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EXISTENCE, UNIQUENESS, AND ASYMPTOTIC STABILITY OF TRAVELING WAVES IN NONLOCAL EVOLUTION EQUATIONS

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Abstract. The existence, uniqueness, and global exponential stability of traveling wave solutions of a class of nonlinear and nonlocal evolution equations are established. It is assumed that there are two stable equilibria so that a traveling wave is a solution that connects them. A basic assumption is the comparison principle: a smaller initial value produces a smaller solution. When applied to differential equations or integro-differential equations, the result recovers and/or complements a number of existing ones.

1. Introduction. In this paper, we are concerned with a one space dimensional evolution equation

$$u_t(x,t) = \mathcal{A}[u(\cdot,t)](x), \qquad x \in \mathbb{R},\, t>0, \tag{1.1}$$

where $\mathcal{A}$ is a nonlinear operator which is independent of the time t, maps functions of space variable $\cdot$ to functions of x, and, via (1.1), generates a semigroup on the Banach space $L^\infty(\mathbb{R})$.

We assume that $\mathcal{A}$ is translation invariant; namely, for any $h \in \mathbb{R}$ and any function $u(x)$,

$$\mathcal{A}[u(\cdot+h)](x) = \mathcal{A}[u(\cdot)](x+h) \qquad \forall x \in \mathbb{R}. \tag{1.2}$$

With this translation invariance, $\mathcal{A}$ maps constant functions to constant functions, so that, denoting by $\mathbf{1}$ the function identically equal to 1, there is a function $f(\cdot)$ such that

$$\mathcal{A}[\alpha\mathbf{1}] = f(\alpha)\mathbf{1} \qquad \forall\, \alpha \in \mathbb{R}. \tag{1.3}$$

We assume that f has the following properties:

$$f \in C^1(\mathbb{R}), \quad f(0)=f(1)=0, \quad f'(0)<0, \quad f'(1)<0. \tag{1.4}$$

Namely, $\mathbf{0}$ and $\mathbf{1}$ are two stable equilibria of $\mathcal{A}$ among all constant functions.

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We are interested in traveling-wave solutions that connect the two stable equilibria **0** and **1**. Throughout this paper, a traveling-wave solution of (1.1) always refers to a pair (U, c), where $U = U(\xi)$ is a function on $\mathbb{R}$ and c is a constant, such that $u(x,t) := U(x - ct)$ is a solution of (1.1) and

$$\lim_{\xi \to \infty} U(\xi) = 1, \qquad \lim_{\xi \to -\infty} U(\xi) = 0. \tag{1.5}$$

We call c the traveling-wave speed and U the profile of the wave front. If $c = 0$, we say U is a standing wave.

Our main assumption on $\mathcal{A}$ is the following comparison principle:

$$\begin{aligned}&\text{if } u_t \geq \mathcal{A}[u],\ v_t \leq \mathcal{A}[v],\ \text{and } u(\cdot,0) \geq (\not\equiv) v(\cdot,0), \text{ then}\\ &u(\cdot,t) > v(\cdot,t) \text{ for all } t > 0.\end{aligned} \tag{1.6}$$

Under certain additional regularity assumptions on $\mathcal{A}$, we shall show that traveling-wave solutions are unique up to a translation (Theorem 2.1) and globally exponentially stable (Theorem 3.1). Also, we provide an a priori estimate for the wave speed (Theorem 3.5) and the existence of a traveling wave (Theorem 4.1).

Our study of (1.1) is motivated by the following traveling-wave problems.

A. Reaction-Diffusion. In their classical paper [12], Fife and McLeod proved the global exponential stability of traveling-wave solutions of the nonlinear reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\, t > 0, \tag{1.7}$$

where $f(\cdot)$ satisfies (1.4). In proving their stability theorem, they used the variational structure

$$\int_{\mathbb{R}} e^{c\xi} u_t^2 + \frac{d}{dt} \int_{\mathbb{R}} e^{c\xi} \Big(u_\xi^2 - 2[F(u) - F(1)H(\xi)] \Big)\, d\xi = 0, \quad \xi := x - ct, \tag{1.8}$$

where $F(u) := \int_0^u f(s)ds$ and $H(\xi)$ is the Heaviside function equal to 1 when $\xi > 0$ and 0 when $\xi < 0$. Here it is crucial and very subtle to include the terms $F(1)H(\xi)$ and $e^{c\xi}$ in (1.8).

B. Neural network. In [10], Ermentrout and McLeod studied, among other things, the existence and uniqueness of monotonic traveling-wave solutions of the integral differential equation

$$u_t(x,t) = -u(x,t) + J * S(u) \quad \big(J * S(u) := \textstyle\int_{\mathbb{R}} J(x-y) S(u(y,t))\, dy\big), \tag{1.9}$$

where $S \in C^1(\mathbb{R})$, $S'(\cdot) > 0$ in $[0,1]$, $S(0) = 0$, $S(1) = 1$, $S'(0) < 1, S'(1) < 1$, and J is a smooth kernel satisfying

$$J \geq 0 \text{ in } \mathbb{R}, \qquad \int_{\mathbb{R}} J(y)\, dy = 1. \tag{1.10}$$

Note that $f(u) := S(u) - u$ satisfies (1.4). Their result is established by using a homotopy connecting (1.9) and (1.7).

C. Ising model. In [7, 8, 9, 14], the authors studied the integro-differential equation

$$u_t = \tanh\{\beta(J * u + h)\} - u, \tag{1.11}$$

where $\beta > 1$ and h are constants and J is a smooth kernel supported on $[-1, 1]$ and satisfies (1.10). When $h = 0$, the existence of a unique standing wave was established by Dal Passo and De Mottoni ([7]), whereas its local exponential stability was shown in [9]. For fixed $\beta > 1$ and every sufficiently small h, De Masi, Gobron, and Presutti ([8]) established the existence, uniqueness, and global exponential stability of traveling waves of (1.11). Since (1.11) does not seem to have an energy identity similar to (1.8), their stability analysis involves a great many technicalities. Recently, Orlandi and Triolo ([14]) extended part of the result of [8] to the general h such that the algebraic equation $\tanh\{\beta(u + h)\} - u = 0$ has three distinguished roots.

D. Phase transition. Recently, Bates, Fife, Ren, and Wang ([2]) studied the traveling waves of

$$u_t = \lambda[J * u - u] + f(u), \tag{1.12}$$

where $\lambda > 0$ is a parameter, f satisfies (1.4) , and J satisfies (1.10). As in [10], a homotopy connecting (1.12) and (1.7) was used to prove the existence of monotonic traveling waves. The uniqueness is established by a method similar to what we shall present here in this paper. As in [8], since an energy identity similar (1.8) is hard to find for $c \neq 0$, asymptotic stability was established only for standing waves.

E. Thalamic model. Very recently, Z. Chen and Ermentrout ([5]) studied traveling waves of

$$u_t = -\beta u + \alpha(1 - u)H^\varepsilon(J * S(u) - \theta), \tag{1.13}$$

where α and β are positive constants, $H^\varepsilon(s) := \frac{1}{2}[1 + \tanh(\frac{s}{\varepsilon})]$ $(0 < \varepsilon \ll 1)$ is the smoothed Heaviside function, $S' > 0$, and θ is a parameter taking values in $(S(0), S(1))$. By using a homotopy, existence and uniqueness of monotonic traveling waves were established.

Our result applies to all these cases (except (1.12) with small λ, which we shall discuss later). More generally, our method applies to traveling-wave solutions of evolution equations of the form

$$u_t = Du_{xx} + G(u, J_1 * S^1(u), \dots, J_n * S^n(u)), \tag{1.14}$$

where $D \geq 0$ is any constant, J_i, $i = 1, \dots, n$, are nonnegative kernels satisfying (1.10), $G_{p_i}(u, p) > 0$, $S^i_u(u) > 0$, $G_u(u, p) < 0$, and $f(u) := G(u, S^1(u), \dots, S^n(u))$ satisfies (1.4).

Our basic strategy is to construct various kinds of super-sub solutions to control the solutions of (1.1). Hence, the comparison principle (1.6) and the stability assumption (1.4) are the only key properties needed in our analysis.

For the reader's convenience, here we describe the structure of the paper and briefly explain our ideas. In Section 2, we prove the uniqueness of traveling-wave solutions by using a moving plane technique ([1, 3, and the references therein]). The idea is very simple: if (U, c) and $(\tilde{U}, \tilde{c})$ are two solutions, then $U(\cdot) \leq \tilde{U}(\cdot + h)$ for a sufficiently large constant h, which implies, by the comparison principle, either $U(\cdot) \equiv \tilde{U}(\cdot + h)$ or $U(\cdot) < \tilde{U}(\cdot + h)$; if $U(\cdot) < \tilde{U}(\cdot + h)$, then h can be decreased to a minimum where an identity holds, so that $\tilde{U}$ is merely a translation of U. Nevertheless, since $\mathbb{R}$ is not compact, it is not trivial to find an h such that $U(\cdot) \leq \tilde{U}(\cdot + h)$ and to decrease h in case $U(\cdot) < \tilde{U}(\cdot + h)$. Here we shall solve this problem by constructing sub-super solutions based on the following principle: A supersolution can be obtained from a monotonically increasing solution by raising the solution and making it move more slowly whereas a subsolution can be obtained in the opposite way. This principle, though not explicitly stated, has been used in [4, 11, 12] and many other places.

In Section 3, we prove the global exponential stability. Namely, if $u_0(x)$ is larger than a certain value for all $x \gg 1$ and is less than a certain value for all $x \ll -1$, then the solution $u(\cdot, t)$ of (1.1) with initial value $u(\cdot, 0) = u_0(\cdot)$ approaches, as $t \to \infty$, a traveling wave. The rate is exponential in the sense that $\|u(\cdot, t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \leq K e^{-\kappa t}$ for some positive κ independent of u_0 and some constants ξ and K depending on u_0. We prove it in three steps.

In Step 1, we imitate Fife-McLeod ([12]), constructing sub- and supersolutions to show that, for large enough t, $u(x, t)$ is close to 1 for $x \gg 1$ and is close to 0 for all $x \ll -1$; i.e.,

$$U(x - ct + \xi) - \delta \leq u(x, t) \leq U(x - ct + \xi + h) + \delta \quad \forall x \in \mathbb{R} \tag{1.15}$$

for arbitrarily fixed $\delta > 0$ and some $\xi \in \mathbb{R}$ and $h \gg 1$. As a by-product, we have an explicit a priori estimate on the traveling-wave speed, which is very much needed in [2, 5, 10] for homotopy.

In Step 2, we show that, for any fixed small positive δ, there is a sufficiently large t such that (1.15) holds for $h = 1$. This step is achieved in [12] by using the compactness of the family $\{u(\cdot, t)\}_{t \geq 1}$, the uniqueness of the traveling-wave solutions, and the energy identity (1.8). Similarly, in [1], this step is done by studying the accumulation points of the sequence $\{u(\cdot + \xi_j, j)\}_{j=1}^{\infty}$ where $u(\xi_j, j) = 1/2$. Here, we prove the assertion by utilizing sub-super solutions, and therefore we do not use the compactness of the family $\{u(\cdot, t)\}_{t \geq 1}$ and the variational structure of the equation. One advantage of our method is that we have an estimate on the time t that is needed to make h small (≤ 1).

In Step 3, we establish the exponential stability. All the previous papers ([1, 2, 8, 10, 12, 14]) used the linear operator $\mathcal{A}'[U](\cdot)$, where $\mathcal{A}'[\cdot](\cdot)$ is the Fréchet derivative of $\mathcal{A}$ defined by

$$\mathcal{A}'[u](v) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\mathcal{A}[u+\varepsilon v]-\mathcal{A}[u]\Big\}. \tag{1.16}$$

It was proved that the linear operator $\mathcal{A}'[U]+c\frac{\partial}{\partial\xi}$ has a vanishing eigenvalue which is simple and corresponds to the translation invariance of $\mathcal{A}$, whereas the real parts of all the other eigenvalues and essential spectra are $\le -\kappa$ for some $\kappa > 0$. Therefore, with the global stability result of Step 2, a local linearization yields the exponential stability. Here, we shall introduce a totally different approach, a "squeezing" technique. We use the sub-supersolutions constructed in Step 2 to show that, when δ and h in (1.15) are small enough (e.g. $\delta \ll 1$ and $h \le 1$), then later on for each fixed increment of t, both h and δ decrease by a fixed factor bigger than 1, thereby establishing the exponential stability. In doing this, we avoided the study of the eigenvalues of the operator $\mathcal{A}'[u]+c\frac{\partial}{\partial\xi}$ and correspondingly the delicate introduction of appropriate function spaces associated with the eigenvalue problem.

In Section 4, we establish the existence of monotonic C^1 traveling waves of (1.1). Since we are working on an abstract operator, we shall not use the extremely powerful homotopy method used in [2, 8, 10, 14]. Instead, we shall introduce a new method (to the author's best knowledge), which is motivated by the asymptotic behavior of solutions of (1.1). We take an arbitrarily fixed monotonic initial data, say, a smoothed Heaviside function, let it be evolved according to (1.1), and show that, as $t \to \infty$, the profile $u(\cdot + \xi(t), t)$, where $u(\xi(t), t) = 1/2$, approaches a limit $U(\cdot)$ which is the profile of a traveling wave. The main effort here is to show that $u(\cdot + \xi(t), t)$ does not become very flat so that the limit U is either a constant function (the unstable equilibrium of (1.1)) or a monotonic function connecting the stable and unstable equilibria of (1.1). To do this, we estimate the quantity $z(1-\delta,t)-z(\delta,t)$ where $z = z(\alpha,t)$, $\alpha \in (0,1)$, is the inverse function of $u(z,t) = \alpha$. Once an upper bound for $z(1-\delta,t)-z(\delta,t)$ is established for all $t > 1$, then $U(\xi)$ has the correct limit, as $\xi \to \pm\infty$, stated in (1.5). To show that U is a traveling wave, we consider $\tilde{U}(\cdot,t)$, the solution of (1.1) starting from $U(\cdot)$. Using sub-super solutions constructed from the solution u, we are able to show that $\tilde{U}(\cdot,T)$, for any $T \in [1,2]$, is a translation of $U(\cdot)$, so that U is actually the profile of a traveling wave.

Section 5 contains two parts. In the first part, we apply our analysis to the example (1.14) to recover and/or to complement the results established in [2, 5, 8, 10, 12, 14].

In the second part, we investigate an "exceptional" case to which our analysis fails to apply. It is (1.12) for λ small so that the function $f(u)-\lambda u$ is not monotonically decreasing. In such a case, it was discovered in [2] that (1.12) has a monotonic standing wave which is discontinuous, whereas our uniqueness and asymptotic stability result applies only to smooth traveling-wave solutions. It was also shown in [2] that this monotonic standing wave is unique in the class of monotonic traveling waves, and is asymptotically stable for monotonic initial data. Hence, to settle down the asymptotic stability problem for general nonmonotonic initial data, we briefly study the equation (1.12) for small λ. We show that there are infinitely many nonmonotonic standing waves of (1.12) and they can be arbitrarily close to the monotonic standing wave established in [2]. This shows that standing waves of (1.12) (with small λ) are neither unique nor asymptotically stable and therefore it explains why our analysis does not apply to this “exceptional” case.

Remark 1.1. In [1], $\mathcal{A}[u] := u_{xx} + f(u,t)$, where $f(u,\cdot)$ is periodic in t with period T independent of x. We expect our analysis, together with some of the techniques used in [1], extends to the case when $\mathcal{A}$ in (1.1) depends on t but is periodic in t.

Remark 1.2. After this paper was accepted, we learned of two problems related to the subject of this paper.

F. An activator-inhibitor model. In [13], Nishiura studied, among other models, the following system:

$$\begin{cases} u_t - \Delta u = f(u) - v, \\ -\Delta v + v = u. \end{cases}$$

Note that if we write the solution of the second equation as $v = J * u$ where $J > 0$ is the fundamental solution of the operator $(-\Delta + \mathbf{1})$, then the above system can be written as (1.14) so that all our analysis here applies. In particular, our existence result implies that there exists a (planner) traveling wave solution to the system.

G. Lattice differential equations. Recently, Chow, Mallet-Paret, and Shen ([6]) studied the following equation:

$$w_t(y,t) = \sum_{i=1}^{k} \alpha_i \Delta_{h_i} w + f(w), \quad y \in \mathbb{R}^N, t > 0,$$

where $f(\cdot)$ is as in (1.4), $\alpha_i > 0$, $\sum_{i=1}^{k} \alpha_i = 1$, and

$$\Delta_{h_i} w = \frac{1}{h_i^2} \sum_{j=1}^{N} (w(y + h_i \vec{e}_j, t) + w(y - h\vec{e}_j, t) - 2w(y,t))$$

is the discrete Laplacian. A traveling wave is a triple $(U, c, \vec{e})$ where U is a function over R, c a constant, and $\vec{e}$ a unit vector, such that $w(y,t) := U(y \cdot \vec{e} - ct)$ is a solution. In other words, one is looking for a traveling-wave solution for the equation (1.12) with

$$\lambda = 2 \sum_{i=1}^{k} \frac{\alpha_i}{h_i^2}$$

and

$$J = \sum_{i=1}^{k} \alpha_i \sum_{j=1}^{N} \frac{1}{2} [\delta(x + h_i \vec{e}_j \cdot \vec{e}) + \delta(x - h_i \vec{e}_j \cdot \vec{e})].$$

Here $\delta(\cdot)$ stands for the Dirac function. Our current analysis, however, does not apply to this problem; see Remark 2.3 (4).

In the sequel, we call $u(x,t)$ a subsolution (or supersolution) of (1.1) if $u_t \le \mathcal{A}[u]$ (or $u_t \ge \mathcal{A}[u]$).

2. Uniqueness. In proving the uniqueness of traveling-wave solutions of (1.1), we impose the following hypotheses:

(A1) $\mathcal{A}$ is translation invariant (cf. (1.2)) and the function $f(\cdot)$ defined in (1.3) satisfies (1.4);

(A2) If $u_t \ge \mathcal{A}[u]$, $v_t \le \mathcal{A}[v]$, and $u(\cdot,0) \ge (\not\equiv) v(\cdot,0)$, then $u(\cdot,t) > v(\cdot,t)$ for all $t > 0$;

(A3) There exist positive constants K_1 and K_2 and a probability measure ν such that, for any function u, v with $-1 \le u, v \le 2$ and every $x \in \mathbb{R}$,

$$|\mathcal{A}'[u+v](\mathbf{1})(x) - \mathcal{A}'[u](\mathbf{1})(x)| \le K_1 \int_{\mathbb{R}} |v(x-y)|\nu(dy) + K_2 \|v(x+\cdot)\|_{C^0([-1,1])},$$

where $\mathcal{A}'[\cdot](\cdot)$ is the Fréchet derivative of $\mathcal{A}$ defined as in (1.16).

Theorem 2.1 (Uniqueness). *Assume that* (A1)–(A3) *hold. Also assume that* (1.1) *has a traveling-wave solution* (U,c) *having the following properties*:

$$U \in C^1(\mathbb{R}), \quad U'(\xi) > 0 \ \ on \ \ \mathbb{R}, \quad \lim_{|\xi|\to\infty} U'(\xi) = 0. \tag{2.1}$$

Then for any traveling-wave solution $(\tilde{U}, \tilde{c})$ *of* (1.1) *with* $\tilde{U} \in C^0(\mathbb{R})$ *and* $0 \le \tilde{U} \le 1$ *on* $\mathbb{R}$, *we have* $\tilde{c} = c$ *and* $\tilde{U}(\cdot) = U(\xi_0 + \cdot)$ *for some* $\xi_0 \in \mathbb{R}$.

We remark here again that throughout this paper, by a traveling-wave solution we mean always to include the boundary condition (1.5).

To prove the theorem, we need the following technical lemma.

Lemma 2.2. *Assume that* (A1) *and* (A3) *hold and let* (U,c) *be as in Theorem* 2.1. *Then there exist a small positive constant* δ_0 *(which is independent of* U*) and a large positive constant* σ_1 *(which depends on* U*) such that, for any* $\delta \in (0, \delta_0]$ *and every* $\xi_0 \in \mathbb{R}$, *the functions* w^+ *and* w^- *defined by*

$$w^{\pm}(x,t) := U\big(x - ct + \xi_0 \pm \sigma_1\delta[1 - e^{-\beta t}]\big) \pm \delta e^{-\beta t} \tag{2.2}$$

are a super-solution and a sub-solution respectively. Here and in the sequel, $\beta := \frac{1}{2}\min\{-f'(0), -f'(1)\}$.

Proof. We only consider w^+. The proof for w^- is analogous and is left to the reader.

We define the small constant $\delta_0 > 0$ and a large positive constant $M_0 \ge 1$ by

$$\delta_0 = \min\Big\{\frac{1}{3}, \frac{\beta}{4(K_1+K_2)}\Big\}, \qquad \nu\big(\{|y| \ge M_0\}\big) := \int_{|y|\ge M_0} \nu(dy) \le \frac{\beta}{4K_1}. \tag{2.3}$$

Let $M_1 = M_1(U)$ be a constant such that

$$U(\xi) > 1 - \delta_0 \ \text{ for all } \ \xi \geq M_1, \qquad U(\xi) < \delta_0 \ \text{ for all } \ \xi \leq -M_1.$$

We define $\sigma_1 = \sigma_1(U)$ by

$$\sigma_1 = \big(\|f'\|_{C^0([-1,2])} + \beta + K_1 + K_2\big)\Big(\min_{\xi\in[-M_1-M_0, M_1+M_0]} \beta U'(\xi)\Big)^{-1}. \tag{2.4}$$

Now we show that w^+ is a supersolution. Denoting $x - ct + \xi_0 + \sum_1 \delta(1 - e^{-\beta t})$ by ξ, we can compute

$$\begin{aligned} w_t^+ - \mathcal{A}[w^+] &= (-c + \sum_1 \beta\delta e^{-\beta t})U'(\xi) - \beta\delta e^{-\beta t} - \mathcal{A}[w^+(\cdot, t)](x) \\ &= \sum_1 \beta\delta e^{-\beta t} U'(\xi) - \beta\delta e^{-\beta t} + \big\{\mathcal{A}[U](\xi) - \mathcal{A}[U + \delta e^{-\beta t}\mathbf{1}](\xi)\big\} \\ &= \delta e^{-\beta t}\big\{\beta \sum_1 U'(\xi) - \beta - \textstyle\int_0^1 \mathcal{A}'[U + \theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi)\, d\theta\big\}, \end{aligned}$$

where we have used the equation $cU'(\xi) = -\mathcal{A}[U](\xi)$ in the second equation and the definition of $\mathcal{A}'[\cdot](\cdot)$ in the third equation. We consider three separate cases:
(i) $|\xi| \leq M_1 + M_0$;
(ii) $\xi > M_1 + M_0$; and
(iii) $\xi < -M_1 - M_0$.

Case (i): Note that, for any $\alpha \in \mathbb{R}$,

$$\mathcal{A}'[\alpha\mathbf{1}](\mathbf{1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\{\mathcal{A}[\alpha\mathbf{1} + \varepsilon\mathbf{1}] - \mathcal{A}[\alpha\mathbf{1}]\} = f'(\alpha)\mathbf{1}.$$

Taking $\alpha = U(\xi) + \theta\delta e^{-\beta}$, we obtain, by the assumption (A3),

$$\begin{aligned} &\big|\mathcal{A}'[U(\cdot) + \theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi) - f'(U(\xi) + \theta\delta e^{-\beta t})\big| \\ &= \big|\mathcal{A}'[U(\cdot) + \theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi) - \mathcal{A}'[U(\xi)\mathbf{1} + \theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi)\big| \\ &\leq K_1 \int_{\mathbb{R}} |U(\xi - y) - U(\xi)|\, \nu(dy) + K_2\|U(\cdot + \xi) - U(\xi)\|_{C^0([-1,1])} \leq K_1 + K_2. \end{aligned}$$

It then follows that

$$|\int_0^1 \mathcal{A}'[U + \theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi)\, d\theta| \leq \|f'\|_{C^0([-1,2])} + K_1 + K_2.$$

Hence, by the definition of $\sum_1$, we have that $w_t^+ - \mathcal{A}[w^+] \geq 0$ when $|\xi| \leq M_1 + M_0$.

Case (ii). Since $\mathcal{A}'[\mathbf{1}](\mathbf{1}) = f'(1)\mathbf{1}$, we have, by (A3),

$$\begin{aligned}
&\big|\mathcal{A}'[U(\cdot)+\theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi) - f'(1)\big| \\
&= \big|\mathcal{A}'[U(\cdot)+\theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi) - \mathcal{A}'[\mathbf{1}](\mathbf{1})(\xi)\big| \\
&\le K_1\Big\{\int_{|y|\ge M_0} + \int_{|y|<M_0}\Big\}|U(\xi-y)+\theta\delta e^{-\beta t}-1|\,\nu(dy) \\
&\qquad + K_2\|U(\xi+\cdot)+\theta\delta e^{-\beta t}-1\|_{C^0([-1,1])} \\
&\le K_1\big\{\nu(\{|y|\ge M_0\})+\delta_0\big\} + K_2\delta_0 \le \beta
\end{aligned}$$

by using $\|U(\xi+\cdot)+\theta\delta e^{-\beta t}-1\|_{C^0([-M_0,M_0])} \le \delta_0$. It then follows from the assumption $f'(1)\le -2\beta$ that

$$-\int_0^1 \mathcal{A}'[U+\theta\delta e^{-\beta t}\mathbf{1}](\mathbf{1})(\xi)\,d\theta \ge -f'(1)-\beta \ge \beta,$$

which implies $w_t^+ - \mathcal{A}[w^+] \ge 0$ when $\xi \ge M_1 + M_0$.

Similarly, we can show that $w_t^+ - \mathcal{A}[w^+] \ge 0$ in the case (iii). This completes the proof of the lemma. $\square$

Proof of Theorem 2.1. We shall prove the theorem in two steps.

Step 1. Since $\tilde{U}(\xi)$ and $U(\xi)$ have the same limit as $\xi \to \pm\infty$, there exist $\xi_1 \in \mathbb{R}$ and $h \gg 1$ such that

$$U(\cdot+\xi_1) - \delta_0 < \tilde{U}(\cdot) < U(\cdot+\xi_1+h) + \delta_0 \qquad \text{on } \mathbb{R}.$$

By a translation, we can assume $\xi_1 = 0$. Comparing $\tilde{U}(x-\tilde{c}t)$ with $w^\pm$ in (2.2) (with $\xi_0 = 0$ for w^- and $\xi_0 = h$ for w^+), we obtain, for all $x\in\mathbb{R}$ and $t>0$,

$$\begin{aligned}
&U(x-ct-\sigma_1\delta_0(1-e^{-\beta t})) - \delta_0 e^{-\beta t} \\
&\qquad < \tilde{U}(x-\tilde{c}t) < U(x-ct+h+\sigma_1\delta_0(1-e^{-\beta t})) + \delta_0 e^{-\beta t}.
\end{aligned}$$

Keeping $\xi := x - \tilde{c}t$ fixed, sending $t\to\infty$, and using (1.5), we then obtain from the first inequality that $c \ge \tilde{c}$ and from the second inequality that $c \le \tilde{c}$, so that $\tilde{c} = c$. In addition,

$$U(\xi-\sigma_1\delta_0) \le \tilde{U}(\xi) \le U(\xi+h+\sigma_1\delta_0) \qquad \forall \xi\in\mathbb{R}. \tag{2.5}$$

Step 2. We define

$$\xi^* := \inf\{\xi \;:\; \tilde{U}(\cdot) \le U(\cdot+\xi)\}, \qquad \xi_* := \sup\{\xi \;:\; \tilde{U}(\cdot) \ge U(\cdot+\xi)\}.$$

From (2.5), both ξ^* and ξ_* are well-defined. To finish the proof, it suffices to show that $\xi_* = \xi^*$. To do this, we use a contradiction argument. Hence, we assume that $\xi_* < \xi^*$ and $\tilde{U}(\cdot) \not\equiv U(\cdot+\xi^*)$.

Since we assume $\lim_{|\xi|\to\infty} U'(\xi) = 0$, there exists a large positive constant $M_2 = M_2(U)$ such that

$$2\sigma_1 U'(\xi) \le 1 \quad \text{if} \ \ |\xi| \ge M_2. \tag{2.6}$$

Note that the definition of ξ^* implies $\tilde{U}(\cdot) \le U(\cdot + \xi^*)$, so that, by the comparison, $\tilde{U}(\cdot) < U(\cdot + \xi^*)$ on $\mathbb{R}$. Consequently, by the continuity of U and $\tilde{U}$, there exists a small constant $\hat{h} \in (0, \frac{1}{2\sum_1}]$ such that

$$\tilde{U}(\xi) < U(\xi + \xi^* - 2\sigma_1\hat{h}) \qquad \forall \xi \in [-M_2 - 1 - \xi^*, M_2 + 1 - \xi^*]. \tag{2.7}$$

When $|\xi + \xi^*| \ge M_2 + 1$,

$$\begin{aligned} U(\xi + \xi^* - 2\sigma_1\hat{h}) - \tilde{U}(\xi) &> U(\xi + \xi^* - 2\sigma_1\hat{h}) - U(\xi + \xi^*) \\ &= -2\sigma_1\hat{h}U'(\xi + \xi^* - 2\theta\sigma_1\hat{h}) > -\hat{h} \end{aligned}$$

by the definition of M_2. Hence, in conjunction with (2.7), $U(\cdot + \xi^* - 2\sigma_1\hat{h}) + \hat{h} \ge \tilde{U}(\cdot)$ on $\mathbb{R}$. Then, by the comparison, for all $x \in \mathbb{R}$ and $t > 0$,

$$U(x - ct + \xi^* - 2\sigma_1\hat{h} + \sigma_1\hat{h}(1 - e^{-\beta t})) + \hat{h}e^{-\beta t} > \tilde{U}(x - ct).$$

Setting $\xi = x - ct$ fixed and sending $t \to \infty$ we obtain $U(\xi + \xi^* - \sigma_1\hat{h}) \ge \tilde{U}(\xi)$ for all $\xi \in \mathbb{R}$. But this contradicts the definition of ξ^*. Hence, $\xi^* = \xi_*$, which completes the proof of the theorem. $\square$

Remark 2.3. (1) Notice that Theorem 2.1 does not require $\tilde{U}$ to be monotonic. That is, our uniqueness is in the class of all traveling waves.

(2) Lemma 2.2 still holds if (2.1) is replaced by the weaker condition

$$\inf_{\xi \in [-m,m], h \ne 0} \frac{U(\xi + h) - U(\xi)}{h} > 0 \quad \text{for all } m > 0\ . \tag{2.8}$$

The proof is unchanged except that one explains properly the positivity of the distributional derivative U'. In particular, from Step 1 of the proof of the theorem, one sees that $\tilde{c} = c$, as long as U satisfies (2.8), regardless of whether $\tilde{U}$ is continuous or not.

(3) The conditions (2.1) and $\tilde{U} \in C^0(\mathbb{R})$ in Theorem 2.1 can be replaced by the following conditions: (i) U satisfies (2.8) and $\lim_{|\xi|\to\infty} U'(\xi) = 0$; and (ii) both U and $\tilde{U}$ have at most one discontinuity which is a jump discontinuity. In fact, in such a situation, one can derive $U(\cdot + \xi_* - 0) < \tilde{U}(\xi) < U(\cdot + \xi^* + 0)$. This implies either $\tilde{U}(\xi) < U(\xi + \xi^* - 2\sigma_1\hat{h})$ or $\tilde{U}(\xi) > U(\xi + \xi_* + 2\sigma_1\hat{h})$ for all $\xi \in [-M_2 - 1 + \xi_*, M_2 + 1 + \xi^*]$; the former inequality is true when the jump of $\tilde{U}$ does not occur at the jump of $U(\cdot + \xi^*)$ and the latter one holds when the jump of $\tilde{U}$ does not occur at the jump of $U(\cdot + \xi_*)$. Following the rest of Step 2, one then

can conclude that $\xi_* = \xi^*$ and $U(\cdot + \xi_* - 0) = \tilde{U}(\cdot - 0)$. This remark recovers the uniqueness theorem of [2] in the class of monotonic traveling-wave solutions.

On the other hand, as shall be seen in Section 5.2, without the assumption that both U and $\tilde{U}$ have at most one jump discontinuity, $\tilde{U}$ may not be a translation of U.

(4) Our generous assumption that ν is only a probability measure may generate some applications for the case when J in (1.9)–(1.14) is merely the density of a probability measure; see Remark 1.2, model **G**. Allowing J to be densities of measures has many impacts on numerical simulations. For example, if $J = \frac{1}{2}\{\delta(x-h) + \delta(x+h)\}$, where δ is the Dirac measure, then (1.12) can be written as

$$u_t = \frac{\lambda h^2}{2} D^h_{xx} u + f(u), \qquad D^h_{xx} u := \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \sim u_{xx},$$

which is a semidiscretization of (1.7) with diffusion coefficient $\lambda h^2/2$. Unfortunately, currently we cannot assume that J in (1.9)–(1.13) is only the density of a probability measure since in such a case, the comparison principle (A2) does not hold. It does hold, however, for (1.14) when $D > 0$, so that our uniqueness Theorem 2.1 applies to (1.14) with $D > 0$ and J_i, $i = 1, \ldots, n$, are the densities of probability measures.

3. Asymptotic stability. In his section, we shall show that a C^1 monotonic traveling wave is globally exponentially stable. To do this, we need the following additional assumptions on the nonlinear operator $\mathcal{A}$.

(B1) $\mathcal{A}$ is translation invariant and, for some positive constants a^- and a^+ with $0 < a^- \le a^+ < 1$, the function f in (1.3) satisfies $f > 0$ in $(-1, 0) \cup (a^+, 1)$ and $f < 0$ in $(0, a^-) \cup (1, 2)$;

(B2) There exists a positive nonincreasing function $\eta(m)$ defined on $[1, \infty)$ such that for any $u(x,t), v(x,t)$ satisfying $-1 \le u, v \le 2$, $u_t \ge \mathcal{A}[u]$, $v_t \le \mathcal{A}[v]$, and $u(\cdot, 0) \ge v(\cdot, 0)$, there holds

$$\min_{x \in [-m,m]} \big\{u(x,1) - v(x,1)\big\} \ge \eta(m) \int_0^1 [u(y,0) - v(y,0)]\, dy \qquad \forall m \ge 1;$$

(B3) With K_1, K_2, ν, u, and v as in **(A3)**, there holds, for every $x \in \mathbb{R}$,

$$\big|\mathcal{A}[u+v](x) - \mathcal{A}[u](x)\big| \le K_1 \int_{\mathbb{R}} |v(x-y)| \nu(dy) + K_2 \|v''\|_{C^0(\mathbb{R})}.$$

Theorem (Global Exponential Stability). *Assume that* (A1)–(A3) *and* (B1)–(B3) *hold. Also assume that* (1.1) *has a traveling-wave solution* (U, c) *satisfying* (2.1). *Then there exists a positive constant* κ *such that for any* $u_0 \in L^\infty(\mathbb{R}^1)$ *satisfying* $0 \le u_0 \le 1$ *and*

$$\liminf_{x \to \infty} u_0(x) > a^+, \qquad \limsup_{x \to -\infty} u_0(x) < a^-,$$

the solution $u(x,t)$ *of* (1.1) *with initial value* $u(\cdot,0) = u_0(\cdot)$ *has the property that*

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le Ke^{-\kappa t} \quad \textit{for all } t \ge 0$$

where ξ *and* K *are constants depending on* u_0*.*

To prove the theorem, we need some preparation.

In the sequel, $\zeta(\cdot) \in C^\infty(\mathbb{R})$ is a fixed function having the following properties:

$$\begin{aligned} &\zeta(s) = 0 \ \text{ if } \ s \le 0; \quad \zeta(s) = 1 \ \text{ if } \ s \ge 4; \\ &0 < \zeta'(s) < 1, \ |\zeta''(s)| \le 1 \ \text{ if } \ s \in (0,4). \end{aligned} \tag{3.1}$$

Lemma 3.2. *Assume that* (B1) *and* (B3) *hold. Then for every* $\delta \in (0, \min\{a^-/2, (1-a^+)/2\}]$*, there exists a small positive constant* $\varepsilon = \varepsilon(\delta)$ *and a large positive constant* $C = C(\delta)$ *such that, for every* $\xi \in \mathbb{R}$*, the function* $w^+(x,t)$ *and* $w^-(x,t)$ *defined by*

$$\begin{aligned} w^+(x,t) &:= (1+\delta) - [1 - (a^- - 2\delta)e^{-\varepsilon t}]\zeta(-\varepsilon(x - \xi + Ct)), \\ w^-(x,t) &:= -\delta + [1 - (1 - a^+ - 2\delta)e^{-\varepsilon t}]\zeta(\varepsilon(x - \xi - Ct)) \end{aligned}$$

are respectively a supersolution and a subsolution of (1.1) *in* $\mathbb{R} \times (0,\infty)$*.*

One observes that the functions w^+ and w^- have the following properties:

$$\begin{cases} w^+(x,0) = 1 + \delta \ \text{ if } \ x \ge \xi, \qquad w^+(x,0) \ge a^- - \delta \text{ for all } \ x \in \mathbb{R}, \\ w^+(x,t) \le \delta + (a^- - 2\delta)e^{-\varepsilon t} \ \text{ for all } \ t > 0, \ x \le \xi - Ct - 4\varepsilon^{-1}, \\ w^-(x,0) = -\delta \text{ if } \ x \le \xi, \quad w^-(x,0) \le a^+ + \delta \ \text{ for all } \ \xi \in \mathbb{R}, \\ w^-(x,t) \ge 1 - \delta - (1 - a^+ - 2\delta)e^{-\varepsilon t} \ \text{ for all } \ t > 0, \ x \ge \xi + Ct + 4\varepsilon^{-1}. \end{cases} \tag{3.2}$$

Proof. We only prove the assertion of the lemma for w^-. The proof for w^+ is analogous and is omitted. By translation invariance, we need only consider the case $\xi = 0$. Since $\mathcal{A}[w^-(x,t)\mathbf{1}] = f(w^-(x,t))\mathbf{1}$,

$$\begin{aligned} &w_t^-(x,t) - \mathcal{A}[w^-(\cdot,t)](x) \\ &= -C\varepsilon[1 - (1 - a^+ - 2\delta)e^{-\varepsilon t}]\zeta' + \varepsilon(1 - a^+ - 2\delta)\zeta e^{-\varepsilon t} - \mathcal{A}[w^-(\cdot,t)](x) \\ &\le -C\varepsilon a^+\zeta' + \varepsilon - f(w^-(x,t)) - \big(\mathcal{A}[w^-(\cdot,t)](x) - \mathcal{A}[w^-(x,t)\mathbf{1}](x)\big). \end{aligned}$$

We can estimate, by (B3),

$$\begin{aligned} &\big|\mathcal{A}[w^-(\cdot,t)](x) - \mathcal{A}[w^-(x,t)\mathbf{1}](x)\big| \\ &\qquad \le K_1 \textstyle\int_{\mathbb{R}} |w^-(x-y,t) - w^-(x,t)|\nu(dy) + K_2\|w^-_{xx}\|_{C^0(\mathbb{R})} \\ &\qquad \le K_1 \displaystyle\int_{\mathbb{R}} \min\{\varepsilon|y|, 1\}\,\nu(dy) + K_2\varepsilon^2 =: \rho(\varepsilon), \end{aligned}$$

where in the second inequality, we have used the estimate

$$|w^-(x-y,t) - w^-(x,t)| \leq \|w_x^-(\cdot,t)\|_{C^1(\mathbb{R})}|y| \leq \varepsilon|y|.$$

It then follows that

$$w_t^- - \mathcal{A}[w^-] \leq -C\varepsilon a^+\zeta' - f(w^-) + [\varepsilon + \rho(\varepsilon)]. \tag{3.3}$$

To find ε and C such that the right-hand side is negative, we consider three cases:
(i) $\zeta < \delta/2$,
(ii) $\zeta > 1 - \delta/2$, and
(iii) $\zeta \in [\delta/2, 1-\delta/2]$.

In the first case, we have $w^- < -\delta/2$ so that $-f(w^-) < -\min_{s\in[-1,-\delta/2]} f(s) < 0$ (by (B1)). Since $\lim_{\varepsilon\searrow 0}\rho(\varepsilon) = 0$, for all sufficiently small positive ε, the right-hand side of (3.3) is negative.

In the second case, we have $1-\delta \geq w^- > -\delta + (1-\delta/2)[1-(1-a^+-2\delta)] \geq a^+ + \delta/2$, so that $-f(w^-) \leq -\min_{s\in[a^++\delta/2,1-\delta]} f(s) < 0$ (also by (B1)). Hence, we can find (and then fix) $\varepsilon > 0$ such that the right-hand side of (3.3) is negative in both cases (i) and (ii).

Finally in the third case, ζ' has a positive lower bound, so that we can take a large enough $C = C(\delta,\varepsilon)$ to make the right-hand side of (3.3) negative. This completes the proof of the lemma. $\square$

Lemma 3.3. *Assume that the hypotheses of Theorem* 3.1 *hold. Then there exist a small positive constant* ε^* *(independent of* u_0*) such that if, for some* $\tau \geq 0$, $\xi \in \mathbb{R}$, $\delta \in (0, \delta_0/2]$, *and* $h > 0$, *there holds*

$$U(x - c\tau + \xi) - \delta \leq u(x,\tau) \leq U(x - c\tau + \xi + h) + \delta \quad \forall\, x \in \mathbb{R}, \tag{3.4}$$

then for every $t > \tau + 1$, *there exist* $\hat{\xi}(t)$, $\hat{\delta}(t)$, *and* $\hat{h}(t)$ *satisfying*

$$\begin{aligned} &\hat{\xi}(t) \in [\xi - \sigma_1\delta, \xi + h + \sigma_1\delta] \quad (\sigma_1 \text{ is as in } (2.4)),\\ &\hat{\delta}(t) \leq e^{-\beta(t-\tau-1)}[\delta + \varepsilon^* \min\{h,1\}],\\ &\hat{h}(t) \leq [h - \sigma_1\varepsilon^* \min\{h,1\}] + 2\sigma_1\delta, \end{aligned}$$

such that (3.4) *holds with* (τ,ξ,δ,h) *replaced by* $(t, \hat{\xi}(t), \hat{\delta}(t), \hat{h}(t))$.

Proof. By a translation, we can assume that $\xi = 0$. Also, by a shift of time, we can assume that $\tau = 0$. First of all, comparing u with $w^\pm$ in (2.2) (with $\xi_0 = 0$ for w^- and $\xi_0 = h$ for w^+) yields, for all $x \in \mathbb{R}$ and $t \geq 0$,

$$\begin{aligned} U(x - ct - \sigma_1\delta(1-e^{-\beta t})) - \delta e^{-\beta t} &\leq u(x,t)\\ &\leq U(x - ct + h + \sigma_1\delta(1-e^{-\beta t})) + \delta e^{-\beta t}. \end{aligned} \tag{3.5}$$

Set $\bar h = \min\{h, 1\}$. Define $\varepsilon_1 := \frac{1}{2}\min_{\xi\in[0,2]} U'(\xi)$. Then

$$\int_0^1 (U(y+\bar h) - U(y))\, dy \geq 2\varepsilon_1 \bar h.$$

It then follows that at least one of the following is true:

$$\text{(i)}\quad \int_0^1 [u(y,0) - U(y)]\, dy \geq \varepsilon_1 \bar h; \qquad \text{(ii)}\quad \int_0^1 [U(y+\bar h) - u(y,0)]\, dy \geq \varepsilon_1 \bar h.$$

Here we consider only the case (i). The case (ii) is similar and is omitted. Comparing u with w^- in (2.2) with $\xi_0 = 0$ and using property (B2) we obtain, for $\eta = \eta(M_2 + 2 + |c|)$ (M_2 is as in (2.6)) and every $x \in [-M_2 - 2 - |c|, M_2 + 2 + |c|]$,

$$u(x,1) - [U(x-\xi_1) - \delta e^{-\beta}] \geq \eta \int_0^1 [u(y,0) - (U(y) - \delta)]\, dy \geq \eta\varepsilon_1 \bar h, \tag{3.6}$$

where $\xi_1 := c + \sigma_1\delta(1 - e^{-\beta})$. We define

$$\varepsilon^* = \min\Big\{\frac{\delta_0}{2}, \frac{1}{2\sigma_1}, \min_{x\in[-M_2-2|c|-2, M_2+2|c|+2]} \frac{\eta\varepsilon_1}{2\sigma_1 U'(x)}\Big\}.$$

Then $U(x - \xi_1 + 2\sigma_1\varepsilon^*\bar h) - U(x-\xi_1) = U'(\theta)2\sigma_1\varepsilon^*\bar h \leq \eta\varepsilon_1\bar h$ for all $x \in [-M_2 - |c| - 1, M_2 + |c| + 1]$. Consequently, from (3.6),

$$u(x,1) \geq U(x - \xi_1 + 2\sigma_1\varepsilon^*\bar h) - \delta e^{-\beta} \quad \forall x \in [-M_2 - |c| - 1, M_2 + |c| + 1]. \tag{3.7}$$

When $|x| \geq M_2 + |c| + 1$, $U(x - \xi_1) \geq U(x - \xi_1 + 2\sigma_1\varepsilon^*\bar h) - \varepsilon^*\bar h$ by the definition of M_2 in (2.6). It then follows from (3.7) and the first inequality in (3.5) with $t = 1$ that

$$u(x,1) \geq U(x - \xi_1 + 2\sigma_1\varepsilon^*\bar h) - [\delta e^{-\beta} + \varepsilon^*\bar h] \quad \forall x \in \mathbb{R}.$$

Noting that $q := \delta e^{-\beta} + \varepsilon^*\bar h \leq \delta_0$, we then can again compare $u(x, 1+t')$ with the function $U(x - ct' - \xi_1 + 2\sigma_1\varepsilon^*\bar h - \sigma_1 q(1 - e^{-\beta t'})) - qe^{-\beta t'}$ to conclude that, for all $t' \geq 0$,

$$\begin{aligned} u(x, 1+t') &\geq U(x - ct' - \xi_1 + 2\sigma_1\varepsilon^*\bar h - \sigma_1 q(1 - e^{-\beta t'})) - qe^{-\beta t'} \\ &\geq U(x - c - ct' + \sigma_1\varepsilon^*\bar h - \sigma_1\delta) - e^{-\beta t'}[\delta + \varepsilon^*\bar h], \end{aligned}$$

where in the second inequality, we have replaced ξ_1 by $c + \sigma_1\delta(1 - e^{-\beta})$ and q by $\delta e^{-\beta} + \varepsilon^*\bar h$. Hence, setting $t = 1 + t'$, $\hat\xi(t) = \sigma_1\varepsilon^*\bar h - \sigma_1\delta$, $\hat h(t) = [h + \sigma_1\delta(1 - e^{-\beta t})] - \hat\xi(t) = h - \sigma_1\varepsilon^* + \sigma_1\delta[2 - e^{-\beta t}]$, and $\hat\delta(t) = e^{-\beta(t-1)}(\delta + \varepsilon^*\bar h)$, we obtain, from the last inequality and (3.5), the assertion of the lemma. □

Proof of Theorem 3.1. Step 1. Comparing u with the functions w^+ and w^- in Lemma 3.2 and using the properties of w^+ and w^- in (3.2), we can derive that, for any $\delta > 0$, there exist large positive constants T and H such that

$$U(x - cT - H/2) - \delta \le u(x,T) \le U(x - cT + H/2) + \delta \qquad \forall\, x \in \mathbb{R}. \tag{3.8}$$

Step 2. We define

$$\delta^* := \min\{\delta_0/2, \varepsilon^*/4\}, \qquad \kappa^* := \sigma_1\varepsilon^* - 2\sigma_1\delta^* \ge \sigma_1\varepsilon^*/2 > 0.$$

Also, we fix $t^* \ge 2$ such that

$$e^{-\beta(t^*-1)}[1 + \varepsilon^*/\delta^*] \le 1 - \kappa^*.$$

We take $\delta = \delta^*$ in (3.8) and denote the corresponding constants H and T by h_0 and T_0. We can assume that $h_0 \ge 1$; otherwise, we directly go to Step 3.

With (3.8), we can apply Lemma 3.3 with $\tau = T_0$, $\xi = -h_0/2$, $h = h_0$, and $\delta = \delta^*$ to conclude that (3.4) holds with $\tau = T_0 + t^*$, some $\xi \in [-h_0/2 - \sigma_1\delta^*, h_0/2 + \sigma_1\delta^*]$, $\delta = \delta^*$, and $h = h_0 - \kappa^*$, since by the definition of t^* and κ^*, $\hat{\delta}(T_0 + t^*) \le e^{-\beta(t^*-1)}[\delta^* + \varepsilon^*] \le \delta^*$ and $\hat{h}(T_0 + t^*) \le h_0 - \sigma_1\varepsilon^* + 2\sigma_1\delta^* \le h_0 - \kappa^*$.

Now repeating the same process we can show that (3.4) holds for $\tau = T_0 + Nt^*$, $\delta = \delta^*$, and $h = h_0 - N\kappa^*$ for all N such that $h_0 - (N-1)\kappa^* \ge 1$. Hence, there exists a finite time $T_1 > T_0$ such that (3.4) holds for $\tau = T_1$, $\delta = \delta^*, h = 1$ and some $\xi \in \mathbb{R}^1$ (which we denote by ξ^0).

Step 3. We now use a mathematical induction to show that for every nonnegative integer k, (3.4) holds for some $\xi = \xi^k \in \mathbb{R}$ and

$$\tau = T^k := T_1 + kt^*, \quad \delta = \delta^k := (1-\kappa^*)^k\delta^*, \quad h = h^k := (1-\kappa^*)^k.$$

Clearly, by Step 2, the assertion is true for $k = 0$. Now assume that the assertion is true for some $k = l \ge 0$. We want to show that it is true for $k = l + 1$. In fact, applying Lemma 3.3 with $\tau = T^l$ and $t = T^{l+1}$ we conclude that (3.4) holds with (τ, ξ, δ, h) replaced by $(T^{l+1}, \hat{\xi}, \hat{\delta}, \hat{h})$ where $(\hat{\xi}, \hat{\delta}, \hat{h})$ satisfies

$$\begin{aligned}
&\hat{\xi} \in [\xi^l - \sigma_1\delta^l, \xi^l + \sigma_1\delta^l],\\
&\hat{\delta} \le e^{-\beta(t^*-1)}(\delta^l + \varepsilon^* h^l) = [1-\kappa^*]^l\delta^* e^{-\beta(t^*-1)}[1 + \varepsilon^*/\delta^*] \le (1-\kappa^*)^{l+1}\delta^*,\\
&\hat{h} \le h^l - \sigma_1\varepsilon^* h^l + 2\sigma_1\delta^l = [1-\kappa^*]^l[1 - \sigma_1\varepsilon^* + 2\sigma_1\delta^*] = [1-\kappa^*]^{l+1}
\end{aligned}$$

by the definition of δ^*, κ^*, and t^*. That is, (3.4) holds for $\tau = T^{l+1}$, some $\xi = \xi^{l+1} \in [\xi^l - \sigma_1\delta^l, \xi^l + \sigma_1\delta^l]$, $\delta = [1-\sigma]^{l+1}\delta^*$, and $h = [1-\kappa^*]^{l+1}$. This completes the mathematical induction.

Step 4. Now we know that (3.4) holds for $(\tau, \xi, \delta, h) = (T^k, \xi^k, \delta^k, h^k)$ for all $k = 0, 1, \ldots$. In addition, from (3.5), (3.4) holds also for all $\tau \in [T^k, \infty), \delta = \delta^k$, $h = h^k + 2\sigma_1\delta^k$ and $\xi = \xi^k - \sigma_1\delta^k$, $k = 0, 1, \ldots$.

We define $\delta(t) = \delta^k$, $\xi(t) = \xi^k - \sigma_1\delta^k, h(t) = h^k + 2\sigma_1\delta^k$ for all t in $[T^k, T^{k+1})$ for all $k = 0, 1, \ldots$. Then

$$U(x - ct + \xi(t)) - \delta(t) \le u(x,t) \le U(x - ct + \xi(t) + h(t)) + \delta(t) \quad \forall\, t \ge T_1,\, x \in \mathbb{R}.$$

From the definition of $\delta(t)$ and $h(t)$, one sees that, denoting by k the largest integer no bigger than $(t - T_1)/t^*$,

$$\delta(t) = \delta^k = [1 - \kappa^*]^k \delta^* \le \delta^* \exp\Big\{\Big(\frac{t - T_1}{t^*} - 1\Big)\ln(1 - \kappa^*)\Big\} \quad \forall\, t \ge T_1,$$
$$h(t) = h^k + 2\sigma_1\delta^k \le [1 + 2\sigma_1\delta^*]\exp\Big\{\Big(\frac{t - T_1}{t^*} - 1\Big)\ln(1 - \kappa^*)\Big\} \quad \forall\, t \ge T_1.$$

In addition, since for any $t \ge \tau \ge T_1$, $\xi(t) \in [\xi(\tau) - \sigma_1\delta(\tau), \xi(\tau) + h(\tau) + \sigma_1\delta(\tau)]$, we deduce that

$$|\xi(t) - \xi(\tau)| \le h(\tau) + 2\sigma_1\delta(\tau),$$

which implies that $\xi(\infty) := \lim_{t\to\infty}\xi(t)$ exists and

$$|\xi(\infty) - \xi(\tau)| \le h(\tau) + 2\sigma_1\delta(\tau) \le [1 + 4\sigma_1\delta^*]\exp\Big\{\Big(\frac{t - T_1}{t^*} - 1\Big)\ln(1 - \kappa^*)\Big\}$$

$\forall\, t \ge T_1$. Hence, defining $\kappa = -\frac{1}{t^*}\ln(1 - \kappa^*)$, we obtain the assertion of the theorem. $\square$

Remark 3.4. If we assume that $\mathcal{A}$ is symmetric, namely, $\mathcal{A}[u](x) = \mathcal{A}[v](-x)$ where $u(x) = v(-x)$, then $U(-x - ct)$ is also a solution of (1.1). In such a case, one can establish more general stability results.

(a) Assume that $c < 0$. Then the following holds:

(i) for any initial data u_0 satisfying $0 \le u_0 \le 1$ and $\liminf_{|x|\to\infty} u_0(x) > a^+$, there exists $K = K(u_0) > 0$ such that

$$\|u(\cdot, t) - \mathbf{1}\|_{L^\infty(\mathbb{R})} \le Ke^{-\beta t} \quad \forall\, t \ge 0;$$

(ii) for every $\delta > 0$, there exists a large positive constant $M(\delta)$ such that if $0 \le u_0 \le 1$ in $\mathbb{R}$, $u_0 \ge a^+ + \delta$ in $[-M(\delta), M(\delta)]$, and $\limsup_{|x|\to\infty} u_0(x) < a^-$, then

$$\|u(\cdot, t) - U(\cdot - ct - \xi_-)\|_{L^\infty((-\infty,1))}$$
$$+ \|u(\cdot, t) - U(-\cdot - ct - \xi_+)\|_{L^\infty((-1,\infty))} \le Ke^{-\kappa t}, \quad \forall\, t \ge 0$$

where $\kappa > 0$ is independent of u_0 and $\xi_\pm$ but K depends on u_0.

(b) If $c > 0$, then an analogous conclusion as in (a) also holds.

(c) If $c = 0$, then for any u_0 satisfying $0 \le u_0 \le 1$ and $\limsup_{|x|\to\infty} u_0(x) < a^-$ (or $\liminf_{|x|\to\infty} u_0(x) > a^+$), the solution of (1.1) with initial data u_0 satisfies, for some large positive constant $K = K(u_0)$,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le Ke^{-\beta t} \qquad \big(\text{ or } \|u(\cdot,t) - \mathbf{1}\|_{L^\infty(\mathbb{R})} \le Ke^{-\beta t}\ \big).$$

The proof of (a) and (b) follows the same idea as in [12] and is omitted. Since (c) is left undiscussed in [12], we sketch the proof here for completeness.

We consider the case $0 \le u_0 \le 1$ and $\limsup_{|x|\to\infty} u_0(x) < a^-$. Let us normalize U so that $U(0) = \frac{a^-}{2}$. Then using the same proof as in Theorem 3.1 we know that there exist ξ_* and ξ^* such that

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} \big[u(x,t) - \min\{U(x-\xi_*), U(-x+\xi^*)\}\big] \le 0. \tag{3.9}$$

We claim that we can take $\xi_* = \xi^*$. In fact, if $\xi_* < \xi^*$, then

$$\begin{aligned}\int_{\xi_*}^{\xi^*-\xi_*+1} &\big(U(y-\xi_*) - \min\{U(y-\xi_*), U(-y+\xi^*)\}\big)\,dy \\ &\ge \int_0^1 (U(y) - U(-y))\,dy =: h_0.\end{aligned}$$

It then follows from the same proof as in Step 2 that (3.9) holds with ξ_* decreased by a fixed quantity and ξ^* increased by a fixed quantity. Hence, (3.9) holds for some $\xi^* = \xi_*$. This implies, for a sufficiently large T, $u(\cdot,T) \le \frac{a^-}{4} + \min\{U(\cdot - \xi_*), U(-\cdot+\xi_*)\} \le \frac{3a^-}{4}$. Consequently, for all $t > 0$, $u(\cdot, T+t) < w(t)$ where $w(t)$ is the solution of the ODE problem $\dot{w} = f(w)$, $w(0) = \frac{3}{4}a^-$. This establishes the assertion of (c).

We end this section with an a priori estimate for the traveling-wave speed c.

Theorem 3.5 (Speed). *Assume that* (B1), (B3)*, and* (A2) *hold. Then for any traveling-wave solution* (U,c) *of* (1.1)*,*

$$|c| \le \bar{C} := \frac{6\|f\|_{C^0([0,1])}}{\bar{\varepsilon}\min\{1-a^+, a^-\}},$$

where $\bar{\varepsilon}$ *is the positive constant defined implicitly by*

$$\begin{aligned}\rho(\bar{\varepsilon}) : &= K_1 \int_{\mathbb{R}} \min\{\bar{\varepsilon}|y|, 1\}\nu(dy) + K_2\bar{\varepsilon}^2 \\ &= \min\{|f(s)| : s \in [\tfrac{a^-}{3}, \tfrac{2a^-}{3}] \cup [\tfrac{1+2a^+}{3}, \tfrac{2+a^+}{3}]\}.\end{aligned}$$

Proof. To obtain an explicit estimate, we examine closely the proof of Lemma 3.2 and keep track of the constants ε and C. Since (U, c) depends only on the values of f on $[0, 1]$, we can assume, without loss of generality, that $\|f\|_{C^0([-1,2])} = \|f\|_{C^0([0,1])}$ and that $f = -f(\frac{a^-}{3})$ in $[-1, -\frac{a^-}{3}]$ and $f = -f(\frac{2+a^+}{3})$ in $[\frac{4-a^+}{3}, 2]$.

For definiteness, we take $\zeta(s) = \frac{1}{2}[1 + \tanh(s)]$. Let $\delta = \frac{a^-}{3}$ and $\bar{\varepsilon}$ and $\bar{C}$ be as in the assertion of the Theorem. Consider the function $w^-(x,t) = -2\delta + (1+\delta)\zeta(\bar{\varepsilon}[x - \bar{C}t])$. The same calculation as in the proof of Lemma 3.2 yields $w_t^- - \mathcal{A}[w^-] \le -\bar{C}(1+\delta)\bar{\varepsilon}\zeta' - f(-2\delta + (1+\delta)\zeta) + \rho(\bar{\varepsilon})$ (noting that $(1+\delta)\zeta'(s) \le 1$ and $(1+\delta)|\zeta''(s)| \le 1$ for all $s \in \mathbb{R}$). By considering the cases (i) $\zeta \in (0, \frac{\delta}{1+\delta}]$, (ii) $\zeta \in [1 - \frac{\delta}{1+\delta}, 1)$, and (iii) $\zeta \in (\frac{\delta}{1+\delta}, 1 - \frac{\delta}{1+\delta})$, we can show that $w_t^- - \mathcal{A}[w^-] \le 0$ in $\mathbb{R} \times [0, \infty)$. Hence, w^- is a subsolution.

Now let $X \gg 1$ be a constant such that $U(\cdot) \ge w^-(\cdot - X, 0)$. Then by comparison, $U(x - ct) \ge w(x - X, t)$ in $\mathbb{R} \times [0, \infty)$, which implies that $x - ct \ge x - \bar{C}t - \bar{X}$ for all $t \ge 0$ and some large $\bar{X}$ that is independent of t. Sending $t \to \infty$ we then obtain that $c \le \bar{C}$. Similarly, we can show that $c \ge -\bar{C}$, thereby completing the proof of the theorem. $\square$

Remark 3.6. From the proof, one sees that the condition (A2) can be weakened to the following:

If $u_t \ge \mathcal{A}[u]$, $v_t \le \mathcal{A}[v]$, and $u(\cdot, 0) \ge v(\cdot, 0)$, then $u \ge v$ in $\mathbb{R} \times [0, \infty)$.

The significance of this improvement is that we can allow J in (1.9), (1.11), (1.12), (1.13), and (1.14) to be densities of probability measures (cf. Remark 2.3(4)).

4. Existence. To show that (1.1) has a traveling-wave solution that satisfies (2.1), we need the following conditions.

(C1) $\mathcal{A}$ is translation invariant and the function f in (1.3) satisfies, for some $a \in (0, 1)$,

$$\begin{aligned} &f(0) > 0 \quad \text{in} \quad (-1, 0) \cup (a, 1), \qquad f < 0 \quad \text{in} \quad (0, a) \cup (1, 2), \\ &f'(0) < 0, f'(1) < 0, f'(a) > 0. \end{aligned} \tag{4.1}$$

(C2) There exists a positive continuous function $\eta(x,t)$ defined on $[0, \infty) \times (0, \infty)$ such that if $u(x,t)$ and $v(x,t)$ satisfy $-1 \le u, v \le 2$, $u_t \ge \mathcal{A}[u]$, $v_t \le \mathcal{A}[v]$, and $u(\cdot, 0) \ge v(\cdot, 0)$, then

$$u(x,t) - v(x,t) \ge \eta(|x|, t) \int_0^1 [u(y, 0) - v(y, 0)]\, dy \quad \forall\, x \in \mathbb{R},\, t > 0.$$

(C3) There exist positive constants K_1, K_2, K_3, and a probability measure ν such that for any $u, v \in L^\infty(\mathbb{R})$ with $-1 \le u, v \le 2$,

$$\begin{aligned} &\big|\mathcal{A}[u+v](x) - \mathcal{A}[u](x)\big| \\ &\qquad \le K_1 \int_{\mathbb{R}} |v(x-y)| \nu(dy) + K_2 \|v_{xx}\|_{C^0([x-1, x+1])}, \quad x \in \mathbb{R}, \end{aligned} \tag{4.2}$$

$$\big|\mathcal{A}[u+v]-\mathcal{A}[u]-\mathcal{A}'[u](v)\big| \le K_3\|v\|^2_{C^0(\mathbb{R})}, \tag{4.3}$$

$$\begin{aligned}&\big|\mathcal{A}'[u+v](\mathbf{1})(x)-\mathcal{A}'[u](\mathbf{1})(x)\big|\\ &\quad\le K_1\int_{\mathbb{R}}|v(x-y)|\nu(dy)+K_2\|v_{xx}\|_{C^0([x-1,x+1])},\quad x\in\mathbb{R},\end{aligned} \tag{4.4}$$

where $\mathcal{A}'[\cdot](\cdot)$ is the Fréchet derivative of $\mathcal{A}$ defined as in (1.16).

(C4) For any function $u_0(\cdot)$ satisfying $0\le u_0\le 1$ and $\|u_0\|_{C^3(\mathbb{R})}<\infty$, the solution $u(x,t)$ of (1.1) with initial data $u(\cdot,0)=u_0(\cdot)$ satisfies $\sup_{t\in[0,\infty)}\|u(\cdot,t)\|_{C^2(\mathbb{R})}<\infty$.

One notices that (C1) implies (A1) which implies (B1), that (C2) implies both (A2) and (B2), and that (C3) implies both (A3) and (B3). Also one can verify (cf. Section 5) that (C1)–(C4) are satisfied by all the models we mentioned in Section 1, except that (C4) is not satisfied by (1.12) with small λ so that $f(u)-\lambda u$ is not strictly decreasing in $[0,1]$. For this exceptional case, as shall be seen in Section 5.2, traveling waves are not continuous, not unique, and not asymptotically stable.

Theorem 4.1 (Existence). *Assume that* (C1)–(C4) *hold. Then problem* (1.1) *admits a traveling-wave solution* (U,c) *that satisfies* (2.1).

Proof. For the reader's convenience, we divide the proof into four steps. Also, for some other possible applications, in the first three steps, we shall restrict ourselves to use only the assumptions (C1)–(C3), though some of the conclusions are trivial if we use (C4).

Step 1. Let $v(x,t)$ be the solution of

$$v_t=\mathcal{A}[v]\quad\text{in }\ \mathbb{R}\times(0,\infty),\qquad v(\cdot,0)=\zeta(\cdot)\ \text{ on }\ \mathbb{R}\times\{0\}.$$

Here and in the sequel, $\zeta(\cdot)$ always refers to the function ζ satisfying (3.1). Our idea of the proof is to show that, for some sequence $\{t_j\}_{j=1}^\infty$, the sequence $\{v(\cdot+\xi(t_j),t_j)\}_{j=1}^\infty$ ($v(\xi(t),t))=a$) has a limit $U(\cdot)$, which is the profile of a traveling-wave front. To do this, we need a number of estimates.

First of all, by comparison, we have $0\le v\le 1$ on $\mathbb{R}\times[0,\infty)$.

We claim that for every $T\ge 0$, $v(\cdot,T)$ is Lipschitz continuous in $\mathbb{R}$. In fact, for any positive $\hat\varepsilon$ such that $\hat\varepsilon e^{K_1T}\le 1$, consider $w:=v+\hat\varepsilon e^{K_1t}$. We can calculate, for $t\in(0,T]$, $w_t-\mathcal{A}[w]=\hat\varepsilon K_1e^{K_1t}+\{\mathcal{A}[v]-\mathcal{A}[v+\hat\varepsilon e^{K_1t}\mathbf{1}]\}\ge 0$ by (4.2). That is, w is a supersolution in $\mathbb{R}\times[0,T]$. Now since $v(\cdot,0)\le v(\cdot+\hat\varepsilon,0)\le v(\cdot,0)+\hat\varepsilon$, it follows by comparison that $v(\cdot,t)\le v(\cdot+\hat\varepsilon,t)\le v(\cdot,t)+\hat\varepsilon e^{K_1t}$ for all $t\in[0,T]$. Hence, $v(\cdot,T)$ is nondecreasing and is Lipschitz continuous in $\mathbb{R}$.

Also, for any $t>0,\tau\ge 0,x\in\mathbb{R}$, and $z\in\mathbb{R}$, by (C2), $[v(x+h,t+\tau)-v(x,t+\tau)]\ge \eta(|x-z|,t)\int_z^{z+1}[v(y+h,\tau)-v(y,\tau)]\,dy$. Sending $h\searrow 0$, we then obtain

$$v_x(x,t+\tau)\ge\eta(|x-z|,t)\int_z^{z+1}v_x(y,\tau)\,dy,\quad x,z\in\mathbb{R},t>0,\tau\ge 0. \tag{4.5}$$

In particular, taking $\tau = z = 0$, we have $v_x(x,t) \geq \eta(|x|,t)\zeta(1) > 0$ in $\mathbb{R} \times (0,\infty)$.

Observe that Lemma 3.2 and (3.2) imply

$$\lim_{x\to\infty} v(x,t) = 1 \quad \text{and} \quad \lim_{x\to-\infty} v(x,t) = 0$$

for all $t > 0$. It then follows that there exists a unique function $z(\alpha,t)$ defined on $(0,1) \times [0,\infty)$ such that $v(z(\alpha,t),t) = \alpha$, $\alpha \in (0,1)$, $t \in [0,\infty)$.

Step 2. From (3.2), one sees that for every small positive $\delta > 0$, there exist $\varepsilon = \varepsilon(\delta)$ and $C = C(\delta)$ such that

$$\begin{cases} z(a+\delta,t) \leq z(a+\delta,\tau) + 4\varepsilon^{-1} + C\,[t-\tau] & \forall\, 0 \leq \tau < t < \infty; \\ z(a-\delta,t) \geq z(a-\delta,\tau) - 4\varepsilon^{-1} - C\,[t-\tau] & \forall\, 0 \leq \tau < t < \infty; \\ z(1-2\delta,t) \leq z(a+\delta,\tau) + 4\varepsilon^{-1} + C\,[t-\tau] & \forall\,\tau \geq 0, t-\tau \geq \varepsilon^{-1}|\ln\delta|; \\ z(2\delta,t) \geq z(a-\delta,\tau) - 4\varepsilon^{-1} - C\,[t-\tau] & \forall\,\tau \geq 0, t-\tau \geq \varepsilon^{-1}|\ln\delta|. \end{cases} \tag{4.6}$$

In addition, we have the following key estimate whose proof will be given at the end of this section.

Lemma 4.2. *Assume* (C1)–(C3). *Then there exist a small positive constant δ_1 and a large positive constant $h_1 \geq 1$ such that $z(a+\delta_1,t) - z(a-\delta_1,t) \leq h_1$ $\forall t \geq 0$.*

Using the assertion of the lemma and (4.6), we can derive the following.

(a) For every $\delta \in (0,\delta_1/2]$, there exists $m_1(\delta) > 0$ such that

$$z(1-\delta,t) - z(\delta,t) \leq m_1(\delta) \qquad \forall t \geq 0. \tag{4.7}$$

In fact, when $t > \varepsilon^{-1}(\delta)|\ln\delta| =: \triangle$, (4.6) implies $z(1-2\delta,t) - z(2\delta,t) \leq z(a+\delta,t-\triangle) - z(a-\delta,t-\triangle) + 8\varepsilon^{-1} + 2C\triangle \leq z(a+\delta_1,t-\triangle) - z(a-\delta_1,t-\triangle) + 8\varepsilon^{-1} + 2C\triangle$, so that, by Lemma 4.2, $z(1-2\delta,t) - z(2\delta,t) \leq h_1 + 8\varepsilon^{-1} + 2C\triangle$.

When $t \in [0,\triangle]$, comparing $v(x,t)$ with $w^{\pm}(x,t+\triangle)$ (w^+ and w^- are as in Lemma 3.3) yields $z(1-2\delta) - z(2\delta) \leq 8\varepsilon^{-1} + 2C\Delta$. Writing 2δ as δ, we obtain (4.7).

(b) For every $M > 0$, there exists a constant $\hat{\eta}(M) > 0$ such that

$$v_x(x + z(a,t),t) \geq \hat{\eta}(M) \qquad \forall\, t \geq 1, x \in [-M,M]. \tag{4.8}$$

Indeed, since there exists $\xi = \xi(t) \in [-h_1, h_1 - 1]$ such that

$$\int_{\xi}^{\xi+1} v_x(y + z(a,\tau),\tau)\,dy = \frac{1}{2h_1}\int_{-h_1}^{h_1} v_x(y + z(a,\tau),\tau)\,dy \geq \delta_1/h_1,$$

it then follows from (4.5), that

$$\min_{x\in[-\hat{M},\hat{M}]} v_x(x + z(a,\tau),\tau+1) \geq \min_{y\in[-\hat{M}-h_1,\hat{M}_1+h_1]} \eta(y,1)\delta_1/h_1.$$

Since $|z(a,\tau+1)-z(a,\tau)| \le h_1+8\varepsilon^{-1}(\delta_1)+2C(\delta_1)$, taking $\hat{M} = M+h_1+8\varepsilon^{-1}(\delta_1)+2C(\delta_1)$ we then obtain (4.8).

(c) Let δ_0 and M_0 be defined as in (2.3). We set

$$M_3 := m_1(\delta_0) \ \big(m_1(\cdot) \text{ is as in (4.7)}\big), \qquad \sigma_2 := \frac{\|f'\|_{C^0([-1,2])} + \beta + K_1 + K_2}{\hat{\eta}(M_3 + M_0)}.$$

Then, for all $t \ge 0$, $v(x + z(a,t), t) \ge 1 - \delta_0$ if $x \ge M_3$ and $v(x + z(a,t), t) \le \delta_0$ if $x < -M_3$. In addition, $\sigma_2 v_x(x + z(a,t), t) \ge \|f'\|_{C^0([-1,2])} + \beta + K_1 + K_2$ for all $t \ge 1$ and all $x \in [-M_3 - M_0, M_3 + M_0]$. Hence, following the proof of Lemma 2.2, one can show that for every $\delta \in (0, \delta_0]$ and every $\xi \in \mathbb{R}$, the function $W^+(x,t)$ and the function $W^-(x,t)$ defined by

$$W^{\pm}(x,t) := v(x + \xi \pm \sigma_2\delta(1 - e^{-\beta t}), t + 1) \pm \delta e^{-\beta t} \tag{4.9}$$

are, respectively, a supersolution and a subsolution of (1.1) in $\mathbb{R} \times (0, \infty)$. Here we need (4.4).

Step 3. Since the family $\{v(\cdot + z(a,t), t)\}_{t\ge0}$ consists of monotonic bounded functions, there exist a sequence $\{t_j\}_{j=1}^{\infty}$ and a nondecreasing function $U(\cdot)$ such that as $j \to \infty$, $t_j \to \infty$ and $v(\xi + z(a,t_j), t_j) \to U(\xi)$ for all $\xi \in \mathbb{R}$.

Clearly, $U(0) = a$ and $0 \le U \le 1$. In addition, from (4.7) we know that for all small $\delta > 0$, $U(m_1(\delta)) \ge 1 - \delta$ and $U(-m_1(\delta)) \le \delta$. This implies that $\lim_{\xi\to\infty} U(\xi) = 1$ and $\lim_{\xi\to-\infty} U(\xi) = 0$. Furthermore, from (4.8), $U(\xi + h) - U(\xi) \ge \hat{\eta}(|\xi| + 1)h$ for all $h \in [0,1]$ and all $\xi \in \mathbb{R}$.

We now show that U is the profile of a traveling-wave front.

Let $\tilde{U}(x,t)$ be the solution of (1.1) with initial data $U(\cdot)$. We want to show that $v(\cdot + z(a,t_j), t_j + t) \to \tilde{U}(\cdot,t)$. In fact, from (4.7) and the monotonicity of $v(\cdot,t)$ and $U(\cdot)$, for any $\hat{\varepsilon} > 0$, there exists a large positive integer $J(\hat{\varepsilon})$ such that if $j > J$, then $v(\cdot - \hat{\varepsilon} + z(a,t_j), t_j) - \hat{\varepsilon} < U(\cdot) < v(\cdot + \hat{\varepsilon} + z(a,t_j), t_j) + \hat{\varepsilon}$, which implies, by comparison functions in (4.9), $v(\cdot - \hat{\varepsilon} - z(a,t_j) - \sigma_2\hat{\varepsilon}(1 - e^{-\beta t}), t_j + t) - \hat{\varepsilon}e^{-\beta t} \le \tilde{U}(\cdot,t) \le v(\cdot + \hat{\varepsilon} + z(a,t_j) + \sigma_2\hat{\varepsilon}(1 - e^{-\beta t}), t_j + t) + \hat{\varepsilon}e^{-\beta t}$ for all $t \ge 0$. Sending $j \to \infty$ first and then $\hat{\varepsilon} \searrow 0$, we then obtain, for all $t \ge 0$,

$$\begin{aligned} &\limsup_{j\to\infty} v(\cdot + z(a,t_j), t_j + t) \le \tilde{U}(\cdot + 0, t), \\ &\liminf_{j\to\infty} v(\cdot + z(a,t_j), t_j + t) \ge \tilde{U}(\cdot - 0, t), \end{aligned} \tag{4.10}$$

where $\tilde{U}(x \pm 0, t) := \lim_{y\to x^{\pm}} \tilde{U}(x,t)$.

Observe that there exists a large positive constant m_0 such that $v(\cdot - m_0, 1) - \delta_0 \le v(\cdot, 0) \le v(\cdot + m_0, 1) + \delta_0$. It then follows from the sub-super solutions in (4.9) that, for all $t \ge 0$,

$$\begin{aligned} &v(\cdot - m_0 - \sigma_2\delta_0(1 - e^{-\beta t}), t + 1) - \delta_0 e^{-\beta t} \\ &\qquad \le v(\cdot, t) \le v(\cdot + m_0 + \sigma_2\delta_0(1 - e^{-\beta t}), t + 1) + \delta_0 e^{-\beta t}. \end{aligned}$$

Setting $t = t_j$, sending $j \to \infty$, and using (4.10), we then obtain

$$\tilde{U}(\xi - m_0 - \sigma_2\delta_0 - 0, 1) \le U(\xi) \le \tilde{U}(\xi + m_0 + \sigma_2\delta_0 + 0, 1), \qquad \forall \xi \in \mathbb{R}.$$

We define

$$\xi_* = \sup\{\xi : \tilde{U}(\cdot + \xi - 0, 1) \le U(\cdot)\}, \quad \xi^* = \inf\{\xi : U(\cdot) \le \tilde{U}(\cdot + \xi + 0, 1)\}.$$

Then, $-m_0 - \sigma_2\delta_0 \le \xi_* \le \xi^* \le m_0 + \sigma_2\delta_0$. So far, we have only used the conditions (C1)–(C3).

Step 4. We shall use the same technique as Step 2 in the proof of Theorem 2.1 to show that $\xi_* = \xi^*$. To do this, we need the assumption (C4).

By (C4), we know that $U \in C^1(\mathbb{R})$, and from (4.7), the convergence $v(\cdot + z(a, t_j), t_j) \to U(\cdot)$ is uniform (in the $C^0(\mathbb{R})$ norm). In addition, since for any $x \in \mathbb{R}$,

$$\|v_x(\cdot, t)\|^2_{C^0([x,x+1])} \le 4\|v(\cdot, t)\|_{C^0([x,x+1])}\|v(\cdot, t)\|_{C^2([x,x+1])},$$

we conclude from (4.7) that

$$\lim_{x\to\infty} \sup_{|\xi|\ge x, t\ge 1} v_x(\xi + z(a, t), t) = 0.$$

Thus, by (4.10), for any fixed $t \ge 0$, $v(\cdot + z(a, t_j), t_j + t) \to \tilde{U}(\cdot, t)$ in $C^1(\mathbb{R})$. In particular, $\lim_{|x|\to\infty} \tilde{U}_x(x, t) = 0$ for all $t \ge 0$. Hence, there exists a large positive constant M_4 such that

$$2\sigma_2 U_x(x, 1) \le 1 \qquad \text{if } \ |x| \ge M_4.$$

Now we are ready to show that $\xi^* = \xi_*$. Assume to the contrary that $\xi^* > \xi_*$. Then, since $\tilde{U}(\cdot + \xi_*, 1) \le (\not\equiv) U(\cdot)$, by comparing $\tilde{U}(\cdot + \xi_*, \cdot + 1)$ with $\tilde{U}(\cdot, \cdot)$, we obtain that $\tilde{U}(\cdot + \xi_*, 2) < \tilde{U}(\cdot, 1)$. Consequently, there exists a small $h > 0$ such that $\tilde{U}(\xi + \xi_* + 2\sigma_2 h, 2) < \tilde{U}(\xi, 1)$ for all $\xi \in [-M_4 - 1, M_4 + 1]$. Hence, the same argument as in the proof of Theorem 2.1 yields

$$\tilde{U}(\cdot + \xi_* + 2\sigma_2 h, 2) - h \le \tilde{U}(\cdot, 1). \tag{4.11}$$

Now, by the definition of U, we can find a large integer j such that

$$v(\cdot + z(a, t_j), t_j) - \tfrac{1}{8}h \le U(\cdot) \le v(\cdot + z(a, t_j), t_j) + \tfrac{1}{8}h.$$

It then follows, by the sub-super solutions in (4.9), that for all $t \ge 0$,

$$v(\cdot + z(a, t_j) - \tfrac{1}{8}\sigma_2 h, t_j + t) - \tfrac{1}{8}h \le \tilde{U}(\cdot, t) \le v(\cdot + z(a, t_j) + \tfrac{1}{8}\sigma_2 h, t_j + t) + \tfrac{1}{8}h.$$

This and (4.11) imply that

$$v(\cdot+\xi_*+\tfrac{15}{8}\sigma_2 h+z(a,t_j),t_j+2)-\tfrac{9}{8}h\le v(\cdot+z(a,t_j)+\tfrac{1}{8}\sigma_2 h,t_j+1)+\tfrac{1}{8}h.$$

That is, $v(\cdot+\xi_*+\frac{7}{4}\sigma_2 h,t_j+2)-\frac{5}{4}h\le v(\cdot,t_j+1)$. It then follows by the comparison principle that

$$v(\cdot+\xi^*+\tfrac{7}{4}\sigma_2 h-\tfrac{5}{4}h(1-e^{-\beta t}),t+1)-\tfrac{5}{4}he^{-\beta t}\le v(\cdot,t)$$

for all $t\ge t_j+1$. Setting $t=t_k$ $(t_k\ge t_j+1)$ and $x=\xi-z(a,t_k)$, and sending $k\to\infty$, we then obtain that

$$\tilde U(\xi+\xi_*+\tfrac{1}{2}\sigma_2 h,1)\le U(\xi)\qquad\forall\xi\in\mathbb{R}.$$

But this contradicts the definition of ξ_*. Hence, we must have $\xi_*=\xi^*$; namely, $\tilde U(\cdot+\xi_*,1)=U(\cdot)$.

Now using the same argument as above, we can compare $\tilde U(\cdot,t)$ with $U(\cdot)$ for all $t\in(1,2]$ to conclude that there exists $c(t)$ such that $\tilde U(\cdot,t)=U(\cdot-c(t))$. The equation $\tilde U_t=\mathcal{A}[\tilde U]$ then gives $-c'(t)U(\xi)=\mathcal{A}[U](\xi)$; that is, $c'(t)$ is a constant and (U,c') is a traveling-wave solution to (1.1). In addition, from the proof, we know that U satisfies (2.1).

Hence, to complete the proof of the theorem, it remains to prove Lemma 4.2, which is a direct consequence of the last part of the following lemma:

Lemma 4.3. *Assume that* (C1)–(C3) *hold. Let $H(x)$ be the Heaviside function equal to* 1 *when $x>0$,* 1/2 *when $x=0$, and* 0 *when $x<0$. Then the following holds.*

(i) *Let ${v_{\rm I}}^1$ and ${v_{\rm I}}^2$ be the solutions to the following linear evolution problems:*

$$\begin{cases} v_{{\rm I},t}^1=\mathcal{A}'[a\mathbf{1}]({v_{\rm I}}^1),\\ {v_{\rm I}}^1(x,0)=H(x),\end{cases}\qquad \begin{cases} v_{{\rm I},t}^2=\mathcal{A}'[a\mathbf{1}]({v_{\rm I}}^2),\\ {v_{\rm I}}^2(x,0)=H(x)-H(-x)=-1+2H(x).\end{cases}$$

Then there exist constants $\tau_1>0$ and $x_1\in\mathbb{R}$ such that

$${v_{\rm I}}^1(x_1+0,\tau_1)\ge 3,\qquad {v_{\rm I}}^2(x_1-0,\tau_1)\le -3,$$

where ${v_{\rm I}}^i(x_1\pm 0,\tau_1)=\lim_{y\to x_1^\pm}{v_{\rm I}}^i(y,\tau_1)$.

(ii) *There exists a small positive constant δ_1 such that the solutions ${v_{\rm II}}^1$ and ${v_{\rm II}}^2$ to*

$$\begin{cases} v_{{\rm II},t}^1=\mathcal{A}[{v_{\rm II}}^1],\\ {v_{\rm II}}^1(x,0)=a+\delta_1 H(x),\end{cases}\qquad \begin{cases} v_{{\rm II},t}^2=\mathcal{A}[{v_{\rm II}}^2],\\ {v_{\rm II}}^2(x,0)=a+\delta_1[H(x)-H(-x)],\end{cases}$$

satisfy

$$v_{\mathrm{II}}{}^1(x_1+0,\tau_1) \geq a+2\delta_1, \qquad v_{\mathrm{II}}{}^2(x_1-0,\tau_1) \leq a-2\delta_1.$$

(iii) *There exists a large positive constant h_2 such that the solutions $v_{\mathrm{III}}{}^1$ and $v_{\mathrm{III}}{}^2$ to*

$$\begin{cases} v_{\mathrm{III}}{}^1_{,t} = \mathcal{A}[v_{\mathrm{III}}{}^1], \\ v_{\mathrm{III}}{}^1(x,0) = a+\delta_1 H(x) - (a-\delta_1)H(-h_2-x), \end{cases}$$

$$\begin{cases} v_{\mathrm{III}}{}^2_{,t} = \mathcal{A}[v_{\mathrm{III}}{}^2], \\ v_{\mathrm{III}}{}^2(x,0) = a+\delta_1[H(x)-H(-x)] + (1-a-\delta_1)H(x-h_2), \end{cases}$$

satisfies

$$v_{\mathrm{III}}{}^1(x_1+0,\tau_1) \geq a+\delta_1, \qquad v_{\mathrm{III}}{}^2(x_1-0,\tau_1) \leq a-\delta_1.$$

(iv) *Let $u(x,t)$ be the solution of* (1.1) *with uniformly continuous and nondecreasing initial data $u_0(\cdot)$ satisfying $0 \leq u_0 \leq 1$ and for some finite $\xi_-(0)$ and $\xi_+(0)$,*

$$u_0(\xi_-(0)) \leq a-\delta_1, \qquad u(\xi_+(0)) \geq a+\delta_1.$$

Then for every $t>0$, there exist $\xi_-(t)$ and $\xi_+(t)$ such that

$$u(\xi_-(t),t) = a-\delta_1, \qquad u(\xi_+(t),t) = a+\delta_1,$$

$$\xi_+(t)-\xi_-(t) \leq \max\{\xi_+(0)-\xi_-(0)+8\varepsilon^{-1}(\delta_1)+2C(\delta_1)\tau_1, 2h_2\}$$

where $\varepsilon(\delta_1)$ and $C(\delta_1)$ are as in Lemma 3.2.

Proof of (i). Since the flow $u_t = \mathcal{A}[u]$ satisfies the comparison principle, one can show that the flow $v_t = \mathcal{A}'[a\mathbf{1}](v)$ also satisfies the comparison principle; namely, if $v^1_t \geq \mathcal{A}'[a\mathbf{1}](v^1)$, $v^2_t \leq \mathcal{A}'[a\mathbf{1}](v^2)$, and $v^1(\cdot,0) \geq v^2(\cdot,0)$, then $v^1(\cdot,t) \geq v^2(\cdot,t)$ for all $t \geq 0$.

Denote $f'(a)$ by γ. Then $\mathcal{A}'[a\mathbf{1}](\mathbf{1}) = f'(a)\mathbf{1} = \gamma\mathbf{1}$, so that $e^{\gamma t}\mathbf{1}$ is an exact solution to $v_t = \mathcal{A}'[a\mathbf{1}](v)$.

Also, one notices that $v_{\mathrm{I}}{}^2(x,t) = -e^{\gamma t} + 2v_{\mathrm{I}}{}^1(x,t)$ in $\mathbb{R}\times[0,\infty)$ since $v_{\mathrm{I}}{}^2(\cdot,0) = -\mathbf{1}+2v_{\mathrm{I}}{}^1(\cdot,0)$. Thus, we need only study $v_{\mathrm{I}}{}^1$.

Since $0 \leq v_{\mathrm{I}}{}^1(\cdot,0) \leq 1$, comparing $v_{\mathrm{I}}{}^1$ with $\mathbf{0}$ and $e^{\gamma t}\mathbf{1}$ then yields $0 \leq v_{\mathrm{I}}{}^1(\cdot,t) \leq e^{\gamma t}$ for all $t \geq 0$. In addition, since for each $h>0$, $v_{\mathrm{I}}{}^1(\cdot+h,0) \geq v_{\mathrm{I}}{}^1(\cdot,0)$, we have $v_{\mathrm{I}}{}^1(\cdot+h,t) \geq v_{\mathrm{I}}{}^1(\cdot,t)$ for all $t \geq 0$. Namely, $v_{\mathrm{I}}{}^1$ is nondecreasing in x. Thus, for any $x \in \mathbb{R}$, $v_{\mathrm{I}}{}^i(x\pm 0,t) := \lim_{y\to x\pm} v_{\mathrm{I}}{}^1(y,t)$ exists.

We shall now use the comparison to show that $\lim_{x\to-\infty} v_{\mathrm{I}}{}^1(x,t) = 0$ and $\lim_{x\to\infty} v_{\mathrm{I}}{}^1(x,t) = e^{\gamma t}$. To do this, we first calculate, for every small positive $\hat{\varepsilon}$ and every $x \in \mathbb{R}$,

$$\begin{aligned} &\big|\mathcal{A}'[a\mathbf{1}](\zeta(\hat{\varepsilon}\cdot))(x) - \gamma\zeta(\hat{\varepsilon}x)\big| = \big|\mathcal{A}'[a\mathbf{1}](\zeta(\hat{\varepsilon}\cdot) - \zeta(\hat{\varepsilon}x)\mathbf{1})(x)\big| \\ &= \Big|\lim_{\hat{\delta}\to 0}\frac{1}{\hat{\delta}}\big\{\mathcal{A}[a\mathbf{1} + \hat{\delta}(\zeta(\hat{\varepsilon}\cdot) - \zeta(\hat{\varepsilon}x))\mathbf{1}](x) - \mathcal{A}[a\mathbf{1}]\big\}\Big| \\ &\le K_1\int_{\mathbb{R}} |\zeta(\hat{\varepsilon}(x-y)) - \zeta(\hat{\varepsilon}x)|\nu(dy) + K_2\|\zeta_{xx}(\hat{\varepsilon}\cdot)\|_{C^0(\mathbb{R})} \quad \big(\text{by } \ (4.2)\big) \\ &\le K_1\int_{\mathbb{R}} \min\{\hat{\varepsilon}|y|, 1\}\nu(dy) + K_2\hat{\varepsilon}^2 =: \rho(\hat{\varepsilon}). \end{aligned} \tag{4.12}$$

Define $w = \rho(\hat{\varepsilon})e^{2\gamma t} + \zeta(\hat{\varepsilon}x)e^{\gamma t}$. Then

$$\begin{aligned} w_t - \mathcal{A}'[a\mathbf{1}](w) &= \gamma e^{\gamma t}[2\rho(\hat{\varepsilon})e^{\gamma t} + \zeta] - e^{\gamma t}\mathcal{A}'[a\mathbf{1}](\rho(\hat{\varepsilon})e^{\gamma t}\mathbf{1} + \zeta(\hat{\varepsilon}x)\mathbf{1}) \\ &\qquad - e^{\gamma t}\mathcal{A}'[a\mathbf{1}](\zeta(\hat{\varepsilon}\cdot) - \zeta(\hat{\varepsilon}x)\mathbf{1})(x) \\ &= e^{\gamma t}\{\rho(\hat{\varepsilon})e^{\gamma t} - \mathcal{A}'[a\mathbf{1}](\zeta(\hat{\varepsilon}\cdot) - \zeta(\hat{\varepsilon}x)\mathbf{1})(x)\} \ge 0 \end{aligned}$$

by the previous estimate (4.12). Hence, w is a supersolution of the linear equation $u_t = \mathcal{A}'[a\mathbf{1}](u)$.

Since $v_{\mathrm{I}}{}^1(x,0) \le w(x + 4/\hat{\varepsilon}, 0)$, the comparison yields $v_{\mathrm{I}}{}^1(x,t) \le w(x+4/\hat{\varepsilon}, t)$ in $\mathbb{R}\times[0,\infty)$. Consequently,

$$\lim_{x\to-\infty} v_{\mathrm{I}}{}^1(x,t) \le \lim_{x\to-\infty} w(x,t) = \rho(\hat{\varepsilon})e^{2\gamma t}.$$

Sending $\hat{\varepsilon} \searrow 0$, we then obtain $\lim_{x\to\infty} v_{\mathrm{I}}{}^1(x,t) = 0$.

Similarly, we can show that $e^{\gamma t} - w(-x,t)$ is a subsolution, and by comparing $v_{\mathrm{I}}{}^1(x,t)$ with $e^{\gamma t} - w(-x,t)$, we can conclude that $\lim_{x\to\infty} v_{\mathrm{I}}{}^1(x,t) = e^{\gamma t}$.

Now set $\tau_1 = \frac{1}{\gamma}\ln 9$, so that $e^{\gamma\tau_1} = 9$. By the monotonicity of $v_{\mathrm{I}}{}^1(\cdot,\tau_1)$ and the limiting behavior of $v_{\mathrm{I}}{}^1(x,\tau_1)$ as $x \to \pm\infty$, there exists $x_1 \in \mathbb{R}$ such that

$$v_{\mathrm{I}}{}^1(x_1+0,\tau_1) \ge \tfrac{1}{3}e^{\gamma\tau_1} = 3, \qquad v_{\mathrm{I}}{}^1(x_1-0,\tau) \le \tfrac{1}{3}e^{\gamma\tau_1} = 3.$$

Using the identity $v_{\mathrm{I}}{}^2(x,t) = -e^{\gamma t} + 2v_{\mathrm{I}}{}^1(x,t)$ on $\mathbb{R}\times[0,\infty)$, one also has

$$v_{\mathrm{I}}{}^2(x_1-0,\tau_1) = -e^{\gamma\tau_1} + 2v_{\mathrm{I}}(x_1-0,\tau_1) \le -3.$$

The first assertion thus follows.

Proof of (ii). Set $K = \frac{4K_3e^{2\gamma\tau_1}}{\gamma}$ (K_3 is as in (4.3)) and $\delta_1 = \min\{a/3, (1-a)/3, 1/K\}$. Consider the function $w(x,t) = a + \delta_1 v_{\mathrm{I}}{}^2(x,t) + K\delta_1^2$. We can calculate, when $t \in [0,\tau_1]$,

$$\begin{aligned} &w_t - \mathcal{A}[w] \\ &\quad = \delta_1 v_{\mathrm{I}}{}^2_t - \mathcal{A}'[a\mathbf{1}](\delta_1 v_{\mathrm{I}}{}^2 + K\delta_1^2) - \{\mathcal{A}[w] - \mathcal{A}[a\mathbf{1}] - \mathcal{A}'[a\mathbf{1}](w - a\mathbf{1})\} \\ &\quad \ge \gamma K\delta_1^2 - K_3\|w - a\mathbf{1}\|^2_{C^0(\mathbb{R})} = \gamma K\delta_1^2 - K_3[\delta_1 e^{\gamma t} + K\delta_1^2]^2 \ge 0, \end{aligned}$$

where in the first inequality, we have used the identities ${v_{\rm I}}^2_t = \mathcal{A}'[a\mathbf{1}]({v_{\rm I}}^2)$ and $\mathcal{A}'[a\mathbf{1}](\mathbf{1}) = \gamma\mathbf{1}$, and the inequality (4.3) (with $u = a\mathbf{1}$ and $u + v = w$), whereas in the second inequality, we have used the definition of K and δ_1. Thus, by comparison, ${v_{\rm II}}^2(x_1 - 0, \tau_1) \le w(x_1 - 0, \tau_1) \le a - 3\delta_1 + K\delta_1^2 \le a - 2\delta_1$.

In a similar manner, one can show that $w = a + \delta_1 {v_{\rm I}}^2(x,t) - K\delta_1^2$ is a subsolution of (1.1) so that ${v_{\rm II}}^1(x_1 + 0, \tau_1) \ge a + 2\delta_1$.

Proof of (iii). Let $\hat\varepsilon$ be any small positive constant. Consider $w = {v_{\rm II}}^1 + \psi(x,t)$ where $\psi = -\rho(\hat\varepsilon)e^{2K_1 t} + a\zeta(-\hat\varepsilon[x - x_1 + C(t-\tau_1)])$, $\rho(\hat\varepsilon)$ is as in (4.12), and $C \gg 1$ is to be determined. Clearly, if $\hat\varepsilon$ is sufficiently small, then $-1 \le w \le 1$ in $\mathbb{R}\times[0,\tau_1]$. Hence, when $t \in [0,\tau_1]$, we can calculate, by (4.2),

$$\begin{aligned}
&\big|\mathcal{A}[w] - \mathcal{A}[{v_{\rm II}}^1]\big| \\
&\qquad \le K_1 \int_{\mathbb{R}} (|\psi(x,t)| + |\psi(x-y,t) - \psi(x,t)|)\nu(dy) + \|\psi_{xx}(\cdot,t)\|_{C^0(\mathbb{R})} \\
&\qquad \le K_1(\rho(\hat\varepsilon)e^{2K_1 t} + a\zeta) + a\rho(\hat\varepsilon).
\end{aligned}$$

Since $w_t - \mathcal{A}[{v_{\rm II}}^1] = \psi_t = -2K_1\rho(\hat\varepsilon)e^{(k_1+1)t} - aC\hat\varepsilon\zeta'$, it then follows that, when $t \in (0,\tau_1]$, $-w_t - \mathcal{A}[w] \le -K_1\rho(\hat\varepsilon)e^{2k_1 t} + a[-\hat\varepsilon C\zeta' + K_1\zeta] \le 0$ if we take $C = C(\hat\varepsilon) := \max_{\rho(\hat\varepsilon)<\zeta<1-\rho(\hat\varepsilon)} \frac{K_1\zeta}{\hat\varepsilon\zeta'}$. Let $\hat\varepsilon > 0$ be sufficiently small such that $\rho(\hat\varepsilon)e^{2K_1\tau_1} \le \delta_1$. Then defining $h_2 = 4\hat\varepsilon^{-1} + |x_1| + C\tau_1$ we have that $w(x,0) \le {v_{\rm III}}^1(x,0)$, so that, by comparison, ${v_{\rm III}}^1(x_1 + 0, \tau) \ge w(x_1 + 0, \tau_1) = {v_{\rm II}}^1(x_1 + 0) - \varepsilon e^{2K_1\tau_1} \ge a + \delta_1$.

Similarly, one can show that for the same $\hat\varepsilon$, C, and h_2 defined as above, ${v_{\rm III}}^2(x,t) \le {v_{\rm II}}^2(x,t) + \rho(\hat\varepsilon)e^{2K_1 t} + (1-a)\zeta(\hat\varepsilon[x - x_1 + C(t-\tau_1)])$ in $\mathbb{R}\times[0,\tau_1]$ so that ${v_{\rm III}}^2(x_1 - 0, \tau) \le a - \delta_1$.

Proof of (iv). First of all, we remark that, for every small positive $\hat\varepsilon$, the function $u(x,t) + \hat\varepsilon e^{K_1 t}$ is a supersolution of (1.1) and the function $u(x,t) - \hat\varepsilon e^{K_1 t}$ is a subsolution, in $\mathbb{R}\times[0, \frac{1}{K_1}|\ln\hat\varepsilon|]$. Hence, for all sufficiently small positive h, by comparing $u(x+h,t)$ with $u(x,t) \pm \rho(\hat\varepsilon)e^{K_1 t}$, one can show that $u(\cdot,t)$ is continuous in $\mathbb{R}$ for every $t \ge 0$. In addition, by (B2), $u(x+h,t) - u(x,t) \ge \max_{z\in\mathbb{R}} \eta(|x - z|, 1)\int_z^{z+1}[u(y+h,0) - u(y,0)]\,dy > 0$, so that $u(\cdot,t)$ is strictly monotonic and $\xi_\pm(t)$ exist and are unique.

By Lemma 3.2 and (3.2), we know that, taking $\delta = \delta_1$ and denoting by $\varepsilon(\delta_1)$ and $C(\delta_1)$ the corresponding constants in Lemma 3.2, $\xi_+(t) - \xi_-(t) \le \xi_+(0) - \xi_-(0) + 8\varepsilon^{-1}(\delta_1) + 2C(\delta_1)t$ for all $t \ge 0$. In particular, the assertion is true for all $t \in [0,\tau_1]$.

To finish the proof, we need only prove the following: for every $t_1 \ge 0$, $\xi_+(t_1 + \tau_1) - \xi_-(t_1 + \tau_1) \le \max\{\xi_+(t_1) - \xi_-(t_1), 2h_2\}$.

By translation, we can assume that $u(0,t_1) = 0$ so that $\xi_-(t_1) < 0 < \xi_+(t_1)$. By symmetry, we need only consider the case $\xi_+(t_1) \ge |\xi_-(t_1)|$.

Set $h_+ = \max\{\xi_+(t_1), h_2\}$. Then, $u(\cdot + h_+, t_1) \ge {v_{\rm III}}^1(\cdot, 0)$ in $\mathbb{R}$, so that, by the comparison, $u(x_1 + h_+, t_1 + \tau_1) \ge {v_{\rm III}}^1(x_1 + 0, \tau_1) \ge a + \delta_1$; namely, $\xi_+(t_1 + \tau_1) \le x_1 + h_+$. (Here we use the continuity of $u(x,t)$.)

Set $h_- = \max\{\xi_+(t_1) - \xi_-(t_2), h_2\}$. Then, $u(\cdot + \xi_-(t_1) - h_-, t_1) \le {v_{\rm III}}^2(\cdot, 0)$, and therefore, by the comparison, $u(x_1 + \xi_-(t_1) - h_-, t_1 + \tau_1) \le v_{\rm III}(x_1 - 0, \tau_1) \le a - \delta_1$. That is, $\xi_-(t_1 + \tau_1) \ge x_1 + \xi_+(t_1) - h_-$.

Combining the two estimates for $\xi_+(t_1 + \tau_1)$ and $\xi_-(\tau_1 + t_1)$, we have

$$\xi_+(t_1 + \tau_1) - \xi_-(t_1 + \tau_1) \le h_+ - \xi_+(t_1) + h_- \le \max\{\xi_+(t_1) - \xi_-(t_1), 2h_2\}.$$

This completes the proof of Lemma 4.3, as well as the proof of Lemma 4.2 and Theorem 4.1. $\square$

Remark 4.4. (1) The condition (C4) is used only in Step 4 of the proof of Theorem 4.1. It is used to show that $U \in C^1$ and $\lim_{|\xi|\to\infty} \tilde{U}'(\xi, t) = 0$. It is purely technical, and can be replaced by weaker conditions.

(2) The condition $f'(a) > 0$, used in the proof of Lemma 4.3(i), is technical. We do not know if we can relax this condition by $f'(a) \ge 0$. However, in general, if $f'(a) = 0$, one can approximate f by f_ε satisfying $f'_\varepsilon(a) > 0$ to establish the existence (assuming necessary a priori estimates can be obtained). See Remark 5.2(4)(5).

(3) If $f'(a) < 0$, then f has at least five zeros in $[0, 1]$. Assume, for simplicity, that f has only one zero a_1 in $(0, a)$ and one zero a_2 in $(a, 1)$ and that $f'(a_1) > 0, f'(a_2) > 0$. Then there are (U_1, c_1) and (U_2, c_2) such that $U_1(x - c_1 t)$ and $U_2(x - c_2 t)$ are solutions of (1.1) and $U_1(-\infty) = 0, U_1(\infty) = a = U_2(-\infty), U_2(\infty) = 1$. For the case of (1.7), Fife and McLeod ([12]) showed that if $c_1 > c_2$, then there is a traveling wave of (1.7) and if $c_2 > c_1$, then there is no traveling wave connecting **0** and **1**; instead, there is a stacking of two traveling waves, approximated by $U_1(x - c_1 t - \xi_1) + [U_2(x - c_2 t - \xi_2) - a]$, where ξ_1 and ξ_2 are any constants. The case $c_1 = c_2$ was left open in [12]. We expect our analysis to extend to the case when $f(u)$ has more than three zeroes, as studied in [12].

5. Examples. In this section, we shall apply our results developed in Sections 2–4 to the examples mentioned in Section 1. Also, we discuss the special case (1.12) for small λ where our analysis fails to apply.

5.1. Applications. We consider the evolution equation

$$u_t = Du_{xx} + G(u, J_1 * S^1(u), \dots, J_n * S^n(u)), \qquad x \in \mathbb{R}, t > 0, \tag{5.1}$$

where $J * S(u)$ stands for the convolution $\int_{\mathbb{R}} J(x - y) S(u(y))\, dy$. We make the following assumptions:

(D1) For some $a \in (0, 1)$, the function $f(u) := G(u, u, \dots, u)$ satisfies (4.1);

(D2) For each $i = 1, \dots, n$, the kernel J_i is C^1 and satisfies $J_i(\cdot) \ge 0$, $\int_{\mathbb{R}} J_i(y)\, dy = 1$, $\int_{\mathbb{R}} |J'_i(y)| < \infty$;

(D3) The functions $G(u, p)$ $(p = (p_1, \dots, p_n))$, and $S^1(u), \dots, S^n(u)$ are smooth functions satisfying for all $u \in [-1, 2]$, $p \in [-1, 2]^n$, and $i = 1, \dots, n$, $G_{p_i}(u, p) \ge 0$, $S^i_u(u) \ge 0$;

(D4) Either $D > 0$ or $G_u(u, p) < 0$ and $G_{p_1}(u, p) S^1_u(u) > 0$ on $[-1, 2]^{n+1}$.

Theorem 5.1. *Assume that* (D1)–(D4) *hold. Then* (5.1) *admits a traveling-wave solution* (U, c) *satisfying* (2.1). *In addition, the traveling-wave solutions of* (5.1) *are unique up to a translation. Furthermore, traveling-wave solutions are globally asymptotically stable in the sense that there exists a positive constant* κ *such that if* $u(x,t)$ *is a solution of* (5.1) *with initial data* u_0 *satisfying* $0 \le u_0 \le 1$ *and*

$$\liminf_{x\to\infty} u_0(x) > a, \qquad \limsup_{x\to-\infty} u_0(x) < a,$$

then, for some constants ξ *and* K *depending on* u_0,

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le Ke^{-\kappa t} \qquad \forall\, t \ge 0.$$

Proof. We need only verify that the assumptions (D1)–(D4) imply (C1)–(C4). Clearly, (D1) implies (C1). Also, since G is smooth, (C3) holds.

Next we verify (C2). In the sequel, we write $G(u, J_1 * u, \ldots, J^n * u)$ as $\mathcal{G}(u)$. Assume that $(u+v)_t \ge \mathcal{G}(u+v)$, $u_t \le \mathcal{G}(u)$, and $v(\cdot,0) = v_0(\cdot) \ge 0$. We want to show that $v(x,t) \ge \eta(x,t)\int_0^1 v_0(y)\,dy$ for some positive function η independent of v. Subtracting the inequality for $(u+v)$ from that for u, we have

$$v_t \ge Dv_{xx} + \mathcal{G}(u+v) - \mathcal{G}(u) = Dv_{xx} + K_0 v + \hat{J} * v \qquad \text{in } \mathbb{R}\times(0,\infty), \tag{5.2}$$

where

$$K_0 = K_0(x,t) := \int_0^1 G_u(w, J_1 * S^1(w), \ldots, J_n * S^n(w))|_{w=u+\theta v}\,d\theta,$$

$$\hat{J} * v := \int_{\mathbb{R}} \hat{J}(x,y,t)v(y)\,dy \quad \text{and}$$

$$\hat{J}(x,y,t) :=$$
$$\int_0^1 \sum_{i=1}^n G_{p_i}(w, J_1 * S_u^1(w), \ldots, J_n * S^n(w))(x,t)S_u'(w)(y,t)\Big|_{w=u+\theta v} J_i(x-y)\,d\theta.$$

One observes that $\hat{J}(x,y,t) \ge 0$ in $\mathbb{R}^2 \times [0,\infty)$. We define

$$K := D + \|G\|_{C^1([-1,2]^{n+1})}\Big(\sum_{i=1}^n \|S^i\|_{C^1([-1,2])}\Big).$$

Then,

$$K \ge D + \|K_0\|_{L^\infty(\mathbb{R}\times(0,\infty))} + \sup_{x\in\mathbb{R}, t\ge 0}\int_{\mathbb{R}} |\hat{J}(x,y,t)|\,dy.$$

First, we claim that $v \geq 0$. In fact, if it is not true, then there exist $\varepsilon > 0$ and $T > 0$ such that $v > -\varepsilon e^{2Kt}$ in $\mathbb{R} \times [0,T)$ and $\inf_{\mathbb{R}} v(\cdot,T) = -\varepsilon e^{2KT}$. Without loss of generality, we assume that $v(0,T) < -\frac{7}{8}\varepsilon e^{2KT}$. Consider the function $w := -\varepsilon(\frac{3}{4} + \sigma z(x))e^{2Kt}$ where σ is a positive parameter and z is a smooth function having the property $z(0) = 1$, $z(\pm\infty) = 3$, and $z \geq 1$ and $|z''| \leq 1$ in $\mathbb{R}$. Gradually decreasing σ from $\frac{1}{4}$ to 0, we can find a minimum $\sigma = \sigma^* \in (\frac{1}{8}, \frac{1}{4}]$ such that $v \geq w$ in $\mathbb{R} \times [0,T]$. For such σ^*, we have $w(\pm\infty, t) = -\varepsilon[\frac{3}{4} + 3\sigma^*]e^{2Kt} < -\frac{9}{8}\varepsilon e^{2kt}$ for all t. Hence, there exists $(x_0, t_0) \in \mathbb{R} \times (0,T]$ such that $v(x_0,t_0) = w(x_0,t_0)$. At this particular point (x_0,t_0), $v_t \leq w_t$ and $v_{xx} \geq w_{xx}$. Consequently, we obtain from (5.2) the chain of inequalities:

$$
\begin{aligned}
-\tfrac{7}{4}\varepsilon K e^{2Kt_0} &\geq w_t(x_0,t_0) \geq v_t(x_0,t_0) \geq D v_{xx}(x_0,t_0) + (\hat{J} * v)(x_0,t_0) \\
&\geq D w_{xx}(x_0,t_0) - \varepsilon e^{2Kt_0} \int_{\mathbb{R}} \hat{J}(x_0,y)\,dy \\
&\geq -\Big(\sigma^* D + \int_{\mathbb{R}} \hat{J}\Big)\varepsilon e^{2Kt_0} \geq -\varepsilon K e^{2Kt_0}.
\end{aligned}
$$

Clearly, this is impossible. Therefore, we must have $v \geq 0$ in $\mathbb{R} \times [0,\infty)$. Consequently, setting $\hat{v} := ve^{Kt}$, we have $\hat{v}_t \geq D\hat{v}_{xx} + \hat{J} * \hat{v}$.

If $D > 0$, then we have $\hat{v}_t \geq D\hat{v}_{xx}$, so that, by the explicit expressions of the solutions of the initial value problem for $u_t = Du_{xx}$, (C2) holds.

Next we consider the case $D = 0$. In this case, we have $\hat{v}_t \geq \hat{J} * \hat{v} \geq 0$, so that $\hat{v}(x,t) \geq v_0(x,0)$ on $\mathbb{R} \times [0,\infty)$. It then follows that $\hat{v}_t \geq \hat{J} * \hat{v} \geq \hat{J} * v_0$, which implies, for any $t_1 > 0$, $\hat{v}(\cdot,t_1) \geq t_1 \hat{J} * v_0$. Now repeating the same process on $[t_1, 2t_1], \ldots, [(N-1)t_1, Nt_1]$, we have $\hat{v}(\cdot, Nt_1) \geq t_1^N \hat{J} * \cdots * \hat{J} * v_0$ for all $N = 1, 2, \ldots$.

Let $T > 0$ and $M > 0$ be any fixed numbers. Since $G_{p_1} S_u^1 > 0$, there exists a positive constant c_0 such that $\hat{J}(x,y,t) \geq c_0 J_1(x-y)$ for all $x \in \mathbb{R}, y \in \mathbb{R}, t \in [0,\infty)$. Also, since $J_1 \geq 0$ and $\int_{\mathbb{R}} J_1 = 1$, there exists a positive integer $N = N(M)$ such that $c_1(M) := \min_{x \in [-M-1, M+1]} (J_1 * \cdots * J_1)(x) > 0$. Therefore, taking $t_1 = T/N$ we have $\hat{v}(x,T) \geq (c_0 T/N)^N J_1 * J_1 * \cdots * J_1 * v_0 \geq (c_0 T/N)^N c_1(M) \int_0^1 v_0(y)\,dy$ for all $x \in [-M, M]$. That is, (C2) holds.

Finally, we verify (C4). Assume that $u_t = Du_{xx} + \mathcal{G}(u)$, $u(\cdot,0) = u_0$ where $u_0(\cdot,0) \in C^3(\mathbb{R})$ and $0 \leq u_0 \leq 1$. By the comparison principle, we know that $u(x,t) \in [0,1]$ on $\mathbb{R} \times [0,\infty)$.

If $D > 0$, (C4) is a direct consequence of the local regularity result of parabolic equations.

Hence, we consider the case $D = 0$. By continuous dependence on initial data, $u \in C^3(\mathbb{R} \times [0,\infty))$. Differentiating $u_t = G(u, J_1 * S^1(u), \ldots, J_n * S^n(u))$ with respect to x, we obtain

$$
(u_x)_t = G_u u_x + \sum_{i=1}^{n} G_{p_i} J_i' * S^i(u). \tag{5.3}
$$

Since $G_u(u,p) < 0$ on $[0,1]^{n+1}$, we have $k := -\max_{[0,1]^{n+1}} G_u(u,p) > 0$. It then follows from (5.3) and Gronwall's inequality that

$$\|u_x(\cdot,t)\|_{L^\infty} \le e^{-kt}\|u_x(\cdot,0)\|_{L^\infty}$$
$$+\sum_{i=1}^n \|G_{p_i}\|_{L^\infty([0,1]^{n+1})}\|S^i\|_{L^\infty([0,1])}\int_{\mathbb{R}}|J_i'(y)|\,dy\int_0^t e^{-k(t-\tau)}$$
$$\le \|u_x(x,0)\|_{L^\infty(\mathbb{R})} + \frac{1}{k}\sum_{i=1}^n \|G_{p_i}\|_{L^\infty([0,1]^{n+1})}\|S^i\|_{L^\infty([0,1])}\int_{\mathbb{R}}|J_i'(y)|\,dy$$

for all $t \ge 0$. Hence $\|u_x\|_{C^0(\mathbb{R}\times[0,\infty))} < \infty$. Differentiating (5.3) with respect to x, we obtain

$$(u_{xx})_t = G_u u_{xx} + 2\sum_{i=1}^n G_{up_i} u_x J_i' * S^i + \sum_{i=1,j=1}^n G_{p_ip_j}(J_i' * S^i(u))(J_j' * S^j(u))$$
$$+\sum_{i=1}^n G_{p_i} J_i' * (S^i_u u_x) =: G_u u_{xx} + G_1,$$

where G_1 is in $L^\infty(\mathbb{R}\times(0,\infty))$. It then follows as before that

$$\|u_{xx}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le e^{-kt}\|u_{xx}(\cdot,0)\|_{L^\infty(\mathbb{R})} + \frac{1}{k}\|G_1\|_{L^\infty(\mathbb{R}\times(0,\infty))}.$$

Therefore, $\|u_{xx}\|_{C^0(\mathbb{R}\times[0,\infty)} < \infty$. This verifies (C4).

Once we know (C1)–(C4) hold, we can apply the results in Sections 2–4 to conclude the assertion of the theorem. □

Remark 5.2. (1) One easily sees that our theorem applies to the Reaction-Diffusion model (1.7), the Neural model (1.9), the Ising model (1.11), the Phase Transition model (1.12) for λ large enough such that $f'(u) - \lambda < 0$ in $[0,1]$, and the Thalamic model (1.13). In addition to our explicit bound on the wave speed stated in Theorem 3.5, our Theorem 5.1 complements the following results:

(i) the result of Ermentrout and McLeod ([10]) for (1.9) where existence and uniqueness (only in the class of monotonic traveling waves) were established (our uniqueness and global asymptotic stability result is new);

(ii) the result of De Masi, Gobron and Presutti ([8]) and Orlandi and Triolo ([14]) (The former ([8]) established the existence, uniqueness, and asymptotic stability of traveling waves for sufficiently small h, by using a perturbation from $h = 0$ studied in [7, 9], and the latter ([14]) established the existence, uniqueness (only in the class of monotonic traveling waves), and linear stability of traveling waves for the general h such that the algebraic equation $u = \tanh\{\beta(u+h)\}$ has three distinguished roots, by using a method similar to that in [10]. Our existence, uniqueness, and global exponential stability theorem applies to all the general h as in [14].);

(iii) the result of [2] for (1.12) where asymptotic stability of traveling-wave solutions is only established for standing waves, by using an argument similar to [12] (our global asymptotic stability result applies to nonstanding traveling-wave solutions);

(iv) the result of [5] for (1.13) where existence and uniqueness of monotonic traveling waves were given.

(2) One may notice that we do not use any of the variational structure of the equation (1.1), so the proof of asymptotic stability is simpler than that presented in [2, 8, 12].

(3) If $D > 0$, we can actually allow J_i to be (densities of) probability measures. See also Remark 1.1.

(4) Note that when $D > 0$, the condition $G_u < 0$ is not needed. Hence, this condition seems unnecessary when $D = 0$. Indeed, for the existence of monotonic traveling-wave solutions, one can replace this condition $G_u < 0$ by the following condition:

Assume that $U(x - ct)$ *is a monotonic* (*in* x) *solution of* (5.1) *with* $D = 0$. *Then* $c > 0$ *if* $U(-\infty) = 0$ *and* $U(\infty) = a$, *and* $c < 0$ *if* $U(-\infty) = a$ *and* $U(\infty) = 1$. (This property is satisfied by (1.12) for any $\lambda > 0$; see [2].)

We sketch the proof as follows. Let (U_D, c_D), $D > 0$, be the traveling-wave solution of (5.1), normalized so that $U_D(0) = a$. From Theorem 3.5, we know that $\{c_D\}_{D\in(0,1]}$ is bounded. Hence, since $\{U_D\}_{D\in(0,1]}$ consists of monotonic bounded functions, there exists a sequence $\{D_j\}_{j=1}^{\infty}$ such that as $j \to \infty$, $D_j \searrow 0$, $c_{D_j} \to c$, and $U_{D_j} \to U$ (pointwise) for some finite $c \in \mathbb{R}$ and some nondecreasing function U satisfying $U(0) = a$, $0 \le U \le 1$. We claim that (U, c) is a traveling-wave solution. First of all, by Lebesgue's dominated convergence theorem, we have $cU' = G(U, J^1 * S^1(U), \cdots, J^n * S^n(U))$ in the distribution sense. Since the right-hand side is bounded, it is also in the classical sense. (This implies that if $c \ne 0$, then U is in C^1, and that U is discontinuous only if $c = 0$.) To finish the proof, we need only show that U is nontrivial. Assume that U is trivial; namely, $U \equiv a$. Without loss of generality, we assume that $c \le 0$. Consider the family $\{U_{D_j}(\cdot + \xi_j)\}$, where $U_{D_j}(\xi_j) = a/2$. Then, one can find a subsequence $\{D_{j_k}\}_{k=1}^{\infty}$ such that $U_{D_{j_k}}$ approaches a limit $\tilde{U}$ which solves $c\tilde{U}' = G(\tilde{U}, J^1 * S^1(\tilde{U}), \ldots, J^n * S^n(\tilde{U}))$. In addition, $\tilde{U}$ is nondecreasing and $0 \le \tilde{U} \le a$, $\tilde{U}(0) = a/2$. Clearly, $\tilde{U} \not\equiv a/2$ so that $\tilde{U}(-\infty) = 0$ and $\tilde{U}(\infty) = a$. But by the hypothesis, we must have $c > 0$, which contradicts the assumption $c \le 0$. This contradiction shows that (U, c) is a traveling-wave solution of (5.1) with $D = 0$.

(5) The condition $f'(a) > 0$ could be relaxed to $f'(a) \ge 0$. In fact, one can perturb f to f_ε satisfying $f'_\varepsilon(a) > 0$ and use the same argument as in the preceding paragraph to establish the existence of a traveling-wave solution.

5.2. Examples of nonuniqueness. At the end of this paper, we provide an example showing that if the profile of a traveling-wave front is monotonic but discontinuous, then it may not be unique and asymptotically stable. The example we constructed below also explains where and why our analysis in Section 2 fails.

Consider the evolution problem

$$u_t = \lambda[J * u - u] + f(u), \qquad f(u) := u - u^3. \tag{5.4}$$

Here J satisfies $J \in C^1(\mathbb{R})$, $J(-s) = J(s) \geq 0$ in $\mathbb{R}$,

$$\int_{\mathbb{R}} J(y) = 1, \quad \int_{\mathbb{R}} [|y|J(y) + |J'(y)|]\, dy < \infty.$$

Notice that the three roots of f are $(-1, 0, 1)$, instead of our previous convention $(0, a, 1)$.

We consider only the case $\lambda \in (0, 1)$. In this case, $f(m) = \lambda m$ has three solutions: $m_- := -\sqrt{1-\lambda}, m_0 = 0, m_+ := \sqrt{1-\lambda}$. Also, if we define $\hat{m}_\pm := \pm\sqrt{(1-\lambda)/3}$, then $f(u) - \lambda u$ is monotonic in $(-\infty, \hat{m}_-)$ and in $(\hat{m}_+, \infty)$. We denote by $m = m^+(g)$ and $m = m^-(g)$ the inverse functions of $g = g(m) := f(m) - \lambda m$ for $m \in (\hat{m}_+, \infty)$ and for $m \in (-\infty, \hat{m}_-)$ respectively. Clearly, $m^\pm(\cdot)$ is smooth and monotonic in its definition domain. We define $\hat{g}_\pm = g(\hat{m}_\pm) = \pm 2(\frac{1-\lambda}{3})^{3/2}$. Clearly,

$$-1 < m_- = m^-(0) < \hat{m}_- = m^-(\hat{g}_-) < 0 < m^+(\hat{g}_+) = \hat{m}_+ < m^+(0) = m_+ < 1.$$

It is proved in [2] that (5.4) has a monotonic standing wave U satisfying $U \in C^1((-\infty, 0]) \cup C^1([0, \infty))$, $U' > 0$ in $(-\infty, 0] \cup [0, \infty)$, $U(\pm 0) = m_\pm$, and $\lambda(J * U - U) + f(U) \equiv 0$. See also Remark 5.2(3). In addition, $\int_{\mathbb{R}} |U(x) - (2H(x) - 1)|\, dx < \infty$.

Now we shall construct nonmonotonic stationary solutions of (5.4). First, we study solutions of (5.4) for some special initial data.

Lemma 5.3. *Let h be an arbitrary small positive constant satisfying*

$$\sup_{x \in \mathbb{R}} |J * U(x + 2h) - J * U(x)| \leq \varepsilon := \frac{1}{4} \min\{g(\hat{m}_+), -g(\hat{m}_-)\} \Big(= \tfrac{1}{2}(\tfrac{1-\lambda}{3})^{3/2} \Big). \tag{5.5}$$

Also let I be any fixed measurable subset of $I_0 := [-h, h]$. Let $u^I(x, t)$ be the solution of (5.4) with initial data

$$u^I(x, 0) = u_0^I(x) := \begin{cases} U(x - h) & \text{if } x \in (h, \infty), \\ U(x + h) & \text{if } x \in (-\infty, -h), \\ m_+ & \text{if } x \in I, \\ m_- & \text{if } x \in I_0 \setminus I. \end{cases}$$

Then the following holds:

1) *For all $x \in \mathbb{R}$ and $t > 0$,*

$$U(x - h - 0) < u^I(x, t) < U(x + h + 0); \tag{5.6}$$

2) *For any* $t \in [0, \infty)$, $u^I(x,t) > m^+(\hat{g}_+ - \varepsilon)$ *for all* $x \in I$ *and* $u^I(x,t) < m^-(\hat{g}_- + \varepsilon)$ *for all* $x \in I_0 \setminus I$*;*

1) *On the set* $I^+ := I \cup [h, \infty)$*, the family* $\{u^I(\cdot,t)\}_{t \geq 0}$ *is uniformly Lipschitz continuous; i.e,*

$$\sup_{x,y \in I^+, x \neq y, t \geq 0} \frac{|u^I(x,t) - u^I(y,t)|}{|x-y|} < \infty.$$

Similarly, on the set $I^- = (-\infty, -h] \cup (I_0 \setminus I)$*, the family* $\{u^I(\cdot,t)\}_{t \geq 0}$ *is uniformly Lipschitz continuous.*

Proof. (1) Since $U(x-h-0) \leq u_0^I(x) \leq U(x+h+0)$, the first assertion follows immediately by the comparison principle.

(2) We first prove that $u^I(x,t) < m^-(\hat{g}_- + \varepsilon)$ for all $x \in I_0 \setminus I$. In fact, if this is not true, then there exist $x \in I_0 \setminus I$ and $T > 0$ such that $u^I(x,t) < m^-(\hat{g}_- + \varepsilon)$ in $[0,T)$, $u^I(x,T) = m^-(\hat{g}_- + \varepsilon)$, and $u_t^I(x,T) \geq 0$. Using (5.6), (5.5), and the fact that $(J * U)(0) = g(U(0\pm)) = g(m_\pm) = 0$, we have $J * u^I(\cdot,T)(x) \leq (J*U)(2h_0) \leq (J*U)(0) + \varepsilon = \varepsilon$. Hence, from the differential equation, we deduce that

$$\begin{aligned} 0 \leq u_t^I(x,T) &= \lambda J * u^I(\cdot,T)(x) + g(m^-(\hat{g}_- + \varepsilon)) \\ &= \lambda J * u^I(\cdot,T)(x) + \hat{g}_- + \varepsilon \leq \hat{g}_- + 2\varepsilon \end{aligned}$$

which is impossible since $\varepsilon \leq -\frac{1}{4}\hat{g}_-$. Hence, we must have $u^I(x,t) < m^-(\hat{g}_- + \varepsilon)$ for all $x \in I_0 \setminus I$. Similarly, $u^I(x,t) > m^+(\hat{g}_+ - \varepsilon)$ for all $x \in I$.

(3) Let $x, y \in I^+$ be any two points. Then by the second assertion of the lemma, $u^I(x,t), u^I(y,t) \in (m^+(g^+ - \varepsilon), \infty)$ for all $t \geq 0$. Setting $v(t) = u^I(x,t) - u^I(y,t)$, we have, from the evolution equation for u^I,

$$\begin{aligned} \dot{v}(t) &= [g(u^I(y,t) + v(t)) - g(u^I(y,t))] + [J * u^I(x) - J * u^I(y)] \\ &= g'(u(y,t) + \theta v(t))v(t) + [J * u^I(x) - J * u^I(y)], \end{aligned}$$

where $\theta \in (0,1)$. Since

$$g'(u(y,t) + \theta v(t)) \leq -k := \max_{u \in [m^+(\hat{g}_+ - \varepsilon), \infty)} \{f'(u) - \lambda\} < 0,$$

and since $|J * u^I(x) - J * u^I(y)| \leq \|J'\|_{L^1(\mathbb{R})}|x-y|$, we obtain, by Gronwall's inequality,

$$|v(t)| \leq e^{-kt}|u^0(x) - u^0(y)| + \int_0^t e^{-k(t-\tau)}\|J'\|_{L^1(\mathbb{R})}|x-y| \leq K|x-y|.$$

Hence, $\{u^I(\cdot,t)\}_{t \geq 0}$ is uniformly Lipschitz continuous on I^+. In a similar manner, we can show that $\{u^I(\cdot,t)\}_{t \geq 0}$ is uniformly Lipschitz continuous on I^-. $\square$

Theorem 5.4. *There exists a small positive constant h_0 such that for any $h \in (0, h_0]$ and every measurable subset I of $I_0 := [-h, h]$ $(0 < |I| < 2h_0)$, there is a standing wave U^I of* (5.4) *having the following properties*:

$$\lambda(J * U^I - U^I) + f(U^I) = 0 \qquad on \ \ \mathbb{R}, \tag{5.7}$$

$$U(\cdot - h - 0) < U^I(\cdot) < U(\cdot + h + 0) \ \ on \ \ \mathbb{R}, \tag{5.8}$$

$$U^I(x) > \hat{m}^+ \ \ on \ \ I, \quad U^I(x) < \hat{m}^- \ \ on \ \ I_0 \setminus I. \tag{5.9}$$

Proof. Define $F(u) := \int_{-1}^{u} f(s)ds$. Then $F(-1) = F(1) = 0$ and $F(u) < 0$ for all $u \in (-1, 1)$. From (5.6) and the properties of U, one can show that the energy of the state $u^I(\cdot, t)$ defined by

$$\mathcal{E}(u^I(\cdot, t)) := \frac{\lambda}{4} \int\int_{\mathbb{R}^2} J(x-y)[u^I(x,t) - u^I(y,t)]^2 \, dx \, dy - \int_{\mathbb{R}} F(u^I(y,t)) \, dy$$

is finite, and in addition

$$\int_0^t \int_{\mathbb{R}} [u_t^I(x,t)]^2 \, dx \, dt + \mathcal{E}(u^I(\cdot, t)) = \mathcal{E}(u_0^I).$$

Since $\mathcal{E}(u^I(\cdot, t)) \geq 0$, $\int_0^\infty \int_{\mathbb{R}} (u_t^I)^2 \leq \mathcal{E}(u_0^I)$.

Notice that a) $\{u^I(\cdot, t)\}_{t \geq 0}$ is equicontinuous on I^+ and I^- respectively, and b) $u_t^I(\cdot, \cdot)$ is uniformly continuous on $I^- \times [0, \infty)$ and $I^+ \times [0, \infty)$ respectively. (Since $u_t^I = \lambda(J * u^I - u^I) + f(u^I)$ is Lipschitz continuous in x in $I^\pm \times [0, \infty)$ and $(u_t)_t = \lambda(J * u_t^I - u_t^I) + f'(u^I) u_t^I$ is bounded on $\mathbb{R} \times [0, \infty)$.) Hence, there exist a function U^I and a sequence $\{t_j\}_{j=1}^\infty$ with $\lim_{j \to \infty} t_j = \infty$ such that $u^I(\cdot, t_j) \to U^I(\cdot)$ and $u_t^I(\cdot, t_j) \to 0$ uniformly on any bounded subset of $\mathbb{R}$.

Using the equation for u^I, we readily see that U^I satisfies (5.7). From the properties of u^I in Lemma 5.3, U^I is Lipschitz continuous on I^- and I^+ respectively and satisfies (5.9) (observing that $\hat{m}^+ < m^+(\hat{g}_+ - \varepsilon)$ and $\hat{m}^- > m^-(\hat{g}_- + \varepsilon)$) and $U(\cdot - h_0 - 0) \leq U^I(\cdot) \leq U(\cdot + h_0 + 0)$ on $\mathbb{R}$. Using the comparison principle, we also have (5.8). This completes the proof the the theorem. $\square$

Remark 5.5. (1) From (5.7)–(5.9), $m^\pm(J * U^I)$ is well-defined on $I^\pm$, and

$$U^I(x) = -m^\pm(J * U^I) \ \text{ on } \ I^\pm.$$

Since $J * U^I$ is C^1 and $m^\pm(g)$ is C^1 on its definition domain, the restriction of U^I on $I^\pm$ is in $C^1(I^\pm)$. The derivative is defined by

$$U^{I\prime}(x) = \lim_{y \to x, x, y \in I^\pm} \frac{U^I(x) - U^I(y)}{x - y} \quad \text{for all } x \in I^\pm.$$

(2) Since $m^+ > 0 > m^-$, if I_1 is not a translation of I_2, then $U^{I_1}(\cdot) \not\equiv U^{I_2}(\cdot + \xi)$ for all $\xi \in \mathbb{R}$. Hence, (5.4) admits infinitely many nonmonotonic standing waves; in particular, it admits solutions which are not odd with respect to any point (i.e., $U^I(\cdot) \not\equiv -U(-\cdot + \xi)$ for any ξ). Also, since one can take arbitrarily small h, U^I can be arbitrarily close to $U(\cdot)$ in $L^p(\mathbb{R})$ for any $p < \infty$.

(3) Although (5.8) is a strict inequality, we cannot "push" either $U(\cdot + h_0)$ or $U(\cdot - h_0)$ to make $U(\cdot - h_0 + \hat{h}) < U^I(\cdot)$ in $[-1, 1]$ or $U(\cdot + h_0 - \hat{h}) > U^I(\cdot)$ in $[-1, 1]$ for any small $\hat{h}$, if h_0 and $-h_0$ are accumulation points of I and $[-h_0, h_0] \setminus I$. This is due to the fact that U^I is discontinuous at both $x = h_0$ and $x = -h_0$.

(4) One sees that our choice of $f(u) = u - u^3$ is only for the convenience of computation. Theorem 5.4 is true for a general f such that $f(u) - \lambda u$ is not strictly decreasing in $[0, 1]$ and there is a discontinuous monotonic standing wave of (5.4).

Remark 5.6. (1) Physically, λ represents the strength of the surface tension of interfaces between two different phases of a material. The stationary solution that we constructed for (5.4) indicates that if the surface tension is too small, then there may not be any phase coarsening.

(2) We do not know if there are stationary solutions of (5.1) with the property that $U(\pm\infty) = \pm 1$, and that the set $\mathcal{M} := \{x : U(x) \in (\hat{m}_-, \hat{m}_+)\}$ has nonzero measure.

(3) We conjecture that there are infinitely many nontrivial equilibria U of (5.1) (with $\lambda \in (0, 1)$) such that $\lim_{|x| \to \infty} U(x) = 1$. This will be in sharp contrast to Remark 3.4(c).

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