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SELF-SIMILAR BLOW-UP FOR A QUASILINEAR PARABOLIC EQUATION WITH GRADIENT DIFFUSION AND EXPONENTIAL SOURCE

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Abstract. We study the asymptotic behaviour near a finite blow-up time $t = T$ of the solutions to the initial-boundary value problem for the quasilinear equation

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + e^u \text{ in } \{|x| < R\} \times (0, T), \quad \sigma > 0,$$

with zero Dirichlet boundary condition and a radial symmetric initial function $u_0(|x|) > 0$ in $\{|x| < R\}$, $u_0'(r) < 0$ in $(0, R)$. We prove single point blow-up and different sharp lower and upper estimates of the solution as $t \to T^-$ and of the final-time profile. For the one-dimensional problem the asymptotic behaviour is proved to be described by nonconstant self-similar solutions of the form $u_*(x, t) = -\log(T - t) + \theta(\xi)$, $\xi = x/(T - t)^{1/(\sigma+2)}$, where $\theta(\xi) \sim -(\sigma + 2)\log\xi$ as $\xi \to \infty$.

1. Introduction and main results. An interesting feature of certain nonlinear parabolic partial differential equations is that, given suitably large initial data, their solutions, $u(x, t)$, can form singularities or blow-up in a finite time T and are bounded and smooth enough up to this time. Typically the blow-up occurs at a single point x^*, with $u(x^*, t) \to \infty$ as $t \to T^-$. Recently, the nature of the solutions for t close to T and x close to x^* has been the subject of much analytical investigation and for the case of linear diffusion many rigorous results established. Many of these results look at self-similar and approximately self-similar structures in the blow-up solutions of those semilinear parabolic equations that admit rescaling invariances. An interesting feature of these solutions is that they only approximately inherit the rescaling invariance of the original equation. In this paper, we extend these studies to the case of nonlinear (gradient or p-Laplacian) diffusion operators and show that there is a class of self-similar blow-up solutions which strictly inherit the rescaling invariance of the original equation.

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Let $B_R \subset \mathbb{R}^N$, $N \geq 1$, be a ball of the radius $R > 0$. In this paper we study the asymptotic behaviour of the blowing-up solution to the initial-boundary value problem for a quasilinear parabolic equation of the form

$$u_t = \mathbf{A}(u) \equiv \nabla \cdot (|\nabla u|^\sigma \nabla u) + e^u \quad \text{for } x \in B_R, \quad t > 0, \tag{1.1}$$

$$u(x,t) = 0 \quad \text{for } x \in \partial B_R = \{|x| = R\}, \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(|x|) \quad \text{in } B_R, \tag{1.3}$$

where $\sigma > 0$ is an arbitrary fixed constant. The initial function given in (1.3) is assumed to satisfy

$$\begin{aligned} &u_0(r) > 0, \quad u_0'(r) < 0 \quad \text{for } r \in (0,R), \quad u_0'(0) = 0, \\ &M_0 = \sup u_0 < \infty, \quad M_1 = \sup |u_0'| < \infty. \end{aligned} \tag{1.4}$$

If $\sigma = 0$, then the semilinear equation (1.1) describes combustion processes in a medium with thermal reactions; see the books [51] and [7] for extensive references. The case $\sigma = 0$ has been recently considered in detail; see references below. In the case $\sigma > 0$ the heat conductivity coefficient is formally assumed to depend on the spatial gradient of the temperature. Such diffusion operators occur in the theory of turbulent filtration and non-Newtonian liquids; see e.g. the book [4] and the references therein.

We will show that for arbitrarily small $\sigma > 0$ the behaviour of large heat structures is quite different. Under hypotheses (1.4), the unique weak solution $u = u(r,t) \geq 0$ is decreasing in $r = |x|$ for any $t \in (0,T)$, $T > 0$ being the maximal existence time. We assume that the solution blows up in a finite time, $T > 0$, so that $u(x,t)$ is uniformly bounded in $B_R \times (0,T')$ with any $T' < T$ and

$$\sup_{x \in B_R} u(x,t) \equiv u(0,t) \to \infty \text{ as } t \to T^-. \tag{1.5}$$

We prove that the asymptotic behaviour of $u(r,t)$ as $t \to T$ essentially differs from that for the semilinear case, $\sigma = 0$. If $\sigma = 0$ in (1.1) and $u(r,t)$ solves the semilinear heat equation

$$u_t = \Delta u + e^u \quad \text{in} \quad B_R \times (0,T), \tag{1.6}$$

then it is known that for arbitrary solutions if $N \leq 2$, and for solutions monotone in time, $u_t \geq 0$ in $B_R \times (0,T)$, if $N \geq 3$, under hypotheses (1.4) the solution satisfies

$$\log(T-t) + u(\xi[(T-t)|\log(T-t)|]^{1/2}, t) \to -\log\big[1 + \frac{\xi^2}{4}\big] \tag{1.7}$$

as $t \to T$ uniformly on any compact subset $\{|\xi| \leq C\}$, where

$$\xi = x[(T-t)|\log(T-t)|]^{-1/2}. \tag{1.8}$$

This space-time rescaling was first introduced in [36]. See the first formal result [17] (where ξ is referred to as the ignition kernel) and recent proofs given in [5,10, 11, 20, 21, 34, 35, 45] (see also the first results in [29, 30, 9], and [49]). In particular, there holds

$$u(x,t) = -\log(T-t) + o(1) \quad \text{as} \quad t \to T \tag{1.9}$$

uniformly on $\{|x| \le C(T-t)^{1/2}\}$ and

$$u(0,t) = -\log(T-t) + o(1) \quad \text{as} \quad t \to T. \tag{1.10}$$

The final-time profile $u(x,T)$ for the case $\sigma = 0$ satisfies as $|x| \to 0$

$$u(x,T) = -2\log|x| + \log|\log|x|| + \log 8 + o(1). \tag{1.11}$$

This asymptotic behaviour is proved to be stable with respect to small perturbations of coefficients to equation (1.6); i.e., (1.7)–(1.11) hold for the equation $u_t = \nabla \cdot (k(u)\nabla u) + Q(u)$, where $k(u) \sim 1,\ Q(u) \sim e^u$ as $u \to \infty$; see [6].

In fact, (1.7), (1.8) implies that $u(x,t)$ behaves as $t \to T$ like a nonconstant self-similar solution to the first-order Hamilton-Jacobi equation

$$u_t = -\frac{1}{2}[(T-t)|\log(T-t)|]^{-1}\nabla u \cdot x + e^u; \tag{1.12}$$

i.e., the heat operator Δu disappears in the limit, see [31] and similar constructions in [10], [11] and [17]. (The fact that (1.12) generates in the limit into a Hamilton-Jacobi problem for the rescaled function with the uniformly stable reduced ω-limit set has been used in [6].) For $N = 1, 2$ the *degeneracy* as $t \to T$ of the equation (1.6) into (1.12) can be explained by the fact of *nonexistence of a nontrivial self-similar solution* to (1.6) of the form $u_*(x,t) = -\log(T-t) + \theta(x/(T-t)^{1/2})$ ([8], [19], see also comments below). (If $N \ge 3$ then such a solution $\theta(\xi)$ could exist but it cannot be stable in a class of solutions $u(x,t)$ satisfying $u_t \ge 0$, so that (1.7), (1.8) is also valid; see [7] and [5]). The fact that for $\sigma > 0$ *stable similar self-similar blowing up solutions exist* plays a key role in future understanding of the equation (1.1) near $t = T$.

We now state the main results for the quasilinear problem (1.1)–(1.3).

1.1 Sharp estimates.

Theorem 1.1 (lower bound of the final profile)**.** *Assume that* (1.4) *holds. Then*

(i) *If* $N \le \sigma + 2$ *then for* $r > 0$ *small enough*

$$\underline{u}(r,T) \equiv \liminf_{t\to T} u(r,t) \ge -(\sigma+2)\log r + C_-, \tag{1.13}$$

where

$$C_- = (\sigma+1)\log\Big[\frac{\sigma+2}{3}N^{1/(\sigma+1)}\Big]. \tag{1.14}$$

(ii) *If* $N > \sigma+2$, *then* (1.13) *holds for any* u_0 *such that* $u(r,t)$ *does not decrease in time in* $B_R \times (0,T)$.

Theorem 1.2 (upper bound of the final profile)**.** *Assume that* (1.4) *holds and also*

$$(|u_0'|^\sigma u_0'')(0) < 0. \tag{1.15}$$

Then for all $r > 0$ *small*

$$\bar{u}(r,T) \equiv \limsup_{t \to T} u(r,t) \le -(\sigma+2)\log r + C_+, \tag{1.16}$$

where

$$C_+ = (\sigma+1)\log\Big[(\sigma+2)\Big[\frac{2+\sigma(2+N)}{\sigma}\Big]^{1/(\sigma+1)}\Big] \ . \tag{1.17}$$

Observe that due to the convenient assumption (1.15), $u_0 \notin C^2$, but $u_0 \in C^{1,1/(\sigma+1)}$, the regularity which is natural for weak solutions of (1.1); see Section 2.

Thus, both estimates (1.13) and (1.16) yield that in the quasilinear case, $\sigma > 0$, the function $-(\sigma+2)\log r$ describes the exact rate of singularity of the blow-up solution occurring near the origin for $t = T$. It follows from (1.11) that this is not true for the semilinear case, $\sigma = 0$. Indeed, if $N = 1$, then as $\sigma \to 0$, $C_+ \approx \log(4/\sigma) \to \infty$. This is consistent with the semilinear case where there is an additional unbounded term $\log|\log r|$ as $r \to 0$, and indeed, the estimate of $\log(4/\sigma)$ is sharp asymptotically; see [14].

Theorem 1.3 (lower bound of the L^∞ norm)**.** *Assume that* (1.4) *and* (1.15) *hold. Then*

$$u(0,t) \ge -\log(T-t) + M_- \quad \textit{as } t \to T, \tag{1.18}$$

where

$$M_- = \log\frac{2+\sigma(2+N)}{2(\sigma+1)} > 0.$$

Theorem 1.4 (upper bound of the L^∞ norm)**.** *Assume that* (1.4) *holds and* $N < \sigma+2$*. Then there exists a constant* $M_+ > 0$ *depending on* σ, R, N, T, M_0 *and* M_1 *such that*

$$u(0,t) \le -\log(T-t) + M_+ \quad \textit{for } t \in [0,T). \tag{1.19}$$

Thus, the function $u(0,t) + \log(T-t)$ is bounded away from zero as $t \to T$ which is a striking difference in comparison with (1.10), $\sigma = 0$. However, as $\sigma \to 0$, if $N = 1$, then $M_- \approx \sigma/2 \to 0$. Thus there is a smooth transition to the semilinear case, and the above estimate is shown in [14] to be sharp asymptotically.

1.2. Asymptotic behaviour, existence of nontrivial self-similar solution in one space dimension. In general, the asymptotic behaviour of the solution as $t \to T$ is expected to depend on the existence or nonexistence of a suitable self-similar solution to equation (1.1). It admits a blowing-up self-similar solution of the form

$$u_*(x,t) = -\log(T-t) + \theta(\xi), \quad \xi = |x|(T-t)^{-1/(\sigma+2)}, \tag{1.20}$$

where the function $\theta(\xi)$ solves the following nonlinear second-order ordinary differential equation:

$$\mathbf{B}(\theta) \equiv \xi^{1-N}(\xi^{N-1}|\theta'|^\sigma \theta')' - \frac{1}{\sigma+2}\theta'\xi + e^\theta - 1 = 0 \quad \text{for } \xi > 0, \tag{1.21}$$

$$\theta'(0) = 0. \tag{1.22}$$

We now state the question of asymptotic behaviour as $t \to T$ for the problem (1.1)–(1.3) as follows. Let us introduce the rescaled function $v(\xi, \tau)$, where $\tau = -\log(T-t) : [0, T) \to [\tau_0, \infty)$, $\tau_0 = -\log T$, is the new time,

$$v(\xi,\tau) = \log(T-t) + u(\xi(T-t)^{1/(\sigma+2)}, t) \tag{1.23}$$

which is defined according to the self-similar space-time structure given by (1.20). Then $v(\xi, \tau)$ solves the quasilinear parabolic equation

$$v_\tau = \mathbf{B}(v) \;\text{ in }\; B_{\ell(\tau)} \times (\tau_0, \infty), \tag{1.24}$$

where the elliptic operator $\mathbf{B}$ is given in (1.21) and $\ell(\tau) = \mathrm{R}e^{\tau/(\sigma+2)}$, with the boundary and initial data

$$v(\xi,\tau) = -\tau \;\text{ for } |\xi| = \ell(\tau), \quad \tau \geq \tau_0, \tag{1.25}$$

$$v(\xi,\tau_0) = \bar{v}_0(\xi) \equiv \log T + u_0(\xi T^{1/(\sigma+2)}) \;\text{ in } B_{\ell(\tau_0)}. \tag{1.26}$$

We denote by $\omega(\bar{v}_0)$ the ω-limit set of the unique global solution $v(\xi, \tau)$ to the problem (1.24)–(1.26):

$$\begin{gathered}\omega(v_0) = \{g \in C(\mathbb{R}^N) : \exists\{\tau_j\} \to \infty \text{ such that } v(\cdot, \tau_j) \to g(\cdot) \\ \text{as } j \to \infty \text{ uniformly on compact subsets of } \mathbb{R}^N\}.\end{gathered} \tag{1.27}$$

The problem of the asymptotic behaviour as $t \to T$ or, which is the same, as $\tau \to \infty$, consists in the comparison of the ω-limit set (1.27) and the set of suitable stationary solutions to the equation (1.24) satisfying (1.21), (1.22).

We now consider the one-dimensional case, $N = 1$. It follows from the estimates given in Theorems 1.1–1.4 and known regularity results for degenerate equations of the form (1.1) (see the survey [38] and general interior regularity results [15], [16]) that

$$\omega(\bar{v}_0) \neq \emptyset. \tag{1.28}$$

Denote by W_s the set of weak stationary solutions θ to equation (1.24), $N = 1$, satisfying

$$\mathbf{B}(\theta) \equiv (|\theta'|^\sigma \theta')' - \frac{1}{\sigma+2}\theta'\xi + e^\theta - 1 = 0 \;\text{ for } \xi > 0, \tag{1.29}$$

$$\theta'(0) = 0, \quad \theta(\xi) \;\text{ decreases, } \theta(\xi) \to -\infty \;\text{ as } \xi \to \infty. \tag{1.30}$$

Theorem 1.5. *Let $N = 1$. Assume that* (1.4) *and* (1.15) *hold. Then*

$$\omega(\bar{v}_0) \subseteq W_s, \tag{1.31}$$

and any $g \in \omega(v_0)$ satisfies

$$g(\xi) \leq -(\sigma + 2)\log\xi + C_+ \quad as\ \xi \to \infty, \tag{1.32}$$

$$\frac{1}{\sigma + 2} g'\xi + 1 \geq 0 \quad for\ \xi \geq 0. \tag{1.33}$$

The next result is the straightforward consequence of (1.28) and (1.31).

Corollary 1.6. *There exists a function $\theta \in W_s$ satisfying*

$$\theta(\xi) = -(\sigma + 2)\log\xi + O(1) \quad as\ \xi \to \infty. \tag{1.34}$$

Thus, a nonconstant monotone in $|x|$ self-similar solution (1.20) exists for an arbitrarily small $\sigma > 0$ and it is asymptotically stable. In the related paper [12] we prove that if $\phi = 0.61803...$ is the positive solution of the quadratic equation $\phi^2 + \phi = 1$, then for $\sigma > \phi$ there exist two different countable sets $\{u_k\}$ of solutions (1.20), each u_k with $\theta \equiv \theta_k(\xi)$ having exactly k maxima and minima for $\xi \geq 0$. These sets differ by their behaviour at the origin and if $\sigma < \phi$ only those solutions which are nonzero in a neighbourhood of the origin appear to exist. If $\sigma < \sigma_\infty \approx 0.6$ the number of such solutions is expected to be finite. The questions of existence of stable blow-up *approximate self-similar solutions* for $\sigma \in (0, \sigma_\infty)$ having a complex space structure are discussed in [13].

The layout of this paper is as follows. In Section 2 we look at the existence and regularity of the solutions of the degenerate quasilinear parabolic problem—studying it as a regularization via a nondegenerate problem. In Section 3 we look at the steady states and derive some estimates which, together with a technique due to Friedman and McLeod ([23]), are used in Sections 4, 5 and 6 to prove Theorems 1.1, 1.2, 1.3 and 1.4. Finally, in Section 7 we prove the convergence of the solution to a self-similar profile in the case of the one-dimensional problem. This section is based upon the derivation of an approximate Lyapunov function for the degenerate problem which is obtained from a related function for the regularised system.

2. Regularity and existence. In general, the quasilinear degenerate equation (1.1) admits weak solutions. A nonnegative bounded continuous function $u(x,t)$ is said to be the weak solution of the problem (1.1)–(1.3) in $Q_{T'} = \mathbb{R}^N \times (0, T')$, $T' > 0$, if there exists the weak derivative $\nabla u(x,t) \in L^{2(\sigma+1)}_{\mathrm{loc}}(Q_{T'})$, and for any compactly supported test function $\varphi \in C_0^\infty(Q_{T'})$ vanishing near $t = T'$, the following identity holds:

$$\int_{Q_{T'}}\int \{|\nabla u|^\sigma(\nabla u \cdot \nabla\varphi) - e^u\varphi - u\varphi_t\}\,dx\,dt - \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx \equiv 0. \tag{2.1}$$

Existence, uniqueness, regularity and different comparison results for equations having a degenerate diffusion term given in (1.1) are well-known; see, e.g., [18], [37], [38], [44], [48] and references in [38]. Weak local-in-time solutions have been proved to have the continuous derivative ∇u; cf. [1], [2] and references in [28], [32], [38].

The fact that the weak solution blows up in a finite time for any large enough initial function u_0 can be proved by standard techniques. In particular, using the fact that for any fixed $\alpha > \sigma$

$$e^u \geq C_\alpha u^{1+\alpha} \quad \text{for} \quad u \geq 0, \qquad C_\alpha = (1+\alpha)^{-(1+\alpha)} e^{1+\alpha}, \tag{2.2}$$

we have that by comparison $u(x,t) \geq v(x,t)$, where the function $v(x,t) \geq 0$ is the unique weak solution to the equation

$$v_t = \nabla \cdot (|\nabla v|^\sigma \nabla v) + C_\alpha v^{1+\alpha} \tag{2.3}$$

with the same boundary and initial data. Nonexistence of a global-in-time solution to (2.3) has been proved by different methods in [25], [43], [48]; see the survey [42].

The weak solution $u(x,t)$ to the problem (1.1)–(1.3) can be constructed as the limit of the sequence $\{u_\epsilon(x,t)\}$ of classical solutions to regularized parabolic equations, see [1], [2], [37] and references in [32], [38]. For a fixed $\epsilon > 0$ set

$$\varphi_\epsilon(p) = (|p|^2 + \epsilon^2)^{\sigma/2} p \quad \text{for} \quad p \in \mathbb{R}^N, \tag{2.4}$$

and denote by $u_\epsilon(x,t)$ the unique local-in-time classical solution to the uniformly parabolic equation

$$(u_\epsilon)_t = \nabla \cdot (\varphi_\epsilon(\nabla u_\epsilon)) + e^{u_\epsilon} \tag{2.5}$$

with the same conditions (1.2), (1.3). Using uniform estimates and known regularity results for parabolic equations ([15], [16]), we may conclude that (see [1], [2] and [38])

$$u_\epsilon \to u, \quad \nabla u_\epsilon \to \nabla u \quad \text{as } \epsilon \to 0 \tag{2.6}$$

uniformly on any compact subset of Q_T. This regularization forms the basis of many of the subsequent arguments used in this paper. Since by the Strong Maximum Principle ([22]) $(u_\epsilon)_r < 0$ in $(0,R) \times (0,T)$ by assumption (1.4), we deduce from (2.6) that $u_r \leq 0$ in Q_T. We also have that

$$(u_\epsilon)_t \to u_t \quad \text{as} \quad \epsilon \to 0 \tag{2.7}$$

in $L^2_{loc}(Q_T)$ and uniformly on any compact subset where $|\nabla u| \neq 0$ and the solution is classical. Notice that in any domain of the form $\omega_\delta = (\delta, R) \times (0,T)$ the derivative $z = u_r$ satisfies the equation having the main diffusion operator of the porous medium type,

$$z_t = (r^{1-N}(r^{N-1}|z|^\sigma z)_r)_r + e^u z, \tag{2.8}$$

and by hypotheses (1.4), $z(0,r) < 0$ in (δ, R) and $z \leq 0$ on the lateral boundary of ω_δ. Hence, by well-known properties of equations of the porous medium type (see references in [38]), we deduce that $z < 0$ in ω_δ. Since $\delta > 0$ is arbitrary, we have that

$$u_r < 0 \text{ in } (B_R\backslash\{0\}) \times (0,T), \tag{2.9}$$

$$u > 0 \text{ in } B_R \times (0,T). \tag{2.10}$$

Finally, we mention that by general regularity results for quasilinear uniformly parabolic equations (see references in [38] and [15], [16]), the solution to the problem (1.1)–(1.3) is classical at any point of nondegeneracy where $|\nabla u| \neq 0$. It then follows from (2.9) that $u(r,t)$ is smooth enough for any $r \in (0,R)$ and $t \in (0,T)$. In fact, by using equation (2.8) and inequality (2.9) we have that equation (1.1) is uniformly parabolic in any domain of the form $[\delta, R] \times (0, T-\delta)$, $\delta > 0$, and hence we may conclude

$$u \in C^\infty([\delta,R] \times (0,T-\delta)) \quad \text{for any small } \delta > 0; \tag{2.11}$$

see also [39] and [40] and the references in [38].

3. Some properties of the stationary solutions. The properties of a one-dimensional family of stationary solutions of (1.1) play an important role in the proof of Theorems 1.1–1.4. We now derive some of them.

We denote by $U(r;\lambda)$, where $\lambda \in \mathbb{R}$ is a fixed parameter, the solution of the stationary equation

$$r^{1-N}(r^{N-1}|U'|^\sigma U')' + e^U = 0 \quad \text{for } r > 0, \tag{3.1}$$

satisfying the boundary conditions

$$U'(0;\lambda) = 0, \quad U(0;\lambda) = \lambda, \tag{3.2}$$

where primes denote differentiation with respect to r. Using the invariance of the equation (3.1) yields the identity

$$U(r;\lambda) = \lambda + U_0(re^{\lambda/(\sigma+2)}), \tag{3.3}$$

where $U_0(r)$ denotes the function $U(r;0)$.

3.1. Upper and lower bounds. Envelope. Since $U_0(r)$ is a strictly decreasing function, by integrating equation (3.1) with $U = U_0$ over $(0,r)$ we have that

$$r^{N-1}|U_0'|^\sigma U_0' = -\int_0^r y^{N-1} e^{U_0(y)}\,dy \leq -e^{U_0(r)}\frac{r^N}{N}\ .$$

Integrating once more yields for any $r > 0$

$$U_0(r) \leq -(\sigma+2)\log r + M_*, \quad M_* = (\sigma+1)\log[(\sigma+2)N^{1/(\sigma+1)}], \tag{3.4}$$

and hence (3.3) implies that for any $\lambda \in \mathbb{R}$

$$U(r;\lambda) \le -(\sigma+2)\log r + M_* \quad \text{for } r > 0. \tag{3.5}$$

Integrating twice another inequality

$$(r^{N-1}|U_0'|^\sigma U_0')' = -e^{U_0} r^{N-1} \ge -r^{N-1}$$

yields the lower estimate

$$U_0(r) \ge -\frac{\sigma+1}{\sigma+2} N^{-1/(\sigma+1)} r^{(\sigma+2)/(\sigma+1)} \quad \text{for } r \ge 0. \tag{3.6}$$

Hence by (3.3) for any $\lambda \in \mathbb{R}$

$$U(r;\lambda) \ge U_*(r;\lambda) \equiv \lambda - \frac{\sigma+1}{\sigma+2} N^{-1/(\sigma+1)} e^{\lambda/(\sigma+1)} r^{(\sigma+2)/(\sigma+1)} \quad \text{for } r \ge 0. \tag{3.7}$$

The uniform with respect to λ upper bound (3.5) implies that for $r > 0$ there exists the *envelope* to the family of functions $\{U(r;\lambda),\ \lambda \in \mathbb{R}\}$,

$$L(r) \equiv \sup_{\lambda\in\mathbb{R}} U(r;\lambda) \ge L_*(r) \equiv \sup_{\lambda\in\mathbb{R}} U_*(r;\lambda). \tag{3.8}$$

The envelope $L_*(r)$ to the set $\{U_*; \lambda \in \mathbb{R}\}$ can be easily calculated explicitly, and finally we arrive at the estimate

$$L(r) \ge L_*(r) = -(\sigma+2)\log r + C_-, \tag{3.9}$$

where the constant C_- is given in (1.14), so that the envelope $L(r)$ to the set of stationary solutions $\{U\}$ occurs in Theorem 1.1 as the lower bound of the final-time profile corresponding to the solution $u(r,t)$ of the nonstationary problem (1.1)–(1.3).

3.2. Asymptotic properties. The invariance (3.3) implies that (3.1) can be reduced to a first-order equation. Setting

$$U_0(r) = -(\sigma+2)\eta + V(\eta), \quad \eta = \log r, \quad V' = H, \tag{3.10}$$

yields the equation

$$\frac{dH}{dV} = -\frac{1}{(\sigma+1)H|(\sigma+2)-H|^\sigma}\{e^V - [N-(\sigma+2)]\,|(\sigma+2)-H|^\sigma((\sigma+2)-H)\}. \tag{3.11}$$

Since

$$U_0(r) = -C_0 r^{(\sigma+2)/(\sigma+1)}(1+o(1)) \quad \text{as } r \to 0, \tag{3.12}$$

where $C_0 = (\sigma+1)N^{-1/(\sigma+1)}/(\sigma+2)$, the corresponding trajectory $\{V_0(\eta),\ \eta \in \mathbb{R}\}$ satisfies

$$V_0(\eta) = (\sigma+2)\eta - C_0 e^{\eta(\sigma+2)/(\sigma+1)}(1+o(1)) \quad \text{as } \eta \to -\infty, \tag{3.13}$$

and hence

$$H = (\sigma+2) - C_0 \frac{\sigma+2}{\sigma+1} e^{V/(\sigma+1)}(1+o(1)) \quad \text{as } V \to -\infty. \tag{3.14}$$

Notice also that $U_0' \equiv (-(\sigma+2)+H)/r < 0$ for $r > 0$, and hence we need to study equation (3.11) in the domain

$$\{V \in \mathbb{R},\ H < (\sigma+2)\}. \tag{3.15}$$

If $N < \sigma+2$ then the trajectory under consideration satisfies $H \to -\infty$ as $V \to -\infty$ and there holds

$$H \equiv V_\eta' = \frac{\sigma+2-N}{\sigma+1} V - (\sigma+2)\log|V|(1+o(1)) \quad \text{as } V \to -\infty. \tag{3.16}$$

Then (3.10) yields that

$$U_0(r) = -A_0 r^{((\sigma+2)-N)/(\sigma+1)}(1+o(1)) \quad \text{as } r \to \infty, \tag{3.17}$$

where $A_0 > 0$ is some fixed constant.

If $N = \sigma+2$, then equation (3.11) has the most simple form

$$\frac{dH}{dV} = \frac{e^V}{(\sigma+1)H[(\sigma+2)-H]^\sigma} \quad (H < \sigma+2). \tag{3.18}$$

For the semilinear case, $\sigma = 0$, it can be easily solved explicitly which yields

$$U_0(r) = -2\log[8+r^2] + 2\log 8 = -4\log r + 2\log 8(1+o(1)) \quad \text{as } r \to \infty. \tag{3.19}$$

If $\sigma > 0$, a similar explicit solution does not exist. It follows from (3.18) that there exists a constant $B_0 > 0$ such that the trajectory given by expansion (3.14) satisfies

$$H \to -B_0 > 0 \quad \text{as } V \to -\infty. \tag{3.20}$$

Using (3.10) then yields

$$U_0(r) = -(\sigma+2+B_0)\log r - D_0 r^{-B_0}(1+o(1)) \quad \text{as } r \to \infty, \tag{3.21}$$

where $D_0 > 0$ is a constant.

Finally, if $N > \sigma + 2$ then there exists the singular point on the $\{V, H\}$-plane,

$$V = \log[(N - (\sigma + 2))(\sigma + 2)^{\sigma+1}] \equiv V_*, \quad H = 0, \tag{3.22}$$

and we have that the above trajectory approaches it. Notice that this singular point corresponds to the singular stationary solution of equation (2.1):

$$U_S(r) = -(\sigma + 2)\log r + V_*, \quad r > 0. \tag{3.23}$$

This yields the following asymptotic behaviour. If

$$N \geq \frac{(\sigma + 2)(\sigma + 5)}{(\sigma + 1)}, \tag{3.24}$$

then we deduce that

$$H = \alpha(V - V_*)(1 + o(1)) \text{ as } V \to V_* \tag{3.25}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}[(\sigma + 2) - N - d^{1/2}] < 0, \\ d &= (N - (\sigma + 2))\big[N - \frac{(\sigma + 2)(\sigma + 5)}{(\sigma + 1)}\big] \geq 0. \end{aligned} \tag{3.26}$$

This implies that for some constant E_0

$$U_0(r) = -(\sigma + 2)\log r + V_* + E_0 r^{\alpha}(1 + o(1)) \text{ as } r \to \infty. \tag{3.27}$$

If

$$N < \frac{(\sigma + 2)(\sigma + 5)}{(\sigma + 1)}, \tag{3.28}$$

then $d < 0$ in (3.26) and hence α is complex. It then follows from (3.25), (3.10) that as $r \to \infty$

$$U_0(r) = -(\sigma + 2)\log r + V_* + E_0 r^{-(N-(\sigma+2))/2}\cos(\nu_0 r + F_0) + o(r^{-(N-(\sigma+2))/2}), \tag{3.29}$$

where $\nu_0 = |d|^{1/2}/2$ and E_0, F_0 are constants.

Finally we state a preliminary result concerning asymptotic properties of the family of stationary solutions $\{U(r;\lambda),\ \lambda \in \mathbb{R}\}$ to be used in the next section. It is a straightforward consequence of (3.3) and expansions (3.17), (3.21) and (3.27) or (3.29).

Proposition 3.1. (i) *If* $N \le \sigma + 2$*, then*

$$U(r;\lambda) \to 0 \quad \text{as } \lambda \to \infty, \tag{3.30}$$

and

$$U_r'(r;\lambda) \to -\infty \quad \text{as } \lambda \to \infty \tag{3.31}$$

uniformly on the set $\{r > 0 : 0 \le U(r;\lambda) \le 2M_0\}$.

(ii) *If* $N > \sigma + 2$*, then as* $\lambda \to \infty$ *uniformly on the set* $\{r \ge \delta\}$, $\delta > 0$,

$$U(r;\lambda) \to -(\sigma+2)\log r + V_*, \quad U_r'(r;\lambda) \to -\frac{(\sigma+2)}{r}\ . \tag{3.32}$$

Notice that (3.30), (3.31) imply that for $N \le \sigma + 2$ the zero point $r = r_0(\lambda) > 0$ such that $U(r_0(\lambda);\lambda) = 0$ satisfies

$$r_0(\lambda) \to 0 \quad \text{as } \lambda \to \infty \quad (N \le \sigma + 2) \tag{3.33}$$

and it follows from (3.32) that

$$r_0(\lambda) \to r_* = e^{V_*/(\sigma+2)} \quad \text{as } \lambda \to \infty \quad (N > \sigma + 2). \tag{3.34}$$

4. The proof of Theorem 1.1 by the method of stationary states. The proof of Theorem 1.1 is based on the method of intersection comparison of the solution $u(r,t)$ as $t \to T$ with the set of stationary solutions $\{U(r;\lambda)\}$ with $\lambda \gg 1$; see a similar analysis of different nonlinear parabolic problems in [24], [27], [46, Chapter VII] and also [32] where the equation

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^\beta, \quad \sigma > 0, \quad \beta > \sigma + 1, \tag{4.1}$$

has been considered.

Proof of Theorem 1.1. Consider first the case (i), $N \le \sigma + 2$. Using (3.30), (3.31) and (3.33) yields that there exists $\lambda_* > M_0$ such that for any fixed $\lambda \ge \lambda_*$ there hold: $r_0(\lambda) < R$ and $U(r;\lambda)$ intersects the function $u_0(r)$ exactly at a single point; see hypotheses (1.4). We now compare two different solutions u and U of (1.1) in $\{r < r_0(\lambda)\} \times (0,T)$. For a fixed $t \in [0,T)$ denote by $I(t;\lambda)$ the number of intersections in $r \in (0, r_0(\lambda))$ of the functions $u(r,t)$ and $U(r;\lambda)$ or, which is the same, the number of sign changes of the difference $w(r,t) = u(r,t) - U(r;\lambda)$. Since $U \equiv 0 < u$ for $r = r_0(\lambda)$, $t \in (0,T)$, the function $I(t;\lambda)$ does not increase; see [47] and references in [29], [33] and [46, Chapter IV]. For the degenerate equation (4.1) such analysis has been made in [32] by using a regularized equation. Since $I(0;\lambda) = 1$ by construction, we deduce that

$$\text{for any } \ \lambda \ge \lambda_* \quad I(t;\lambda) \le 1 \ \text{ for } t \in [0,T). \tag{4.2}$$

Proposition 4.1. *Let $N \leq \sigma + 2$. For any $\lambda > \lambda_*$ there exists a $t_\lambda \in (0, T)$ such that*

$$u(r,t) > U(r;\lambda) \quad \text{in } B_{r_0(\lambda)} \times (t_\lambda, T). \tag{4.3}$$

Proof. By (1.5) there exists $t_\lambda \in (0,T)$ such that $u(0,t_\lambda) > \lambda$. Then we conclude that

$$u(r,t_\lambda) \geq U(r;\lambda) \quad \text{in } B_{r_0(\lambda)}. \tag{4.4}$$

Indeed, if (4.4) is false and there exists $r' \in (0, r_0(\lambda))$ such that

$$u(r',t_\lambda) < U(r';\lambda), \tag{4.5}$$

then using the fact that $u(0,t_\lambda) > U(0;\lambda)$ and $u(r_0(\lambda),t_\lambda) > U(r_0(\lambda);\lambda) = 0$ we arrive at the inequality $I(t_\lambda;\lambda) \geq 2$ contradicting (4.2) with $t = t_\lambda$. Thus, (4.4) is valid, and hence (4.3) follows by comparison. □

Remark. One can see by continuity that (4.3) is valid if t_λ is such that $u(0,t_\lambda) = \lambda$. Then we deduce that

$$u(r,t) \geq U(r;u(0,t)) \quad \text{in } B_R \times (t_{\lambda_*}, T), \tag{4.6}$$

and also we conclude that

$$u(0,t) \text{ is a nondecreasing function in } (t_{\lambda_*}, T). \tag{4.7}$$

Hence, (4.7) implies that the L^∞-norm of $u(r,t)$ as $t \to T$ does not decrease. (Cf. a more general result for the one-dimensional equation given in Proposition 7.3.)

In order to finish the proof of Theorem 1.1 with $N \leq \sigma + 2$ we make a passage to the limit $t \to T^-$ in (4.3) which yields for any $r \in (0,R)$

$$u(r,T) \geq \tilde{L}(r) \equiv \sup_{\lambda > \lambda_*} U(r;\lambda). \tag{4.8}$$

Since $\tilde{L}(r) \equiv L(r)$ for small $r > 0$, by using (3.9) we arrive at (1.13).

In the case (ii), $N > \sigma + 2$, the proof is quite similar with $r_0(\lambda)$ replaced by $r_*(\lambda)$, where $r_*(\lambda)$ is the first point such that $U(r;\lambda) = u_0(r)$. Indeed, such point $r_*(\lambda) \in (0,R)$ exists for any large $\lambda > \lambda_*$. If not and hence $u_0(r) \leq U(r;\lambda)$ in $(0,R)$, then by comparison $u(r,t) \leq U(r;\lambda)$ in $B_R \times (0,T)$ contradicting (1.5). Finally, since by assumption $u(r_*(\lambda),t)$ does not decrease, we have that $u(r_*(\lambda),t) \geq U(r_*(\lambda);\lambda)$ for any $t \in (0,T)$. Therefore, the difference $w = u - U$ does not change sign on the lateral boundary of the domain $\{r < r_*(\lambda)\} \times (0,T)$. Hence (4.2) is valid, and the rest of the proof is quite analogous. □

Finally, we notice that the class of initial functions $u_0(r)$ such that $u(r,t)$ does not decrease in time is wide enough. By using an approach similar to [32] we can prove that this is true for any sufficiently smooth u_0 such that $u_0 = 0$ for $r = R$ and

$$A(u_0) \geq 0 \quad \text{in } B_R. \tag{4.9}$$

In particular, for $N > \sigma + 2$ we can take any function $u_0(r) = (U(r;\lambda))_+ \equiv \max\{0, U(r;\lambda)\}$, where $\lambda > 0$ is large enough provided that $R > r_*$; see (3.34).

5. The proof of Theorem 1.2. We now use the idea which was first given in [23] for semilinear parabolic equations of the form $u_t = \Delta u + f(u)$; see also a generalization for different quasilinear heat equations in [30], [31], [32].

For convenience we set $f(u) \equiv e^u$, $\varphi_0(p) = |p|^\sigma p$ for $p \in \mathbb{R}$ and rewrite equation (1.1) for $u = u(r,t) \geq 0$ in the form

$$u_t = r^{1-N}(r^{N-1}\varphi_0(u_r))_r + f(u) \quad \text{in } B_R \times (0,T). \tag{5.1}$$

By using the Maximum Principle we study the sign of the function

$$J(r,t) = r^{N-1}\varphi_0(u_r) + r^N F(u), \tag{5.2}$$

where $F(u)$ is a smooth nonnegative function to be determined later. Since by (2.9) equation (5.1) is not degenerate in $(B_R\backslash\{0\}) \times (0,T)$ we can calculate the explicit form of the parabolic equation satisfied by function (5.2). (Another technical approach has been used in [32] in the Cauchy problem for (4.1) where the final result has been proved by a regularization method and the passage to the limit.) We now state the main estimate.

Proposition 5.1. *Assume that* (1.4) *and* (1.15) *hold. Then there exist constants* $A \in (0,1)$ *and* $\alpha > 0$ *small enough such that*

$$|u_r|^\sigma u_r + r\frac{\sigma}{2+\sigma(1+N)}e^u(1 - Ae^{-\alpha u}) \leq 0 \quad \text{in } B_R \times (0,T). \tag{5.3}$$

Proof. The function J given in (5.2) solves in $B_R \times (0,T)$ the following parabolic equation:

$$\begin{aligned} J_t = \varphi_0' J_{rr} - \frac{(N-1)}{r}\varphi_0' J_r + [(\sigma+1)(f' - (N+1)F') + (N-1)F']J \\ - r^N F'' \varphi_0' (u_r)^2 + r^N q(u), \end{aligned} \tag{5.4}$$

where

$$q(u) = \{([2+\sigma(N+1)]FF' + F'f - (\sigma+1)Ff')\}(u). \tag{5.5}$$

Assume now for a moment that the function $F(u)$ satisfying

$$F, F', F'' \geq 0 \quad \text{for } u \geq 0, \quad F(0) < f(0)/N, \tag{5.6}$$

has been chosen so that

$$q(u) \leq 0 \quad \text{for } u > 0. \tag{5.7}$$

Then if follows from (5.4) that $J(r,t)$ satisfies in $(B_R\backslash\{0\}) \times (0,T)$ the following parabolic differential inequality:

$$J_t \leq \varphi_0' J_{rr} - \frac{(N-1)}{r}\varphi_0' J_r + [(\sigma+1)(f' - (N+1)F') + (N-1)F']J, \tag{5.8}$$

and

$$J(0,t) = 0 \quad \text{for } t \in (0,T). \tag{5.9}$$

Since $F(0) < f(0)/N$, we have that by (2.9)

$$\begin{aligned} J_r|_{r=R} &= (r^{N-1}\varphi_0)_r + NR^{N-1}F(0) + R^N F'(0)u_r \\ &\le (r^{N-1}\varphi_0)_r + NR^{N-1}F(0) = R^{N-1}[NF(0) - f(0)] < 0. \end{aligned} \tag{5.10}$$

Using now regularity properties of $u(r,t)$ and the continuity of $J(r,t)$ in $\bar{B}_R \times (0,T)$, we conclude that by the Maximum Principle

$$J(r,t) \le 0 \quad \text{in } B_R \times (0,T) \tag{5.11}$$

provided that

$$J(r,0) \equiv r^{N-1}[\varphi_0(u_0') + rF(u_0)] \le 0 \quad \text{in } (0,R). \tag{5.12}$$

Consider now inequality (5.7) with $f(u) = e^u$. Set

$$F(u) = \rho(u)e^u. \tag{5.13}$$

Then the function $\rho(u)$ has to satisfy

$$\rho'(1+\gamma\rho) + \rho(\gamma\rho - \sigma) \le 0 \quad \text{for } u > 0, \tag{5.14}$$

where $\gamma = 2 + \sigma(1+N)$. Let

$$\rho(u) = \frac{\sigma}{\gamma}(1 - Ae^{-\alpha u}), \quad \alpha > 0, \quad A > 0. \tag{5.15}$$

Then (5.14) is valid if

$$\alpha(\sigma+1) - \sigma \le \sigma(\alpha - 1)Ae^{-\alpha u} \quad \text{for } u > 0.$$

This yields

$$A \in (0,1), \quad \alpha \in \big(0, \ \frac{\alpha(1-A)}{\sigma + 1 - A\sigma}\big] \ . \tag{5.16}$$

The inequality $F(0) < f(0)/N = 1/N$ implies that $F(0) = \rho(0) = \sigma(1-A)/\gamma < 1/N$ which is true if $A \in (0,1)$. Finally, consider inequality (5.12) for the initial function. Using (5.13)–(5.15) yields

$$|u_0'|^\sigma u_0' + r\frac{\sigma}{\gamma}(1 - Ae^{-\alpha u_0})e^{u_0} \le 0 \quad \text{in } (0,R). \tag{5.17}$$

By choosing $\alpha > 0$ small enough and $A < 1$ to be close to 1 so that (5.16) is valid we deduce that (5.17) holds. Finally, we notice that (5.6) is valid provided that (5.16) holds.

Thus, (5.8)–(5.10) and (5.17) imply that function (5.2) with $F(u)$ given by (5.13), (5.15), (5.16) satisfies (5.11) which completes the proof of Proposition 5.1. □

Proof of Theorem 1.2. It follows from (5.3) that if for some small $r' > 0$ and $t \in (0, T)$ the function $u(r', t)$ is large enough, then

$$w(r,t) \equiv |u_r|^\sigma u_r + \mu r e^u \leq 0 \quad \text{for any } r \in [0, r'], \tag{5.18}$$

where

$$\mu = \frac{\sigma}{2 + \sigma(2+N)} < \frac{\sigma}{2 + \sigma(1+N)}. \tag{5.19}$$

Hence

$$e^{-u(r,t)/(\sigma+1)} u_r(r,t) \leq -(\mu r)^{1/(\sigma+1)} \quad \text{for } r \in [0, r']. \tag{5.20}$$

Integrating this inequality over $(0, r)$, $r \leq r'$, yields

$$u(r,t) \leq -(\sigma+1) \log\big[\frac{\mu^{1/(\sigma+1)}}{(\sigma+2)} r^{(\sigma+2)/(\sigma+1)} + e^{-u(0,t)/(\sigma+1)}\big] \tag{5.21}$$

and hence for any $r > 0$ small enough (1.16) is valid. □

6. Bounds on the L^∞ norm: Proof of Theorems 1.3 and 1.4.

6.1. Proof of Theorem 1.3. The lower bound (1.18) is a direct consequence of Proposition 5.1. Indeed, using (5.18) and the fact that $w(0, t) = 0$ we conclude that as $t \to T$

$$w_r \equiv (|u_r|^\sigma u_r)_r + \mu e^u + \mu r e^u u_r \leq 0 \quad \text{for } r = 0. \tag{6.1}$$

It follows from equation (5.1) and regularity properties of the solution $u(r, t)$ at the origin that for $r = 0$

$$u_t = N(|u_r|^\sigma u_r)_r + e^u. \tag{6.2}$$

Substituting $(|u_r|^\sigma u_r)$ given by (6.2) into (6.1) we obtain the ordinary differential inequality

$$u_t \leq (1 - \mu N) e^u \quad \text{for } r = 0 \quad \text{as } t \to T. \tag{6.3}$$

By integrating this inequality over (t, T) we arrive at (1.18) with $M_- = -\log(1 - \mu N)$ which completes the proof. □

It is interesting to point out that formal asymptotic calculations reported in [14] show that the estimates obtained in Theorems 1.2 and 1.3 are sharp in the limit of small σ. Thus the method of proof of these estimates gives a good insight into the form of the solution. We now look at the upper bound—here the estimates are not explicit!

6.2. Preliminary analysis of the self-similar profiles. The proof of Theorem 1.4 is based on the method of intersection comparison of the solution $u(r, t)$ with some particular self-similar solutions of the form (1.20) having the same blow-up time T. Fix arbitrary $\mu > 0$ and consider the self-similar solution (1.20)

$$u_*(|x|, t; \mu) = -\log(T - t) + \theta(\xi; \mu), \tag{6.4}$$

where the function $\theta(\xi; \mu)$ satisfies (1.21), (1.22) and also

$$\theta(0; \mu) = \mu. \tag{6.5}$$

We notice firstly that by standard local analysis for any fixed $\mu > 0$ there exists a unique weak local-in-ξ solution $\theta(\xi; \mu)$ satisfying near the origin

$$\theta(\xi; \mu) = \mu - \frac{\sigma + 1}{(\sigma + 2)N}[(e^\mu - 1)/N]^{1/(\sigma+1)}\xi^{(\sigma+2)/(\sigma+1)}(1 + o(1)) \tag{6.6}$$

as $\xi \to 0$. By local regularity results for degenerate ordinary differential equations we have that $\theta \in C^1$ and $\theta \in C^\infty$ at any point of nondegeneracy where $\theta' \neq 0$. It can be easily seen that the weak solution θ is extended to whole half-line $\xi \in \mathbb{R}_+$. Indeed, let us rewrite equation (1.21) in the form

$$(\xi^{N-1}|\theta'|^\sigma\theta')' = \frac{1}{\sigma + 2}\theta'\xi^N + \xi^{N-1}(1 - e^\theta) \quad \text{for } \xi > 0. \tag{6.7}$$

Fix an arbitrary compact set $K = [\xi_1, \xi_2] \subset \bar{\mathbb{R}}_+$. Assume without loss of generality that $V = \theta'$ has a constant sign on K. Then by (6.7) we have that

$$(\xi^{N-1}|V|^{\sigma+1})' \leq \frac{1}{\sigma + 2}|V|\xi^N + C_1\xi^{N-1},$$

where constants $C_i > 0$ do not depend on ξ_2. Therefore, if $\xi_1 \gg 1$ and $|V| \gg 1$ by setting $\xi^{N-1}|V|^{\sigma+1} = U$ we deduce that

$$U' \leq C_2 U^{1/(\sigma+1)}\xi^{(1+N\sigma)/(\sigma+1)}.$$

Integrating over K yields the estimate

$$U^{\sigma/(\sigma+1)} \leq C_3(1 + \xi_2^{(1+N\sigma)/(\sigma+1)+1})$$

which implies global existence of the solution θ.

It follows from (6.7) that if $\xi = \xi_1 > 0$ is a local strict maximum (minimum) of $\theta(\xi)$, i.e., $\theta' = 0$ and $(|\theta'|^\sigma\theta')' < 0$ for $\xi = \xi_1$ (respectively $(|\theta'|^\sigma\theta')' > 0$), then $\theta(\xi_1) > 0$ ($\theta(\xi_1) < 0$).

Since (1.21) is a degenerate equation, we do not have automatically the uniqueness of the solution $\theta(\xi; \mu)$ on a given compact subset $K = [0, \xi_1]$. By using the structure of the right-hand side of equation (6.7) and local uniqueness results, we conclude that $\theta(\xi; \mu)$ is the unique solution on K if

$$\theta^2 + {\theta'}^2 \neq 0 \quad \text{on } K \tag{6.8}$$

(cf. [28] and also [26]). We now prove that (6.8) is valid provided that $\mu > 0$.

Proposition 6.1. *Let $\theta(\xi)$ be an arbitrary weak solution to* (1.21) *on* $[0, \xi_*]$ *such that* $\theta(\xi_*) = \theta'(\xi_*) = 0$. *Then* $\theta \equiv 0$ *on* $[0, \xi_*]$.

Proof. Assume for a contradiction that $\theta(\xi) \neq 0$ in a small left neighbourhood of $\xi = \xi_*$ and more exactly there exists a sequence $\{\xi_k\}$ such that $\xi_k \to \xi_*$, $\{\theta(\xi_k)\}$ decreases and $\theta(\xi_k) = \sup_{(\xi_k,\xi_*)} \theta(\xi)$. Then we may also assume that $\theta'(\xi_k) < 0$. Integrating (6.7) over (ξ_k, ξ_*) yields

$$\frac{1}{\sigma+2}\theta(\xi_k)\xi_k^N + \int_{\xi_k}^{\xi_*} \xi^{N-1}\big[\frac{N\theta}{\sigma+2} + e^{\theta} - 1\big]\, d\xi = \xi_k^{N-1}|\theta'(\xi_k)|^{\sigma}\theta'(\xi_k). \tag{6.9}$$

By the assumptions we have that as $k \to \infty$

$$\int_{\xi_k}^{\xi_*} \xi^{N-1}\big[\frac{N\theta}{\sigma+2} + e^{\theta} - 1\big]\, d\xi \geq -C\theta(\xi_k)(\xi_* - \xi_k) \tag{6.10}$$

where $C \geq 0$ is a constant. Then (6.9) yields

$$\frac{1}{\sigma+2}\theta(\xi_k)\xi_k^N - C\theta(\xi_k)(\xi_* - \xi_k) \leq \xi_k^{N-1}|\theta'(\xi_k)|^{\sigma}\theta'(\xi_k), \tag{6.11}$$

whence the contradiction, since for large k the left- and the right-hand sides of (6.11) have different signs. $\square$

It follows from Proposition 6.1 that if (6.8) is not valid on K then $\theta \equiv 0$ in a small neighbourhood of the origin which contradicts the assumption $\mu > 0$. Thus, we conclude that on any compact subset K the solution $\theta(\xi;\mu)$ is unique and both $\theta(\xi;\mu)$ and $\theta'_\xi(\xi;\mu)$ depend continuously on μ on K.

We now study the behaviour of $\theta(\xi;\mu)$ for large $\mu > 0$.

Proposition 6.2. *As* $\mu \to \infty$

$$\theta(\xi;\mu) = \mu + U_0(\xi e^{\mu/(\sigma+2)}) + o(1) \tag{6.12}$$

uniformly on compact subsets $\{0 \leq \xi \leq ce^{-\mu/(\sigma+2)}\}$, $c > 0$.

Proof. The rescaled function

$$V_\mu(\eta) = -\mu + \theta(\eta e^{-\mu/(\sigma+2)};\mu), \quad \eta > 0,$$

satisfies the following problem:

$$\eta^{1-N}(\eta^{N-1}|V'_\mu|^{\sigma}V'_\mu)' + e^{V_\mu} = e^{-\mu}\big[1 + \frac{1}{\sigma+2}V'_\mu\eta\big], \quad \eta > 0, \tag{6.13}$$

$$V'_\mu(0) = 0, \quad V_\mu(0) = 0. \tag{6.14}$$

Hence (6.12) is the direct consequence of the continuous dependence of the solution to the stationary equation (cf. (3.1)) upon small for $\mu \gg 1$ perturbations of the right-hand side of (6.13); see also [28]. $\square$

Set $\xi_\mu = e^{-\mu/(\sigma+2)}$. Then by Proposition 6.2 we have that as $\mu \to \infty$

$$\theta(\xi_\mu;\mu) = \mu + U_0(1) + o(1) < \mu \tag{6.15}$$

and

$$\theta'(\xi_\mu;\mu) < -a_0 e^{\mu/(\sigma+2)}, \quad a_0 = -\frac{1}{2}U_0'(1) > 0. \tag{6.16}$$

Denote by $\xi_0 = \xi_0(\mu)$ the first vanishing point of $\theta(\xi;\mu)$. Then $\theta' < 0$ on $(0,\xi_0)$ and hence by (6.7) we have that

$$(\xi^{N-1}|\theta'|^\sigma\theta')' \le 0 \quad \text{on } (0,\xi_0). \tag{6.17}$$

Integrating (6.17) over (ξ_μ,ξ), $\xi \in (\xi_\mu,\xi_0)$, and using (6.16) yield

$$\xi^{N-1}|\theta'|^\sigma\theta' \le -a_0^{\sigma+1}e^{\mu(\sigma+2-N)/(\sigma+2)}$$

and hence

$$\theta' \le -a_0 e^{\mu(\sigma+2-N)/(\sigma+1)(\sigma+2)}\xi^{(1-N)/(\sigma+1)}. \tag{6.18}$$

Integrating again, we conclude by using (6.15) that if $N < \sigma+2$, then

$$\theta(\xi;\mu) \le \mu - b_0 e^{\mu(\sigma+2-N)/(\sigma+1)(\sigma+2)}[\xi^{(\sigma+2-N)/(\sigma+1)} - \xi_\mu^{(\sigma+2-N)/(\sigma+1)}], \tag{6.19}$$

where $b_0 = a_0\frac{\sigma+1}{\sigma+2-N}$. If follows from (6.19) that as $\mu \to \infty$

$$\xi_0(\mu) < \xi_*(\mu) \equiv \left[\frac{2\mu}{b_0}\right]^{(\sigma+1)/(\sigma+2-N)} e^{-\mu/(\sigma+2)} \to 0 \quad (N < \sigma+2). \tag{6.20}$$

We now study the behaviour of $\theta(\xi;\mu)$ with $\mu \gg 1$ as $\xi \to \infty$. We begin with introducing the family of strict weak lower solutions to equation (1.21).

Proposition 6.3. *Let $N < \sigma + 2$. Fix arbitrary*

$$A_0 > A_* = \frac{\sigma(\sigma+2)+N}{(\sigma+2-N)} - \log\frac{(\sigma+1)(\sigma+2)}{\sigma+2-N}\ . \tag{6.21}$$

Then for any given constant $\alpha > 0$ the function

$$g_\alpha(\xi) = A_0 - \alpha\xi^{(\sigma+2-N)/(\sigma+1)} \tag{6.22}$$

satisfies

$$\mathbf{B}(g_\alpha(\xi)) > 0 \quad \textit{for } \xi > 0. \tag{6.23}$$

Proof. Since the heat diffusion operator vanishes identically on functions (6.22) we have that $\mathbf{B}(g_\alpha(\xi)) = \delta_0 z + e^{A_0 - z} - 1 \equiv G(z)$, where $z = \alpha\xi^{(\sigma+2-N)/(\sigma+1)}$, $\delta_0 = (\sigma+2-N)/(\sigma+1)(\sigma+2)$. It is easily seen that $z_* = A_0 - \log\delta_0$ is the point of absolute minimum of $G(z)$ and hence $G(z) > 0$ provided that $G(z_*) = \delta_0[A_0 + 1 - \log\delta_0] - 1 > 0$ (cf. (6.21)), whence the result. □

We now state the main result about comparison of an arbitrary solution $\theta(\xi)$ to equation (1.21) with the family $\{g_\alpha\}$ of lower solutions.

Lemma 6.4. *Fix arbitrary $\alpha > 0$. Assume that $\theta(\xi)$ intersects $g_{\alpha_0}(\xi)$ at a point $\xi = \xi_1 > 0$ so that*

$$\theta(\xi_1) = g_{\alpha_0}(\xi_1) \quad \text{and} \quad \theta'(\xi_1) < g'_{\alpha_0}(\xi_1). \tag{6.24}$$

Then

$$\theta(\xi) < g_{\alpha_0}(\xi) \quad \text{for any } \xi > \xi_1 \tag{6.25}$$

and

$$\theta'(\xi) \le -\frac{\sigma + 2 - N}{\sigma + 1} \, \frac{\theta(\xi) - A_0}{\xi} \quad \text{for } \xi > \xi_1. \tag{6.26}$$

Proof. Assume for the contradiction that (6.25) is false and hence there exists $\xi_2 > \xi_1$ such that

$$\theta(\xi_2) \ge g_{\alpha_0}(\xi_2). \tag{6.27}$$

Since the family $\{g_\alpha\}$ satisfies

$$\begin{aligned} &g_\alpha(\xi) \;\text{ is decreasing in } \alpha \;\text{ for } \xi \in \mathbb{R}_+, \\ &g_\alpha(\xi) \to -\infty \;\text{ as } \alpha \to \infty \;\text{ on any compact } K = [\xi_1, \xi_2], \quad \xi_1 > 0, \end{aligned} \tag{6.28}$$

there exists

$$\alpha_* = \inf\{\alpha > 0 : g_\alpha(\xi) \le \theta(\xi) \;\text{ on } [\xi_1, \xi_2]\}. \tag{6.29}$$

Then by (6.24), (6.27) we have that $\alpha_* > \alpha_0$ and by definition of α_*

$$g_{\alpha_*}(\xi) \le \theta(\xi) \;\text{ on } [\xi_1, \xi_2], \tag{6.30}$$

and there exists $\xi_* \in (\xi_1, \xi_2)$ such that

$$g_{\alpha_*} = \theta, \; g'_{\alpha_*} = \theta' \;\text{ for } \xi = \xi_*. \tag{6.31}$$

Since $g'_{\alpha_*}(\xi_*) \ne 0$, the second derivatives $g''_{\alpha_*}(\xi_*)$, $\theta''(\xi_*)$ are well-defined and by (6.30), (6.31)

$$g''_{\alpha_*} \le \theta'' \;\text{ for } \xi = \xi_*. \tag{6.32}$$

Hence, (6.31), (6.32) and (6.23) imply that for $\xi = \xi_*$

$$\mathbf{B}(\theta) \ge \mathbf{B}(g_{\alpha_*}) > 0 \tag{6.33}$$

contradicting the fact that θ is a solution of (1.21). The proof of (6.26) is quite similar since this inequality means that $\theta' \le g'_\alpha$ if $\theta = g_\alpha$ with arbitrary $\alpha > \alpha_0$. Indeed if on the contrary $\theta'(\xi_2) > g'_\alpha(\xi_2)$ at some point $\xi = \xi_2$, where $\theta = g_\alpha$, then we arrive at the situation which has been considered above. □

We have proved estimates of $\theta(\xi; \mu)$ with $\mu \gg 1$ on small compact subsets in ξ near the origin; see (6.15) and (6.19). By using Lemma 6.4 we now give an upper estimate for large ξ.

Proposition 6.5. *For* $\mu \gg 1$ *there holds*

$$\theta(\xi;\mu) \le g_{\alpha_*(\mu)}(\xi) \quad \textit{for} \;\; \xi \ge \xi_*(\mu), \tag{6.34}$$

where

$$\alpha_*(\mu) = A_0(\xi_*(\mu))^{-(\sigma+2-N)/(\sigma+1)} \sim \mu^{-1} e^{\mu(\sigma+2-N)/(\sigma+1)(\sigma+2)}. \tag{6.35}$$

Proof. By choosing $\alpha_*(\mu)$ from the equality $g_\alpha(\xi_*) = 0$ which indeed yields (6.35), it follows from (6.19), (6.20) that $\theta(\xi;\mu)$ intersects $g_{\alpha_*}(\xi)$ within the interval $(0, \xi_*(\mu))$. By Lemma 6.4 this yields (6.34). $\square$

Finally, we consider the self-similar solution (6.4) for large μ. It follows from the estimates of $\theta(\xi;\mu)$ given by (6.15), (6.19) and (6.34) that

$$u_*(r,t;\mu) < -\log(T-t) + \mu \;\; \text{for} \;\; r \le r_\mu(t) = \xi_\mu (T-t)^{1/(\sigma+2)}, \tag{6.36}$$

$$u_*(r,t;\mu) \le -\log(T-t) + \mu\big[1 - \big[\frac{r}{r_*(t)}\big]^{(\sigma+2-N)/(\sigma+1)}\big] \tag{6.37}$$

for

$$r_\mu(t) \le r \le r_*(t) = \xi_*(\mu)(T-t)^{1/(\sigma+2)}$$

and

$$u_*(r,t;\mu) < -\log(T-t) + A_0 - \alpha_*(\mu)(T-t)^{-(\sigma+2-N)/(\sigma+1)(\sigma+2)} r^{(\sigma+2-N)/(\sigma+1)} \tag{6.38}$$

if $r \ge r_*(t)$. Using these estimates together with corresponding estimates of $(u_*)_r$ following from (6.16), (6.18) and (6.26), we arrive at the uniform estimates.

Proposition 6.6. *For any given small* $\delta > 0$ *as* $\mu \to \infty$

$$u_*(x,t;\mu) \to -\infty \;\; \textit{uniformly on} \;\; [\delta, R] \times [0,T) \tag{6.39}$$

and

$$(u_*)_r(x,0;\mu) \to -\infty \;\; \textit{uniformly on} \;\; \{r \in (0,R) : 0 \le u_*(r,0;\mu) \le 2M_0\}. \tag{6.40}$$

6.3. Proof of Theorem 1.4: intersection comparison with self-similar solutions. The idea of the proof is similar to that used in Section 4. We compare $u(r,t)$ with the self-similar solution $u_*(r,t;\mu)$, $\mu \gg 1$, having the same blow-up time. As in Section 4, we introduce the number of intersections $I(t;\mu)$, $t \in [0,T)$, of these solutions. It follows from (6.39) and (6.40) with $t = 0$, that $u_0(0) < u_*(0,0;\mu)$ and $I(0;\mu) = 1$ provided that μ is large enough. Since by (6.39) with $r = R$

$$u_*(R,t;\mu) < 0 \equiv u(R,t) \;\; \text{for} \;\; t \in [0,T), \tag{6.41}$$

by the Maximum Principle (see some details in [32]) we have that

$$I(t;\mu) \le 1 \;\; \text{for} \;\; t \in [0,T), \tag{6.42}$$

cf. (4.2). We now prove the following result.

Proposition 6.7. *If $\mu \gg 1$, then for any $t \in (0,T)$*

$$\sup_{r\in(0,R)} u(r,t) \equiv u(0,t) \le \sup_{r\in(0,R)} u_*(r,t;\mu) \equiv u_*(0,t;\mu). \tag{6.43}$$

Proof. Assume for a contradiction that (6.43) is false and there exists $t_0 \in (0,T)$ such that $u(0,t_0) > u_*(0,t_0;\mu)$. This leads to the contradiction of the fact that both the solutions $u(r,t)$ and $u_*(r,t;\mu)$ have the same blow-up time T. Indeed (6.41) and (6.42) with $t = t_0$ yield that $I(t_0;\mu) = 0$ and hence $u(r,t_0) \ge u_*(r,t_0;\mu)$ in $[0,R]$. Moreover by the Strong Maximum Principle ([22]), applied to the parabolic equation for the difference $w = u - u_*$ in the domain $[\delta,R]\times(t_0-\delta,t_0+\delta)$ ($\delta > 0$ is a small constant) where $|u_r|$ and $|(u_*)_r|$ are bounded away from zero, we may conclude that $u(r,t_0) > u_*(r,t_0;\mu)$ in $[0,R]$. Then by (6.41) and by the continuity of u_* we have that $u(r,t_0) \ge u_*(r,t_0+\tau;\mu)$ provided that $\tau > 0$ is chosen small enough. Hence $u(r,t) \ge u_*(r,t+\tau;\mu)$ in $[0,R]\times(t_0,T)$. Setting here $t = T-\tau$ yields $u(r,T-\tau) \ge u_*(r,T;\mu)$ in $[0,R]$ whence the contradiction. □

Since $u_*(0,t;\mu) = -\log(T-t)+\mu$, (6.43) yields (1.19) with $M_+ = \mu$. This completes the proof of Theorem 1.4. □

7. Asymptotic self-similar behaviour in one-dimensional problem: Proof of Theorem 1.5. In this section we use all of the earlier estimates to prove that in one dimension the self-similar solution is globally attracting for problems with monotone initial data. The method of proof that we use is based upon the construction of an approximate Lyapunov function derived from that used in the regularised problem.

We first rewrite the one-dimensional problem (1.24)–(1.26) satisfied by the function (1.24) in the form

$$v_\tau = \mathbf{B}(v) \equiv a(v_\xi)v_{\xi\xi} + b(\xi,v,v_\xi) \;\text{ in }\; \omega_0 = B_{\ell(\tau)} \times (\tau_0,\infty), \tag{7.1}$$

$$v(\xi,\tau) = -\tau \;\text{ for }\; |\xi| = \ell(\tau),\; \tau \ge \tau_0, \tag{7.2}$$

$$v(\xi,\tau_0) = \bar{v}_0(\xi) \;\text{ in }\; B_{\ell(\tau_0)}, \tag{7.3}$$

where $B_{\ell(\tau)} = \{|\xi| < \ell(\tau) = Re^{\tau/(\sigma+2)}\}$ and

$$a(w) = (\sigma+1)|w|^\sigma, \tag{7.4}$$

$$b(\xi,v,w) = -\frac{1}{\sigma+2}\xi w + e^v - 1. \tag{7.5}$$

7.1. First estimates. We begin with some upper and lower estimates of the solution $v(\xi,\tau)$.

Proposition 7.1. *Let $N = 1$ and assume that* (1.4), (1.15) *hold. Then*

$$v_\xi < 0 \;\; \textit{in} \;\; (0,\ell(\tau))\times(\tau_0,\infty), \tag{7.6}$$

$$-\tau \le v(\xi,\tau) \le M_+ \quad in \quad \omega_0, \tag{7.7}$$

$$v(0,\tau) \ge M_- > 0 \quad for \quad \tau > \tau_*, \tag{7.8}$$

where $\tau_* \gg \tau_0$ *is some constant,*

$$v(\xi,\tau) \le -(\sigma+2)\log\xi + \bar{C}_+ \quad in \quad \omega_0, \tag{7.9}$$

where $\bar{C}_+ \ge C_+$ *is a large enough constant, and*

$$v(\xi,\tau) \ge M_- - \nu\xi^{(\sigma+2)/(\sigma+1)}, \quad \nu = \frac{\sigma+1}{\sigma+2}N^{-1/(\sigma+1)}e^{M_+/(\sigma+1)} \quad in\ B_{\ell(\tau)}\ for\ \tau > \tau_*. \tag{7.10}$$

Proof. (7.6) follows from (1.23) and (2.9). Estimates (7.7) are a straightforward consequence of (7.6) and Theorem 1.4, and (7.8) is the result of Theorem 1.3. (7.9) follows from (1.23) and Theorem 1.2. In order to prove the lower estimate (7.10) we use in the estimate (4.6), where the stationary profile $U(r;\lambda)$ is given by (3.3), namely,

$$u(r,t) \ge u(0,t) + U_0(re^{u(0,t)/(\sigma+2)}) \quad \text{in} \quad B_R \times (t_{\lambda_*}, T), \tag{7.11}$$

the lower and the upper (in the second term in the right-hand side) bounds of $u(0,t)$ from Theorems 1.3 and 1.4. Then we deduce that

$$u(r,t) \ge -\log(T-t) + M_- + U_0(r(T-t)^{-1/(\sigma+2)}e^{M_+/(\sigma+2)}). \tag{7.12}$$

Using estimate (3.7) yields (7.10). (Notice that (7.10) is not sharp and we will prove another better estimate.) $\square$

We need also some uniform estimates of the derivatives. The function $z = v_\xi$ solves the following quasilinear equation with the diffusion operator of the porous medium type:

$$z_\tau = (|z|^\sigma z)_{\xi\xi} - \frac{1}{\sigma+2}z_\xi\xi + a(\xi,\tau)z \quad \text{for} \quad \xi \in (0,\ell(\tau)),\ \tau > \tau_0, \tag{7.13}$$

where the coefficient $a(\xi,\tau) = e^v - 1/(\sigma+2)$ is uniformly bounded by (7.7). We have that

$$z(0,\tau) = 0 \quad \text{for} \quad \tau \ge \tau_0. \tag{7.14}$$

We now estimate z at the boundary point $\xi = \ell(\tau)$. Since by Theorem 1.2 the solution $u(r,t)$ is uniformly bounded in $[R-\delta,R]\times[0,T)$, $\delta \in (0,R)$ is a constant, we have that by the Bernstein-type estimates $|u_r| \le C_1$ there, see references in [38] (we now denote by $C_i > 0$ different constants which do not depend on τ). Evidently, by (1.23) this implies that

$$|v_\xi| \equiv |z| \le C_1e^{-\tau/(\sigma+2)} \quad \text{near} \quad \xi = \ell(\tau) \quad \text{for} \quad \tau \ge \tau_0. \tag{7.15}$$

Using again Bernstein-type techniques as in [3] on equation (7.13) with boundary estimates (7.14), (7.15), we arrive at the uniform bound

$$-C_2 \le v_\xi \le 0 \quad \text{in} \quad \omega_0. \tag{7.16}$$

This estimate implies that the ω-limit set given by (1.27), $N = 1$, is well-defined. Since $|u_r|$ is also bounded away from zero near $r = R$ as $t \to T$ and hence equation (1.1) is uniformly parabolic with bounded smooth coefficients, we have that $|u_{rr}| < C_3$ there. Hence, $|(|z|^\sigma z)_\xi| \le C_3 e^{-\tau}$ near the boundary point $\xi = \ell(\tau)$ for large $\tau \gg \tau_0$. By a known regularity result for equations having porous medium operator, see [3], [15] and also [38], we conclude that the heat flux $(|z|^\sigma z)_\xi$ corresponding to equation (7.13) is uniformly bounded which yields the following bound:

$$|(|v_\xi|^\sigma v_\xi)_\xi| \le C_4 \quad \text{in} \quad B_{\ell(\tau)} \quad \text{for} \quad \tau > \tau_*. \tag{7.17}$$

It then follows from equation (7.1) and estimates (7.7) and (7.17) that there exists a constant $C_5 > 0$ such that

$$|v_\tau| \le C_5(1 + |\xi|) \quad \text{in} \quad B_{\ell(\tau)} \quad \text{for} \quad \tau > \tau_*. \tag{7.18}$$

We can prove more exact estimates of $|v_\tau|$ on the boundary. Since $u_t(R, t) \equiv 0$ and by (1.23)

$$u_t \equiv (T - t)^{-1}\big[v_\tau + \frac{1}{\sigma + 2} v_\xi \xi + 1\big], \tag{7.19}$$

we have that $|v_\tau| = \big|1 + \frac{1}{\sigma+2} v_\xi \xi\big|$ on the boundary and hence by (7.15) we have that

$$|v_\tau| \le 1 + \frac{1}{\sigma + 2} C_1 R = C_6 \quad \text{near } \xi = \ell(\tau) \quad \text{for} \quad \tau > \tau_0. \tag{7.20}$$

Finally, we prove some important estimates concerning sharp lower bounds of any profile from the ω-limit set, $\omega(v_0)$.

Lemma 7.2. *As $\tau \to \infty$*

$$v_\tau + \frac{1}{\sigma + 2} v_\xi \xi + 1 \ge 0 \quad \textit{on any compact subset.} \tag{7.21}$$

Lemma 7.2 is a direct consequence of the following result about monotonicity in time of an arbitrary large solution of a quasilinear heat equation with source.

Proposition 7.3. *Let $N = 1$. Assume that* (1.4) *holds. Then there exists a constant $M_k \ge M_0$ depending on σ, M_0 and M_1 such that if*

$$u(r_0, t_0) > M_k \quad \textit{at some point} \quad (r_0, t_0) \in Q_T, \tag{7.22}$$

then

$$u(r_0, t) \quad \textit{does not decrease for} \quad t \in [t_0, T]. \tag{7.23}$$

Proof of Proposition 7.3. First, we notice that for $r_0 = 0$ this result has been proved in Section 4; cf. (4.7). For $r_0 > 0$ (7.23) is proved by the technique [29] and [33] for general quasilinear equations of the form $u_t = (\varphi(u))_{xx} + Q(u)$. In fact, using this approach based on the intersection comparison of $u(r,t)$ with the two-parametric family of stationary solutions $\{U(|x-a|;\lambda),\ \lambda > 0,\ a \in \mathbb{R}\}$ (see Section 3), we do not need to study first the regularized equation (2.5), $N = 1$, having a classical solution $u_\epsilon(r,t)$ for which (7.23) can be written as follows:

$$(u_\epsilon)_t \geq 0 \quad \text{for} \quad t \in [t_0, T). \tag{7.24}$$

(Then (7.23) follows from (7.24) by passage to the limit $\epsilon \to 0$.) Indeed, by the regularity given in (2.11) we can make the direct proof of (7.23) with $r_0 > 0$, which is the same as those which have been used for uniformly parabolic equations in [29], [33]. □

Proof of Lemma 7.2. (7.21) is the immediate consequence of (7.23). Indeed, it follows from (1.5) that there exist $r_0 > 0$ and $t_0 \in (0,T)$ such that $u(r_0,t_0) > M_k$. Then $u(r,t_0) > M_k$ for any $r \in [0,r_0]$ and hence by Proposition 7.3 and (2.11) we have that

$$u_t \geq 0 \quad \text{in} \quad (0,r_0) \times (t_0,T) \tag{7.25}$$

(more exactly, $u_t > 0$ by the Strong Maximum Principle). Then (7.21) follows from (7.25) and (7.19). □

7.2. An approximate Lyapunov function. Since problem (7.1)–(7.3) is stated in a domain with unsteady moving boundaries, we cannot construct a Lyapunov function which is exactly nonincreasing on the evolution trajectory $\{v(\cdot,\tau),\ \tau > \tau_0\}$. We now prove the existence of some approximate Lyapunov function by using the general idea for the uniformly parabolic equation ([50]).

Regularized problem estimates. We begin with introducing the regularized version of equation (7.1). For a fixed small $\epsilon > 0$ denote by $v_\epsilon(\xi,\tau)$ the classical solution of the problem (7.1)–(7.5) for the uniformly parabolic equation with the coefficient $a(v_\xi)$ replaced by

$$a_\epsilon(w) = (\sigma+1)(\epsilon^2 + |w|^2)^{\sigma/2}. \tag{7.26}$$

Then by general regularity results (see [15], [16] and [38]) we may conclude that

$$v_\epsilon \to v,\ (v_\epsilon)_\xi \to v_\xi \quad \text{as} \quad \epsilon \to 0 \tag{7.27}$$

uniformly on any compact subset of ω_0 and also

$$(v_\epsilon)_\tau \to v_\tau \quad \text{as} \quad \epsilon \to \infty \tag{7.28}$$

in $L^2_{\text{loc}}(\omega_0)$. Since $v_\xi < 0$ in $(0,\ell(\tau)) \times (\tau_0,\infty)$, see (2.9), we may also deduce that for any $\xi \in (0,\ell(\tau))$, $\tau > \tau_0$, the function $v_\epsilon(\xi,\tau)$ converges to $v(\xi,\tau)$ as $\epsilon \to 0$ with derivatives.

Fix $\tau_* \gg 1$ large enough so that (7.6)–(7.10) and (7.15)–(7.18), (7.20) are valid. Take an arbitrary large $S > \tau_*$. It then follows from (7.29) and known regularity results that for any $\epsilon > 0$ small enough

$$-\tau \le v_\epsilon \le 2M_+ \quad \text{in} \quad q_* = B_{\ell(\tau)} \times [\tau_*, S], \tag{7.29}$$

$$v_\epsilon(0,\tau) \ge M_-/2 > 0 \quad \text{in} \quad [\tau_*, S], \tag{7.30}$$

$$v_\epsilon \le -(\sigma+2)\log\xi + 2\bar{C}_+ \quad \text{in} \quad q_*, \tag{7.31}$$

$$v_\epsilon \ge M_-/2 - \nu\xi^{(\sigma+2)/(\sigma+1)} \quad \text{in} \quad q_*, \tag{7.32}$$

$$-2C_2 \le (v_\epsilon)_\xi \le 0 \quad \text{in} \quad q_*, \tag{7.33}$$

$$|(v_\epsilon)_\xi| \le 2C_1 e^{-\tau/(\sigma+2)} \quad \text{near } \xi = \ell(\tau) \quad \text{for} \quad \tau \in [\tau_*, S], \tag{7.34}$$

$$|(|(v_\epsilon)_\xi|^\sigma (v_\epsilon)_\xi)_\xi| \le 2C_4 \quad \text{in} \quad q_*, \tag{7.35}$$

$$|(v_\epsilon)_\tau| \le 2C_5(1+|\xi|) \quad \text{in} \quad q_*, \tag{7.36}$$

$$|(v_\epsilon)_\tau| \le 2C_6 \quad \text{near} \quad \xi = \ell(\tau) \quad \text{for} \quad \tau \in [\tau_*, S]. \tag{7.37}$$

An approximate Lyapunov function for the regularized problem. For a given $\xi_0 > 0$, v_0, w_0 denote by $\varphi_\epsilon(\xi_0, \xi, v_0, w_0)$ the unique classical solution to the nonlinear ordinary differential equation

$$a_\epsilon((\varphi_\epsilon)')(\varphi_\epsilon)'' + b(\xi, \varphi_\epsilon, (\varphi_\epsilon)') = 0 \tag{7.38}$$

where primes denote differentiation with respect to ξ, either for $\xi \in [0, \xi_0]$ or for $\xi \in [\xi_0, \infty)$, with boundary conditions

$$\varphi_\epsilon|_{\xi=\xi_0} = v_0, \quad (\varphi_\epsilon)'_\xi|_{\xi=\xi_0} = w_0. \tag{7.39}$$

As in Section 6, we prove that φ_ϵ is well-defined for any $\xi \ge 0$. For arbitrary $(\xi, v, w) \in \mathbb{R}^3_+ = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ we introduce two C^∞ functions:

$$\rho_\epsilon(\xi, v, w) = \exp\big\{-\gamma \int_0^\xi \zeta[\epsilon^2 + (\varphi'_\epsilon(0, \zeta, v_0, w_0))^2]^{-\sigma/2}\, d\zeta\big\}, \tag{7.40}$$

where $v_0 = \varphi_\epsilon(\xi, 0, v, w)$, $w_0 = \varphi'_\epsilon(\xi, 0, v, w)$, $\gamma = 1/(\sigma+1)(\sigma+2) > 0$,

$$\Phi_\epsilon(\xi, v, w) = \int_0^w (w-\eta) a_\epsilon(\eta) \rho_\epsilon(\xi, v, \eta)\, d\eta + \int_v^{-1} \rho_\epsilon(\xi, \mu, 0)(e^\mu - 1)\, d\mu. \tag{7.41}$$

It is easily seen that the function $\rho_\epsilon(\xi, v, w)$ solves the first-order equation (see [26])

$$b(\rho_\epsilon)'_w - a_\epsilon w(\rho_\epsilon)'_v - a_\epsilon(\rho_\epsilon)'_\xi = -\rho_\epsilon b'_w, \tag{7.42}$$

and both the functions ρ_ϵ and Φ_ϵ satisfy

$$\rho_\epsilon a_\epsilon \equiv (\Phi_\epsilon)''_{ww}, \;\; \rho_\epsilon b \equiv -(\Phi_\epsilon)'_v + (\Phi_\epsilon)''_{w\xi} + w(\Phi_\epsilon)''_{wv}. \tag{7.43}$$

We now introduce the approximate Lyapunov function

$$L_\epsilon[v_\epsilon](\tau) = \int_0^{\ell(\tau)} \Phi_\epsilon(\xi, v_\epsilon, (v_\epsilon)_\xi)\, d\xi, \tag{7.44}$$

which will be shown to be "almost" nonincreasing on the evolution trajectory $\{v_\epsilon(\cdot,\tau),\, \tau \in (s_*, S)\}$ with $S > \tau_* \gg 1$.

We need some estimates of ρ_ϵ and Φ_ϵ. First, one can see from (7.40) that

$$0 < \rho_\epsilon(\xi, v, w) < 1 \tag{7.45}$$

and hence by (7.41)

$$\Phi_\epsilon(\xi, v, w) \le (\sigma+1)(\epsilon^2 + w^2)^{\sigma/2}\frac{w^2}{2} + \Big|\int_v^{-1} \rho_\epsilon(\xi, \mu, 0)(e^\mu - 1)\, d\mu\Big|\,. \tag{7.46}$$

We have also from (7.39) that

$$|(\Phi_\epsilon)_w| \le \Big|\int_0^w a_\epsilon(\eta)\, d\eta\Big| \le (\sigma+1)(\epsilon^2 + w^2)^{\sigma/2}|w| \tag{7.47}$$

and

$$\Phi_\epsilon \ge \int_v^{-1} \rho_\epsilon(\xi, \mu, 0)(e^\mu - 1)\, d\mu. \tag{7.48}$$

Properties of the approximate Lyapunov function for the degenerate equation. We now consider the function $\rho(\xi, v, w)$ and $\Phi(\xi, v, w)$ which formally correspond to (7.40), (7.41) with $\epsilon = 0$. Denote by $\varphi(\xi_0, \xi, v_0, w_0)$, $\xi_0 > 0$, the weak solution to the equation

$$a_0(\varphi')\varphi'' + b(\xi, \varphi, \varphi') = 0 \;\text{ for }\; \xi \in [0, \xi_0) \tag{7.49}$$

with boundary conditions (7.39). By Proposition 6.1 this solution exists and is unique on $[0, \xi_0]$ for any $\xi_0 > 0, v_0, w_0$. By standard results for ordinary differential equations we have that $\varphi(\xi_0, \xi, v_0, w_0)$ is smooth enough with respect to the parameters (ξ_0, v_0, w_0) if

$$\varphi^2 + (\varphi')^2 \ne 0 \;\text{ on }\; [0, \xi_0]; \tag{7.50}$$

i.e., there is no point of strong degeneracy of equation (7.49) on $[0, \xi_0]$; see details in [26] and [28]. In this case we automatically conclude that under assumption (7.50)

$$\varphi_\epsilon \to \varphi,\; \varphi_\epsilon' \to \varphi' \;\text{ as }\; \epsilon \to 0 \;\text{ uniformly on }\; [0, \xi_0]. \tag{7.51}$$

It is interesting to note that by the uniqueness result given in Proposition 6.1 we have the following.

Proposition 7.4. *For any fixed* ξ_0, v_0, w_0 (7.51) *is valid.*

Proof. We need to consider the case where (7.50) does not hold, i.e., there exists $\xi_* \in [0, \xi_0]$ such that

$$\varphi^2(\xi_*) + (\varphi'(\xi_*))^2 = 0. \tag{7.52}$$

We assume that $\xi_* \in (0, \xi_0)$. Then $\varphi = \varphi' \equiv 0$ on $[0, \xi_*]$ by Proposition 6.1. We have shown that for arbitrarily small $\delta > 0$

$$\varphi_\epsilon(\xi_* + \delta) \to \varphi(\xi + \delta),\ \varphi_\epsilon'(\xi_* + \delta) \to \varphi_\epsilon'(\xi_* + \delta) \text{ as } \epsilon \to 0. \tag{7.53}$$

On the other hand, we have that

$$\varphi_\epsilon \to 0,\ \varphi_\epsilon' \to 0 \text{ as } \epsilon \to 0 \text{ uniformly on } [0, \xi_*]. \tag{7.54}$$

Indeed, if (7.54) is false and hence by compactness argument we have that for some sequence $\{\epsilon_k\} \to, 0\ \varphi_\epsilon \to \varphi,\ \varphi_\epsilon', \varphi_\epsilon' \to \bar{\varphi}'$, as $\epsilon = \epsilon_k \to 0$ and $\bar{\varphi} \not\equiv 0$ on $[0, \xi_*]$. Using the fact that $\bar{\varphi}(\xi_*) = 0,\ \bar{\varphi}'(\xi_*) = 0$ by continuity and (7.53) we then arrive at the contradiction of Proposition 6.1. Hence, the result follows from (7.51), (7.53) and continuity of φ, φ' at $\xi = \xi_*$. The analysis of the case $\xi_* = 0$ is quite similar. □

For arbitrary $\xi \geq 0, v, w$ we now introduce the functions (cf. (7.40) and (7.41))

$$\varphi(\xi, v, w) = \exp\big\{-\gamma \int_0^\xi \zeta |\varphi'(\xi, \zeta, v, w)|^{-\sigma}\, d\zeta\big\}, \tag{7.55}$$

$$\Phi(\xi, v, w) = (\sigma + 1) \int_0^w (w - \eta)|\eta|^\sigma \rho(\xi, v, \eta)\, d\zeta + \int_v^{-1} \rho(\xi, \mu, 0)(e^\mu - 1)\, d\mu. \tag{7.56}$$

One can see from (7.55) that (cf. (7.45) and (7.46))

$$0 \leq \rho(\xi, v, w) < 1, \quad \Phi(\xi, v, w) \leq \frac{1}{\sigma + 2}|w|^{\sigma+2} + \Big|\int_v^{-1} \rho(\xi, \mu, 0)(e^\mu - 1)\, d\mu\Big|, \tag{7.57}$$

and

$$\rho(\xi_0, v_0, w_0) = 0 \tag{7.58}$$

if and only if the integral

$$P(\xi_0, v_0, w_0) = \int_0^{\xi_0} \zeta |\varphi'(\xi_0, \zeta, v_0, w_0)|^{-\sigma}\, d\zeta \text{ diverges .} \tag{7.59}$$

It is easily seen that $P(\xi_0, v_0, w_0) < \infty$ if (7.50) holds. Indeed, (7.59) can be true if $\varphi'(\zeta) \equiv \varphi_\zeta'(\xi_0, \zeta, v_0, w_0)$ vanishes on $[0, \xi_0]$. If $\varphi'(\zeta_*) = 0$ for some $\zeta_* \in [0, \xi_0]$ but $\mu_* = \varphi(\zeta_*) \neq 0$, for instance, $\mu_* > 0$, then by the local analysis of equation (7.49) near $\zeta = \zeta_*$ we have that if $\zeta_* > 0$ (cf. (6.6)),

$$\varphi(\zeta) = \mu_* - \frac{\sigma + 1}{\sigma + 2}(e^{\mu_*} - 1)^{1/(\sigma+1)}|\zeta_* - \zeta|^{(\sigma+2)/(\sigma+1)}(1 + o(1)), \tag{7.60}$$
$$\varphi'(\zeta) = -(e^{\mu_*} - 1)^{1/(\sigma+1)}|\zeta_* - \zeta|^{1/(\sigma+1)}\mathrm{sign}(\zeta_* - \zeta)(1 + o(1)) \text{ as } \zeta \to \zeta_*.$$

Substituting (7.60) into the integral in (7.59) shows that it is finite.

On the other hand, the following result is valid.

Proposition 7.5. *There exists the maximal set* $B_* \subset \mathbb{R}^3_+$ *such that*

$$\rho(\xi, v, w) = 0 \quad \textit{if} \quad (\xi, v, w) \in B_*, \tag{7.61}$$

and

$$\rho > 0 \quad \textit{in} \quad G_* = \mathbb{R}^3_+ \backslash B_*. \tag{7.62}$$

Proof. We have shown that $\rho(\xi_0, v_0, w_0) > 0$ if (7.50) is valid for the function $\varphi(\xi_0, \zeta, v_0, w_0)$. We now prove that

$$B_* = \{(\xi_0, v_0, w_0) \in \mathbb{R}^3_+ : \text{ (7.50) is not valid}\}. \tag{7.63}$$

Assume that (7.50) is false for $\zeta = \zeta_* \in [0, \xi_0]$, $\varphi \not\equiv 0$ in a small right neighbourhood of $\zeta = \zeta_*$ and $\varphi(\zeta_*) = \varphi'(\zeta_*) = 0$. If $\zeta_* > 0$ then $\varphi \equiv 0$ on $[0, \zeta_*]$ by Proposition 6.1. By a local analysis near $\zeta = \zeta_*$ it is easy to show that there exist exactly two such nontrivial trajectories $\varphi = \varphi_*(\zeta)$ or $\varphi = -\varphi_*(\zeta)$ for $\zeta > \zeta_*$ with the behaviour as $\zeta \to \zeta_*$

$$\begin{aligned} \varphi_*(\zeta) &= (\sigma+2)^{-1/\sigma} \Big[\frac{\sigma}{\sigma+1}\Big]^{(\sigma+1)/\sigma} \zeta_*^{1/\sigma} (\zeta - \zeta_*)_+^{(\sigma+1)/\sigma} (1 + o(1)), \\ \varphi_*'(\zeta) &= (\sigma+2)^{-1/\sigma} \Big[\frac{\sigma}{\sigma+1}\Big]^{1/\sigma} \zeta_*^{1/\sigma} (\zeta - \zeta_*)_+^{1/\sigma} (1 + o(1)). \end{aligned} \tag{7.64}$$

One can see that for functions $\varphi = \pm\varphi_*$ satisfying (7.64) with some $\zeta_* \in (0, \xi_0]$, (7.59) is valid and hence (7.61) follows by (7.54). Thus we have that for any $\zeta_* > 0$

$$\{(\xi, v, w) \in \mathbb{R}^3_+ : \xi > \zeta_*, \ v = \pm\varphi_*(\xi), \ w = \pm\varphi_*'(\xi)\} \subseteq B_*;$$

i.e., the "bad" set B_* is at least two-dimensional. (In fact, B_* is more complicated).

In order to finish the proof of (7.63), consider the case $\zeta_* = 0$, i.e., $\varphi \neq 0$ near $\zeta = 0$ and $\varphi(0) = \varphi'(0) = 0$. Then integrating equation (7.49),

$$(|\varphi'|^\sigma \varphi')' = \frac{1}{\sigma+2} \varphi' \xi + 1 - e^\varphi \equiv \frac{1}{\sigma+2} \varphi' \xi - \varphi(1 + o(1)) \quad \text{as} \quad \xi \to 0$$

over a small interval $(0, \xi)$ yields

$$|\varphi'|^\sigma \varphi' = \frac{1}{\sigma+2} \varphi \xi - \frac{\sigma+1}{\sigma+2} \int_0^\xi \varphi(\eta)(1 + o(1))\, d\eta$$

and hence for small $\xi > 0$

$$|\varphi'|^{\sigma+1} \le \frac{1}{\sigma+2} |\varphi| \xi + \frac{\sigma+1}{\sigma+2} \int_0^\xi |\varphi(\eta)| \, |1 + o(1)| \, d\eta. \tag{7.65}$$

Therefore, by comparison $|\varphi(\xi)| \le Y(\xi)$, where $Y > 0$, $Y' > 0$ for small $\xi > 0$ satisfies $(Y')^{\sigma+1} = 2Y\xi$; i.e., $Y = \text{const}\, \xi^{(\sigma+2)/\sigma}$. Finally, we have $|\varphi'(\xi)| \le \text{const}\, \xi^{2/\sigma}$ as $\xi \to 0$ and hence (7.59) is valid again. $\square$

It follows from (7.63) that the “good” set G_* given in (7.62) can be defined as follows:

$$G_* = \{(\xi_0, v_0, w_0) \in \mathbb{R}^3_+ : (7.50) \text{ is valid}\}. \tag{7.66}$$

Then by the continuous dependence of φ' on parameters $(\xi_0, v_0, w_0) \in G_*$ we deduce that G_* is open. By standard regularity results we have that

$$\rho \;\text{ and }\; \Phi \;\text{ are smooth on }\; G_*. \tag{7.67}$$

Hence, by construction, these functions satisfy (cf. (7.42) and (7.43))

$$b\rho_w - aw\rho_w - a\rho_\xi = -\rho b_w \;\text{ on }\; G_*, \tag{7.68}$$

and

$$\rho a = \Phi_{ww}, \;\; \rho b = -\Phi_v + \Phi_{w\xi} + w\Phi_{wv} \;\text{ on }\; G_*. \tag{7.69}$$

Regularity of ρ and Φ on the bad set B_* is unknown. By the uniqueness result given in Proposition 6.1 we have only the following.

Proposition 7.6. *The functions ρ and Φ are continuous in $\bar{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R}$.*

Proof. Fix arbitrary $(\xi_0, v_0, w_0) \in B_*$. Then by (7.63) there exists $\xi_* \in [0, \xi_0]$ such that (7.52) holds and hence $\varphi(\xi_0) \equiv \varphi(\xi_0, \xi_*, v_0, w_0) = \varphi'(\xi_*) = 0$ and hence $\varphi = \varphi' \equiv 0$ on $[0, \xi_*]$ by Proposition 6.1. The rest of the proof is quite similar to the proof of Proposition 7.4 with the limit $\epsilon \to 0$ replaced by $(\xi, v, w) \to (\xi_0, v_0, w_0)$. $\square$

Since the functions ρ and Φ are expected to be nonsmooth on the set B_* wide enough we cannot use directly the approximate Lyapunov function

$$L[v](\tau) = \int_0^{\ell(\tau)} \Phi(\xi, v, v_\xi)\, d\xi \tag{7.70}$$

corresponding to the problem (7.1)–(7.3) for the degenerate equation. Therefore we first consider the approximate Lyapunov function (7.44) for the regularized problem, and the final result will be proved by the limit $\epsilon \to 0$. It follows from the existence of the bad set that we cannot expect that $\rho_\epsilon \to \rho$ and $\Phi_\epsilon \to \Phi$ as $\epsilon \to 0$ on $\bar{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R}$. We now prove the following.

Proposition 7.7. *There holds*

$$\rho_\epsilon \to \rho \;\;\textit{as}\;\; \epsilon \to 0 \;\;\textit{on}\;\; G_*. \tag{7.71}$$

Proof. Take arbitrary $(\xi_0, v_0, w_0) \in G_*,\; \xi_0 > 0$. Compare functions (7.40) and (7.55). We have by (7.51) that

$$[\epsilon^2 + (\varphi'_\epsilon)]^{-\sigma/2} \to |\varphi'|^{-\sigma} \;\text{ as }\; \epsilon \to 0 \tag{7.72}$$

uniformly on any subset where $|\varphi'|$ is uniformly bounded away from zero. Fix an arbitrary point $\zeta_* \in [0, \xi_0]$ such that $\varphi'(\zeta_*) = 0$, $\varphi(\zeta_*) = \mu_* \neq 0$. Assume without loss of generality that $\zeta_* \in (0, \xi_0)$, $\mu_* > 0$, and consider the integral over a small neighbourhood $(\zeta_* - \lambda,\ \zeta_* + \lambda)$, $\lambda > 0$, $\lambda \ll 1$

$$I_0(\zeta_*;\lambda) \equiv \int_{\zeta_*-\lambda}^{\zeta_*+\lambda} \zeta|\varphi'(\zeta)|^{-\sigma}\,d\zeta = \zeta_*(1+o(1)) \int_{\zeta_*-\lambda}^{\zeta_*+\lambda} |\varphi'(\zeta)|^{-\sigma}\,d\zeta \tag{7.73}$$

as $\lambda \to 0$, which is finite; see (7.60). It follows from (7.51) that for small $\epsilon > 0$ φ_ϵ' vanishes at some point $\zeta = \zeta_\epsilon$ such that

$$\zeta_\epsilon \to \zeta_*,\ \mu_\epsilon \equiv \varphi_\epsilon(\zeta_\epsilon) \to \mu_* \ \text{ as } \ \epsilon \to 0. \tag{7.74}$$

Consider the corresponding integral

$$\begin{aligned} I_\epsilon(\zeta_\epsilon;\lambda) &\equiv \int_{\zeta_\epsilon-\lambda}^{\zeta_\epsilon+\lambda} \zeta[\epsilon^2 + (\varphi_\epsilon'(\zeta))^2]^{-\sigma/2}\,d\zeta \\ &= \zeta_\epsilon(1+o(1)) \int_{\zeta_\epsilon-\lambda}^{\zeta_\epsilon+\lambda} [\epsilon^2 + (\varphi_\epsilon'(\zeta))^2]^{-\sigma/2}\,d\zeta \end{aligned} \tag{7.75}$$

as $\lambda \to 0$. By a local analysis of equation (7.38) near the point $\zeta = \zeta_\epsilon$,

$$(\sigma+1)(\epsilon^2 + (\varphi_\epsilon')^2)^{\sigma/2}\varphi_\epsilon'' = -C_\epsilon + o(1) \ \text{ as } \ \zeta \to \zeta_\epsilon,\ C_\epsilon = e^{\mu_\epsilon} - 1,$$

we have that $F_\epsilon(\varphi_\epsilon') = -C_\epsilon(\zeta - \zeta_\epsilon)(1+o(1))$, where

$$F_\epsilon(z) = (\sigma+1)\int_0^z (\epsilon^2 + s^2)^{\sigma/2}\,ds.$$

Using the fact that $F_\epsilon(z) \equiv \epsilon^{\sigma+1}G(z/\epsilon)$, where

$$G(\eta) = (\sigma+1)\int_0^\eta (1+\eta^2)^{\sigma/2}\,d\eta = |\eta|^\sigma \eta(1+o(1)) \ \text{ as } \ \eta \to \infty \tag{7.76}$$

we finally obtain that as $\zeta \to \zeta_\epsilon$

$$(\varphi_\epsilon'(\xi))^2 = \epsilon^2[G^{-1}(C_\epsilon \epsilon^{-(\sigma+1)}|\zeta - \zeta_\epsilon|(1+o(1)))]^2. \tag{7.77}$$

Substituting (7.77) into (7.75) and making the transformation in the integral,

$$C_\epsilon \epsilon^{-(\sigma+1)}(\zeta - \zeta_\epsilon) = y,$$

yields that for $\lambda \ll 1$

$$I_\epsilon(\zeta_\epsilon;\lambda) \equiv \zeta_\epsilon C_\epsilon(1+o(1))\epsilon \int_{-\lambda C_\epsilon \epsilon^{-(\sigma+1)}}^{\lambda C_\epsilon \epsilon^{-(\sigma+1)}} [1 + (G^{-1}(|y|(1+o(1))))^2]^{-\sigma/2}\,dy, \tag{7.78}$$

and hence using the expansion given in (7.76), $G^{-1}(|y|) = |y|^{1/(\sigma+1)}(1 + o(1))$ as $|y| \to \infty$, we deduce that

$$I_\epsilon(\zeta_\epsilon; \lambda) = 2(\sigma+1)C_\epsilon^{-\sigma/(\sigma+1)}\lambda^{1/(\sigma+1)}\zeta_\epsilon(1 + o(1)) \tag{7.79}$$

provided that ϵ and λ are small enough. By substituting expansion (7.60) into (7.73) and by using (7.74) we conclude that

$$I_0(\zeta_*; \lambda) = I_\epsilon(\zeta_\epsilon; \lambda)(1 + o(1)) \tag{7.80}$$

as $\epsilon \to 0$ and $\lambda \to 0$. It then follows from (7.72) and local result (7.80) that $P_\epsilon(\xi_0, v_0, w_0) \to P(\xi_0, v_0, w_0)$ as $\epsilon \to 0$ in G_*, and

$$P_\epsilon(\xi_0, v_0, w_0) \equiv \int_0^{\xi_0} \zeta[\epsilon^2 + (\varphi_\epsilon'(\zeta))^2]^{-\sigma/2}\, d\zeta \to P(\xi_0, v_0, w_0) \tag{7.81}$$

as $\epsilon \to 0$ in G_*, and hence by (7.42) and (7.56), (7.71) is valid, whence the result. $\square$

Finally, we prove the following upper bound of ρ_ϵ.

Proposition 7.8. *There exist constants $C_* > 0$ and $\bar{\xi}_0 \gg 1$ such that*

$$\varphi_\epsilon(\xi_0, v_0, 0) \le e^{-C_*\xi_0} \tag{7.82}$$

on any subset

$$\xi_0 \ge \bar{\xi}_0, \ \ -(\sigma+2)\log\frac{\xi_0}{R} \le v_0 - 1, \quad \epsilon > 0 \quad \textit{is small.} \tag{7.83}$$

Proof. Fix arbitrary ξ_0, v_0 and ϵ satisfying (7.83), and consider the integral given in (7.81) with $w_0 = 0$ where φ_ϵ is the solution to (7.38), (7.39). Setting $z(\zeta) = \varphi_\epsilon'(\zeta)$ we obtain from the equation (7.38) that

$$a_\epsilon(z)z' = \frac{1}{\sigma+2}z\zeta + 1 - e^{\varphi_\epsilon} \quad \text{for} \quad \zeta < \zeta_0. \tag{7.84}$$

Since $z(\xi_0) = w_0 = 0$ and hence $z < 0$ in a small left neighbourhood of the point $\zeta = \xi_0$ we have from (7.84) $a_\epsilon(z)z' \le 1$. Integrating over (ζ, ξ_0) yields

$$\int_z^0 a_\epsilon(s)\, ds \le (\xi_0 - \zeta)$$

and since $\int_z^0 a_\epsilon(s)\, ds \ge \int_z^0 a_0(s)\, ds = -|z|^\sigma z$ if $z \le 0$, we deduce that

$$|\varphi_\epsilon'(\zeta)| \le (\xi_0 - \zeta)^{1/(\sigma+1)}. \tag{7.85}$$

Since $v_0 \le -1$, we may conclude that (7.85) is valid on some left neighbourhood $(\xi_0 - \delta_0, \xi_0)$ with small $\delta_0 > 0$ which does not depend on ξ_0, v_0 and small ϵ. Thus, we have from (7.81) and (7.85)

$$P_\epsilon(\xi_0, v_0, 0) \ge (\xi_0 - \delta_0) \int_{\xi_0-\delta_0}^{\xi_0} [\epsilon^2 + (\xi_0 - \zeta)^{2/(\sigma+1)}]^{-\sigma/2} \, d\zeta.$$

Transformation $(\xi_0 - \zeta)/\epsilon^{\sigma+1} = y$ in the above integral yields

$$\begin{aligned} P_\epsilon(\xi_0, v_0, 0) &\ge (\xi_0 - \delta_0)\epsilon \int_0^{\delta_0 \epsilon^{-(\sigma+1)}} (1 + y^{2/(\sigma+1)})^{-\sigma/2} \, dy \\ &= (\xi_0 - \delta_0)(\sigma+1)\delta_0^{1/(\sigma+1)}(1 + o(1)) \ge \frac{\sigma+1}{2}\delta_0^{1/(\sigma+1)}\xi_0 \end{aligned} \tag{7.86}$$

as $\epsilon \to 0$ provided that (7.83) is valid. Then (7.86) yields (7.82) with $C_* = (\sigma + 1)\delta_0^{1/(\sigma+1)}/2$.

7.3. The main estimate. We now prove a uniform estimate which plays a key role in future studies of the structure of the ω-limit set.

Lemma 7.9. *For any $S > \tau_* \gg 1$ as $\epsilon \to 0$*

$$\int_{\tau_*}^{s} d\tau \int_0^{\ell(\tau)} \rho_\epsilon(\xi, v_\epsilon, (v_\epsilon)_\xi)(\partial_\tau v_\epsilon)^2 \, d\xi \le C_7, \tag{7.87}$$

where the constant $C_7 > 0$ does not depend on S and ϵ.

Proof. By differentiating the approximate Lyapunov function (7.44) we have that

$$\begin{aligned} &-\frac{\partial}{\partial\tau} L_\epsilon[v_\epsilon](\tau) = -\int_0^{\ell(\tau)} [(\Phi_\epsilon)_v (v_\epsilon)_\tau + (\Phi_\epsilon)_w (v_\epsilon)_{\xi\tau}] \\ &- \Phi(\xi, v_\epsilon, (v_\epsilon)_\xi)|_{\xi=\ell(\tau)} \ell'(\tau) \equiv N_1(\tau) + n_1(\tau). \end{aligned} \tag{7.88}$$

By using the upper estimate (7.46) and (7.2) we have that

$$|n_1(\tau)| \le \ell_1(\tau) + \ell_2(\tau), \tag{7.89}$$

where by the boundary estimate (7.34)

$$\ell_1(\tau) = 2\frac{(\sigma+2)}{(\sigma+1)} C_1^2 R e^{-\tau/(\sigma+2)} (\epsilon^2 + 4C_1^2 e^{-2\tau/(\sigma+2)})^{\sigma/2} \le C_8 e^{-\tau/(\sigma+2)}$$

(by $C_i > 0$ we now will denote different constants which are independent of S and ϵ) and

$$\ell_2(\tau) = \Big| \int_{-\tau}^{1} \rho_\epsilon(\ell(\tau), \mu, 0)(e^\mu - 1) \, d\mu \Big|.$$

Since $\tau \geq \tau_* \gg 1$ we may suppose that $\ell(\tau) > \bar{\xi}_0$ and hence by Proposition 7.8 there holds

$$\rho_\epsilon(\ell(\tau), \mu, 0) \leq e^{-C_* Re^{\tau/(\sigma+2)}} \quad \text{for any } \tau \in [\tau_*, S]$$

and $\mu \in [-\tau, -1]$. Thus we have that $\ell_1(\tau) \leq C_9 e^{-\tau/(\sigma+2)}$, and (7.89) yields

$$|n_1(\tau)| \leq C_{10} e^{-\tau/(\sigma+2)}. \tag{7.90}$$

Integrating by parts in the integral term given in (7.88) yields

$$\begin{aligned} N_1(\tau) &= -\int_0^{\ell(\tau)} [(\Phi_\epsilon)_v - \frac{d}{d\xi}(\Phi_\epsilon)_w](v_\epsilon)_\tau \, d\xi - (\Phi_\epsilon)_w (v_\epsilon)_\tau \big|_0^{\ell(\tau)} \\ &\equiv N_2(\tau) + n_2(\tau). \end{aligned} \tag{7.91}$$

Since $(\Phi_\epsilon)_w = 0$ if $v_\xi = 0$, see (7.47), we have from the boundary estimates (7.34) and (7.37) that

$$|n_2(\tau)| = |(\Phi_\epsilon)_w (v_\epsilon)_\tau|\big|_{\xi=\ell} \leq C_{11} e^{-\tau/(\sigma+2)}. \tag{7.92}$$

One can see from (7.91) that

$$N_2(\tau) = \int_0^{\ell(\tau)} [-(\Phi_\epsilon)_v + (\Phi_\epsilon)_{w\xi} + (\Phi_\epsilon)_{wv}(v_\epsilon)_\xi + (\Phi_\epsilon)_{ww}(v_\epsilon)_{\xi\xi}](v_\epsilon)_\tau \, d\xi,$$

and hence by the definition of the functions ρ_ϵ and Φ_ϵ (see identities (7.43)) we deduce that

$$N_2(\tau) = \int_0^{\ell(\tau)} (\rho_\epsilon b + \rho_\epsilon a_\epsilon (v_\epsilon)_{\xi\xi})(v_\epsilon)_\tau \, d\xi \equiv \int_0^{\ell(\tau)} \rho_\epsilon(\xi, v, (v_\epsilon)_\xi)(\partial_\tau v_\epsilon)^2 \, d\xi. \tag{7.93}$$

Finally, it follows from (7.88)–(7.93) that

$$-\frac{\partial}{\partial \tau} L_\epsilon[v_\epsilon](\tau) \geq \int_0^{\ell(\tau)} \rho_\epsilon (\partial_\tau v_\epsilon)^2 \, d\xi + C_{12} e^{-\tau/(\sigma+2)}.$$

Integrating this inequality over (τ_*, S) yields

$$\int_{\tau_*}^S d\tau \int_0^{\ell(\tau)} \rho_\epsilon (\partial_\tau v_\epsilon)^2 \, d\xi + L_\epsilon[v_\epsilon](S) \leq C_{13} + L_\epsilon[v_\epsilon](\tau_*) \leq C_{14}. \tag{7.94}$$

We now prove the uniform estimate of $L_\epsilon[v_\epsilon](S)$ from below. Since $\ell(S) \gg \bar{\xi}_0$, we can write

$$L_\epsilon[v_\epsilon](S) = \int_0^{\bar{\xi}_0} \Phi_\epsilon \, d\xi + \int_{\bar{\xi}_0}^{\ell(S)} \Phi_\epsilon d\xi \equiv I_1 + I_2.$$

Then $I_1 \geq -C_{15}$ by (7.48) and the lower bound (7.32). As for I_2, we have by Proposition 7.8 and (7.48), (7.32)

$$I_2 \geq -\int_{\bar{\xi}_0}^{\ell(S)} e^{-C_*\xi}|v_\epsilon(\xi,\tau)|\,d\xi \geq -\nu\int_{\bar{\xi}_0}^{\ell(S)} e^{-C_*\xi}\xi^{(\sigma+2)/(\sigma+1)}\,d\xi \geq -C_{16}.$$

Hence, for any $S > \tau_*$

$$L_\epsilon[v_\epsilon](S) \geq -C_{17}, \tag{7.95}$$

and then (7.94) yields estimate (7.87) which completes the proof.

7.4. A local variant of Theorem 1.5. Fix an arbitrary $L > 0$. We may assume that $\ell(\tau_*) \gg L$ and hence by Lemma 7.9 we have that

$$\int_{\tau_*}^{S} d\tau \int_0^L \rho_\epsilon(\xi, v_\epsilon, (v_\epsilon)_\xi)(\partial_\tau v_\epsilon)^2\,d\xi < C_7, \tag{7.96}$$

where $C_7 > 0$ is independent of L and S.

We now assume that L is small enough so that $v = v_\epsilon$ and $w = (v_\epsilon)_\xi$ satisfies in $[0, L] \times [\tau_*, S]$ (see (7.29)–(7.33))

$$\xi \in [0, L], \quad 0 < \frac{M_-}{4} \leq v \leq 2M_+, \quad -2C_2 \leq w \leq 0. \tag{7.97}$$

We denote this set of (ξ, v, w) by $\mathcal{M}_L \subset \bar{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R}$.

Proposition 7.10. *There holds*

$$\mathcal{M}_L \subset G_* \quad \text{for small} \quad L > 0. \tag{7.98}$$

Proof. One can see using equation (7.38) that $\varphi_\epsilon > 0$ for any $0 \leq \xi_0 \leq L \ll 1$ and for boundary data (7.41) satisfying (7.97). Hence (7.50) is valid and (7.98) follows from (7.62). □

Thus, we can make a passage to the limit $\epsilon \to 0$ in (7.96), whence

Proposition 7.11. *For any small $L > 0$*

$$\int_{\tau_*}^{S} d\tau \int_0^L (v_\tau)^2 d\xi \leq C_{18}, \tag{7.99}$$

where the constant $C_{18} > 0$ does not depend on S.

Proof. Using (7.98), (7.71), (7.67) and (7.27), (7.28) in passing to the limit in (7.96) as $\epsilon \to 0$ we deduce that

$$\int_{\tau_*}^{S} d\tau \int_0^L \rho(\xi, v, v_\xi)(v_\tau)^2 d\xi \leq C_{19}.$$

Since the set $\mathcal{M}_L$ is closed, by (7.98), (7.67) and (7.62) there exists a positive constant $\rho_0 > 0$ such that $\rho \geq \rho_0$ in $\mathcal{M}_L$, which yields (7.99). □

We now state the local (in ξ) variant of Theorem 1.5.

Lemma 7.12. *For any small $L > 0$*

$$\omega(\bar v_0) \subset \mathcal{W}_S(L) \equiv \big\{g \in C^1, g' \le 0 : \mathbf{B}(g) = 0 \quad on \quad [0, L], \\ g'(0) = 0, \quad g(0) \in \big[\frac{M_-}{4}, 2M_+\big]\big\}. \tag{7.100}$$

Proof. (7.100) is the direct consequence of (7.99); see general approach [41] and a similar analysis in [26].

7.5. Continuation in L, Proof of Theorem 1.5. Denote

$$L_* = \sup\{L > 0 : \ (7.100) \text{ holds}\}. \tag{7.101}$$

Then $L_* > 0$ by Lemma 7.12. Evidently, if $L_* = \infty$, then (1.30) is valid. We now assume for a contradiction that

$$L_* < \infty. \tag{7.102}$$

Then by regularity

$$\omega(\bar v_0) \subseteq \mathcal{W}_S(L_*). \tag{7.103}$$

We now give a simple important property of arbitrary stationary profiles from $\mathcal{W}_S(L_*)$.

Proposition 7.13. *For an arbitrary $g \in \mathcal{W}_S(L_*)$ and arbitrary $\xi_0 \in [0, L_*]$ there exists a neighbourhood $Q(R_0) \subset \mathbb{R}^3_+$ of the point $R_0 = (\xi_0, g(\xi_0), g'(\xi_0))$ such that*

$$Q(R_0) \subset G_*. \tag{7.104}$$

Proof. By the definition of $\mathcal{W}_S(L_*)$, (7.67) and Proposition 6.1 we have that $R_0 \subset G_*$. Since G_* is open, (7.104) follows. $\square$

Notice that since an arbitrary $g \in \mathcal{W}_S(L_*)$ satisfies $g' \le 0$ and $g(0) > 0$, we have that $g'(\xi) < 0$ for $\xi > 0$ and hence $g \in C^\infty$ for $\xi > 0$.

Proof of Theorem 1.5. We first prove that the function

$$\rho_*(\tau; L_*) \equiv \inf_{\xi \in [0, L_*]} \rho(\xi, v(\xi, \tau), v_\xi(\xi, \tau)) \tag{7.105}$$

satisfies

$$\lim \inf_{\tau \to \infty} \rho_*(\tau; L_*) = C_{20} > 0. \tag{7.106}$$

Indeed, assume for a contradiction that there exist sequences $\{\xi_k\} \subset [0, L_*]$ and $\{\tau_k\} \to \infty$ such that

$$\rho(\xi_k, v(\xi_k, \tau_k), v_\xi(\xi_k, \tau_k)) \to 0 \quad \text{as} \quad k \to \infty. \tag{7.107}$$

We can also assume that $v(\cdot, \tau_k) \to g(\cdot) \in \mathcal{W}_S(L_*)$ and $\xi_k \to \xi_0 \in [0, L_*]$. Then by compactness we deduce that $v(\xi_k, \tau_k) \to g(\xi_0)$, $v_\xi(\xi_k, \tau_k) \to g'(\xi_0)$, and hence by Proposition 7.6 we have that as $k \to \infty$

$$\rho(\xi_k, v(\xi_k, \tau_k), v_\xi(\xi_k, \tau_k)) \to \rho(\xi_0, g(\xi_0), g'(\xi_0)) = 0.$$

By (7.61) this implies that $(\xi_0, g(\xi_0), g'(\xi_0)) \in B_*$ which contradicts Proposition 7.13.

We now prove that there exists $\delta_0 > 0$ small enough such that (cf. (7.106))

$$\lim \inf_{\tau \to \infty} \rho_*(\tau; L_* + \delta_0) = C_{21} > 0. \tag{7.108}$$

If (7.108) is false then there exist a decreasing sequence $\{\delta_k > 0\} \to 0$ and $\{\xi_k\} \subset (L_*, L_* + \delta_k)$ (see (7.106)), $\{\tau_k\} \to \infty$ such that (7.107) holds. We have that $v(\cdot, \tau_k) \to f(\cdot) \in \omega(\bar{v}_0)$ and by (7.103) there exists $g \in \mathcal{W}_s(L_*)$ such that

$$f \equiv g \quad \text{on} \quad [0, L_*], \tag{7.109}$$

and by estimates (7.15) and (7.17)

$$f(\xi), f'(\xi) \quad \text{are continuous on} \quad [0, L_* + \delta_1]. \tag{7.110}$$

Since $\{\xi_k\} \to L_* + 0$, by passing to the limit $k \to \infty$ we deduce that

$$\rho(\xi_k, v(\xi_k, \tau_k),\ v_\xi(\xi_k, \tau_k)) \to \rho(L_*, f(L_*), f'(L_*)) = 0,$$

which by (7.109) and (7.110) contradicts Proposition 7.13.

Thus, (7.108) holds for some $\delta_0 > 0$. Then we have that $(\xi, v, v_\xi) \in G_*$ for any $(\xi, \tau) \in [0, L_* + \delta_0] \times [\tau_*, S]$, and hence (7.71) implies that after passing to the limit $\epsilon \to 0$ in (7.96) with $L = L_* + \delta_0$, from (7.108) we arrive at estimate (7.99) with $L = L_* + \delta_0$. By using the technique of the proof of Lemma 7.12 we then conclude that $\omega(\bar{v}_0) \subset \mathcal{W}_s(L_* + \delta_0/2)$ which contradicts the definition of L_* in (7.101) and (7.102). Hence, $L_* = \infty$ and (1.30) holds.

Estimate (1.31) for any $g \in \omega(\bar{v}_0) \subseteq W_s$ follows from (7.9). (1.32) is the result of weak limit as $\tau \to \infty$ in the inequality (7.21). □

Proof of Corollary 1.6. The existence of a nontrivial self-similar function $\theta \in W_s$ follows from (1.30) and (1.27). Integrating (1.32) with $g = \theta$ over $(1, \xi)$ yields

$$\theta(\xi) \geq \theta(1) - (\sigma + 2) \log \xi \quad \text{for} \quad \xi \geq 1.$$

Using the uniform (with respect to θ) lower estimate (7.10) we deduce that

$$\theta(\xi) \geq -\nu - (\sigma + 2) \log \xi \quad \text{as} \quad \xi \to \infty.$$

Thus, any stationary solution $\theta \in \omega(\bar{v}_0)$ satisfies

$$-\nu \leq \theta(\xi) + (\sigma + 2) \log \xi \leq C_+ \tag{7.111}$$

for large ξ which completes the proof of (1.33).

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