and define a subsequence $(\varphi_{0q}, v_{0q}) \in V \cap H_{\beta\gamma}$ such that $(\varphi_{0q}, v_{0q}) \to (\varphi_0, v_0)$ in H as $q \to \infty$. We deduce from the uniqueness proof below that two solutions (φ_q, v_q) and (φ_p, v_p) of Problem P with initial functions (φ_{0q}, v_{0q}) and (φ_{0p}, v_{0p}) respectively satisfy

$$\begin{aligned}
&\frac{\tau}{2}\|(\varphi_q - \varphi_p)(t)\|_{-1}^2 + \frac{1}{2}\int_\Omega (v_q - v_p)^2(t) \\
&\quad \le \left(\frac{\tau}{2}\|\varphi_{0q} - \varphi_{0p}\|_{-1}^2 + \frac{1}{2}\int_\Omega (v_{0q} - v_{0p})^2\right) e^{dt} + Ce^{dt}\|\varphi_{0q} - \varphi_{0p}\|_{-1}.
\end{aligned}$$

Thus $\{(\varphi_q, v_q)\}_{q\ge 1}$ is a Cauchy sequence in $C([0,T];H)$ and there exists $(\varphi, v) \in C([0,T];H)$ such that

$$(\varphi_q, v_q) \to (\varphi, v) \quad \text{in } C([0,T];H).$$

It remains to show that (φ, v) is a solution of Problem P_1. We have that

$$\|(\varphi_q, v_q)\|_{L^2(0,T;(H^1(\Omega))^2)} \le C(T + \|(\varphi_{0q}, v_{0q})\|_{H_{\beta\gamma}}) \le \tilde{C}(T + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}),$$

and we may conclude from (3.12) that

$$\|\varphi_q\|_{L^{2p}(Q_T)}, \|\psi_{qt}\|_{L^{\hat{p}}(0,T;(H^1(\Omega))')}, \|v_{qt}\|_{L^2(0,T;(H^1(\Omega))')} \le C(1 + \|(\varphi_0, v_0)\|_{H_{\beta\gamma}}),$$

where $\psi_q = N\varphi_q$. Thus

$$\begin{aligned}
(\varphi_q, v_q) &\rightharpoonup (\varphi, v) \;\; \text{weakly in } L^2(0,T;(H^1(\Omega))^2), \\
g(\varphi_q) &\rightharpoonup \chi \;\; \text{weakly in } L^{\hat{p}}(Q_T) \qquad \big(\hat{p} = 2p/(2p-1)\big)
\end{aligned}$$

for some function $\chi \in L^{\hat{p}}(Q_T)$,

$$(\varphi_q, v_q) \to (\varphi, v) \;\; \text{strongly in } (L^2(Q_T))^2,$$

$$(\varphi_q, v_q) \rightharpoonup (\varphi, v) \quad \text{weak}^* \text{ in } L^\infty(\delta, T; V) \quad \text{and weakly in } L^2(\delta, T; H^4(\Omega) \times H^2(\Omega))$$

for all $\delta \in (0,T)$.

It follows that the pair (φ, v) satisfies the equations

$$\begin{cases} \tau\varphi_t + \xi^2\Delta(\xi^2\Delta\varphi - \chi - \ell\varphi + 2v) = 0 \\ v_t = K\Delta v - \dfrac{K\ell}{2}\Delta\varphi \end{cases}$$

in $\mathcal{D}'(\Omega \times (0,T))$, together with the initial and boundary conditions of Problem P_1.

In order to check that $\chi = g(\varphi)$, we use the standard monotonicity method. We set

$$X_q = \xi^2 \int_0^T \int_\Omega (g(\varphi_q) + C\varphi_q - g(\phi) - C\phi)(\varphi_q - \phi),$$

where C is the constant from (3.6). Note that $X_q \geq 0$ for $\phi \in H^1(\Omega) \subset L^{2p}(\Omega)$. Using an integral equality similar to (3.7), we deduce that

$$\begin{aligned} X_q = \int_0^T \{ &-\frac{\tau}{2}\frac{d}{dt}\|\bar{\varphi}_q\|_{-1}^2 - \xi^4 \int_\Omega (\nabla\varphi_q)^2 - \ell \int_\Omega {\varphi_q}^2 + 2\xi^2 \int_\Omega v_q \bar{\varphi}_q \\ &+ \beta\xi^2 \int_\Omega g(\varphi_q) + C \int_\Omega {\varphi_q}^2 \ \} - \xi^2 \int_0^T \int_\Omega (g(\phi) + C\phi)\varphi_q \\ &- \xi^2 \int_0^T \int_\Omega (g(\varphi_q) + C\varphi_q - g(\phi) - C\phi)\phi \geq 0. \end{aligned}$$

Letting $q \to \infty$, we deduce that

$$0 \leq \xi^2 \int_0^T \int_\Omega (\chi + C\varphi - g(\phi) - C\phi)(\varphi - \phi).$$

Taking $\phi = \varphi - \lambda w$, $\lambda > 0$ and letting $\lambda \to 0$, we obtain $\chi = g(\varphi)$. This completes the existence proof.

Finally we show that the map $S(t)$ from $\mathcal{H}_\alpha$ into $\mathcal{H}_\alpha$ is Hölder continuous of exponent $\frac{1}{2}$. Let (φ_i, v_i), $i = 1, 2$, be solutions of Problem P with initial functions $(\varphi_{0i}, v_{0i}) \in \mathcal{H}_\alpha$. They satisfy the partial differential equations in $(\mathcal{D}'(\Omega \times (0,T)))^2$ and in $(L^2(Q_\delta^T))^2$ for all $\delta \in (0,T)$. The functions $\varphi_1 - \varphi_2$ and $v_1 - v_2$ satisfy the system

$$\tau(\varphi_1 - \varphi_2)_t + \xi^4\Delta^2(\varphi_1 - \varphi_2) - \xi^2\Delta(g(\varphi_1) - g(\varphi_2)) + \ell\xi^2\Delta(\varphi_1 - \varphi_2) - 2\xi^2\Delta(v_1 - v_2) = 0 \tag{3.34}$$

$$(v_1 - v_2)_t = K\Delta(v_1 - v_2) - \frac{K\ell}{2}\Delta(\varphi_1 - \varphi_2). \tag{3.35}$$

For all $0 < t < T$ we multiply (3.34) by $\psi_1 - \psi_2$, where $\psi_i = N\bar{\varphi}_i$, $i = 1, 2$, to obtain

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\|\bar{\varphi}_1 - \bar{\varphi}_2\|_{-1}^2 + \xi^4 \int_\Omega (\nabla(\varphi_1 - \varphi_2))^2 + \ell\xi^2 \int_\Omega (\bar{\varphi}_1 - \bar{\varphi}_2)^2 \\ &= -\xi^2 \int_\Omega (g(\varphi_1) - g(\varphi_2))(\bar{\varphi}_1 - \bar{\varphi}_2) + 2\xi^2 \int_\Omega (v_1 - v_2)(\bar{\varphi}_1 - \bar{\varphi}_2) \\ &= -\xi^2 \int_\Omega (g(\varphi_1) - g(\varphi_2))(\varphi_1 - \varphi_2) + 2\xi^2 \int_\Omega (v_1 - v_2)(\varphi_1 - \varphi_2) \\ &+ \xi^2 \int_\Omega (g(\varphi_1) - g(\varphi_2))m(\varphi_1 - \varphi_2) - 2\xi^2 \int_\Omega (v_1 - v_2)m(\varphi_1 - \varphi_2) \end{aligned} \tag{3.36}$$

$$\leq C\xi^2\int_\Omega(\varphi_1-\varphi_2)^2+\frac{\ell\xi^2}{2}\int_\Omega(\varphi_1-\varphi_2)^2+\frac{2\xi^2}{\ell}\int_\Omega(v_1-v_2)^2$$
$$+\xi^2|m(\varphi_1-\varphi_2)|(C_1\int_\Omega(|\varphi_1|^{2p}+|\varphi_2|^{2p})+C_2)+2\xi^2|\Omega||m(\varphi_1-\varphi_2)||m(v_1-v_2)|$$
$$\leq\frac{\xi^4}{4}\int_\Omega(\nabla(\varphi_1-\varphi_2))^2+C\|\varphi_1-\varphi_2\|_{-1}^2+\frac{\ell\xi^2}{2}\int_\Omega(\bar\varphi_1-\bar\varphi_2)^2+C\int_\Omega(v_1-v_2)^2$$
$$+\xi^2|m(\varphi_1-\varphi_2)|(C_1\int_\Omega(|\varphi_1|^{2p}+|\varphi_2|^{2p})+C_2)$$

and multiply (3.36) by v_1-v_2 which gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(v_1-v_2)^2+\frac{K}{2}\int_\Omega(\nabla(v_1-v_2))^2\leq\frac{K\ell^2}{8}\int_\Omega(\nabla(\varphi_1-\varphi_2))^2. \tag{3.37}$$

Next we add (3.36) to (3.37) multiplied by $\frac{2\xi^4}{K\ell^2}$ to obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi_1-\varphi_2\|_{-1}^2+\frac{\xi^4}{K\ell^2}\frac{d}{dt}\int_\Omega(v_1-v_2)^2$$
$$\leq d(\frac{\tau}{2}\|\varphi_1-\varphi_2\|_{-1}^2+\frac{1}{2}\int_\Omega(v_1-v_2)^2)+|m(\varphi_1-\varphi_2)|(C_1\int_\Omega(|\varphi_1|^{2p}+|\varphi_2|^{2p})+C_2)$$

to which we apply the Gronwall lemma. This gives for $t\geq\delta>0$

$$\frac{\tau}{2}\|(\varphi_1-\varphi_2)(t)\|_{-1}^2+\frac{\xi^4}{K\ell^2}\int_\Omega(v_1-v_2)^2(t)$$
$$\leq(\frac{\tau}{2}\|(\varphi_1-\varphi_2)(\delta)\|_{-1}^2+\frac{\xi^4}{K\ell^2}\int_\Omega(v_1-v_2)^2(\delta))\;e^{dt}$$
$$+|m(\varphi_{01}-\varphi_{02})|C(\int_\delta^t ds\;e^{d(t-s)}(\int_\Omega(|\varphi_1|^{2p}+|\varphi_2|^{2p})\,dx)).$$

Using the continuity of (φ_i,v_i), $i=1,2$, in $\mathcal{H}_\alpha$ and letting $\delta\downarrow 0$, we deduce using also (3.12)), that

$$\frac{\tau}{2}\|(\varphi_1-\varphi_2)(t)\|_{-1}^2+\frac{\xi^4}{K\ell^2}\int_\Omega(v_1-v_2)^2(t)$$
$$\leq(\frac{\tau}{2}\|(\varphi_{01}-\varphi_{02})\|_{-1}^2+\frac{\xi^4}{K\ell^2}\int_\Omega(v_{01}-v_{02})^2)\;e^{dt}+Ce^{dt}\|\varphi_{01}-\varphi_{02}\|_{-1},$$

which completes the proof of Theorem 3.1.

Finally, even though we will not really need this property in a further proof, we find it interesting to show that one can also associate to Problem P_1 a semigroup in $(L^2(\Omega))^2$. The proofs of the results that we give below are due in part to A. Debussche.

Theorem 3.3. *The following results hold.*
(i) *Let* $(\varphi_0, v_0) \in (L^2(\Omega))^2$*; then* $(\varphi, v) \in C([0,\infty); (L^2(\Omega))^2)$*;*
(ii) *The mapping* $S(t)$ *is continuous from* $(L^2(\Omega))^2$ *into itself.*

Proof. (i) It remains to show that

$$\lim_{t\to 0}(\varphi(t), v(t)) = (\varphi(0), v(0)),$$

in the sense of $(L^2(\Omega))^2$. We deduce from inequalities similar to (3.9) and (3.13) that

$$\frac{d}{dt}\int_\Omega (\frac{\tau}{2}\varphi_m^2(t) + \frac{2\xi^2}{K\ell}v_m^2(t)) \le \int_\Omega (C^2\varphi_m^2(t) + 4v_m^2(t)) \le \tilde{C}\int_\Omega (\frac{\tau}{2}\varphi_m^2(t) + \frac{2\xi^2}{K\ell}v_m^2(t))$$

for all $t > 0$.

Applying Gronwall's lemma and letting $m \to \infty$, we obtain that for all $t > 0$

$$\int_\Omega (\frac{\tau}{2}\varphi^2(t) + \frac{2\xi^2}{K\ell}v^2(t)) \le \int_\Omega (\frac{\tau}{2}\varphi_0^2 + \frac{2\xi^2}{K\ell}v_0^2)\ e^{\tilde{C}t},$$

which implies in particular that

$$\limsup_{t\to 0}\int_\Omega (\frac{\tau}{2}\varphi^2(t) + \frac{2\xi^2}{K\ell}v^2(t)) \le \int_\Omega (\frac{\tau}{2}\varphi_0^2 + \frac{2\xi^2}{K\ell}v_0^2).$$

Furthermore, since $L^2(\Omega) \subset (H^1(\Omega))'$ and since

$$\varphi \in L^\infty(0,\infty; L^2(\Omega)), \quad \varphi \in C([0,\infty); (H^1(\Omega))'),$$

we deduce from [14, Lemma 3.3, page 72] that φ is weakly continuous with values in $L^2(\Omega)$. We recall also that $v \in C([0,\infty);\ L^2(\Omega))$. Then

$$\begin{aligned} 0 &\le \limsup_{t\to 0}\int_\Omega (\frac{\tau}{2}(\varphi(t)-\varphi_0)^2 + \frac{2\xi^2}{K\ell}(v(t)-v_0)^2) \\ &\le \limsup_{t\to 0}\int_\Omega (\frac{\tau}{2}\varphi^2(t) + \frac{2\xi^2}{K\ell}v^2(t)) \\ &\quad - \tau\int_\Omega \varphi(t)\varphi_0 - \frac{4\xi^2}{K\ell}\int_\Omega v(t)v_0 + \frac{\tau}{2}\int_\Omega {\varphi_0}^2 + \frac{2\xi^2}{K\ell}\int_\Omega v_0^2 \\ &\le 0, \end{aligned}$$

which completes the proof of (i).

(ii) We suppose as $n \to \infty$, $(\varphi_{0n}, v_{0n}) \to (\varphi_0, v_0)$ in $(L^2(\Omega))^2$.

Let $t > 0$ be arbitrary. Then as $n \to \infty$, $(\varphi_n(t), v_n(t)) \to (\varphi(t), v(t))$ in $H_{\beta\gamma}$, and since, by Theorem 3.1(iii)

$$\|(\varphi_n, v_n)\|_{C([\frac{t}{2},\infty);V)} \le C$$

we deduce that as $n \to \infty$

$$(\varphi_n(t), v_n(t)) \rightharpoonup (\varphi(t), v(t)) \quad \text{weakly in } V$$

and finally that

$$(\varphi_n(t), v_n(t)) \to (\varphi(t), v(t)) \quad \text{strongly in } (L^2(\Omega))^2.$$

4. The maximal attractor. In this section we show the existence of bounded absorbing sets in $\mathcal{H}_\alpha$ and in $(H^2(\Omega))^2 \cap \mathcal{H}_\alpha$ for the semigroup $S(t)$ associated with Problem P_1 so that $S(t)$ possesses a maximal attractor in $\mathcal{H}_\alpha$. Furthermore it turns out that there exists an absorbing set in $(C^m(\bar{\Omega}))^2$ for each $m \in \mathbb{N}^+$ so that the attractor is bounded in all spaces $(C^m(\bar{\Omega}))^2$.

First we show the existence of an absorbing set in $\mathcal{H}_\alpha$. We multiply the equation for φ by $\psi = N\varphi$ as defined in (2.3) and the equation for v by $\frac{4\xi^4}{K\ell^2}v$. Integrating by parts and adding the resulting equations we get (see (3.11))

$$\begin{aligned} &\frac{d}{dt}(\frac{\tau}{2}\|\varphi\|_{-1}^2 + \frac{2\xi^4}{K\ell^2}\int_\Omega v^2) + \frac{\xi^4}{2}\int_\Omega (\nabla\varphi)^2 + \frac{\xi^4}{4\ell^2 C(\Omega)}\int_\Omega v^2 + \frac{\xi^4}{4\ell^2}\int_\Omega (\nabla v)^2 \\ &\quad + \ell\xi^2\int_\Omega \varphi^2 + \frac{\xi^2 a_{2p-1}}{4}\int_\Omega \varphi^{2p} \le C(\alpha) \end{aligned} \tag{4.1}$$

for all $t > 0$.

Applying Gronwall's lemma we deduce that there exists positive constants $d,\ c_1$, and $c_2(\alpha)$ such that

$$\|\varphi(t)\|_{-1}^2 + \int_\Omega (v(t))^2 \le C_1(\|\varphi_0)\|_{-1}^2 + \int_\Omega v_0^2\ )\ e^{-dt} + C_2(\alpha) \tag{4.2}$$

for all $t > 0$.

Taking $R > 0$ and $(\varphi_0, v_0) \in \mathcal{H}_\alpha$ such that $\|(\varphi_0, v_0)\|_{\mathcal{H}_\alpha} \le R$ we deduce from (4.2) that

$$\|\varphi(t)\|_{-1}^2 + \int_\Omega (v(t))^2 \le 1 + C_2(\alpha)$$

for all $t \ge t_0 = t_0(R) = \frac{1}{d} ln(R^2 C_1)$.

We have thus proven the following result:

Theorem 4.1. *For any $\alpha \ge 0$, there exists a constant $R = R(\alpha) > 0$ such that $B_{\mathcal{H}_\alpha}(0, R(\alpha))$ is an absorbing set for $S(t)$ in $\mathcal{H}_\alpha$.*

Next we show the existence of an absorbing set in $(H^2(\Omega))^2 \cap \mathcal{H}_\alpha$. We have that

$$J(\varphi, v) \le \frac{\xi^4}{2}\int_\Omega (\nabla\varphi)^2 + \ell\xi^2\int_\Omega \varphi^2 + \frac{\xi^2 a_{2p-1}}{p}\int_\Omega \varphi^{2p} + \frac{4\xi^2}{\ell}\int_\Omega v^2 + C \tag{4.3}$$

so that by (4.1) there exists a positive constant μ such that

$$\frac{d}{dt}\Big(\frac{\tau}{2}\|\varphi\|_{-1}^2 + \frac{2\xi^4}{K\ell^2}\int_\Omega v^2\Big) + \mu J(\varphi, v) \le \tilde{C}(\alpha). \tag{4.4}$$

Then for all $t \ge t_0$ and $r > 0$ we have that

$$\int_t^{t+r} J(\varphi, v)(s)\, ds \le \frac{1}{\mu}\Big(\tilde{C}(\alpha) r + \Big(\frac{\tau}{2} + \frac{2\xi^4}{K\ell^2}\Big)(R(\alpha))^2\Big).$$

Since $J(\varphi, v)$ decreases along orbits, we deduce that

$$rJ(\varphi, v)(t + r) \le \frac{1}{\mu}\Big(\tilde{C}(\alpha) r + \Big(\frac{\tau}{2} + \frac{2\xi^4}{K\ell^2}\Big)(R(\alpha))^2\Big)$$

for all $t \ge t_0$ and since furthermore

$$J(\varphi, v) \ge \frac{\xi^4}{2}\int_\Omega (\nabla\varphi)^2 + \frac{\xi^2 a_{2p-1}}{4p}\int_\Omega \varphi^{2p} - C,$$

we deduce that

$$\int_\Omega (\nabla\varphi)^2 + \int_\Omega \varphi^2 \le C(\alpha) \quad \text{for all } t \ge t_0 + r. \tag{4.5}$$

In order to estimate ∇v we multiply the equation for φ by φ and the equation for v by $-\Delta v$. This gives (cf. (3.13) and (3.21))

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega \varphi^2 + \frac{\xi^4}{2}\int_\Omega (\Delta\varphi)^2 \le C^2\int_\Omega \varphi^2 + 4\int_\Omega v^2 \tag{4.6}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla v)^2 + \frac{K}{2}\int_\Omega (\Delta v)^2 \le \frac{K\ell^2}{8}\int_\Omega (\Delta\varphi)^2. \tag{4.7}$$

Integrating (4.1) and (4.6) between t and $t + r$ we deduce that

$$\int_t^{t+r}\int_\Omega (\nabla v)^2,\ \int_t^{t+r}\int_\Omega (\Delta\varphi)^2 \ \le C \qquad \text{for } t \ge t_0 \tag{4.8}$$

so that by the uniform Gronwall lemma we deduce from (4.7) that

$$\int_\Omega (\nabla v)^2 \le C(\alpha) \ \text{ for } \qquad t \ge t_o + r. \tag{4.9}$$

Finally we estimate $\Delta\varphi$ and Δv. To that purpose we multiply the equation for φ by $\Delta^2\varphi$ and the equation for v by $\Delta^2 v$ to obtain (cf. (3.25))

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta\varphi)^2 + \frac{5\xi^4}{8}\int_\Omega (\Delta^2\varphi)^2 \le 8\int_\Omega (\Delta v)^2 + \int_\Omega (\Delta g(\varphi))^2 \tag{4.10}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta v)^2 + K\int_\Omega (\nabla\Delta v)^2 \le \frac{\xi^4}{16}\int_\Omega (\Delta^2\varphi)^2 + \frac{K^2\ell^2}{\xi^4}\int_\Omega (\Delta v)^2. \tag{4.11}$$

As in (3.27), we have that

$$\int_\Omega (\Delta g(\varphi))^2 \le \frac{\xi^4}{2}\int_\Omega (\Delta^2\varphi)^2 + C(\alpha). \tag{4.12}$$

Substituting (4.12) into (4.10) and adding it to (4.11) gives

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta\varphi)^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta v)^2 + \frac{\xi^4}{16}\int_\Omega (\Delta^2\varphi)^2 + K\int_\Omega (\nabla\Delta v)^2 \le C_1\int_\Omega (\Delta v)^2 + C(\alpha). \tag{4.13}$$

We deduce from (4.7) and (4.8) that we can apply the uniform Gronwall lemma to (4.13). This gives

$$\int_\Omega (\Delta\varphi)^2(t),\ \int_\Omega (\Delta v)^2(t) \le \tilde{C}(\alpha) \quad \text{for } t \ge t_0 + 2r. \tag{4.14}$$

It then follows from Theorem 4.1, (4.5) and (4.14) that (φ, v) enters an absorbing set of $(H^2(\Omega))^2 \cap \mathcal{H}_\alpha$.

Next we deduce from [14] the following result:

Theorem 4.2. *For every $\alpha \ge 0$ the semigroup $S(t)$ associated with Problem P_1 maps $\mathcal{H}_\alpha$ into itself. It possesses in $\mathcal{H}_\alpha$ a maximal attractor $\mathcal{A}_\alpha$ that is connected.*

Remark 4.3. In fact (φ, v) enters an absorbing set of $(C^m(\bar\Omega))^2 \cap \mathcal{H}_\alpha$ for each $m \ge 1$, so that the attractor $\mathcal{A}_\alpha$ is contained in $(C^\infty(\bar\Omega))^2$ and bounded in all the spaces $(C^m(\bar\Omega))^2$, $m \in \mathbb{N}^+$. The proof is similar to that of Theorem 6.3 (i) below but the idea is to apply the uniform Gronwall lemma instead of the standard Gronwall lemma.

Remark 4.4. Let E_α denote the set of stationary solutions. Then $\mathcal{A}_\alpha = W^u(E_\alpha) := \{(\varphi, v) \in \mathcal{H}_\alpha : S(-t)(\varphi, v) \text{ is defined for all } t \ge 0 \text{ and } S(-t)(\varphi, v) \to E_\alpha \text{ as } t \to \infty\}$. This follows from the fact that $\{S(t)\}_{t\ge 0}$ is a gradient system (cf. [12, Theorem 3.8.5, page 51]).

5. Existence of inertial sets. In this section we show the existence of inertial sets for the semigroup $S(t)$ corresponding to Problem P on $\mathcal{H}_\alpha$ where α is any given fixed nonnegative constant.

Let B_α be the absorbing ball in $(H^2(\Omega))^2 \cap \mathcal{H}_\alpha$ from Section 4 and define the positively invariant set

$$X_\alpha = \overline{\bigcup_{t\ge t_0} S(t)B_\alpha}, \quad \tilde{t}_0 = 1 + t_0(R(\alpha)). \tag{5.1}$$

We remark that X_α is bounded in $(C(\bar\Omega))^2$.

Lemma 5.1. *The semigroup $S(t)$ satisfies a Lipschitz property from X_α into itself.*

Proof. Let (φ_i, v_i), $i = 1, 2$, be the solutions of Problem P with initial functions $(\varphi_{0i}, v_{0i}) \in X_\alpha$. As in the uniqueness proof we have that

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\|\bar\varphi_1 - \bar\varphi_2\|_{-1}^2 + \xi^4 \int_\Omega (\nabla(\varphi_1 - \varphi_2))^2 + \frac{\ell\xi^2}{2}\int_\Omega (\bar\varphi_1 - \bar\varphi_2)^2 \\ &\qquad \le \frac{\xi^2}{\ell}\int_\Omega (g(\varphi_1) - g(\varphi_2))^2 + \frac{4\xi^2}{\ell}\int_\Omega (v_1 - v_2)^2. \end{aligned} \tag{5.2}$$

Using the fact that the positively invariant set X_α is contained in B_α, we deduce that

$$\|\varphi_1 - \varphi_2\|_{C(\bar\Omega\times[0,\infty))} \le M, \tag{5.3}$$

where M is a positive constant, so that we have

$$\begin{aligned} \frac{\xi^2}{\ell}\int_\Omega (g(\varphi_1) - g(\varphi_2))^2 &\le \frac{\xi^2}{\ell} C(M)\int_\Omega (\varphi_1 - \varphi_2)^2 \\ &\le \frac{\xi^4}{4}\int_\Omega (\nabla(\varphi_1 - \varphi_2))^2 + \tilde C(M)\|\varphi_1 - \varphi_2\|_{-1}^2. \end{aligned} \tag{5.4}$$

Substituting (5.4) into (5.2) and adding the result to (3.37) multiplied by $\frac{2\xi^4}{K\ell^2}$ we deduce that there exists a positive constant k such that

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\|\varphi_1 - \varphi_2\|_{-1}^2 + \frac{\xi^4}{K\ell^2}\frac{d}{dt}\int_\Omega (v_1 - v_2)^2 + k\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}^2 \\ &\qquad \le C(M)\|\varphi_1 - \varphi_2\|_{-1}^2 + \frac{4\xi^2}{\ell}\int_\Omega (v_1 - v_2)^2. \end{aligned} \tag{5.5}$$

Applying the Gronwall lemma to (5.5) we deduce that $S(t)$ is Lipschitz continuous from X_α to X_α, namely, there exists a constant $D > 0$ such that

$$\|\varphi_1 - \varphi_2\|_{-1}^2 + \int_\Omega (v_1 - v_2)^2 \le (\|\varphi_{01} - \varphi_{02}\|_{-1}^2 + \int_\Omega (v_{01} - v_{02})^2\ )e^{Dt}. \tag{5.6}$$

Next we introduce some notation. We define $H_N = \operatorname{span}\{w_1, \dots, w_N\}$ and we define p_N as the orthogonal projector from $(H^1(\Omega))'$ onto H_N and $q_N = I - p_N$. We remark that for all $u \in q_N(H^1(\Omega))$,

$$\|u\|_{-1}^2 \le \frac{1}{\lambda_{N+1}}|u|_{L^2(\Omega)}^2 \le \frac{1}{{\lambda_{N+1}}^2}|\nabla u|_{L^2(\Omega)}^2. \tag{5.7}$$

We also introduce the corresponding product projections $P_N,\ Q_N$ on H by

$$P_N(\varphi, v) := (p_N\varphi, p_N v) \quad \text{and } Q_N = I - P_N.$$

In what follows we shall prove the following result:

Theorem 5.2. *The semigroup $S(t)$ corresponding to Problem P_1 satisfies the squeezing property: for any $t^* > 0$, there exists $N_0 = N_0(t^*)$ such that for any $U_1 = (\varphi_1, v_1)$, $U_2 = (\varphi_2, v_2)$ in X_α satisfying*

$$\|P_{N_0}(S(t^*)(U_1) - S(t^*)(U_2))\|_H < \|(I - P_{N_0})(S(t^*)(U_1) - S(t^*)(U_2))\|_H$$

one has the inequality

$$\|S(t^*)(U_1) - S(t^*)(U_2)\|_H \leq \frac{1}{8}\|U_1 - U_2\|_H. \tag{5.8}$$

The squeezing property expresses the fact that either the lower modes are dominated by the higher modes or else the flow is contracting exponentially. Before proving Theorem 5.2 we show the following auxiliary result.

Lemma 5.3. *There exists a positive constant C such that for any two solutions (φ_1, v_1), (φ_2, v_2) of Problem P_1 with initial functions in X_α, for all $t > 0$ the pair $(\phi, V) = Q_N(\varphi_1 - \varphi_2,\, v_1 - v_2)$ satisfies*

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\|\phi\|_{-1}^2 + \frac{\xi^4}{K\ell^2}\frac{d}{dt}\int_\Omega V^2 + \frac{\xi^4{\lambda_{N+1}}^2}{2}\|\phi\|_{-1}^2 \\ + (\frac{\xi^4}{\ell^2}\lambda_{N+1} - \frac{4\xi^2}{\ell^2})\int_\Omega V^2 \leq C\int_\Omega(\varphi_1 - \varphi_2)^2.\end{aligned} \tag{5.9}$$

Proof. Applying the operator Q_N to the differences of the equations for (φ_1, v_1) and (φ_2, v_2) we obtain

$$\begin{cases}\tau\phi_t + \xi^2\Delta(\xi^2\Delta\phi - q_N(g(\varphi_1) - g(\varphi_2)) - \ell\phi + 2V) = 0 \\ V_t = K\Delta V - \dfrac{K\ell}{2}\Delta\phi.\end{cases} \tag{5.10}$$

We multiply the first equation in (5.10) by $N\phi \in q_N(H^2(\Omega))$ and the second by $\frac{2\xi^4}{K\ell^2}V$. Adding up those equations, we obtain after integrations by parts

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\|\phi\|_{-1}^2 + \frac{\xi^4}{K\ell^2}\frac{d}{dt}\int_\Omega V^2 + \frac{3\xi^4}{4}\int_\Omega(\nabla\phi)^2 + \frac{\xi^4}{\ell^2}\int_\Omega(\nabla V)^2 \\ + \frac{\ell\xi^2}{4}\int_\Omega\phi^2 \leq \frac{4\xi^2}{\ell}\int_\Omega V^2 + \frac{\xi^2}{2\ell}\int_\Omega(g(\varphi_1) - g(\varphi_2))^2.\end{aligned} \tag{5.11}$$

As in (5.4) we have that

$$\frac{\xi^2}{2\ell}\int_\Omega(g(\varphi_1) - g(\varphi_2))^2 \leq \frac{\xi^2}{2\ell}C(M)\int_\Omega(\varphi_1 - \varphi_2)^2,$$

which we substitute into (5.11). Using also (5.7) we finally deduce (5.9).

Next we prove the squeezing property, Theorem 5.2.

Proof of Theorem 5.2. We fix $t^* > 0$ and let $(\varphi_{01}, v_{01}), (\varphi_{02}, v_{02}) \in X_\alpha$. For all $t \geq 0$ we set

$$(\psi(t), W(t)) = (\varphi_1(t) - \varphi_2(t), v_1(t) - v_2(t)).$$

We deduce from (5.9) that there exist positive constants $\mu,\ C_1,\ C_2,\ \tilde{C}$ such that

$$\frac{d}{dt}\|(\phi, \mu V)\|_H^2 + (C_1\lambda_{N+1} - C_2)\|(\phi, \mu V)\|_H^2 \leq \tilde{C}\int_\Omega (\varphi_1 - \varphi_2)^2. \tag{5.12}$$

Applying the Gronwall lemma to (5.12) we deduce that

$$\begin{aligned}\|(\phi, \mu V)(t)\|_H^2 &\leq \|(\phi(0), \mu V(0))\|_H^2\ e^{-(C_1\lambda_{N+1} - C_2)t} \\ &\quad + \tilde{C}e^{-(C_1\lambda_{N+1} - C_2)t}\int_0^t ds\, e^{(C_1\lambda_{N+1} - C_2)s}\int_\Omega (\varphi_1 - \varphi_2)^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|(\phi, \mu V)(t)\|_H^2 &\leq \|(\phi(0), \mu V(0))\|_H^2 e^{-(C_1\lambda_{N+1} - C_2)t} \\ &\quad + C_\varepsilon e^{-(C_1\lambda_{N+1} - C_2)t}\int_0^t ds\ e^{(C_1\lambda_{N+1} - C_2)s}\|(\varphi_1 - \varphi_2)\|_{-1}^2 \\ &\quad + \varepsilon e^{-(C_1\lambda_{N+1} - C_2)t}\int_0^t ds\ e^{(C_1\lambda_{N+1} - C_2)s}\|(\varphi_1 - \varphi_2)\|_{H^1(\Omega)}^2.\end{aligned} \tag{5.13}$$

We deduce from the Lipschitz property (5.6) that

$$\begin{aligned}&C_\varepsilon\ e^{-(C_1\lambda_{N+1} - C_2)t}\int_0^t ds\ e^{(C_1\lambda_{N+1} - C_2)s}\|(\varphi_1 - \varphi_2)\|_{-1}^2 \\ &\quad \leq \|(\psi(0), W(0))\|_H^2 \frac{C_\varepsilon}{C_1\lambda_{N+1} - C_2 + D}\ e^{Dt}.\end{aligned} \tag{5.14}$$

Then we multiply (5.5) by $e^{(C_1\lambda_{N+1} - C_2)s}$ and integrate between 0 and t. This gives, using also (5.6),

$$\begin{aligned}&k\int_0^t ds\, e^{(C_1\lambda_{N+1} - C_2)s}\|(\varphi_1 - \varphi_2)\|_{H^1(\Omega)}^2 \leq \frac{\tau}{2}\|\psi(0)\|_{-1}^2 + \frac{\xi^4}{K\ell^2}\int_\Omega (W(0))^2 \\ &\quad + (C(M)\lambda_{N+1} + \tilde{C})\int_0^t e^{(C_1\lambda_{N+1} - C_2)s}\|(\psi(s), W(s))\|_H^2\, ds \\ &\quad \leq C\|(\psi(0), W(0))\|_H^2 + \|(\psi(0), W(0))\|_H^2 \frac{C(M)\lambda_{N+1} + \tilde{C}}{C_1\lambda_{N+1} - C_2 + D}\ e^{Dt}.\end{aligned} \tag{5.15}$$

Substituting (5.14) and (5.15) in (5.13) we obtain

$$\begin{aligned}\|(\phi,\mu V)(t)\|_H^2 \le \|(\psi(0),W(0))\|_H^2 (Ce^{-(C_1\lambda_{N+1}-C_2)t} \\ + \frac{C_\varepsilon}{C_1\lambda_{N+1}-C_2+D}\ e^{Dt} + \frac{\varepsilon}{k}\frac{C(M)\lambda_{N+1}+\tilde{C}}{C_1\lambda_{N+1}-C_2+D}\ e^{Dt}).\end{aligned} \tag{5.16}$$

We now suppose that at the time t^*

$$\|P_{N_0}(\psi(t^*),W(t^*))\|_H < \|Q_{N_0}(\psi(t^*),W(t^*))\|_H\,; \tag{5.17}$$

then

$$\begin{aligned}\|(\psi,W)(t^*)\|_H^2 &= \|Q_{N_0}(\psi,W)(t^*)\|_H^2 + \|P_{N_0}(\psi,W)(t^*)\|_H^2 \\ &\le 2\,\|Q_{N_0}(\psi,W)(t^*)\|_H^2 \le 2\left(1+\frac{1}{\mu}\right)\|(\phi,\mu V)(t^*)\|_H^2.\end{aligned}$$

Using (5.16) we deduce that

$$\begin{aligned}\|(\psi,W)(t^*)\|_H^2 \le 2\left(1+\frac{1}{\mu}\right)\|(\psi(0),W(0))\|_H^2 \\ (Ce^{-(C_1\lambda_{N_0+1}-C_2)t^*} + \frac{C_\varepsilon}{C_1\lambda_{N_0+1}-C_2+D}e^{Dt^*} + \frac{\varepsilon}{k}\frac{C(M)\lambda_{N_0+1}+\tilde{C}}{C_1\lambda_{N_0+1}-C_2+D}e^{Dt^*}).\end{aligned} \tag{5.18}$$

Finally, we choose $\varepsilon > 0$ so that

$$2\left(1+\frac{1}{\mu}\right)\frac{\varepsilon}{k}\sup_{N_0\in\mathbb{N}}\frac{C(M)\lambda_{N_0+1}+\tilde{C}}{C_1\lambda_{N_0+1}-C_2+D}\ e^{Dt^*} \le \frac{1}{128}$$

and we choose N_0 such that the first terms on the right-hand side of (5.18) are bounded by $\frac{1}{128}$. Then

$$\|(\psi,W)(t^*)\|_H^2 \le \frac{1}{64}\|(\psi(0),W(0))\|_H^2.$$

This completes the proof of Theorem 5.2.

Applying the result of [8], we obtain the following.

Theorem 5.4. *Let X_α be defined as in* (5.1). *Then there exists an inertial set $\mathcal{M}_\alpha$ for $(S(t)_{t\ge 0},\ X_\alpha)$ which has fractal dimension $\le$ constant $\cdot\, N_0$.*

Remark 5.5. Suppose that N_0 is large. We have that $\lambda_{N_0} \sim C|\Omega|^{-\frac{2}{n}}{N_0}^{\frac{2}{n}}$ (see [14]). Suppose that in the proof above, we had chosen N_0 such that $\lambda_{N_0} \le C(t^*) \le \lambda_{N_0+1}$. Then $C|\Omega|^{-\frac{2}{n}}{N_0}^{\frac{2}{n}} \le 2C(t^*)$ and thus $N_0 \le \left(\frac{2C(t^*)}{C}\right)^{\frac{2}{n}}|\Omega|$ so that

$$\text{fractal dimension} \le \text{constant } \cdot|\Omega|.$$

6. Extra regularity of the solutions. In this section we prove that the solution (φ, v) of Problem P becomes smooth at once. As a consequence, the functions in the attractor are smooth as well.

We first define spaces of smooth functions and then give a precise statement of results on the parabolic regularization of the solutions.

6.1. Some spaces of smooth functions. We define the bilinear form $a(u,v) = \int_\Omega \nabla u \nabla v$ which is continuous on $H^1(\Omega)$ but not coercive. Next we associate with $a(\cdot,\cdot)$ a linear unbounded operator A on $H^1(\Omega)$ with domain

$$D(A) = \{u \in H^2(\Omega) : \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega \ \}.$$

For $u \in D(A)$, Au is defined by

$$(Au, v) = a(u,v) \quad \text{for all } v \in H^1(\Omega)$$

so that

$$Au = -\Delta u.$$

Alternatively A is given by

$$Au = \sum_{j=2}^{\infty} \lambda_j (u, w_j) w_j$$

and

$$D(A) = \{u \in L^2(\Omega) : \ \sum_{j=2}^{\infty} {\lambda_j}^2 (u, w_j)^2 < \infty\},$$

so that for $s \in \mathbb{R}^+$ the operator

$$A^s u = \sum_{j=2}^{\infty} {\lambda_j}^s (u, w_j) w_j$$

is defined on the domain

$$D(A^s) = \{u \in L^2(\Omega) : \ \sum_{j=2}^{\infty} {\lambda_j}^{2s} (u, w_j)^2 < +\infty\}.$$

A norm on $D(A^s)$ will then be given by

$$\|u\|^2_{D(A^s)} = \|u\|^2_{L^2(\Omega)} + \sum_{j=2}^{\infty} {\lambda_j}^{2s} (u, w_j)^2. \tag{6.1}$$

We shall prove the following result.

Theorem 6.1. *Let $\partial\Omega$ be sufficiently smooth and let $k \in \mathbb{N}^+$. Then*
(i)

$$D(A^k) = \{u \in H^{2k}(\Omega) : \ \frac{\partial \Delta^p u}{\partial n} = 0 \quad \text{on } \partial\Omega,\ p = 0, \dots, k-1\} \tag{6.2}$$

and

$$\|u\|^2_{D(A^k)} = \|u\|^2_{L^2(\Omega)} + \|\Delta^k u\|^2_{L^2(\Omega)}. \tag{6.3}$$

Furthermore $\|\cdot\|_{D(A^k)}$ is equivalent to $\|\cdot\|_{H^{2k}(\Omega)}$ on $D(A^k)$.
(ii)

$$D(A^{k+\frac{1}{2}}) = \{u \in H^{2k+1}(\Omega) : \ \frac{\partial \Delta^p u}{\partial n} = 0 \quad \text{on } \partial\Omega,\ p = 0, \dots, k-1\} \tag{6.4}$$

and

$$\|u\|^2_{D(A^{k+\frac{1}{2}})} = \|u\|^2_{L^2(\Omega)} + \|\nabla \Delta^k u\|^2_{L^2(\Omega)}. \tag{6.5}$$

Furthermore $\|\cdot\|_{D(A^{k+\frac{1}{2}})}$ is equivalent to $\|\cdot\|_{H^{2k+1}(\Omega)}$ on $D(A^{k+\frac{1}{2}})$.

Before proving Theorem 6.1, we give a standard result which is a consequence of Brezis ([1, Theorem IX, page 182]).

Lemma 6.2. *Let $m \in \mathbb{N}$ and let Ω be an open set of $\mathbb{R}^N$ of class C^{m+2} with $\partial\Omega$ bounded. If $u \in H^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = f, & \text{in the sense of distributions in } \ \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{in the sense of distributions on } \ \partial\Omega \end{cases}$$

with $f \in H^m(\Omega)$; then $u \in H^{m+2}(\Omega)$, and

$$\|u\|_{H^{m+2}(\Omega)} \le C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{6.6}$$

Proof of Theorem 6.1. (i) We use below an induction argument. For $k = 1$, the result is standard (see for instance Temam ([14, equalities (2.59), (2.60), page 63 and Lemma 4.2, page 150]).

Next we suppose that the result holds for $k \ge 1$ and show that it is true for $k+1$. Let $\mathcal{F}_k$ denote the right-hand side of (6.2) and suppose that $u \in \mathcal{F}_{k+1}$. Then $\Delta^{k+1} u \in L^2(\Omega)$ and

$$\|\Delta^{k+1} u\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} (\Delta^{k+1} u, w_j)^2 = \sum_{j=1}^{\infty} (u, \Delta^{k+1} w_j)^2$$

since u satisfies the homogeneous Neumann boundary conditions from the definition of $\mathcal{F}_{k+1}$, so that

$$\|\Delta^{k+1} u\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} {\lambda_j}^{2(k+1)} (u, w_j)^2 \tag{6.7}$$

and hence $u \in D(A^{k+1})$. Conversely let $u \in D(A^{k+1})$. By induction hypothesis $u \in \mathcal{F}_k$. We first show that $u \in H^{2k+2}(\Omega)$. Since

$$\sum_{j=1}^{\infty} {\lambda_j}^{2k+2}(u, w_j)^2 < \infty$$

we have, by integration by parts, that

$$\sum_{j=1}^{\infty} {\lambda_j}^{2k}(\Delta u, w_j)^2 < \infty$$

so that $\Delta u \in D(A^k) = \mathcal{F}_k \subset H^{2k}(\Omega)$, which, by Lemma 6.2, implies that $u \in H^{2k+2}(\Omega)$ and

$$\|u\|_{H^{2k+2}(\Omega)} \le C(\|\Delta u\|_{H^{2k}(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{6.8}$$

Next we show that $\frac{\partial \Delta^k u}{\partial n} = 0$ on $\partial\Omega$. Since $u \in \mathcal{F}_k$ we have

$$\sum_{j=1}^{\infty} {\lambda_j}^{2k+2}(u, w_j)^2 = \sum_{j=1}^{\infty} {\lambda_j}^{2}(u, \Delta^k w_j)^2 = \sum_{j=1}^{\infty} {\lambda_j}^{2}(\Delta^k u, w_j)^2 < +\infty;$$

hence, $\Delta^k u \in D(A)$, so that

$$\frac{\partial \Delta^k u}{\partial n} = 0 \quad \text{ on } \partial\Omega.$$

Since $\Delta u \in \mathcal{F}_k$, we deduce from (6.8) that

$$\|u\|_{H^{2k+2}(\Omega)} \le C(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} + \|\Delta^{k+1} u\|_{L^2(\Omega)})$$

and since

$$\int_\Omega (\Delta u)^2 = \sum_{j=1}^{\infty} {\lambda_j}^2 (u, w_j)^2 \le C \sum_{j=1}^{\infty} {\lambda_j}^{2k+2}(u, w_j)^2 = C\|\Delta^{k+1} u\|^2_{L^2(\Omega)},$$

we have that

$$\|u\|_{H^{2k+2}(\Omega)} \le \tilde{C}(\|u\|_{L^2(\Omega)} + \|\Delta^{k+1} u\|_{L^2(\Omega)}),$$

which completes the proof of (i).

(ii) Next we prove (ii). Let $\mathcal{F}_{k+\frac{1}{2}}$ denote the right-hand side of (6.4). We first show that

$$\mathcal{F}_{k+\frac{1}{2}} \subset D(A^{k+\frac{1}{2}}). \tag{6.9}$$

Let $u \in \mathcal{F}_{k+\frac{1}{2}}$; then $\nabla(\Delta^k u) \in L^2(\Omega)$. Now

$$\int_\Omega (\nabla u)^2 = \sum_{j=1}^{\infty} \lambda_j (u, w_j)^2$$

for $u \in H^1(\Omega)$, hence

$$\int_\Omega (\nabla(\Delta^k u))^2 = \sum_{j=1}^{\infty} \lambda_j (\Delta^k u, w_j)^2 = \sum_{j=1}^{\infty} \lambda_j (u, \Delta^k w_j)^2 = \sum_{j=1}^{\infty} {\lambda_j}^{2k+1} (u, w_j)^2 < \infty, \tag{6.10}$$

which completes the proof of (6.9).

Next we show that

$$D(A^{k+\frac{1}{2}}) \subset \mathcal{F}_{k+\frac{1}{2}}. \tag{6.11}$$

Since $D(A^{k+\frac{1}{2}}) \subset D(A^k)$, we deduce from (i) that if $u \in D(A^{k+\frac{1}{2}})$, it satisfies the boundary conditions in the definition of $\mathcal{F}_{k+\frac{1}{2}}$. Also $u \in H^{2k}(\Omega)$ and by (6.10)

$$\sum_{j=1}^{\infty} {\lambda_j}^{2k+1} (u, w_j)^2 = \int_\Omega (\nabla(\Delta^k u))^2 = \int_\Omega (\Delta^k \nabla u)^2 = \sum_{j=1}^{\infty} {\lambda_j}^{2k} (\nabla u, w_j)^2 < \infty,$$

so that we deduce from (i) that $\nabla u \in H^{2k}(\Omega)$. This completes the proof of (6.11). Using again (6.10) we deduce (6.5). Finally we prove by induction that $\|\cdot\|_{D(A^{k+\frac{1}{2}})}$ is equivalent to $\|\cdot\|_{H^{2k+1}(\Omega)}$ on $D(A^{k+\frac{1}{2}})$. For $k = 0$, we have that

$$\|u\|_{D(A^{\frac{1}{2}})} = \|u\|_{H^1(\Omega)}.$$

The proof then follows as in case (i).

6.2 Parabolic regularization. In what follows we show the main result of this section.

Theorem 6.3. *Let (φ, v) be the solution of Problem P for the initial condition (φ_0, v_0). Then we have that for all $T > 0$:*

(i) *If $(\varphi_0, v_0) \in D(A^{k+1}) \times D(A^{k+\frac{1}{2}}) \cap H_{\beta\gamma}$ with $k \in \mathbb{N}$, then*

$$(\varphi, v) \in L^\infty(0, T; D(A^{k+1}) \times D(A^{k+\frac{1}{2}})) \cap L^2(0, T; D(A^{k+2}) \times D(A^{k+1})).$$

If furthermore $v_0 \in D(A^{k+1})$, then

$$v \in L^\infty(0, T; D(A^{k+1})) \cap L^2(0, T; D(A^{k+\frac{3}{2}})).$$

(ii) *If $(\varphi_0, v_0) \in (D(A))^2$, then for each $\delta > 0$ and each $(j, k) \in \mathbb{N} \times \mathbb{N}^+$,*

$$(\varphi^{(j)}, v^{(j)}) \in L^\infty(\delta, +\infty; (D(A^k))^2),$$

where $u^{(j)}$ denotes $\frac{\partial^j u}{\partial t^j}$.

Corollary 6.4. *Let* $(\varphi_0, v_0) \in H_{\beta\gamma}$. *Then*

$$(\varphi, v) \in (C^\infty(\bar{\Omega} \times [\delta, \infty)))^2 \quad \text{for all } \delta > 0.$$

Proof. We deduce from Theorem 3.1 that if $(\varphi_0, v_0) \in H_{\beta\gamma}$, then $(\varphi(t), v(t)) \in (D(A))^2$ for almost every $t > 0$. The result of Corollary 6.4. is then a direct consequence of Theorem 6.3.

Proof of Theorem 6.3. (i) We use an induction argument on $k \in \mathbb{N}$ and deduce the following result:

If $(\varphi_0, v_0) \in D(A^{k+1}) \times D(A^{k+\frac{1}{2}}) \cap H_{\beta\gamma}$ then

$$\|(\varphi_m, v_m)\|_{L^\infty(0,T;D(A^{k+1})\times D(A^{k+\frac{1}{2}}))}, \ \|(\varphi_m, v_m)\|_{L^2(0,T;D(A^{k+2})\times D(A^{k+1}))} \le C, \quad (6.12k)$$

and if $(\varphi_0, v_0) \in (D(A^{k+1}))^2 \cap H_{\beta\gamma}$ then

$$\|v_m\|_{L^\infty(0,T;D(A^{k+1}))}, \ \|v_m\|_{L^2(0,T;D(A^{k+\frac{3}{2}}))} \le C. \quad (6.13k)$$

1) Initialization of the induction ($k = 0$):

For $k = 0$, (6.12k) follows from Theorem 3.1(ii); one shows that (6.13k) holds for $k = 0$ by multiplying the differential equations in (3.2) by ${\lambda_j}^2 v_{jm}(t)$.

2) We now assume that (6.12k) and (6.13k) hold for some $k \in \mathbb{N}$ and $(\varphi_0, v_0) \in D(A^{k+2}) \times D(A^{k+\frac{3}{2}}) \cap H_{\beta\gamma}$. Multiplying (3.1) by ${\lambda_j}^{2k+4} \varphi_{jm}(t)$ and (3.2) by $-{\lambda_j}^{2k+3} v_{jm}(t)$, and summing on $j = 1, \dots, m$, we obtain

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta^{k+2}\varphi_m)^2 + \xi^4 \int_\Omega (\Delta^{k+3}\varphi_m)^2 \\ &\qquad \le \frac{\xi^4}{2}\int_\Omega (\Delta^{k+3}\varphi_m)^2 + \int_\Omega (\Delta^{k+2} g(\varphi_m))^2 + 4\int_\Omega (\Delta^{k+2} v_m)^2 \end{aligned} \quad (6.14)$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla\Delta^{k+1} v_m)^2 + K\int_\Omega (\Delta^{k+2} v_m)^2 \\ &\qquad \le \frac{K}{2}\int_\Omega (\Delta^{k+2} v_m)^2 + \frac{K\ell^2}{8}\int_\Omega (\Delta^{k+2}\varphi_m)^2. \end{aligned} \quad (6.15)$$

Using an inequality from [13, page 271], we have that, since $\varphi_m \in L^\infty(Q_T)$,

$$\int_\Omega (\Delta^{k+2} g(\varphi_m))^2 \le C\|D^{2k+4}\varphi_m\|^2_{L^2(\Omega)} \le C\|\varphi_m\|^2_{H^{2k+4}(\Omega)} \le \tilde{C}(1 + \int_\Omega (\Delta^{k+2}\varphi_m)^2),$$

where we have used Theorem 6.1. We multiply (6.14) by $\frac{K}{16}$ and we add (6.15); applying Gronwall's lemma we then deduce (6.12$(k+1)$).

Next we show (6.13($k+1$)). We now suppose that $(\varphi_0, v_0) \in (D(A^{k+2}))^2$, then multiply the equations in (3.2) by ${\lambda_j}^{2k+4} v_{jm}(t)$ and sum on $j = 1, \dots, m$. After integrating the resulting inequality on $(0,t)$, and using (6.12($k+1$)), we obtain (6.13($k+1$)), which concludes the induction argument.

(ii) Next we give the estimates necessary for the proof of (ii). To begin with we give an induction argument on $l \in \mathbb{N}$ to prove the following result: For $(\varphi_0, v_0) \in (D(A))^2$ and $r > 0$

$$\|({\varphi_m}^{(i)}, {v_m}^{(i)})\|_{L^\infty(t_i,+\infty;(D(A))^2)} \leq C \quad \text{for } i = 0, \dots, l, \tag{6.16l}$$

and $t_i = 3ir$, and

$$\int_t^{t+r} \int_\Omega \{({\varphi_m}^{(l+1)})^2 + ({v_m}^{(l+1)})^2\} \leq C \quad \text{for } t \geq t_l. \tag{6.17l}$$

1) Initialization of the induction argument ($l = 0$). The result follows as in the proof of (i) with $T = \infty$, and as in Theorem 3.1 (ii).

2) The induction argument. We now assume that (6.16l), (6.17l) hold for some $l \in \mathbb{N}$ and prove that these inequalities hold for $l + 1$. To that purpose we differentiate the equations (3.1) and (3.2) $(l + 1)$ times with respect to t. This gives

$$\begin{aligned}\tau \int_\Omega {\varphi_m}^{(l+2)} w_j + \xi^2 \int_\Omega (\xi^2 \Delta {\varphi_m}^{(l+1)} - g'(\varphi_m){\varphi_m}^{(l+1)} \\ - \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)}) - \ell {\varphi_m}^{(l+1)} + 2{v_m}^{(l+1)})\Delta w_j = 0,\end{aligned} \tag{6.18}$$

$$\int_\Omega {v_m}^{(l+2)} w_j + K \int_\Omega \nabla {v_m}^{(l+1)} \nabla w_j = \frac{K\ell}{2} \int_\Omega \nabla {\varphi_m}^{(l+1)} \nabla w_j, \tag{6.19}$$

where $\mathcal{G}\ : \mathbb{R}^{l+1} \to \mathbb{R}$ is polynomial.

We multiply (6.18) by ${\varphi_{jm}}^{(l+1)}(t)$, and (6.19) by $v_{jm}^{(l+1)}(t)$ and sum for $j = 1, \dots, m$ to obtain

$$\begin{aligned}&\frac{\tau}{2}\frac{d}{dt} \int_\Omega ({\varphi_m}^{(l+1)})^2 + \xi^4 \int_\Omega (\Delta {\varphi_m}^{(l+1)})^2 + \ell\xi^2 \int_\Omega (\nabla {\varphi_m}^{(l+1)})^2 \\ &\quad = \xi^2 \int_\Omega (g'(\varphi_m){\varphi_m}^{(l+1)} + \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)}) - 2{v_m}^{(l+1)})\Delta {\varphi_m}^{(l+1)} = 0\end{aligned} \tag{6.20}$$

and

$$\frac{1}{2}\frac{d}{dt} \int_\Omega ({v_m}^{(l+1)})^2 + K \int_\Omega (\nabla {v_m}^{(l+1)})^2 = \frac{K\ell}{2} \int_\Omega \nabla {\varphi_m}^{(l+1)} \nabla {v_m}^{(l+1)}. \tag{6.21}$$

We have that

$$\begin{aligned}&\xi^2 \int_\Omega (g'(\varphi_m){\varphi_m}^{(l+1)} + \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)}) - 2{v_m}^{(l+1)})\Delta {\varphi_m}^{(l+1)} \\ &\quad \leq \frac{\xi^4}{4} \int_\Omega (\Delta {\varphi_m}^{(l+1)})^2 + \int_\Omega (g'(\varphi_m){\varphi_m}^{(l+1)} + \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)}) - 2{v_m}^{(l+1)})^2 \\ &\quad \leq \frac{\xi^4}{4} \int_\Omega (\Delta {\varphi_m}^{(l+1)})^2 + C(1 + \int_\Omega ({\varphi_m}^{(l+1)})^2) + 12 \int_\Omega ({v_m}^{(l+1)})^2\end{aligned}$$

for $t \geq t_l$. Also

$$\frac{K\ell}{2}\int_\Omega \nabla{\varphi_m}^{(l+1)}\nabla{v_m}^{(l+1)} \leq \frac{\xi^4}{4}\int_\Omega (\Delta{\varphi_m}^{(l+1)})^2 + \frac{K^2\ell^2}{4\xi^4}\int_\Omega ({v_m}^{(l+1)})^2$$

so that we deduce from (6.20), (6.21) that

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_\Omega ({\varphi_m}^{(l+1)})^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega ({v_m}^{(l+1)})^2 + \frac{\xi^4}{2}\int_\Omega (\Delta{\varphi_m}^{(l+1)})^2 + K\int_\Omega (\nabla{v_m}^{(l+1)})^2 \\ \leq C(1 + \int_\Omega ({\varphi_m}^{(l+1)})^2 + \int_\Omega ({v_m}^{(l+1)})^2\ ).\end{aligned} \tag{6.22}$$

Next we use the induction hypothesis to deduce from the uniform Gronwall lemma that

$$\begin{aligned}&\int_\Omega ({\varphi_m}^{(l+1)})^2,\ \int_\Omega ({v_m}^{(l+1)})^2\ \leq C \quad \text{and} \\ &\int_t^{t+r}\int_\Omega (\Delta{\varphi_m}^{(l+1)})^2,\ \int_t^{t+r}\int_\Omega (\nabla{v_m}^{(l+1)})^2\ \leq C\end{aligned} \tag{6.23}$$

for $t \geq t_l + r$. Then we multiply the equations in (6.18) and (6.19) by ${\lambda_j}^2{\varphi_{jm}}^{(l+1)}(t)$ and $-\lambda_j v_{jm}^{(l+1)}(t)$ respectively and sum on $j = 1, \dots, m$ to obtain

$$\begin{aligned}&\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta{\varphi_m}^{(l+1)})^2 + \frac{\xi^4}{2}\int_\Omega (\Delta^2{\varphi_m}^{(l+1)})^2 \\ &\leq \int_\Omega (\Delta(g'(\varphi_m){\varphi_m}^{(l+1)} + \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)})))^2 + 4\int_\Omega (\Delta{v_m}^{(l+1)})^2\end{aligned} \tag{6.24}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla{v_m}^{(l+1)})^2 + \frac{K}{2}\int_\Omega (\Delta{v_m}^{(l+1)})^2 \leq \frac{K\ell^2}{8}\int_\Omega (\Delta{\varphi_m}^{(l+1)})^2. \tag{6.25}$$

Since for $n \leq 3$, $H^2(\Omega)$ is a multiplicative algebra, we infer from (6.16l) that

$$\begin{aligned}&\int_\Omega (\Delta(g'(\varphi_m){\varphi_m}^{(l+1)} + \mathcal{G}(\varphi_m, {\varphi_m}^{(1)}, \dots, {\varphi_m}^{(l)})))^2 \\ &\quad \leq C(1 + \|{\varphi_m}^{(l+1)}\|^2_{H^2(\Omega)}) \leq \tilde{C}(1 + \int_\Omega (\Delta{\varphi_m}^{(l+1)})^2) \qquad \text{(by (6.23))}.\end{aligned}$$

We add (6.24) to (6.25) multiplied by $\frac{16}{K}$ and we apply the uniform Gronwall lemma, using (6.23) to deduce that

$$\begin{aligned}&\int_\Omega (\Delta{\varphi_m}^{(l+1)}(t))^2,\ \int_\Omega (\nabla{v_m}^{(l+1)}(t))^2\ \leq C \\ &\int_t^{t+r}\int_\Omega (\Delta^2{\varphi_m}^{(l+1)})^2,\ \int_t^{t+r}\int_\Omega (\Delta{v_m}^{(l+1)})^2\ \leq C\end{aligned} \tag{6.26}$$

for $t \geq t_l + 2r$. In order to obtain an estimate for $\int_\Omega (\Delta {v_m}^{(l+1)})^2$, we multiply (6.19) by ${\lambda_j}^2 v_{jm}^{(l+1)}(t)$ and sum on $j = 1, \ldots, m$. This gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta {v_m}^{(l+1)})^2 + K\int_\Omega (\nabla(\Delta {v_m}^{(l+1)}))^2 \\ \leq \frac{K\ell}{4}\int_\Omega (\Delta^2 {\varphi_m}^{(l+1)})^2 + \frac{K\ell}{4}\int_\Omega (\Delta {v_m}^{(l+1)})^2, \end{aligned} \tag{6.27}$$

to which we apply the uniform Gronwall lemma to obtain

$$\int_\Omega (\Delta {v_m}^{(l+1)}(t))^2 \leq C \quad \text{ for } t \geq t_l + 3r = t_{l+1}.$$

Finally we multiply the equations (6.18) and (6.19) by ${\varphi_{jm}}^{(l+2)}(t)$ and $v_{jm}^{(l+2)}(t)$ respectively, and sum from $j = 1, \ldots, m$ to obtain (6.17($l+1$)).

In what follows we derive by induction on $l \in \mathbb{N}$ the following result:

If $(\varphi_0, v_0) \in (D(A))^2$ then for each $\delta > 0$ and each $k \in \mathbb{N}^+$

$$\|({\varphi_m}^{(l)}, {v_m}^{(l)})\|_{L^\infty(\delta,+\infty;(D(A^k))^2)} \leq C(l, k, \delta). \tag{6.28l}$$

1) Initialization of the induction: $l = 0$. The proof is similar to that of Theorem 6.3 (i), except for the fact that we now apply the uniform Gronwall lemma.

2) The induction argument. We now suppose that $(\varphi_0, v_0) \in (D(A))^2$ and that (6.28i) holds for $i \leq l$, $l \in \mathbb{N}$. We use again an induction argument on $k \in \mathbb{N}^+$ to prove that for $s > 0$

$$\int_t^{t+s}\int_\Omega (\Delta^{k+1}{\varphi_m}^{(l+1)})^2, \ \int_t^{t+s}\int_\Omega (\Delta^{k+1}{v_m}^{(l+1)})^2 \ \leq C \tag{6.29k}$$

$$\int_\Omega (\Delta^k {\varphi_m}^{(l+1)})^2, \ \int_\Omega (\Delta^k {v_m}^{(l+1)})^2 \ \leq C \tag{6.30k}$$

for $t \geq t_k^* = 2sk$.

a) We first check that (6.29k) and (6.30k) are true for $k = 1$: choosing $r = \frac{s}{3(l+1)}$, (6.16($l+1$)) gives (6.30k) for $k = 1$ while (6.26) gives the first part of (6.29k) for $k = 1$. In order to check the second part of (6.29k) for $k = 1$, we multiply the equation in (6.18) by $-{\lambda_j}^3 v_{jm}^{(l+1)}(t)$ and sum on $j = 1, \ldots, m$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla\Delta {v_m}^{(l+1)})^2 + \frac{K}{2}\int_\Omega (\Delta^2 {v_m}^{(l+1)})^2 \leq \frac{K\ell^2}{8}\int_\Omega (\Delta^2 {\varphi_m}^{(l+1)})^2. \tag{6.31}$$

By (6.27) and (6.29k) for $k = 1$, we can apply the uniform Gronwall lemma to (6.31) to deduce that

$$\int_\Omega (\nabla\Delta {v_m}^{(l+1)}(t))^2 \leq C$$

for $t \geq t_{l+1} + r = s + \frac{s}{3(l+1)}$ and

$$\int_t^{t+s} \int_\Omega (\Delta^2 v_m{}^{(l+1)})^2 \leq C$$

for $t \geq 2s$, which completes the proof of (6.29k) for $k = 1$.

b) We now suppose that (6.29k) and (6.30k) are satisfied for $k \geq 1$ and prove them below for $k+1$. We multiply the equations in (6.18)–(6.19) by $\lambda_j{}^{2k+2}\varphi_{jm}^{(l+1)}(t)$ and $\lambda_j{}^{2k+2}v_{jm}^{(l+1)}(t)$, and sum on $j = 1, \dots, m$ to obtain

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_\Omega (\Delta^{k+1}\varphi_m{}^{(l+1)})^2 + \frac{\xi^4}{4}\int_\Omega (\Delta^{k+2}\varphi_m{}^{(l+1)})^2 \leq \int_\Omega (\Delta^{k+1}(g'(\varphi_m)\varphi_m{}^{(l+1)}))^2 \\ + \int_\Omega (\Delta^{k+1}\mathcal{G}(\varphi_m, \varphi_m{}^{(1)}, \dots, \varphi_m{}^{(l)}))^2 + 4\int_\Omega (\Delta^{k+1} v_m{}^{(l+1)})^2\end{aligned} \tag{6.32}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta^{k+1} v_m{}^{(l+1)})^2 + K\int_\Omega (\nabla(\Delta^{k+1} v_m{}^{(l+1)}))^2 \\ \leq \frac{\xi^4}{8}\int_\Omega (\Delta^{k+2}\varphi_m{}^{(l+1)})^2 + \frac{K^2\ell^2}{2\xi^4}\int_\Omega (\Delta^{k+1} v_m{}^{(l+1)})^2.\end{aligned} \tag{6.33}$$

Using (6.28l) and the embedding $H^2(\Omega) \subset L^\infty(\Omega)$ for $n = 1, 2, 3$, we deduce that

$$\int_\Omega (\Delta^{k+1}\mathcal{G}(\varphi_m, \varphi_m{}^{(1)}, \dots, \varphi_m{}^{(l)}))^2 \leq C. \tag{6.34}$$

Using also (6.30j) for $j \leq k$, we have that

$$\begin{aligned}\int_\Omega (\Delta^{k+1}(g'(\varphi_m)\varphi_m{}^{(l+1)}))^2 \leq C(\int_\Omega (\Delta^{k+1}\varphi_m{}^{(l+1)})^2 + |D^{2k+1}\varphi_m{}^{(l+1)}|^2_{L^2(\Omega)} \\ + |D^{2k}\varphi_m{}^{(l+1)}|^2_{L^2(\Omega)} + |D^{2k-1}\varphi_m{}^{(l+1)}|^2_{L^2(\Omega)} + 1) \\ \leq C(\|\varphi_m{}^{(l+1)}\|^2_{H^{2k+2}(\Omega)} + 1) \leq \tilde{C}(\int_\Omega (\Delta^{k+1}\varphi_m{}^{(l+1)})^2 + 1).\end{aligned} \tag{6.35}$$

We substitute (6.34) and (6.35) in (6.32) to which we add (6.33) and apply the uniform Gronwall lemma. We obtain

$$\int_\Omega (\Delta^{k+1}\varphi_m{}^{(l+1)}(t))^2, \int_\Omega (\Delta^{k+1} v_m{}^{(l+1)}(t))^2 \leq C$$

and

$$\int_t^{t+s}\int_\Omega (\Delta^{k+2}\varphi_m{}^{(l+1)})^2, \int_t^{t+s}\int_\Omega (\nabla(\Delta^{k+1} v_m{}^{(l+1)}))^2 \leq C$$

for $t \geq t_k^* + s$. In order to show the second inequality in (6.29k) for $k := k+2$ we multiply the equations in (6.19) by $-{\lambda_j}^{2k+3} v_{jm}^{(l+1)}(t)$ and sum on $j = 1, \ldots, m$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla\Delta^{k+1} {v_m}^{(l+1)})^2 + \frac{K}{2}\int_\Omega (\Delta^{k+2} {v_m}^{(l+1)})^2 \leq \frac{K\ell^2}{8}\int_\Omega (\Delta^{k+2} {\varphi_m}^{(l+1)})^2$$

to which we apply the uniform Gronwall lemma to obtain that

$$\int_t^{t+s}\int_\Omega (\Delta^{k+2} {v_m}^{(l+1)})^2 \leq C \qquad \text{for } t \geq t_k^* + 2s = t_{k+1}^*.$$

This completes the proof of (6.29k) and (6.30k). If $\delta > 0$ is chosen such that for each $k \in \mathbb{N}^+$ we take $s = \frac{\delta}{2k}$, then we have that

$$\|({\varphi_m}^{(l+1)}, {v_m}^{(l+1)})\|_{L^\infty(\delta,+\infty;(D(A^k))^2)} \leq C(l+1, k, \delta)$$

which completes the induction argument on l.

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