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LARGE DEVIATIONS ESTIMATES FOR THE EXIT PROBABILITIES OF A DIFFUSION PROCESS THROUGH SOME VANISHING PARTS OF THE BOUNDARY

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Introduction. In the study of random perturbations of dynamical systems, the aim of the theory of Large Deviations is to give estimates on the events of "small probabilities"; we refer the reader to the works of Wentzell-Freidlin ([22]), Donsker-Varadhan ([13]) and Azencott ([1]) for an introduction and for the basic results of this theory.

In general, these events of "small probabilities" are the ones for which the behavior of the perturbed trajectories does not look like the behavior of the unperturbed trajectories. We are going in this article to investigate another possible source of "small probabilities" by estimating the exit probabilities of perturbed trajectories of a deterministic dynamical systems through "small" parts of the boundary of a domain in $\mathbb{R}^N$ which typically collapse to a point when the perturbation goes to zero; this is what we call "vanishing parts" of the boundary.

In order to be more specific, we consider Ω a smooth bounded domain of $\mathbb{R}^N$ and we denote by $(X_t^\varepsilon)_t$ the solution of the stochastic differential equation

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \varepsilon\sigma(X_t^\varepsilon)dW_t\,, \qquad X_0^\varepsilon = x \in \Omega\,, \tag{1}$$

where $\varepsilon > 0$ and where b and σ are Lipschitz-continuous functions with values respectively in $\mathbb{R}^N$ and in the space of $N \times p$ matrices. Finally, $(W_t)_t$ denotes a standard p-dimensional Wiener process.

Then we consider $(\Gamma^\varepsilon)_\varepsilon$ a sequence of open subsets of $\partial\Omega:$ the aim is to estimate the probability for a sample path of the solution of (1) to exit Ω through Γ^ε. To do so, we introduce the real-valued function u^ε defined on $\overline{\Omega}$ by

$$u^\varepsilon(x) = \mathbb{P}_x(X^\varepsilon_{\tau^\varepsilon_x} \in \Gamma^\varepsilon) = \mathbb{E}_x(\mathbb{1}_{\Gamma^\varepsilon}(X^\varepsilon_{\tau^\varepsilon_x}))\,,$$

where we denote respectively by $\mathbb{P}_x$ and $\mathbb{E}_x$ the conditional probability and the conditional expectation with respect to the event $\{X_0^\varepsilon = x\}$ and where τ_x^ε stands for the first exit time of $(X_t^\varepsilon)_t$ from Ω, i.e.,

$$\tau_x^\varepsilon = \inf\{t \geq 0,\ X_t^\varepsilon \notin \Omega\}\,,$$

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where the infimum of the empty set is $+\infty$.

The classical results concerning exit probabilities in the theory of Large Deviations read

$$-\varepsilon^2 \ln(u^\varepsilon(x)) \to I(x) \quad \text{as } \varepsilon \to 0, \ \text{ for } x \in \Omega\,, \tag{2}$$

where the function I is the so-called Action Functional; this implies that

$$u^\varepsilon(x) = \exp\left(-\frac{I(x)+o(1)}{\varepsilon^2}\right) \quad \text{in } \Omega\,,$$

and, since in general $I > 0$ in Ω, this shows that the exit probabilities are exponentially small when $\varepsilon \to 0$. We are looking here for such an asymptotic behavior of u^ε where the function I has to be determined.

This type of exponential estimate was first obtained in different contexts by a probabilistic approach; we refer again the reader to Wentzell-Freidlin ([22]), Donsker-Varadhan ([13]) and Azencott ([1]) for the classical results in this direction. The partial differential equation (PDE) approach to these problems was first introduced by Fleming ([15]). The PDE method to obtain such kind of results is based on the fact that u^ε is a solution of a linear elliptic PDE and it can be described as follows:

(1) To perform the logarithmic change of variable

$$I^\varepsilon(x) \ := \ -\varepsilon^2 \ln(u^\varepsilon(x)) \quad \text{in } \Omega\,.$$

The function I^ε solves a nonlinear elliptic PDE.

(2) To pass to the limit $\varepsilon \to 0$, in this nonlinear PDE: the aim is to show that I^ε converges (in some sense) to a function I which is a solution of a certain first-order PDE (or in certain cases a second-order PDE). It is worth noticing that this is a passage to the limit in a singular perturbation problem.

(3) To interpret this limiting first-order PDE by using control theory in order to obtain the formula of representation for I.

The limiting step of the method was for a long time the second step; indeed, the passage to the limit in the nonlinear singular perturbation problem by using classical PDE theory requires the strong convergence of DI^ε, the gradient of I^ε, and therefore, in order to pass to the limit, one needs rather strong estimates on the function I^ε and these estimates are not true in general.

This problem was solved by the introduction of the notion of viscosity solutions by Crandall-Lions ([12]) which allows passages to the limit in nonlinear singular perturbation problems without the strong convergence of the gradient but with only the local uniform convergence of the solutions. For a presentation of this notion of solution, we also refer the reader to Crandall-Evans-Lions ([10]), Lions ([18]) and the "Users' Guide" of Crandall-Ishii-Lions ([11]). In all the following, we will use this notion of solutions and, to simplify, we will write solutions for viscosity solutions.

Using viscosity solutions' theory, Evans-Ishii ([14]) gave a new justification of the PDE method for Large Deviations problems and, in particular, of the steps 2 and 3. In the Evans-Ishii ([14]) work, the local uniform convergence of the solutions was obtained by proving some gradient estimates; these estimates are tedious to obtain and they can be false in problems where the diffusion is degenerate.

Then a new method—the so-called "half-relaxed limits" procedure—introduced by Perthame and the first author ([5]) allows one to avoid these gradient estimates; this new method both simplifies the PDE approach and also permits one to treat Large Deviations problems for degenerated diffusions (cf. Barles-Perthame, [7]). The key ingredient in this method is a so-called strong comparison result for the limiting PDE, i.e., a Maximum Principle type result for discontinuous (viscosity) solutions—which implies *a posteriori* the uniform convergence of I^ε to I.

In this article, we follow the method of Barles-Perthame ([7]) but it is worth mentioning that no strong comparison result will be available in our context since we meet possibly discontinuous action functionals or at least we will have to handle discontinuous Dirichlet boundary data. Instead we will use "weak" comparison results inspired by Blanc ([9]).

Coming back to the problem of estimating exit probabilities, we consider here two types of behavior for the Γ^ε: the first one is when Γ^ε converges in some sense to a "regular" subset Γ of $\partial\Omega$ and the second one is typically the case when Γ^ε collapses to a point of $\partial\Omega$. Precise formulations will be given later.

The first case may be simplified by considering only fixed subsets; i.e., $\Gamma^\varepsilon \equiv \Gamma \subset \partial\Omega$. In this simpler case, Evans-Ishii ([14]) obtained the asymptotic behavior of u^ε for nondegenerated diffusions. Then Barles-Perthame ([7]) extended this result to the case when the diffusion is degenerated but for a subset which is a connected component of $\partial\Omega$. In this article, we generalize these results by allowing both the diffusion to be degenerated and $(\Gamma^\varepsilon)_\varepsilon$ to converge to any open subset of $\partial\Omega$.

To the best of our knowledge, the second case—when typically the Γ^ε collapse to a point—has not been considered in the literature yet. We assume in this case that the diffusion is not degenerated and more precisely that σ is an $N\times N$ invertible matrix. To give a flavor of our results, let us consider the case when $\Gamma^\varepsilon := \mathcal{B}(x_0,\rho_\varepsilon)\cap\partial\Omega$ where $x_0\in\partial\Omega$, $\rho_\varepsilon>0$, $\rho_\varepsilon\to 0$ when $\varepsilon\to 0$ and $\mathcal{B}(x_0,\rho_\varepsilon)$ is the ball of center x_0 and of radius ρ_ε. Then the behavior of u^ε depends strongly on the connections between ε and ρ_ε. More precisely, if

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln\rho_\varepsilon \;=\; 0\,,$$

the function u^ε has the expected behavior; i.e., the function I is given by

$$\begin{aligned} I(x) = \inf\Big\{ \frac{1}{2}\int_0^{\tau_x} |\sigma^{-T}(y_x(t))[\dot{y}_x(t) - b(y_x(t))]|^2 dt;\; y_x(\cdot)\in H^1_{loc}(\mathbb{R}^+;\mathbb{R}^N),\\ y_x(0)=x,\; y_x(\tau_x)=x_0,\; y_x(t)\in\overline{\Omega}\ \text{ for } t<\tau_x \Big\}, \end{aligned} \tag{3}$$

where σ^T is the transpose matrix of σ and σ^{-T} is the inverse of σ^T.

The main new feature of the proof of this result—in addition to the "weak" comparison result we need—is the fact that the Dirichlet boundary condition satisfied by I does not come automatically from the passage to the limit; it requires a specific treatment which is one of the main difficulties in the proof.

In the case when

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln \rho_\varepsilon = +\infty,$$

the convergence to zero of u^ε is faster and we show that

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln u^\varepsilon(x) = +\infty \quad \text{ in } \Omega.$$

This result is obtained by considering the adjoint problem to the Dirichlet problem satisfied by u^ε.

In the limiting case, i.e., when

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln \rho_\varepsilon = \rho > 0,$$

we can only prove that

$$\liminf_{\varepsilon\to 0} -\varepsilon^2 \ln u^\varepsilon(x) \ \geq\ \sup\{(N-1)\rho, I(x)\} \quad \text{ in } \Omega\,.$$

The problem of knowing if this inequality is optimal is open.

This paper is organized as follows. The first section is devoted to the case when the limiting subset is "regular." In the second, we consider the case of a sequence of "vanishing parts" of the boundary. The third section contains the proof of the "weak" comparison result. Finally, in the appendix, we give a result on a gradient estimate used in the second part.

1. The probability estimates in the case of a "regular" limiting subset. In this section, we consider the case when the sequence $(\Gamma^\varepsilon)_\varepsilon$ converges to a "regular" subset of $\partial\Omega$. We first explain what we mean by a "regular" subset of $\partial\Omega$ and what kind of convergence we use for the sequence $(\Gamma^\varepsilon)_\varepsilon$.

Definition 1.1. $\Gamma \subset \partial\Omega$ is said to be a "regular" subset of the boundary if and only if

$$\overline{\Gamma} = \overline{\mathrm{int}(\Gamma)}, \tag{4}$$

where we denote by $\mathrm{int}(\Gamma)$ the interior of Γ.

Notice that the property (4) is always satisfies if Γ is an open subset of $\partial\Omega$. Another way to express (4) is in terms of the characteristic function of the set Γ that we denote by $1\!\!1_\Gamma$. We recall that $1\!\!1_\Gamma(x) = 1$ if $x \in \Gamma$ and 0 otherwise. The property (4) is equivalent to the fact that the function $z := 1\!\!1_\Gamma$ satisfies

$$z^* = (z_*)^* \quad \text{ on } \ \partial\Omega\,, \tag{5}$$

where z^* and z_* stand respectively for the upper and lower semi-continuous envelope of the function z. These notations will be used throughout the paper for different functions.

In order to define the convergence of the sequence $(\Gamma^\varepsilon)_\varepsilon$, we first recall the definition of the so-called "half-relaxed limits" in the theory of viscosity solutions. If $(z_\varepsilon)_\varepsilon$ is a sequence of uniformly bounded functions, we set

$$\limsup{}^* z_\varepsilon(x) = \limsup_{\substack{y\to x\\ \varepsilon\to 0}} z_\varepsilon(y) \quad \text{and} \quad \liminf{}_* z_\varepsilon(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} z_\varepsilon(y)\,.$$

Definition 1.2. The sequence $(\Gamma^\varepsilon)_\varepsilon$ is said to converge to $\Gamma \subset \partial\Omega$ if and only if

$$(\mathbb{1}_\Gamma)^* = \limsup{}^*\ \mathbb{1}_{\Gamma^\varepsilon} \quad \text{and} \quad (\mathbb{1}_\Gamma)_* = \liminf{}_*\ \mathbb{1}_{\Gamma^\varepsilon}\,. \tag{6}$$

The convergence property can also be expressed respectively by the following two equalities:

$$\overline{\Gamma} = \bigcap_{\varepsilon_0} (\overline{\bigcup_{\varepsilon\,<\,\varepsilon_0} \Gamma^\varepsilon}) \quad \text{and} \quad \text{int}(\Gamma) = \bigcup_{\varepsilon_0} \Big[\text{int}(\bigcap_{\varepsilon\,<\,\varepsilon_0} \Gamma^\varepsilon)\Big]\,.$$

Our main result is the following.

Theorem 1.1. *We assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b and σ are Lipschitz-continuous functions on $\overline{\Omega}$ and that the sequence $(\Gamma^\varepsilon)_\varepsilon$ converges to a regular subset of $\partial\Omega$. If, in addition, the two following properties hold*

$$\exists\zeta > 0,\ \forall x \in \partial\Omega,\ \textit{either}\ \ |\sigma^T(x)n(x)| \ge \zeta > 0\ \textit{ or }\ b(x)\cdot n(x) \ge \zeta > 0, \tag{7}$$

where $n(x)$ denotes the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$ and where σ^T denotes the transpose of the matrix σ, and

$$\begin{cases} \exists\, T < \infty,\ \forall\, y(\cdot)\ \textit{solution of}\ \dot y(t) = b(y(t))\ \textit{with}\ y(0) \in \overline{\Omega}, \\ \exists\, s \le T\ \textit{such that}\ y(s) \notin \overline{\Omega}, \end{cases} \tag{8}$$

then we have

$$\forall A > 0,\quad \liminf{}_*\,(-\varepsilon^2 \ln u^\varepsilon) \wedge A = (\limsup{}^*\,(-\varepsilon^2 \ln u^\varepsilon))_* \wedge A = I \wedge A \quad \textit{in } \Omega, \tag{9}$$

where $\alpha \wedge \beta = \min\{\alpha, \beta\}$ for $\alpha,\ \beta \in \mathbb{R}$ and where I is the value function of the control problem

$$\frac{dy_x}{dt}(t) = 2a(y_x(t))\alpha(t) + b(y_x(t)), \quad y_x(0) = x \in \Omega,$$

$$I(x) = \inf\Big\{\frac{1}{2}\int_0^\theta a(y_x(t))\alpha(t)\cdot\alpha(t)\,dt;\ \alpha(\cdot)\in L^\infty(\mathbb{R}^+,\mathbb{R}^N),\quad y_x(\theta)\in\Gamma,\ y_x(t)\in\overline{\Omega}\ \ \textit{for } t<\theta\Big\}, \tag{10}$$

where $a = \sigma\sigma^T$.

We first remark that, if the action functional I given by (10) is bounded, the property (9) reduces to

$$\liminf_* \,(-\varepsilon^2 \ln u^\varepsilon) = (\limsup{}^* \,(-\varepsilon^2 \ln u^\varepsilon))_* = I \quad \text{ in } \Omega\,.$$

Indeed this equality is obtained by taking A large enough in (9). Under the assumptions of Theorem 1.1, in particular because of the possible degeneracy of σ, the function I can be discontinuous and therefore one cannot expect the local uniform convergence of $-\varepsilon^2 \ln u^\varepsilon$ to I in Ω. This equality is a way to express, in some weak sense, that this convergence holds.

But, in general, the function I can be equal to $+\infty$ at certain points of $\overline{\Omega}$ since nothing ensures that, from any point $x\in\overline{\Omega}$, there is a trajectory y_x which exits Ω through Γ. In the same way, the function I can be equal to 0 at a certain point of $\overline{\Omega}$: so Theorem 1.1 is not really a Large Deviation-type result since *a priori* at the points x where $I(x)=0$, the probability given by u^ε does not tend to zero.

Of course, if σ is an $N\times N$ positive-definite matrix and if $b(x)\cdot n(x) < 0$ on Γ, then

$$0 < I(x) < +\infty \quad \text{ in } \Omega\,,$$

and so we have a classical Large Deviations-type result. The assumptions (7) and (8) are classical in this context: (7) is some “nondegeneracy”-type condition like the ones introduced in [7] to prove strong comparison results for the generalized Dirichlet problem (in the viscosity sense); such a type of assumption is necessary to get such strong comparison results. We remark that (7) is less restrictive than the corresponding assumption in [7] because of the alternative case with b.

The condition (8) is a classical assumption to obtain the strict subsolution one needs in order to have a comparison result for the limiting PDE (cf. [7], [14], [16]). The importance of such assumptions in getting a comparison result will be explained in Section 3 (see also Lemma 2.5 in Section 2).

Proof of Theorem 1.1. We prove the theorem in several steps which follow the basic plan explained in the introduction. The first one is devoted to showing that u^ε is a viscosity solution of a second-order linear equation with a “generalized” Dirichlet boundary condition in the viscosity sense; indeed, since we consider degenerated diffusion, there is no reason for u^ε to be a classical solution of this Dirichlet problem and, moreover, losses of boundary data may occur on the boundary because again of this possible degeneracy.

In the second step, we perform a logarithmic-type change of variable and we let ε go to zero by using the "half-relaxed limits" method. Finally, we conclude by proving a weak comparison result which allows us to identify the action functional.

Step 1. It consists in proving the

Lemma 1.1. *The function u^ε defined on $\overline{\Omega}$ by*

$$u^\varepsilon(x) = \mathbb{E}_x(\varphi^\varepsilon(X^\varepsilon_{\tau^\varepsilon_x}))\,,$$

where $\varphi^\varepsilon := \mathbb{1}_{\Gamma^\varepsilon}$, is a viscosity solution of the "generalized" Dirichlet problem

$$\begin{aligned} \mathcal{L}u^\varepsilon := -\frac{1}{2}\varepsilon^2 \mathrm{Tr}(a(x)D^2u^\varepsilon) - b(x)\cdot Du^\varepsilon = 0 \qquad & \text{in } \Omega, \\ u^\varepsilon = \varphi^\varepsilon \qquad & \text{on } \partial\Omega\,, \end{aligned} \tag{11}$$

where Tr *denotes the trace operator.*

We recall that, in (11), the Dirichlet boundary condition has to be understood in the viscosity sense; i.e.,

$$\min\{\mathcal{L}u^\varepsilon, u^\varepsilon - \varphi^*\} \le 0 \quad \text{ on } \partial\Omega\,,$$

and

$$\max\{\mathcal{L}u^\varepsilon, u^\varepsilon - \varphi_*\} \ge 0 \quad \text{ on } \partial\Omega\,.$$

We refer to Crandall-Ishii-Lions ([11]) and Barles-Burdeau ([4]) for details on these types of "generalized" boundary conditions.

Proof of Lemma 1.1. We extend a result of Barles-Burdeau ([4]) where a similar result was proved in the case when the function on the boundary φ is continuous. The ideas are essentially the same as in [4] but we give the proof for the convenience of the reader. We just show how to obtain the supersolution property on the boundary since the other properties are easier to derive.

Let ϕ be a C^2-function on $\overline{\Omega}$ and let $x_0 \in \partial\Omega$ be a global minimum point of $u^\varepsilon_* - \phi$ on $\overline{\Omega}$. We may assume that $u^\varepsilon_*(x_0) = \phi(x_0)$ by changing if necessary ϕ. Then $u^\varepsilon_*(x) \ge \phi(x)$ on $\overline{\Omega}$.

If $u^\varepsilon_*(x_0) \ge \varphi^\varepsilon_*(x_0)$, we are done. Otherwise, $u^\varepsilon_*(x_0) < \varphi^\varepsilon_*(x_0)$. We may change ϕ outside a neighborhood of x_0 to have $\phi \le \varphi^\varepsilon_*$ on $\partial\Omega$ if this is not the case. We consider a sequence $(x_p)_p$ of points of $\overline{\Omega}$ such that $x_p \to x_0$ and

$$\lim_p u^\varepsilon(x_p) = u^\varepsilon_*(x_0)\,.$$

If there exists a subsequence of $(x_p)_p$ denoted by $(x_{p'})_{p'}$ such that $x_{p'} \in \partial\Omega$, then, by definition of u^ε, we get $u^\varepsilon(x_{p'}) = \varphi^\varepsilon(x_{p'})$. And, since

$$\liminf_{p'} \varphi^\varepsilon(x_{p'}) \ge \varphi^\varepsilon_*(x_0)\,,$$

letting p' tend to $+\infty$, we have $u^\varepsilon_*(x_0) \geq \varphi^\varepsilon_*(x_0)$. But since $u^\varepsilon_*(x_0) < \varphi^\varepsilon_*(x_0)$, this cannot happen and therefore $x_p \in \Omega$ for p large enough. We define

$$h_p^2 := |u^\varepsilon(x_p) - \phi(x_p)|,$$

then $h_p \to 0$ as $p \to \infty$ and $\phi(x_p) = u^\varepsilon(x_p) + o(h_p)$. Using the Dynamic Programming Principle, we get

$$u^\varepsilon(x_p) = \mathbb{E}_{x_p}\Big[\mathbb{1}_{\{h_p<\tau_p\}} u^\varepsilon(X^\varepsilon_{h_p}) + \mathbb{1}_{\{h_p\geq\tau_p\}}\varphi^\varepsilon(X^\varepsilon_{\tau_p})\Big]\,,$$

where $(X^\varepsilon_t)_t$ is a solution of (1) with $X^\varepsilon_0 = x_p$ and τ_p is the first exit time of Ω for X^ε_t.

By the definition of h_p and since $\varphi^\varepsilon \geq \varphi^\varepsilon_*$ on $\partial\Omega$, $u^\varepsilon \geq u^\varepsilon_*$ and $u^\varepsilon_* \geq \phi$ on $\overline{\Omega}$, we have

$$\phi(x_p) - o(h_p) = u^\varepsilon(x_p) \ \geq\ \mathbb{E}_{x_p}\Big[\mathbb{1}_{\{h_p\,<\,\tau_p\}}\phi(X^\varepsilon_{h_p}) + \mathbb{1}_{\{h_p\geq\tau_p\}}\varphi^\varepsilon_*(X^\varepsilon_{\tau_p})\Big]\,. \qquad (12)$$

Moreover, since $\varphi^\varepsilon_* \geq \phi$ on $\partial\Omega$, we get

$$\phi(x_p) \ \geq\ \mathbb{E}_{x_p}[\phi(X^\varepsilon_{h_p\wedge\tau_p})] + o(h_p)\,. \qquad (13)$$

Using Itô's Lemma, we obtain

$$\begin{aligned}\phi(X^\varepsilon_{h_p\wedge\tau_p}) = \phi(x_p) &+ \int_0^{h_p\wedge\tau_p} \varepsilon\sigma(X^\varepsilon_t)D\phi(X^\varepsilon_t)\,dW_t \\ &+ \int_0^{h_p\wedge\tau_p}\Big(\frac{\varepsilon^2}{2}\mathrm{Tr}(a(X^\varepsilon_t)D^2\phi(X^\varepsilon_t)) + b(X^\varepsilon_t)\cdot D\phi(X^\varepsilon_t)\Big)/,ds\,.\end{aligned}$$

Combining this with (13), we have

$$\mathbb{E}_{x_p}\Big[\int_0^{h_p\wedge\tau_p}\Big(-\frac{\varepsilon^2}{2}\mathrm{Tr}(aD^2\phi) - b\cdot D\phi\Big)(X^\varepsilon_s)/,ds\Big] \ \geq\ o(h_p)\,.$$

Because of the regularity of the functions ϕ, a and b, we get

$$\Big(-\frac{\varepsilon^2}{2}\mathrm{Tr}(aD^2\phi) - b\cdot D\phi\Big)(x_p)\mathbb{E}_{x_p}[h_p\wedge\tau_p] \geq o(h_p)\,.$$

It remains to show that

$$\mathbb{E}_{x_p}\Big[\frac{h_p\wedge\tau_p}{h_p}\Big] \ \to\ 1\,.$$

To do so, we prove that $\mathbb{P}[\tau_p \le h_p] \to 0$. We subtract $\phi(x_p)$ in (12) and we compute

$$u^\varepsilon(x_p) - \phi(x_p) \ge \mathbb{E}_{x_p}\Big[1\!\!1_{\{h_p<\tau_p\}}(\phi(X^\varepsilon_{h_p}) - \phi(x_p)) + 1\!\!1_{\{\tau_p \le h_p\}}(\varphi^\varepsilon_*(X^\varepsilon_{\tau_p}) - \varphi^\varepsilon_*(x_0))\Big] \\ + \mathbb{P}[\tau_p \le h_p](\varphi^\varepsilon_*(x_0) - \phi(x_p))\,.$$

Since ϕ is continuous,

$$\mathbb{E}_{x_p}\Big[1\!\!1_{\{h_p<\tau_p\}}(\phi(X^\varepsilon_{h_p}) - \phi(x_p))\Big] = o_p(1)\,.$$

Moreover, φ^ε_* is l.s.c. , then

$$\mathbb{E}_{x_p}\Big[1\!\!1_{\{\tau_p \le h_p\}}(\varphi^\varepsilon_*(X^\varepsilon_{\tau_p}) - \varphi^\varepsilon_*(x_0))\Big] \;\ge\; o_p(1)\,.$$

Thus

$$u^\varepsilon(x_p) - \phi(x_p) \;\ge\; \mathbb{P}[\tau_p \;\le\; h_p]\,(\varphi^\varepsilon_*(x_0) - \phi(x_p)) + o_p(1)\,.$$

Finally, since $u^\varepsilon(x_p) - \phi(x_p) \to 0$ and since $\varphi^\varepsilon_*(x_0) > \phi(x_0)$, we have

$$\limsup_p \mathbb{P}[\tau_p \le h_p] = 0\,.$$

And the proof is complete. $\square$

Step 2. We perform the change of variable

$$I^{A,\varepsilon}(x) = -\varepsilon^2 \ln\Big(u^\varepsilon(x) + e^{-\frac{A}{\varepsilon^2}}\Big) \quad \text{ for } x \in \overline{\Omega}\,,$$

where A is a large positive constant. The term $e^{-\frac{A}{\varepsilon^2}}$ is a trick introduced in [7] which avoids the difficulty when u^ε is equal to zero and gives a straightforward upper bound for $I^{A,\varepsilon}$; indeed, since $0 \le u^\varepsilon \le 1$, we have

$$o(1) \;\le\; I^{A,\varepsilon} \;\le\; A \quad \text{ on } \overline{\Omega}\,.$$

The function $I^{A,\varepsilon}$ is a solution of

$$\begin{aligned} -\frac{1}{2}\varepsilon^2 \mathrm{Tr}(a(x)D^2 I^{A,\varepsilon}) + \frac{1}{2}|\sigma^T(x) I^{A,\varepsilon}|^2 - b(x)\cdot DI^{A,\varepsilon} = 0 \quad &\text{ in } \Omega\,, \\ I^{A,\varepsilon} + \varepsilon^2 \ln\Big(\varphi^\varepsilon(x) + e^{-\frac{A}{\varepsilon^2}}\Big) = 0 \quad &\text{ on } \partial\Omega\,. \end{aligned} \tag{14}$$

Now, we apply the “half-relaxed limit” procedure; we define

$$\underline{I}^A = \liminf{}_* I^{A,\varepsilon} \quad \text{ and } \quad \bar{I}^A = \limsup{}^* I^{A,\varepsilon}\,.$$

If we set $I^\varepsilon := -\varepsilon^2 \ln(u^\varepsilon)$, it is a classical remark and an easy exercise to show that

$$\underline{I}^A = \inf\{\liminf_* I^\varepsilon, A\} \quad \text{and} \quad \bar{I}^A = \inf\{\limsup^* I^\varepsilon, A\}\,.$$

By classical results in the viscosity solutions' theory, $\bar{I}^A$ and $\underline{I}^A$ are respectively subsolution and supersolution of

$$\frac{1}{2}|\sigma^T(x)I^A|^2 - b(x)\cdot DI^A = 0 \quad \text{in } \Omega\,. \tag{15}$$

Moreover, on the boundary, an easy computation shows that, in the viscosity sense,

$$\bar{I}^A \;\leq\; \overline{\psi} = A(1-\liminf_* \varphi^\varepsilon) \quad \text{on } \partial\Omega\,,$$

and

$$\underline{I}^A \;\geq\; \underline{\psi} = A(1-\limsup^* \varphi^\varepsilon) \quad \text{on } \partial\Omega\,.$$

Moreover, the convergence of the sequence $(\Gamma^\varepsilon)_\varepsilon$ to the subset Γ implies

$$\limsup^* \varphi^\varepsilon = (1\!\!1_\Gamma)^* \quad \text{and} \quad \liminf_* \varphi^\varepsilon = (1\!\!1_\Gamma)_*$$

and the regularity of Γ reads $((1\!\!1_\Gamma)_*)^* = (1\!\!1_\Gamma)^*$; therefore, we have

$$(\overline{\psi})_* = \underline{\psi} \quad \text{on } \partial\Omega\,,$$

and the condition on the boundary reduces to

$$I^A = \overline{\psi} \quad \text{on } \partial\Omega\,. \tag{16}$$

Step 3. It remains to compare the half-relaxed limits $\underline{I}^A$ and $\bar{I}^A$. To do so, we give now a "weak" comparison result for the general Dirichlet problem

$$\begin{aligned} \frac{1}{2}|\sigma^T(x)Du|^2 - b(x)\cdot Du = 0 \quad &\text{in } \Omega\,,\\ u = \varphi \quad &\text{on } \partial\Omega\,, \end{aligned} \tag{17}$$

where the function φ is bounded, possibly discontinuous.

The result is the

Theorem 1.2 (Weak Comparison Result)**.** *We assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b and σ are Lipschitz-continuous functions on $\overline{\Omega}$ and that the assumptions* (7) *and* (8) *hold. Let u be an u.s.c. bounded subsolution of* (17) *and let v be a l.s.c. bounded supersolution of* (17)*.*

1. *If the function φ satisfies*

$$(\varphi^*)_* = \varphi_* \quad \textit{on } \partial\Omega\,, \tag{18}$$

then $u_* \le v$ *in* Ω.

2. *If the function* u *satisfies*

$$\liminf_{y\to x, y\in\Omega} u(y) \;\le\; \varphi_*(x)\,, \tag{19}$$

for any $x \in \partial\Omega$, *then* $u_* \le v$ *in* Ω. $\square$

The proof of this result is given in Section 3. We are going to use here the first part of this result; the second part will be used in the case of sequences of "vanishing parts" of the boundary.

We conclude the proof of Theorem 1.1 by using Theorem 1.2. The problem (15)–(16) satisfies the assumptions (7) and (8) and the function on the boundary $\overline{\psi}$ is u.s.c. and therefore it satisfies (18) as we have shown before. Hence the first part of Theorem 1.2 implies

$$(\bar{I}^A)_* \;\le\; \underline{I}^A \quad \text{in } \Omega\,.$$

Since $\underline{I}^A \le \bar{I}^A$ in Ω by their definition, we obtain $(\bar{I}^A)_* = \underline{I}^A$ in Ω. And the first part of the theorem is proved.

It remains to show that $\underline{I}^A = (\bar{I}^A)_* = I$ in Ω. To do so, we follow the classical method by interpreting the PDE (15)–(16) as the dynamic programming equation of a deterministic control problem. By representation formulae (cf. [18], [9]), the l.s.c. value function of the exit time problem associated to (15)–(16) is given by

$$I^A(x) = \inf\Big\{\int_0^\theta a(y_x(t))\alpha(t)\cdot\alpha(t)\,dt + \underline{\psi}(y_x(\theta));\ \alpha(\cdot)\in L^\infty(\mathbb{R}^+,\mathbb{R}^N),$$
$$y_x(\theta)\in\partial\Omega,\ y_x(t)\in\overline{\Omega}\ \text{ for } t<\theta\Big\}$$

with the dynamic of y_x given by (10). Indeed in that case, I^A is equal to the value function associated to relaxed controls, since the dynamic equation is linear and since the space of controls is $\mathbb{R}^N$. We apply again the weak comparison result and we get $\underline{I}^A = (\bar{I}^A)_* = I^A$ in Ω since I^A is a l.s.c. solution of (15)–(16) (cf. [9]).

Finally, it remains to prove that $I^A = I \wedge A$ in Ω. To do so, we first remark that, by their definition, $I^A \le I$ in Ω and that $I^A \le A$ in Ω. The first property is obtained by reducing the space of controls to the controls for which $y_x(\theta)\in\Gamma$ and for the second one, it is enough to choose the control $\alpha(\cdot)\equiv 0$. Therefore

$$I^A \;\le\; I\wedge A \quad \text{in } \Omega\,.$$

To prove the opposite inequality, we consider $x\in\Omega$. If $I(x)\ge A$, this means that, for any control $\alpha(\cdot)$ such that $y_x(\theta)\in\Gamma$, we have

$$\int_0^\theta a(y_x(t))\alpha(t)\cdot\alpha(t)\,dt \;\ge\; A\,.$$

So an optimal control associated to $I^A(x)$ is obviously to choose $\alpha(\cdot) \equiv 0$; indeed, the associated trajectory does not exit Ω through Γ but the integral cost is zero and we only pay the terminal cost A. Hence

$$I^A(x) \ = \ A = I(x) \wedge A\,.$$

If $I(x) < A$, this means that there exist controls $\alpha(\cdot)$ such that $y_x(\theta) \in \Gamma$ and

$$\int_0^\theta a(y_x(t))\alpha(t)\cdot\alpha(t)\,dt < A\,.$$

And we take for I^A the same minimizing sequence of controls as for I and we get

$$I^A(x) = I(x) = I(x) \wedge A\,.$$

And the proof is complete. $\square$

Remark 1.1. In the proof of Theorem 1.1, one can replace the use of the weak comparison result by a slightly different argument inspired from Barles-Perthame ([5]): let us denote by $I^A[\overline{\psi}]$ the function defined as I^A above but replacing $\underline{\psi}$ by $\overline{\psi}$. Approximating $\overline{\psi}$ and $\underline{\psi}$ respectively from above and below by continuous functions, one easily shows using the arguments of [9] relying on representation formulae that $I^A[\overline{\psi}]$ and I^A are respectively the maximal subsolution and the minimal supersolution of (15)–(16). Therefore one has

$$I^A \ \leq \ \underline{I}^A \ \leq \ \bar{I}^A \ \leq \ I^A[\overline{\psi}] \quad \text{ on } \overline{\Omega}\,.$$

And one concludes by proving that

$$I^A = (I^A[\overline{\psi}])_* \quad \text{ on } \overline{\Omega}\,,$$

which leads to

$$I^A = \underline{I}^A = (\bar{I}^A)_* = (I^A[\overline{\psi}])_* \quad \text{ on } \overline{\Omega}\,.$$

But this argument relies heavily on the fact that $(\overline{\psi})_* = \underline{\psi}$ on $\overline{\Omega}$ and this property will not be satisfied in the case of sequences of "vanishing parts" of the boundary. Theorem 1.2 provides similar proofs in these two different cases.

2. Probability estimates of exiting through vanishing parts of the boundary. In order to emphasize the main ideas of our approach, we first consider a simple example where everything can be computed explicitly. This example is the following:

$$\frac{\partial u^\varepsilon}{\partial t} - \frac{\varepsilon^2}{2}\Delta u^\varepsilon = 0 \quad \text{ in } \mathbb{R}^N \times (0,+\infty)\,,$$

with the initial condition

$$u^\varepsilon(x,0) = \mathbb{1}_{\mathcal{B}(0,\rho_\varepsilon)}(x) \quad \text{in } \mathbb{R}^N ,$$

where $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. It is well-known that the solution is given by

$$u^\varepsilon(x,t) = \frac{1}{(2\pi t\varepsilon^2)^{n/2}} \int_{\mathbb{R}^N} \mathbb{1}_{\mathcal{B}(0,\rho_\varepsilon)}(y) \exp\Big(-\frac{|y-x|^2}{2t\varepsilon^2}\Big) dy .$$

It is clear that, since $\rho_\varepsilon \to 0$,

$$u^\varepsilon(x,t) \simeq \frac{1}{(2\pi t\varepsilon^2)^{n/2}} \text{meas}\,(\mathcal{B}(0,\rho_\varepsilon)) \exp\Big(-\frac{|x|^2}{2t\varepsilon^2}\Big) .$$

So, in order to get the logarithmic equivalent of the function u^ε, one has to compare the terms $-\varepsilon^2 \ln(\text{meas}\,(\mathcal{B}(0,\rho_\varepsilon)))$, i.e., essentially $-n\varepsilon^2 \ln(\rho_\varepsilon)$ and $\frac{|x|^2}{2t}$. Therefore, it is natural to investigate three cases.

1. If $-\varepsilon^2 \ln(\rho_\varepsilon) \to 0$, then

$$-\varepsilon^2 \ln(u^\varepsilon) \;\to\; \frac{|x|^2}{2t} \quad \text{in } \mathbb{R}^N \times (0,+\infty) .$$

2. If $-\varepsilon^2 \ln(\rho_\varepsilon) \to \rho > 0$, then

$$-\varepsilon^2 \ln(u^\varepsilon) \;\to\; N\rho + \frac{|x|^2}{2t} \quad \text{in } \mathbb{R}^N \times (0,+\infty) .$$

3. If $-\varepsilon^2 \ln(\rho_\varepsilon) \to +\infty$, then

$$-\varepsilon^2 \ln(u^\varepsilon) \;\to\; +\infty \quad \text{in } \mathbb{R}^N \times (0,+\infty) .$$

This example shows that, in our problem, one has to expect three types of behavior for the probability u^ε according to the three possible limits of $-\varepsilon^2 \ln(\text{meas}\,(\Gamma^\varepsilon))$, i.e., zero, a finite positive value or infinity. We are now going to investigate these three cases in our problem. In order to simplify the notation, we denote by m_ε the measure of the set Γ^ε; i.e.,

$$m_\varepsilon = \text{meas}(\Gamma^\varepsilon) .$$

In all the following, we assume that the matrix a is uniformly elliptic; i.e.,

$$\exists \gamma > 0 \;\text{ such that } \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \;\geq\; \gamma|\xi|^2 \;\text{ for any } x \in \overline{\Omega},\ \xi \in \mathbb{R}^N . \tag{20}$$

We recall that $a \equiv \sigma\sigma^T$ and that, in this case, σ is assumed to be an $N \times N$ matrix.

2.1. The case when $-\varepsilon^2 \ln(\text{meas}(\Gamma^\varepsilon)) \to 0$. The result is the

Theorem 2.1. *We assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b and σ are Lipschitz-continuous functions on $\overline{\Omega}$, and that the assumptions (8) and (20) hold. We assume in addition that the sequence $(\Gamma^\varepsilon)_\varepsilon$ converges to a closed subset Γ of $\partial\Omega$ and that, for any $x \in \Gamma$, there exist two sequences $(x_\varepsilon)_\varepsilon$ and $(\rho_\varepsilon)_\varepsilon$ such that, for any ε, $x_\varepsilon \in \partial\Omega$, $\rho_\varepsilon > 0$ and*

- $x_\varepsilon \to x$ *when* $\varepsilon \to 0$,
- $\mathcal{B}(x_\varepsilon, \rho_\varepsilon) \cap \partial\Omega \subset \Gamma^\varepsilon$,
- ρ_ε *satisfies*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\rho_\varepsilon) = 0\,. \tag{21}$$

Then we have

$$-\varepsilon^2 \ln u^\varepsilon \;\to\; I \quad \text{in } \; C(\Omega)\,,$$

where I is given by (3).

We first want to point out that the action functional I is defined by (3) instead of (10) since here the diffusion is not degenerated.

We recall that the convergence in $C(\Omega)$ is the uniform convergence on every compact subset of Ω.

Proof of the Theorem 2.1. To simplify the proof, we first consider the model problem described in the Introduction when $\Gamma^\varepsilon = \mathcal{B}(x_0, \rho_\varepsilon) \cap \partial\Omega$, for some $x_0 \in \partial\Omega$. Of course, we assume that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ in addition to (21) to have a vanishing part.

The beginning of the proof is exactly the same as in Theorem 1.1. We perform the change of variable

$$I^{A,\varepsilon}(x) = -\varepsilon^2 \ln\Big(u^\varepsilon(x) + e^{-\frac{A}{\varepsilon^2}}\Big) \qquad \text{for } x \in \overline{\Omega}\,.$$

The function $I^{A,\varepsilon}$ is a solution of

$$\begin{aligned} -\frac{1}{2}\varepsilon^2 \mathrm{Tr}(a(x)D^2 I^{A,\varepsilon}) + \frac{1}{2}|\sigma^T(x) I^{A,\varepsilon}|^2 - b(x)\cdot DI^{A,\varepsilon} = 0 \quad & \text{in } \Omega\,, \\ I^{A,\varepsilon} + \varepsilon^2 \ln\Big(\varphi^\varepsilon(x) + e^{-\frac{A}{\varepsilon^2}}\Big) = 0 \quad & \text{on } \partial\Omega\,. \end{aligned} \tag{22}$$

Letting ε go to 0, by the half-relaxed limits technique, we obtain that

$$\bar{I}^A = \limsup{}^* I^{A,\varepsilon} \qquad \text{and} \qquad \underline{I}^A = \liminf{}_* I^{A,\varepsilon}$$

are respectively the sub- and supersolution of

$$\begin{aligned} \frac{1}{2}|\sigma^T(x) DI^A|^2 - b(x)\cdot DI^A = 0 \qquad & \text{in } \Omega\,, \\ I^A = A\mathbb{1}_{\partial\Omega\setminus\{x_0\}} \qquad & \text{on } \partial\Omega\,. \end{aligned} \tag{23}$$

The main difficulty here is that the boundary condition for the subsolution $\bar{I}^A$ is in fact empty. Indeed we have

$$(A1\!\!1_{\partial\Omega\setminus\{x_0\}})^* = A \quad \text{ on } \partial\Omega \,,$$

and the inequality $\bar{I}^A \leq A$ on $\partial\Omega$ is just a straightforward consequence of the definition of $I^{A,\varepsilon}$. In the passage to the limit, some information on the function $\bar{I}^A$ around x_0 was lost and we have to recover this information. More precisely, to apply Theorem 1.2, we have to prove that $\bar{I}^A$ satisfies (19); i.e.,

$$\liminf_{y\to x, y\in\Omega} \bar{I}^A(y) \;\leq\; A1\!\!1_{\partial\Omega\setminus\{x_0\}}(x)\,, \tag{24}$$

for any $x \in \partial\Omega$. Of course, the only difficulty occurs at the point x_0 where the function $A1\!\!1_{\partial\Omega\setminus\{x_0\}}$ is equal to 0; everywhere else, as we already mentioned above, the inequality (24) is obviously satisfied.

To prove this condition at x_0, we first define the points $x_t := x_0 - tn(x_0)$, for $t > 0$ small enough. Strictly inside Ω, the function $I^{A,\varepsilon}$ is smooth since u^ε is smooth (recall that here the diffusion is nondegenerate). For $\rho_\varepsilon \leq \alpha$, we may write

$$I^{A,\varepsilon}(x_\alpha) = I^{A,\varepsilon}(x_{\rho_\varepsilon}) - \int_{\rho_\varepsilon}^{\alpha} DI^{A,\varepsilon}(x_0 - tn(x_0))\cdot n(x_0)\,dt\,. \tag{25}$$

To estimate the right-hand side of this equality, we need the two following lemmas.

Lemma 2.1. *We have $I^{A,\varepsilon}(x_{\rho_\varepsilon}) = o(1)$.*

We postpone the proof of this lemma.

Lemma 2.2. *There exist two constants C_1 and C_2 such that, for any $x \in \Omega$,*

$$|DI^{A,\varepsilon}(x)| \;\leq\; C_1 + C_2\frac{\varepsilon^2}{d(x)}\,,$$

where $d(\cdot)$ denotes the distance function to the boundary $\partial\Omega$.

The proof of this second (classical) lemma is given in the appendix for the reader's convenience. It is worth noticing that it implies the existence of subsequences of $(I^{A,\varepsilon})_\varepsilon$ which converge in $C(\Omega)$ by Ascoli's Theorem.

Using Lemmas 2.1 and 2.2, we obtain from (25)

$$I^{A,\varepsilon}(x_\alpha) \leq o(1) + \int_{\rho_\varepsilon}^{\alpha} (C_1 + C_2\frac{\varepsilon^2}{t})\,dt \leq o(1) + C_1(\alpha-\rho_\varepsilon) + C_2\varepsilon^2(\ln\alpha - \ln\rho_\varepsilon)\,. \tag{26}$$

Now we take the lim sup in this inequality. Because of the gradient bound of Lemma 2.2, we have

$$\limsup_{\varepsilon}\; I^{A,\varepsilon}(x_\alpha) = \bar{I}^A(x_\alpha)\,,$$

and using the assumption (21) in the right-hand side of (26), we get $\bar{I}^A(x_\alpha) \leq C_1\alpha$. Hence, recalling the fact that $\bar{I}^A \geq 0$ in Ω, we have built a sequence $(x_\alpha)_\alpha$ of points of Ω such that $x_\alpha \to x_0$ as $\alpha \to 0$ and $\bar{I}^A(x_\alpha) \to 0$. Thus $\bar{I}^A$ satisfies (19).

So we can use the weak comparison result for $\bar{I}^A$ and $\underline{I}^A$ which yields

$$\underline{I}^A = (\bar{I}^A)_* \quad \text{in } \Omega\,.$$

In fact, by the gradient estimate, the functions $\underline{I}^A$ and $\bar{I}^A$ are Lipschitz continuous in Ω and therefore

$$\underline{I}^A = \bar{I}^A \quad \text{in } \Omega\,.$$

On an other hand, it can easily be shown that the function I defined by (3) is bounded (recall that Ω is connected) and Lipschitz continuous on $\overline{\Omega}$. Moreover, following readily the ideas of Blanc ([9]) and using the weak comparison result, one can also easily prove that, for $A > \|I\|_\infty$, the function I satisfies (19) and it is the unique l.s.c. solution of (23) satisfying (19).

Applying again the weak comparison result and using the continuity of I, we obtain for $A > \|I\|_\infty$

$$\underline{I}^A = \bar{I}^A \;=\; I \quad \text{in } \Omega\,.$$

But, since $A > \|I\|_\infty$, this immediately implies the result.

In the general case when Γ^ε is not only the ball $\mathcal{B}(x_0,\rho_\varepsilon)\cap\partial\Omega$, the proof relies essentially on the same arguments as before. The only new point comes from the proof of the property

$$\liminf_{y\to x, y\in\Omega} \bar{I}^A(y) \;\leq\; A1\!\!1_{\partial\Omega\setminus\Gamma}(x)\,,$$

for any $x \in \partial\Omega$. But, by assumption, for any $x_0 \in \Gamma$, there exists a sequence $(x_\varepsilon)_\varepsilon$ of points of $\partial\Omega$ such that $x_\varepsilon \to x_0$ and such that $\mathcal{B}(x_\varepsilon,\rho_\varepsilon)\cap\partial\Omega \subset \Gamma^\varepsilon$. Then we consider the points $x_t^\varepsilon := x_\varepsilon - tn(x_\varepsilon)$. One sees easily that a result analogous to Lemma 2.2 is still valid in this situation and therefore we have

$$\limsup_{\varepsilon} I^{A,\varepsilon}(x_\alpha^\varepsilon) = \bar{I}^A(x_\alpha) \;\leq\; C_1\alpha\,.$$

The remainder of the proof follows the one of Theorem 1.1 with the use of the second part of Theorem 1.2 and therefore it is left to the reader. □

Proof of Lemma 2.1. The proof relies on a scaling argument. We introduce the function J^ε defined by

$$J^\varepsilon(x) \;:=\; I^{A,\varepsilon}(x_0+\rho_\varepsilon x) \quad \text{for } x \in \overline{\Omega}\,.$$

The function J^ε is a solution of

$$\begin{aligned} -\tfrac{1}{2}\varepsilon^2\mathrm{Tr}(a(x_0+\rho_\varepsilon x)D^2J^\varepsilon) + \tfrac{1}{2}|\sigma^T(x_0+\rho_\varepsilon x)DJ^\varepsilon|^2 & \\ - \rho_\varepsilon b(x_0+\rho_\varepsilon x)\cdot DJ^\varepsilon = 0 &\quad \text{in } \Omega_{\rho_\varepsilon}\,, \\ J^\varepsilon = A(1 - 1\!\!1_{\mathcal{B}(0,1)\cap\partial\Omega_{\rho_\varepsilon}}) &\quad \text{on } \partial\Omega_{\rho_\varepsilon}\,, \end{aligned}$$

where

$$\Omega_{\rho_\varepsilon} = \{x \in \mathbb{R}^N \text{ such that } \ x_0 + \rho_\varepsilon x \in \Omega\}\,.$$

In order to pass to the limit, we denote by $\Omega_0 = \lim_\varepsilon \Omega_{\rho_\varepsilon}$, the half-space containing 0 and whose normal direction is $n(x_0)$, and we set $\bar{J} = \limsup^* J^\varepsilon$. The u.s.c. function $\bar{J}$ is a subsolution of

$$\begin{aligned} |\sigma^T(x_0)DJ|^2 = 0 \qquad\qquad & \text{in } \Omega_0, \\ J = A(1 - \mathbb{1}_{\mathcal{B}(0,1)\cap\partial\Omega_0}) \quad & \text{on } \partial\Omega_0\,. \end{aligned} \tag{27}$$

We first remark that the subsolution condition on the boundary is satisfied in the classical sense because of the coercivity of the Hamiltonian in (27) (cf. [2]). Hence we obtain

$$\bar{J}(x) \ \leq\ 0 \quad \text{ for any } x \in \mathcal{B}(0,1) \cap \partial\Omega_0\,.$$

But we showed before that $I^{A,\varepsilon} \geq o(1)$ on $\overline{\Omega}$ and thus the function $\bar{J}$ which is nonnegative is equal to 0 on $\mathcal{B}(0,1) \cap \partial\Omega_0$. Therefore, the function $\bar{J}$ is equal to 0 in Ω_0; indeed, the equation implies that $D\bar{J} \equiv 0$ in Ω_0 and this means that $\bar{J}$ is a nonnegative constant (again because $I^{A,\varepsilon} \geq o(1)$ on $\overline{\Omega}$); finally, this constant is necessarily 0 because $\bar{J} = 0$ on the boundary $\mathcal{B}(0,1) \cap \partial\Omega_0$.

Now, by classical arguments, J^ε converges uniformly to 0 in any compact subset of Ω_0, in particular on $\overline{\mathcal{B}}(-n(x_0), \frac{1}{2})$, the closure of $\mathcal{B}(-n(x_0), \frac{1}{2})$. In order to conclude, we come back to the original function $I^{A,\varepsilon}$; we set

$$D_\varepsilon \ :=\ \overline{\mathcal{B}}(x_0 - \rho_\varepsilon n(x_0), \frac{\rho_\varepsilon}{2})\,.$$

The above property implies that

$$o(1) \leq I^{A,\varepsilon}(x_{\rho_\varepsilon}) \leq \max_{D_\varepsilon} I^{A,\varepsilon} = \max_{\overline{\mathcal{B}}(-n(x_0), \frac{1}{2})} J^\varepsilon \to 0 \quad \text{when } \varepsilon \to 0,$$

and the result is proved. □

Remark 2.1. In fact, since we assume that the matrix a is uniformly elliptic, the use of the weak comparison result can be avoided and replaced by more elementary arguments. Indeed, the inequality

$$I \ \leq\ \underline{I}^A \quad \text{ in } \Omega\,,$$

for A large enough, can be obtained by approximating the l.s.c. boundary condition $A\mathbb{1}_{\partial\Omega\setminus\Gamma}$ from below (recall that Γ is closed) and by passing to the limit in the representation formulae. This inequality is the easiest to get and it can always be obtained in this way.

The other inequality, namely

$$\bar{I}^A \ \leq\ I \quad \text{ in } \Omega\,,$$

for A large enough, can be proved by first extending $\bar{I}^A$ from Ω to $\overline{\Omega}$ into a Lipschitz-continuous function on $\overline{\Omega}$, still denoted by $\bar{I}^A$. Then $\bar{I}^A$ is a subsolution of

$$\begin{aligned} \frac{1}{2}|\sigma^T(x)Dw|^2 - b(x)\cdot Dw = 0 \qquad & \text{in } \Omega\,, \\ w = \bar{I}^A \qquad & \text{on } \partial\Omega\,, \end{aligned} \tag{28}$$

while the function I can be proved to be a supersolution of this problem since, in particular, it satisfies a state-constraint boundary condition in $\partial\Omega\backslash\Gamma$ and since $I = \bar{I}^A = 0$ on Γ. From the comparison result of [7], we obtain the desired inequality.

Another way to understand this argument, without using directly the fact that I is a supersolution of (28) is to remark that the comparison result of [7] implies, for $x \in \Omega$,

$$\begin{aligned} \bar{I}^A(x) \leq \inf\Big\{ \frac{1}{2}\int_0^{\tau_x} |\sigma^{-T}(y_x(t))[\dot{y}_x(t) - b(y_x(t))]|^2 dt;\ y_x(\cdot) \in H^1_{loc}(\mathbb{R}^+;\mathbb{R}^N), \\ y_x(0) = x,\ y_x(\tau_x) \in \partial\Omega, \ \text{ and } \ y_x(t) \in \overline{\Omega} \ \text{ for } t < \tau_x \Big\}, \end{aligned}$$

since the right-hand side is the unique solution of (28). And we conclude by noticing that the right-hand side of this inequality is less than I since, in the minimization for I, one considers only the trajectories y_x which exit Ω through Γ.

It is worth remarking that, in both cases, the elementary arguments to obtain the second inequality rely strongly on the continuity of $\bar{I}^A$ and I, and on the simple form of the Dirichlet boundary data. On the contrary, the weak comparison result allows us to avoid having to use such ad hoc arguments on each example and it also permits us to treat more complicated problems. We have in particular in mind the extension of Theorem 2.1 to the case of degenerated diffusions but, of course, the general assumptions ensuring that (24) holds, have still to be found.

2.2. The case when $-\varepsilon^2 \ln(\mathbf{meas}(\Gamma^\varepsilon)) \to +\infty$. We first recall that we denote by m_ε the measure of Γ^ε. The result is the

Theorem 2.2. *Assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b is a Lipschitz-continuous function on $\overline{\Omega}$, that $a \in W^{2,\infty}(\Omega)$ and that the assumptions* (8) *and* (20) *hold. If*

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln(m_\varepsilon) = +\infty\,. \tag{29}$$

Then $-\varepsilon^2 \ln u^\varepsilon \to +\infty$ as $\varepsilon \to 0$ in Ω.

It is worth remarking that, in this case, we do not need to assume the convergence of the sequence $(\Gamma^\varepsilon)_\varepsilon$; only the fact that the measures of the Γ^ε are very small plays a role.

The linearity of the equation (11) allows us to mix the two behaviors of the Γ^ε considered in Theorems 2.1 and 2.2. The precise result is the

Theorem 2.3. *Assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b is a Lipschitz-continuous function on $\overline{\Omega}$, that $a \in W^{2,\infty}(\Omega)$ and that the assumptions* (8) *and* (20) *hold. If $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$ where the sequence $(\Gamma_1^\varepsilon)_\varepsilon$ converges to Γ_1 and satisfies the assumptions of Theorem 2.1 and where*

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln(\operatorname{meas}(\Gamma_2^\varepsilon)) = +\infty\,,$$

then $-\varepsilon^2 \ln u^\varepsilon \to I_1$, in $C(\Omega)$, where I_1 is given by (3) *with $\Gamma = \Gamma_1$.*

We first give the very short proof of Theorem 2.3. We set $u_i^\varepsilon = \mathbb{P}(X^\varepsilon_{\tau_x^\varepsilon} \in \Gamma_i^\varepsilon)$, for $i = 1, 2$. Since we do not assume that Γ_1^ε and Γ_2^ε are disjoint subsets of $\partial\Omega$, we only have

$$(u_1^\varepsilon + u_2^\varepsilon)(x) \le u^\varepsilon(x) \ \le\ 2(u_1^\varepsilon + u_2^\varepsilon)(x) \quad \text{on } \overline{\Omega}\,,$$

but this property is enough to say that the logarithmic behavior of u^ε is the same as the logarithmic behavior of $u_1^\varepsilon + u_2^\varepsilon$. To proceed, we set for $i = 1, 2$

$$I_i^{A,\varepsilon}(x) = -\varepsilon^2 \ln\Big(u_i^\varepsilon(x) + e^{-\frac{A}{\varepsilon^2}}\Big) \quad \text{on } \overline{\Omega}\,,$$

and we set

$$J^{A,\varepsilon}(x) = -\varepsilon^2 \ln\Big(u_1^\varepsilon(x) + u_2^\varepsilon(x) + 3e^{-\frac{A}{\varepsilon^2}}\Big) \quad \text{on } \overline{\Omega}\,.$$

Then, we have

$$J^{A,\varepsilon}(x) = -\varepsilon^2 \ln\Big(\exp\Big(-\frac{I_1^{A,\varepsilon}(x)}{\varepsilon^2}\Big) + \exp\Big(-\frac{I_2^{A,\varepsilon}(x)}{\varepsilon^2}\Big) + e^{-\frac{A}{\varepsilon^2}}\Big) \quad \text{on } \overline{\Omega}\,,$$

and a classical result yields

$$\liminf\nolimits_* J^{A,\varepsilon} = \min\{\liminf\nolimits_* I_1^{A,\varepsilon}\,,\, \liminf\nolimits_* I_2^{A,\varepsilon}\,,\, A\}\,,$$

and

$$\limsup\nolimits^* J^{A,\varepsilon} = \min\{\limsup\nolimits^* I_1^{A,\varepsilon}\,,\, \limsup\nolimits^* I_2^{A,\varepsilon}\,,\, A\}\,.$$

Taking into account the nondegeneracy of σ, we deduce from Theorem 2.1 and 2.2 that

$$\liminf\nolimits_* I_1^{A,\varepsilon} = \limsup\nolimits^* I_1^{A,\varepsilon} = I_1 \quad \text{in } \Omega\,,$$

and

$$\liminf\nolimits_* I_2^{A,\varepsilon} = \limsup\nolimits^* I_2^{A,\varepsilon} = +\infty \quad \text{in } \Omega\,.$$

Therefore, for A large enough,

$$\liminf\nolimits_* J^{A,\varepsilon} = \limsup\nolimits^* J^{A,\varepsilon} = I_1 \quad \text{in } \Omega$$

(recall that since the diffusion is nondegenerate, I_1 is bounded). One concludes easily since the behavior of $J^{A,\varepsilon}$ gives exactly the behavior of $-\varepsilon^2 \ln(u_1^\varepsilon + u_2^\varepsilon)$.

Now we turn to the

Proof of Theorem 2.2. We recall that the function u^ε is a solution of

$$-\frac{\varepsilon^2}{2}\mathrm{Tr}(a(x)D^2u^\varepsilon) - b(x)\cdot Du^\varepsilon = 0 \quad \text{in } \Omega\,, \tag{30}$$

with the boundary condition

$$u^\varepsilon = \mathbb{1}_{\Gamma^\varepsilon} \quad \text{in } \partial\Omega\,. \tag{31}$$

Since the equation is uniformly elliptic, the function u^ε is in $C^2(\Omega)$. We set

$$\Omega^\delta = \{x \in \Omega \text{ such that } \operatorname{dist}(x,\partial\Omega) \;>\; \delta\}\,.$$

We apply the Green's formula twice to (30) in Ω^δ for δ small enough, with a test function $v^{\varepsilon,\delta}$ which will be chosen later.

$$\begin{aligned}0 &= \int_{\Omega^\delta}\Big(-\frac{\varepsilon^2}{2}\mathrm{Tr}(aD^2u^\varepsilon) - b\cdot Du^\varepsilon\Big)v^{\varepsilon,\delta}\\ &= \int_{\Omega^\delta} u^\varepsilon\Big(-\frac{\varepsilon^2}{2}\mathrm{div}(aDv^{\varepsilon,\delta}) + \mathrm{div}((b-\frac{\varepsilon^2}{2}\tilde{A})v^{\varepsilon,\delta})\Big)\\ &+\frac{\varepsilon^2}{2}\int_{\partial\Omega^\delta}(\tilde{A}\cdot nu^\varepsilon v^{\varepsilon,\delta} + aDv^{\varepsilon,\delta}\cdot nu^\varepsilon - Du^\varepsilon\cdot anv^{\varepsilon,\delta} - b\cdot nu^\varepsilon v^{\varepsilon,\delta})\,,\end{aligned} \tag{32}$$

where $\tilde{A} = ((\tilde{A})_i)_{1\le i\le n}$ is the vector whose i-th component is given by

$$\tilde{A}_i = \sum_j \frac{\partial a_{i,j}}{\partial x_j}\,.$$

Now, we introduce the function $v^{\varepsilon,\delta}$ defined by the following lemma.

Lemma 2.3. *For ε and δ small enough, there exists a unique solution $v^{\varepsilon,\delta}$ in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ for all $1<p<\infty$, of the Dirichlet problem*

$$\begin{aligned}\mathrm{div}\Big(-\frac{\varepsilon^2}{2}aDv^{\varepsilon,\delta} + (b-\frac{\varepsilon^2}{2}\tilde{A})v^{\varepsilon,\delta}\Big) &= 1 \qquad \text{in } \Omega^\delta\,,\\ v^{\varepsilon,\delta} &= 0 \qquad \text{on } \partial\Omega^\delta\,.\end{aligned} \tag{33}$$

Moreover, we have the following estimate on the boundary:

$$\|Dv^{\varepsilon,\delta}\|_{L^\infty(\partial\Omega^\delta)} \;\le\; \frac{D}{\varepsilon^2}\,, \tag{34}$$

where D is a positive constant independent of δ and ε.

We postpone the proof of this lemma.

Using the properties of $v^{\varepsilon,\delta}$ in (32), we obtain

$$0 = \int_{\Omega^\delta} u^\varepsilon + \frac{\varepsilon^2}{2} \int_{\partial\Omega^\delta} aDv^{\varepsilon,\delta} \cdot n u^\varepsilon \,.$$

Moreover, using the estimate (34), we get

$$\int_{\Omega^\delta} u^\varepsilon \;\le\; C \int_{\partial\Omega^\delta} u^\varepsilon \,,$$

for some constant C independent of ε and δ.

Since u^ε is smooth on $\overline{\Omega}\backslash\Gamma^\varepsilon$ and since Γ^ε is an open subset of $\partial\Omega$, one has $u^\varepsilon(y) \to 0$ when $y \to x \in \partial\Omega\backslash\Gamma^\varepsilon$. Using this remark together with the fact that $0 \le u^\varepsilon \le 1$ on $\overline{\Omega}$, we get

$$\int_{\Omega^\delta} u^\varepsilon \;\le\; C(o(1) + m_\varepsilon) \,. \tag{35}$$

Notice that the $o(1)$ may depend on ε. Letting δ tend to 0, we deduce

$$\int_{\Omega} u^\varepsilon \;\le\; Cm_\varepsilon \,. \tag{36}$$

In order to deduce some properties for the function $I^{A,\varepsilon}$, we first add $\int_\Omega e^{-\frac{A}{\varepsilon^2}}$ to each side of the inequality (36):

$$\int_\Omega (u^\varepsilon + e^{-\frac{A}{\varepsilon^2}}) \le Cm_\varepsilon + \int_\Omega e^{-\frac{A}{\varepsilon^2}} \,.$$

We divide this inequality by the measure of the set Ω in order to be able to apply Jensen's inequality and we perform the logarithmic change

$$-\varepsilon^2 \ln\Big(\frac{1}{|\Omega|}\int_\Omega (u^\varepsilon + e^{-\frac{A}{\varepsilon^2}})\Big) \;\ge\; -\varepsilon^2 \ln\Big(\frac{C}{|\Omega|} m_\varepsilon + e^{-\frac{A}{\varepsilon^2}}\Big) \,.$$

Then, by Jensen's inequality,

$$\frac{1}{|\Omega|}\int_\Omega I^{A,\varepsilon} \;\ge\; -\varepsilon^2 \ln\Big(\frac{C}{|\Omega|} m_\varepsilon + e^{-\frac{A}{\varepsilon^2}}\Big) \,. \tag{37}$$

Since the assumption (29) holds, the right-hand side of (37) tends to A.

Now we are going to show that this implies

$$\underline{I}^A \;\ge\; A \quad \text{in } \Omega \,.$$

To do so, we argue by contradiction assuming that there exists a point $x_0 \in \Omega$ such that $\underline{I}^A(x_0) < A$. By the definition of the $\liminf_*$, there exists a subsequence $(I^{A,\varepsilon'})_{\varepsilon'}$ of $(I^{A,\varepsilon})_\varepsilon$ and a sequence $(x_{\varepsilon'})_{\varepsilon'}$ of points of Ω such that

$$\lim_{\varepsilon' \to 0} I^{A,\varepsilon'}(x_{\varepsilon'}) = \underline{I}^A(x_0)\,.$$

By Lemma 2.2, we can use Ascoli's Theorem and we may extract a subsequence of $(I^{A,\varepsilon'})_{\varepsilon'}$, still denoted $(I^{A,\varepsilon'})_{\varepsilon'}$, which converges to a function I in $C(\Omega)$. Using this fact together with the global boundedness of $I^{A,\varepsilon}$ by A, one obtains from Lebesgue's Theorem

$$\int_\Omega I^{A,\varepsilon'} \;\to\; \int_\Omega I\,.$$

Hence, from (37), we get

$$\frac{1}{|\Omega|}\int_\Omega I \;\geq\; A\,.$$

But $I^{A,\varepsilon} \leq A$ in Ω and therefore $I \leq A$ in Ω. Therefore this inequality implies that $I \equiv A$ in Ω. We have reached a contradiction since $I(x_0) = \underline{I}^A(x_0) < A$ and the proof is complete. $\square$

Proof of Lemma 2.3. We divide the proof into four steps.

Step 1. The Maximum Principle for the equation (33).

Lemma 2.4. *Let $u, v \in W^{2,p}(\Omega^\delta) \cap W_0^{1,p}(\Omega^\delta)$ be respectively a subsolution and a supersolution of* (33). *Then*

$$u \;\leq\; v \quad \textit{in } \Omega^\delta\,.$$

Proof. For $\mu \in (0,1)$, we set $w = \mu u - v$ where we have dropped the dependency of w with respect to μ for the sake of simplicity of notation. Since the equation (33) is linear, w is a subsolution of

$$\mathrm{div}\Big(-\frac{\varepsilon^2}{2}aDw + (b - \frac{\varepsilon^2}{2}\tilde{A})w\Big) \;\leq\; \mu - 1 \quad \text{ in } \Omega^\delta\,. \tag{38}$$

The aim is to prove that $w \leq 0$ in Ω^δ for any $0 < \mu < 1$ and then to let μ tend to 1 to obtain $u \leq v$ in Ω^δ. To do so, we consider the test function

$$S(w) = \inf\Big\{w^+, 2\gamma\varepsilon^2\frac{1-\mu}{C^2}\Big\}\,,$$

where $C = \|b\|_\infty + \|\tilde{A}\|_\infty$. Using Green's formula, we obtain

$$\frac{\varepsilon^2}{2}\int_{\Omega^\delta} S'(w)aDw\cdot Dw - \int_{\Omega^\delta} wS'(w)\Big(b - \frac{\varepsilon^2}{2}\tilde{A}\Big)\cdot Dw + \int_{\Omega^\delta}(1-\mu)S(w) \;\leq\; 0\,.$$

But, on one hand, $S'(w)Dw = 0$ almost everywhere on the set

$$\left\{x \in \Omega^\delta\,; S(w) \;\geq\; 2\gamma\varepsilon^2 \frac{1-\mu}{C^2}\right\} \cup \{x \in \Omega^\delta\,; S(w) \;\leq\; 0\}$$

and therefore the above inequality still holds true if, in the two first integrals, we integrate on

$$D \;:=\; \left\{x \in \Omega^\delta\,; 0 \;<\; S(w) \;<\; 2\gamma\varepsilon^2 \frac{1-\mu}{C^2}\right\}.$$

On the other hand, we can use the uniform ellipticity of the matrix a to get

$$\frac{\varepsilon^2}{2}\int_D \gamma S'(w)|Dw|^2 - \int_D wS'(w)C|Dw| + \int_{\Omega^\delta}(1-\mu)S(w) \;\leq\; 0\,.$$

Then applying Young's inequality

$$ab \;\leq\; \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2\,,$$

for $\eta > 0$, leads to

$$\int_D \frac{1}{2}(\gamma\varepsilon^2 - \eta)S'(w)|Dw|^2 - \int_D \frac{C^2}{2\eta}S'(w)w^2 + \int_{\Omega^\delta}(1-\mu)S(w) \;\leq\; 0\,.$$

Now we choose $\eta = \gamma\varepsilon^2$; the above inequality reads

$$\int_D \left[(1-\mu)w - \frac{C^2}{2\eta}w^2\right] + \int_{D^c}(1-\mu)S(w) \;\leq\; 0\,.$$

But in D, we have

$$1-\mu - \frac{C^2}{2\eta}w = 1-\mu - \frac{C^2}{2\gamma\varepsilon^2}w \;>\; 0\,.$$

Therefore the integrands of the two above integrals are nonnegative and necessarily $S(w) \equiv 0$ almost everywhere; i.e.,

$$w \;\leq\; 0 \quad \text{ a.e. in } \Omega^\delta\,.$$

Step 2. The L^∞ estimate for $v^{\varepsilon,\delta}$. It is clear enough that 0 is a subsolution of (33), so using Lemma 2.4, we have $0 \leq v^{\varepsilon,\delta}$ in Ω^δ.

To prove the upper estimate, we are going to build a supersolution. To do so, we need the

Lemma 2.5. *Assume that b and σ are Lipschitz-continuous functions on $\overline{\Omega}$ and that (8) holds; then there exist a function $\xi \in C^2(\overline{\Omega})$ and $\nu > 0$ such that*

$$\frac{1}{2}|\sigma^T(x)D\xi|^2 - b(x)\cdot D\xi \ \le\ -\nu \quad \text{on } \overline{\Omega}\ . \tag{39}$$

The proof of this lemma can be found in Ishii-Koike ([16]) (see also [2]).

Now we use Lemma 2.5 to build a supersolution. We first remark that we may suppose that the function ξ is positive since we may add a constant to ξ.

We introduce the function

$$w(x) \ :=\ e^{K\xi(x)} \quad \text{on } \overline{\Omega}^{\delta}\,,$$

where K is a large constant to be chosen later, and we plug this function in the equation

$$\begin{aligned}\operatorname{div}\Big(-\frac{\varepsilon^2}{2}aDw + (b-\frac{\varepsilon^2}{2}\tilde{A})w\Big) = w\Big(&-\frac{\varepsilon^2}{2}K\mathrm{Tr}(aD^2\xi) - \frac{\varepsilon^2}{2}K^2 aD\xi\cdot D\xi \\ &+ Kb\cdot D\xi - K\varepsilon^2\tilde{A}\cdot D\xi + \operatorname{div}(b-\frac{\varepsilon^2}{2}\tilde{A})\Big)\,.\end{aligned}$$

From (39), we get $b(x)\cdot D\xi \ge \nu$ on $\overline{\Omega}$ and using this property, we obtain

$$\begin{aligned}\operatorname{div}\Big(-\frac{\varepsilon^2}{2}aDw + (b-\frac{\varepsilon^2}{2}\tilde{A})w\Big) \ge w\Big(&K\nu - \|\operatorname{div}(b)\|_\infty - \frac{\varepsilon^2}{2}\|\operatorname{div}(\tilde{A})\|_{L^\infty} \\ &- \frac{\varepsilon^2}{2}K\|\mathrm{Tr}(aD^2\xi) + KaD\xi\cdot D\xi + 2\tilde{A}\cdot D\xi\|_\infty\Big)\,.\end{aligned} \tag{40}$$

We choose the constant K such that

$$K \ :=\ \frac{2}{\nu}\Big(\|\operatorname{div}(b)\|_\infty + \frac{1}{2}\|\operatorname{div}(\tilde{A})\|_\infty\Big)\,.$$

Then, for ε small enough, the right side of the inequality (40) is positive. Therefore the function w is a supersolution of (33); moreover, $w \ge 0$ on $\partial\Omega^\delta$ by definition.

Hence, by the Maximum Principle proved in Step 1, $v^{\varepsilon,\delta} \le w$ on $\overline{\Omega}^{\delta}$ and therefore

$$0 \ \le\ v^{\varepsilon,\delta} \ \le\ M \quad \text{on } \overline{\Omega}^{\delta}\,,$$

where $M := \|w\|_\infty$. Notice that this bound depends neither on ε nor on δ.

Step 3. Existence of $v^{\varepsilon,\delta}$. We are going to show that we can apply the Schauder fixed-point theorem to the map

$$T : W^{1,p}_0(\Omega^\delta) \to W^{1,p}_0(\Omega^\delta), \qquad v \mapsto u\,,$$

where u is the unique solution of

$$-\frac{\varepsilon^2}{2}\mathrm{Tr}(aD^2u) = 1 - \frac{\varepsilon^2}{2}\tilde{A}\cdot Dv - \mathrm{div}((b-\frac{\varepsilon^2}{2}\tilde{A})v)$$

in $W^{2,p}(\Omega^\delta)\cap W_0^{1,p}(\Omega^\delta)$. By classical results, one has

$$\|u\|_{W^{2,p}} \le C\|1-\frac{\varepsilon^2}{2}\tilde{A}\cdot Dv+\mathrm{div}((b-\frac{\varepsilon^2}{2}\tilde{A})v)\|_{L^p} \le C_1+C_2\|v\|_{L^p}+C_3\|Dv\|_{L^p}\,. \quad (41)$$

We use an interpolation inequality. Thus, there exists a positive constant C such that

$$\|Dv\|_{L^p} \;\le\; \frac{1}{2C_3}\|v\|_{W^{1,p}} + C\|v\|_{L^p}\,.$$

Combining this with (41), we get a $W^{1,p}$-estimate for u:

$$\|u\|_{W^{1,p}} \;\le\; C(1+\|v\|_{L^p}) + \frac{1}{2}\|v\|_{W^{1,p}}\,.$$

Now we set $R \;:=\; 2C(1+\|w\|_{L^p})$, where w is the supersolution built in Step 2, and we introduce the set $\mathcal{C}$ defined by

$$\mathcal{C} \;:=\; \{v\in W_0^{1,p}(\Omega^\delta);\ \|v\|_{W^{1,p}}\le R \ \text{ and } \ \|v\|_{L^p}\le\|w\|_{L^p}\}\,.$$

From the above estimates, it is clear that T maps the closed convex set $\mathcal{C}$ into itself. Moreover, T is a compact map. Therefore Schauder's fixed-point theorem applies and combining this with Step 1, we obtain that there exists a unique solution $v^{\varepsilon,\delta}$ of (33).

Step 4. Estimate of $Dv^{\varepsilon,\delta}$ on the boundary of Ω^δ. The proof consists in building a suitable barrier function. Since Ω is smooth, Ω^δ is also smooth for δ small enough and therefore it satisfies the exterior sphere condition. Then, for $x_0\in\partial\Omega^\delta$, there exist a point $y_0\in\mathbb{R}^N\backslash\Omega^\delta$ and a constant $R>0$ such that $\{x_0\}=\overline{\mathcal{B}}(y_0,R)\cap\overline{\Omega^\delta}$. We set $\psi(x) := |x-y_0|-|x_0-y_0|$ for $x\in\Omega^\delta$, and we define the function w by $w(x) := g(\psi(x))$ for $x\in\Omega^\delta$, where g is a smooth function which will be chosen later.

We plug this function in the equation

$$\begin{aligned}\mathrm{div}\Big(-\frac{\varepsilon^2}{2}aDw+(b-\frac{\varepsilon^2}{2}\tilde{A})w\Big) &= -\frac{\varepsilon^2}{2}aD\psi\cdot D\psi g''(\psi) \\ &+\Big(-\frac{1}{2}\mathrm{Tr}(aD^2\psi)+(b-\varepsilon^2\tilde{A})\cdot D\psi\Big)g'(\psi)+\mathrm{div}(b-\frac{\varepsilon^2}{2}\tilde{A})g(\psi)\,.\end{aligned} \quad (42)$$

We choose the function g given by the

Lemma 2.6. *For ε small enough and for any constants C, M such that $CM > 1$, there exist a constant $\eta > 0$ and a C^∞-function g, a solution of the differential ordinary equation*

$$-\frac{\varepsilon^2\gamma}{2}g'' - Cg' - Cg = 1 \quad \text{in } (0,\eta)\,, \tag{43}$$

such that $g(0) = 0$ and $g(\eta) = M$, $g, g' > 0$ and $g'' < 0$ in $(0,\eta)$, and $0 \le g'(0) \le \frac{D}{\varepsilon^2}$.

We first conclude the proof of the gradient bound before giving the proof of this lemma. Since a is uniformly elliptic and since ψ is smooth in Ω^δ, we have from (42)

$$\operatorname{div}\Big(-\frac{\varepsilon^2}{2}aDw + (b - \frac{\varepsilon^2}{2}\tilde{A})w\Big) \;\ge\; -\frac{\varepsilon^2}{2}\gamma g''(\psi) - Cg'(\psi) - Cg(\psi) \quad \text{in } \Omega^\delta\,,$$

for some constant $C > 0$ depending only on the functions b, σ and ψ.

We choose the function g given by Lemma 2.6 with such a constant C and the constant M given in the second step above. With this choice, the function w is a supersolution of (33) in the domain $\mathcal{N} := \Omega^\delta \cap \mathcal{B}(y_0, R+\eta)$ and $w \ge v^{\varepsilon,\delta}$ on the boundary $\partial\mathcal{N}$.

Hence, by similar arguments to the ones used in Step 1, the Maximum Principle holds in $\mathcal{N}$ and therefore $w \ge v^{\varepsilon,\delta}$ in $\overline{\mathcal{N}}$. But since $v^{\varepsilon,\delta}(x_0) = w(x_0) = 0$, we have

$$\frac{\partial v^{\varepsilon,\delta}}{\partial n}(x_0) \;\ge\; \frac{\partial w}{\partial n}(x_0)\,.$$

But, by Lemma 2.6

$$\frac{\partial w}{\partial n}(x_0) = g'(0)\frac{\partial \psi}{\partial n}(x_0) \;\ge\; -\frac{D}{\varepsilon^2}\,.$$

Moreover, since $v^{\varepsilon,\delta} \ge 0$ in Ω^δ,

$$\frac{\partial v^{\varepsilon,\delta}}{\partial n}(x_0) \;\le\; 0\,,$$

and therefore

$$|Dv^{\varepsilon,\delta}(x_0)| \;\le\; \frac{D}{\varepsilon^2}\,.$$

And, since it is true for any $x_0 \in \partial\Omega^\delta$, the gradient estimate is proved. □

Proof of Lemma 2.6. We consider the function g given by $g(t) := \mu e^{-r_1 t} - \lambda e^{-r_2 t} - \frac{1}{C}$, where μ and λ are positive constants and where r_1 and r_2 are given by

$$r_1 \;:=\; \frac{C - \sqrt{C^2 - 2C\varepsilon^2\gamma}}{\varepsilon^2\gamma} \quad \text{and} \quad r_2 \;:=\; \frac{C + \sqrt{C^2 - 2C\varepsilon^2\gamma}}{\varepsilon^2\gamma}\,.$$

Notice that r_1 and r_2 are real numbers if we suppose that the parameter ε is small enough.

We first remark that $r_1 < r_2$ and then we choose the value η to be such that

$$\mu r_1 e^{-r_1\eta} - \lambda r_2 e^{-r_2\eta} = 0\,,$$

which corresponds to the maximum point of the function g in $\mathbb{R}^+$. Then we fix the constants μ and λ to satisfy the condition at 0 and η; i.e.,

$$\mu \;:=\; \frac{CM - e^{-r_2\eta}}{C(e^{-r_1\eta} - e^{-r_2\eta})} \quad \text{and} \quad \lambda \;:=\; \frac{CM - e^{-r_1\eta}}{C(e^{-r_1\eta} - e^{-r_2\eta})}\,.$$

It remains to prove that $g, g' > 0$ in $(0,\eta)$ and $g'' < 0$ in $(0,\eta)$ and to check the estimate of $g'(0)$. Using the definition of η, we get

$$g'(t) = -\mu r_1 e^{-r_1 t} + \lambda r_2 e^{-r_2 t} = \mu r_1 e^{-r_1 t}\left(-1 + e^{(r_1-r_2)(t-\eta)}\right)$$

and it is clear enough that the right-hand side is positive for $0 < t < \eta$ since $r_1 - r_2 < 0$. Since g' is positive in $(0,\eta)$, g is increasing and, thus, positive. The fact that $g'' < 0$ in $(0,\eta)$ is just a consequence of the ODE. Finally, the estimate on $g'(0)$ comes from straightforward computations. $\square$

2.3. The case when $-\varepsilon^2 \ln(\mathbf{meas}(\Gamma^\varepsilon)) \to m$, $(0 < m < +\infty)$. For this limiting case, we have only a partial result.

Theorem 2.4. *Assume that the boundary $\partial\Omega$ is of class $C^{1,1}$, that b is a Lipschitz-continuous function on $\overline{\Omega}$, that $a \in W^{2,\infty}(\Omega)$ and that the assumptions* (8) *and* (20) *hold. If*

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln(m_\varepsilon) = m\,, \tag{44}$$

then we have

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln u^\varepsilon \;\geq\; \max\{m, I\} \quad \text{in } \Omega\,.$$

It is an open question to know if we have in this case

$$-\varepsilon^2 \ln u^\varepsilon \;\to\; m + I$$

where I is given as above by (3) as it is expected in view of the example of the Heat Equation given at the beginning of the section.

Since the result is partial, we may not write a nice result which mixes the three behaviors of the Γ^ε considered in Theorems 2.1, 2.2 and 2.4 as in Theorem 2.3.

Proof of Theorem 2.4. First, we prove that

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \ln u^\varepsilon \;\geq\; m\,.$$

The beginning of the proof is exactly the same as in the proof of Theorem 2.2 except that we choose differently the test function $v^{\varepsilon,\delta}$. More precisely, the function $v^{\varepsilon,\delta} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $1 < p < +\infty$ is a solution of

$$\operatorname{div}\Big(-\frac{\varepsilon^2}{2} a D v^{\varepsilon,\delta} + (b - \frac{\varepsilon^2}{2}\tilde{A}) v^{\varepsilon,\delta}\Big) = \exp\Big(-\frac{J(x)}{\varepsilon^2}\Big) \quad \text{in } \Omega^\delta ,$$

where J is a positive function in Ω such that

$$\lim_{x \to \partial\Omega} J(x) = +\infty .$$

We may note that, in the domain Ω^δ where $\delta > 0$ is small enough, $0 < \exp(-\frac{J(x)}{\varepsilon^2}) \leq 1$, and then a similar result to Lemma 2.3 holds; the proof of this claim is left to the reader. From the inequality (36), we deduce

$$\int_\Omega e^{-\frac{I^{A,\varepsilon}+J}{\varepsilon^2}} \leq C m_\varepsilon + \int_\Omega e^{-\frac{J+A}{\varepsilon^2}} . \tag{45}$$

We may consider a function J such that $J(x) \geq 2A$ for $d(x) < \alpha$ and $J \equiv 0$ for $d(x) > 2\alpha$, where $\alpha > 0$ is a small parameter. This is clearly possible since $J(x) \to \infty$ as $x \to \partial\Omega$. Then we divide Ω in Ω^α and $\Omega \backslash \Omega^\alpha$ in (45):

$$\int_{\Omega^\alpha} e^{-\frac{I^{A,\varepsilon}+J}{\varepsilon^2}} \leq C m_\varepsilon + |\Omega| e^{-\frac{A}{\varepsilon^2}} - \int_{\Omega\backslash\Omega^\alpha} e^{-\frac{I^{A,\varepsilon}+J}{\varepsilon^2}} \leq C m_\varepsilon + |\Omega| e^{-\frac{A}{\varepsilon^2}} + |\Omega\backslash\Omega^\alpha| e^{-\frac{A}{\varepsilon^2}} .$$

We take the logarithmic change: for $A >> m$, we get

$$-\varepsilon^2 \ln\Big(\int_{\Omega^\alpha} e^{-\frac{I^{A,\varepsilon}+J}{\varepsilon^2}}\Big) \;\geq\; -\varepsilon^2 \ln m_\varepsilon + o(\varepsilon) . \tag{46}$$

By the gradient estimate of Lemma 2.2, we can extract from $(I^{A,\varepsilon})_\varepsilon$ a subsequence—still denoted by $(I^{A,\varepsilon})_\varepsilon$—which converges uniformly in Ω^α to a Lipschitz function denoted by I^A. Then, a classical result shows that

$$-\varepsilon^2 \ln\Big(\int_{\Omega^\alpha} e^{-\frac{I^{A,\varepsilon}+J}{\varepsilon^2}}\Big) \;\to\; \min_{\Omega^\alpha}\{I^A + J\} .$$

And passing to the limit in (46) yields $\min_{\Omega^\alpha}\{I^A+J\} \geq m$. In particular, we deduce $I^A \geq m$ in $\Omega^{2\alpha}$. Letting α go to 0 and remarking that this property holds true for any converging subsequence of $(I^{A,\varepsilon})_\varepsilon$ show that $\underline{I}^A := \liminf_* I^{A,\varepsilon} \geq m$ in Ω.

To conclude the proof, we have to show that $\underline{I}^A \geq I$ in Ω. To do so, we enlarge the sets Γ^ε to $\tilde{\Gamma}^\varepsilon$ where

$$\tilde{\Gamma}^\varepsilon \;:=\; \{x \in \partial\Omega \text{ such that } x \in \mathcal{B}(x_0, \varepsilon) \text{ for some } x_0 \in \Gamma^\varepsilon\} ,$$

or in other words

$$\tilde{\Gamma}^\varepsilon = \bigcup_{x_0\in\Gamma^\varepsilon} \mathcal{B}(x_0,\varepsilon)\cap\partial\Omega\,.$$

We set

$$\tilde{u}_\varepsilon(x) = \mathbb{P}_x(X^\varepsilon_{\tau^\varepsilon_x}\in\tilde{\Gamma}^\varepsilon) = \mathbb{E}_x(1\!\!1_{\tilde{\Gamma}^\varepsilon}(X^\varepsilon_{\tau^\varepsilon_x}))\,,$$

and

$$\tilde{I}^{A,\varepsilon}(x) = -\varepsilon^2\ln\Big(\tilde{u}_\varepsilon(x)+e^{-\frac{A}{\varepsilon^2}}\Big)\quad \text{on }\overline{\Omega}\,.$$

By their definitions, since $\Gamma^\varepsilon\subset\tilde{\Gamma}^\varepsilon$, it is clear that $u^\varepsilon\le\tilde{u}_\varepsilon$ on $\overline{\Omega}$ and therefore $\tilde{I}^{A,\varepsilon}(x)\le I^{A,\varepsilon}(x)$ on $\overline{\Omega}$. But Theorem 2.1 applies to $\tilde{u}_\varepsilon$ and we obtain

$$I(x) = \liminf{}_*\ \tilde{I}^{A,\varepsilon}(x)\ \le\ \liminf{}_*\ I^{A,\varepsilon}(x) = \underline{I}^A(x)\quad\text{on }\overline{\Omega}$$

(recall again that a is uniformly elliptic), and the result is proved. $\square$

3. The weak comparison result. We recall that the Hamiltonian of the equation (17) is given by

$$H(x,p)\ :=\ \frac{1}{2}|\sigma^T(x)p|^2 - b(x)\cdot p\quad\text{on }\overline{\Omega}\times\mathbb{R}^N\,, \tag{47}$$

and that the equation (17) reads, in the viscosity sense,

$$\begin{cases} H(x,Du)=0 & \text{in }\Omega\,,\\ \min\{H(x,Du),u-\varphi^*\}\le 0 & \text{on }\partial\Omega\,,\\ \max\{H(x,Du),u-\varphi_*\}\ge 0 & \text{on }\partial\Omega\,.\end{cases}$$

The proof of Theorem 1.2 presents several difficulties; the most classical ones are the fact that there is no dependence on u in the Hamiltonian of the equation and that one has to take care of the Dirichlet boundary condition in the viscosity sense. The first difficulty is solved by considering assumption (8) which is classical in this context and which implies the existence of a smooth strict subsolution (cf. Lemma 2.5 and the references [2], [15], [16], [19]). The second difficulty is solved by introducing the assumption (7) which ensures a "nondegeneracy property" of the Hamiltonian along the normal of the boundary as in [7]. It is obviously fulfilled if the diffusion is nondegenerated.

But there is here an additional difficulty coming from the discontinuity of the Dirichlet boundary data. Indeed, if, for example, $\varphi = 1\!\!1_{\partial\Omega\setminus\{x_0\}}$ then one expects the solutions u to satisfy $0\le u\le 1$ on $\overline{\Omega}$ and therefore the viscosity subsolution property is not restrictive enough on the boundary since obviously $u\le 1\equiv\varphi^*$ on $\partial\Omega$.

There are two ways to solve this problem: either by considering only "regular" boundary data φ, i.e., discontinuous functions satisfying (18) for which this difficulty does not exist anymore, or by imposing additional conditions on the solutions, namely (19) on $\partial\Omega$ and to be a Barron-Jensen solution of (17) in Ω. For the sake of completeness, we devote the following subsection to explaining this concept of viscosity solutions which was introduced by E.N. Barron and R. Jensen ([8]) for equations with convex Hamiltonians.

3.1. The concept of Barron-Jensen solutions. As we already mentioned above, the notion of Barron-Jensen solutions was introduced for equations whose Hamiltonians are convex in the gradient variable. Before making more comments on this notion of solutions and, in particular, on the connections with the classical notion of viscosity solutions, we recall the definition.

Definition 3.1. We say that the locally bounded function u is a Barron-Jensen solution of

$$H(x,u,Du)=0 \quad \text{in } \Omega\,, \tag{48}$$

where H is a convex continuous real-valued function in $\Omega\times\mathbb{R}\times\mathbb{R}^N$, if it satisfies

$$\begin{gathered}\forall\,\phi\in C^1(\Omega), \text{ at each minimum point } x_0\in\Omega \text{ of } u_*-\phi, \text{ we have}\\ H(x_0,u_*(x_0),D\phi(x_0))=0\,.\end{gathered} \tag{49}$$

For continuous solutions, to be a Barron-Jensen solution of (48) is equivalent to being a classical viscosity solution of (48). In fact, when the Hamiltonian is convex, a continuous viscosity solution u of (48) is also a supersolution of

$$-H(x,u,Du)\ \geq\ 0 \quad \text{in } \Omega\,. \tag{50}$$

And since u is also a supersolution of

$$H(x,u,Du)\ \geq\ 0 \quad \text{in } \Omega\,, \tag{51}$$

it is clear enough that u is a Barron-Jensen solution of (48).

For discontinuous solutions, the connections between classical viscosity solutions and Barron-Jensen solutions are far less simple because we have to consider the semicontinuous envelopes of the functions. If u is a locally bounded viscosity solution of (48), then u_* satisfies (51), but this is the function $(u^*)_*$ which satisfies (50). Hence one can show that u is a Barron-Jensen solution of (48) only if $u_*=(u^*)_*$ in Ω.

Nevertheless, the concept of Barron-Jensen solutions is interesting for uniqueness problems since the condition (49) which characterizes the solution u concerns only the l.s.c. envelope of u whereas classical viscosity solutions deal with u_* and u^* and therefore it is not immediately clear how to interpret uniqueness of these semicontinuous envelopes. Several uniqueness results attest to the significance of this new concept of viscosity solutions; we refer the reader to the pioneering work of

Barron-Jensen ([8]) for the Cauchy problem in $\mathbb{R}^N$, to Barles ([3]) for the stationary stopping time problem in $\mathbb{R}^N$, to Soravia ([21]) and to Blanc ([9]) for Dirichlet problems. Of course, since the concept of Barron-Jensen solutions concerns only the equation inside the domain, we may have to add conditions on the boundary such as (19), in particular to treat discontinuous boundary conditions.

In Theorem 1.2, since we compare an u.s.c. subsolution and a l.s.c. supersolution, we do not need explicitly the concept of Barron-Jensen solutions but only the additional property corresponding to the equation (50). In fact, when the Hamiltonian is given by (47), we have

Theorem 3.1. *If u is a bounded subsolution of $H(x, Dw) = 0$ in Ω then the function $(u^*)_*$ is a supersolution of*

$$-\frac{1}{2}|\sigma^T(x)Dw|^2 + b(x) \cdot Dw \ \geq \ 0 \qquad in\ \Omega\,.$$

Proof of Theorem 3.1. Since the proof is based on the ideas introduced in [3], we only point out the modifications we need to solve the additional difficulties we meet here; these difficulties come from the quadratic term of the Hamiltonian since H does not satisfy the assumptions used in [3], namely

$$\Big|\frac{\partial H}{\partial x}\Big| \leq C(1+|p|) \quad \text{ and } \quad \Big|\frac{\partial H}{\partial p}\Big| \leq C \quad \text{ in } \Omega \times \mathbb{R}^N\,,$$

for some constant $C > 0$. For every $\beta > 0$, we introduce the function u^β defined, for $x \in \Omega$ and $t \geq 0$, by

$$u^\beta(x,t) = \sup_{y\in\overline{\Omega}}\Big\{\mu u^*(y) - e^{K_\mu t}\frac{|x-y|^2}{\beta^2}\Big\}$$

with a parameter $\mu \in (0,1)$ devoted to tend to 1 and a constant $K_\mu > 0$ depending on μ to be chosen later.

We first recall that the inf- and sup-convolution procedures were introduced in the framework of Hamilton-Jacobi equations by Lasry and Lions ([17]); we refer the reader to [17] for the proof of the properties we use below. The trick of using the parameter μ was introduced in [7] while the introduction of the t-dependence seems to be necessary to perform the Barron-Jensen approach for such stationary problems (cf. [3]).

We have the

Lemma 3.1. *The function u^β is in $W^{1,\infty}(\Omega \times \mathbb{R}^+)$ and, for K_μ large enough, it is a viscosity subsolution of*

$$\frac{\partial u^\beta}{\partial t} + \frac{1}{2}|\sigma^T(x)Du^\beta|^2 - b(x)\cdot Du^\beta = 0 \quad in\ \ \Omega^\beta \times (0,T)\,, \tag{52}$$

for any $T > 0$, where $\Omega^\beta := \{x \in \Omega,\ d(x) > \sqrt{2\|u\|_\infty}\beta\}$.

Proof. We first remark that u^β is in $W^{1,\infty}(\Omega\times\mathbb{R}^+)$ by classical arguments (cf. [17]). Therefore, by Rademacher's Theorem, u^β is differentiable almost everywhere. Then, since the Hamiltonian of the equation (52) is convex with respect to the gradient variables $\big(\frac{\partial u}{\partial t}, Du\big)$, in order to prove that u^β is a viscosity subsolution of (52), it suffices to prove that the equation is satisfied in the almost everywhere sense, i.e., that the equation holds in the classical sense at any point of differentiability of u^β.

Using a classical result in nonsmooth analysis theory, it is easy to see that, if u^β is differentiable at $(x_0, t_0) \in \Omega^\beta \times (0, T)$ and if the supremum defining $u^\beta(x_0, t_0)$ is achieved at y_β, i.e.,

$$u^\beta(x_0, t_0) = \mu u^*(y_\beta) - e^{K_\mu t_0}\frac{|x_0 - y_\beta|^2}{\beta^2}\,,$$

then one has

$$Du^\beta(x_0, t_0) = 2e^{K_\mu t_0}\frac{y_\beta - x_0}{\beta^2}\,,$$

and

$$\frac{\partial u^\beta}{\partial t}(x_0, t_0) = -K_\mu e^{K_\mu t_0}\frac{|x_0 - y_\beta|^2}{\beta^2}\,.$$

Indeed, one easily shows that the quantities in the right-hand side are in the subdifferential of u^β at (x_0, t_0) but since u^β is differentiable at (x_0, t_0), this subdifferential is reduced to $\{(\frac{\partial u^\beta}{\partial t}(x_0, t_0), Du^\beta(x_0, t_0))\}$. Then it is easy to check that y_β is in the domain Ω and not on $\partial\Omega$ since x_0 is in Ω^δ and, by the definition of u^β, y_β is a maximum point of the map

$$y \mapsto u^*(y) - \frac{1}{\mu}e^{K_\mu t_0}\frac{|x_0 - y|^2}{\beta^2} \quad \text{in } \overline{\Omega}\,.$$

Thus, since the function u is a subsolution of the equation (17) and since $y_\beta \in \Omega$, we have

$$\frac{1}{2\mu}|\sigma^T(y_\beta)p|^2 - b(y_\beta)\cdot p \ \le\ 0 \tag{53}$$

with $p := 2\frac{y_\beta - x_0}{\beta^2}e^{K_\mu t_0} = Du^\beta(x_0, t_0)$. Then we deduce from (53)

$$\begin{aligned}\tfrac12|\sigma^T(x_0)p|^2 - b(x_0)\cdot p \le \tfrac12\big(1 - \frac{1}{\mu}\big)|\sigma^T(y_\beta)p|^2 + \tfrac12\big(|\sigma^T(x_0)p|^2 - |\sigma^T(y_\beta)p|^2\big)\\ + \big(b(y_\beta) - b(x_0)\big)\cdot p.\end{aligned}$$

Using the equality $a^2 - b^2 = 2b(a-b) + (a-b)^2$, we obtain

$$\begin{aligned}&\tfrac12|\sigma^T(x_0)p|^2 - b(x_0)\cdot p\\ &\le \tfrac12\big(1 - \frac{1}{\mu}\big)|\sigma^T(y_\beta)p|^2 + |\sigma^T(y_\beta)p|\big(|\sigma^T(x_0)p| - |\sigma^T(y_\beta)p|\big)\\ &\quad + \tfrac12\big(|\sigma^T(x_0)p| - |\sigma^T(y_\beta)p|\big)^2 + \big(b(y_\beta) - b(x_0)\big)\cdot p.\end{aligned} \tag{54}$$

And since σ and b are Lipschitz continuous, we get

$$|\sigma^T(x_0)p| - |\sigma^T(y_\beta)p| \ \leq \ C\frac{|x_0 - y_\beta|^2}{\beta^2}e^{K_\mu t_0}$$

and

$$\Big(b(y_\beta) - b(x_0)\Big) \cdot p \ \leq \ C\frac{|x_0 - y_\beta|^2}{\beta^2}e^{K_\mu t_0}\,.$$

We denote by M the right side of these inequalities. To estimate M, we remark that the supremum defining $u^\beta(x_0, t_0)$ being achieved at y_β, we have

$$\mu u^*(x_0) \leq \mu u^*(y_\beta) - e^{K_\mu t_0}\frac{|x_0 - y_\beta|^2}{\beta^2}\,,$$

and we deduce from this inequality

$$e^{K_\mu t_0}\frac{|x_0 - y_\beta|^2}{\beta^2} \leq 2\mu\|u\|_\infty \leq 2\|u\|_\infty\,,$$

and finally $M \leq 2C\|u\|_\infty$. Then the inequality (54) implies

$$\frac{1}{2}|\sigma^T(x_0)p|^2 - b(x_0)\cdot p \ \leq \ \frac{1}{2}\Big(1 - \frac{1}{\mu}\Big)|\sigma^T(y_\beta)p|^2 + M|\sigma^T(y_\beta)p| + \frac{1}{2}M^2 + M\,. \quad (55)$$

Moreover, the time derivative of u^β is

$$\frac{\partial u^\beta}{\partial t}(x_0, t_0) = -K_\mu e^{K_\mu t_0}\frac{|x_0 - y_\beta|^2}{\beta^2}, \quad \text{i.e.,}\ -\frac{K_\mu}{C}M\,.$$

Plugging this into (55), we get

$$\begin{aligned}\frac{\partial u^\beta}{\partial t}(x_0, t_0) &+ \frac{1}{2}|\sigma^T(x_0)Du^\beta(x_0, t_0)|^2 - b(x_0)\cdot Du^\beta(x_0, t_0)\\ &\leq \frac{1}{2}(1 - \frac{1}{\mu})|\sigma^T(y_\beta)p|^2 + M|\sigma^T(y_\beta)p| + \frac{1}{2}M^2 + M - \frac{K_\mu}{C}M\,.\end{aligned}$$

But, by the Cauchy-Schwarz inequality,

$$M|\sigma^T(y_\beta)p| \leq \frac{1}{2}(\frac{1}{\mu} - 1)|\sigma^T(y_\beta)p|^2 + \frac{1}{2}\frac{\mu}{1-\mu}M^2\,,$$

and, thanks to the estimate on M, we deduce that the right-hand side of the above inequality is negative for $K_\mu \geq C(1 + 2C\|u\|_\infty \frac{1}{1-\mu})$, and the lemma is proved. □

The end of the proof of Theorem 3.1 consists in following exactly the proof of [3] : a standard regularization procedure by convolution on u^β leads to a C^1 subsolution of (52) which is therefore a viscosity supersolution of

$$-\frac{\partial w}{\partial t} - \frac{1}{2}|\sigma^T(x)Dw|^2 + b(x)\cdot Dw = 0 \quad \text{in } \Omega^\beta \times (0,T)$$

(since it is C^1), and one concludes by letting the regularization parameters tend to 0 and then by letting μ tend to 1.

3.2. Proof of Theorem 1.2. The proof is based on the same ideas as in [9]. We first remark that the assumption (7) allows us to prove that the subsolution u satisfies the condition on the boundary in classical sense; i.e.,

$$u \;\le\; \varphi^* \quad \text{on } \partial\Omega\,.$$

We refer the reader to [2] and [7] for the proof of this claim. Therefore, since u is u.s.c. , this implies, for any $x \in \partial\Omega$,

$$\liminf_{y\to x, y\in\Omega} u(y) = u_*(x) \le \liminf_{y\to x, y\in\partial\Omega} u(y) \le (\varphi^*)_*(x) = \varphi_*(x),$$

because the function φ satisfies (18). Hence u satisfies the condition (19). As a consequence of this property, we have just to prove the second part of the theorem.

In order to do it, we consider $d(\cdot)$, the distance function to the boundary; since the boundary is of class $C^{1,1}$, we know that d is a $C^{1,1}$ function in some neighborhood of the boundary and more precisely in $\Omega^{2\delta}$ defined by $\Omega^{2\delta} := \{x \in \Omega\,;\; d(x) < 2\delta\}$, for some $\delta > 0$. Then we still denote by d a $C^{1,1}$ function on $\overline{\Omega}$ which is equal to the distance function to $\partial\Omega$ in $\Omega^\delta := \{x \in \Omega\,;\; d(x) < \delta\}$. We may assume that d is still Lipschitz continuous with a Lipschitz constant equal to 1 and we use the notation $n(x)$ for $-Dd(x)$ for $x \in \overline{\Omega}$ even if, for $x \notin \Omega^\delta$, the vector $n(x)$ may not be unitary, but $|n(x)| \le 1$.

We introduce the function u^α defined, for every $(x,t) \in \Omega \times \mathbb{R}^+$ and $\alpha > 0$, by

$$u^\alpha(x,t) = \inf_{y\in\overline{\Omega}} \Big\{ \mu u_*(y) + e^{-Kt}\phi_\alpha(x,y) + L(d(x) - d(y)) \Big\}$$

where μ is a parameter in $(0,1)$ and with

$$\phi_\alpha(x,y) := \frac{|x-y|^4}{\alpha} + R\frac{|x-y|^3}{\alpha}(d(x)-d(y)) + S\frac{|d(x)-d(y)|^4}{\alpha}\,,$$

where L, K, R and S are positive constants to be chosen later. In order to simplify the notation, we drop the dependence in μ for the function u^α and for the constant K.

We extend the function u^α up to the boundary $\partial\Omega$ by setting

$$u^\alpha(x,t) = \limsup_{y\to x, y\in\Omega} u^\alpha(y,t) \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}^+\,.$$

The properties of the function u^α are described in the following lemma.

Lemma 3.2. *For any $\mu \in (0,1)$, there exist constants L, K, R and S such that, for any $T > 0$ and for α small enough, the function u^α is Lipschitz continuous on $\overline{\Omega} \times [0,T]$ and it is a viscosity subsolution of*

$$\frac{\partial u}{\partial t} + H(x, Du) - Be^{\frac{Kt}{4}} \sqrt{[4]}\alpha = 0 \quad \text{in } \Omega \times (0,T]\,, \tag{56}$$

for some constant $B > 0$, independent of α. Moreover, on the boundary, u^α satisfies, in the classical sense,

$$u^\alpha \ \le \ \varphi^\alpha \quad \text{on } \partial\Omega \times [0,T]\,, \tag{57}$$

where the function φ^α is defined, for $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$ and $\alpha > 0$, by

$$\varphi^\alpha(x,t) \ = \ \inf_{y \in \partial\Omega} \Big\{ \varphi_*(y) + e^{-Kt} \frac{|x-y|^4}{\alpha} \Big\}\,.$$

The introduction of such functions u^α follows ideas explained in the preceding subsection; since u_* is a supersolution of $-H(x, Du) \ge 0$ by Theorem 3.1, the inf-convolution procedure allows us to obtain an approximate subsolution of the equation.

Moreover, the key property of the function u^α and one of the main consequences of the inf-convolution is the fact that the inequality (57) holds; indeed, we have replaced the discontinuous boundary data φ_* by φ^α which is Lipschitz continuous and which satisfies $\varphi^\alpha \le \varphi_*$ on $\partial\Omega$.

Of course, the main difficulty here comes from the boundary of the domain. This is why we introduce the terms with the distance function d in the definition of u^α; they allow us to obtain the classical properties of the inf-convolution procedure in a Barron-Jensen solutions context; the function u^α is indeed a subsolution of (56) in $\Omega \times (0,T)$ and not in a smaller domain as it would be expected. This fact relies strongly on the assumption (7) and the function ϕ_α is built in order to take advantage of this assumption.

We finally remark that the parameter μ and the time-dependent formulation in the definition of u^α are introduced for technical reasons as in the proof of Theorem 3.1. Roughly speaking, the time-dependent formulation of the inf-convolution allows us to treat discontinuous solutions for stationary problems (cf. [3]) and μ is used to solve the difficulties coming from the quadratic term of the Hamiltonian (cf. Theorem 3.1 and [7]).

We first conclude the proof of Theorem 1.2 using Lemma 3.2. To do so, we are going essentially to compare the functions u^α and v. We first remark that, since the function v does not depend on t and since $\varphi_* \ge \varphi^\alpha$, v is a supersolution of

$$\begin{aligned} \frac{\partial w}{\partial t} + H(x, Dw) = 0 \qquad & \text{in } \Omega \times (0,T]\,, \\ w = \varphi^\alpha \qquad & \text{on } \partial\Omega \times (0,T]\,, \end{aligned} \tag{58}$$

for any $T > 0$. In fact, we are not going to compare directly u^α and v because, in order to be able to take advantage of this comparison, the similar argument in [3] uses in an essential way the dependence on u of the Hamiltonian but, in our case, there is not such a dependence. In order to solve this difficulty, we are going to consider a strict subsolution of (17). Using such a strict subsolution is natural here since, despite the artificial timedependence, we are really proving a comparison result for a stationary problem. And for stationary problems, it is a classical method to use such strict subsolutions to take care of the lack of dependence on u in the equation.

Thanks to Lemma 2.5, there exist a function $\xi \in C^2(\overline{\Omega})$ and a constant $\nu > 0$ such that

$$H(x, D\xi(x)) \;\le\; -\nu \quad \text{ in } \overline{\Omega}\,. \tag{59}$$

Subtracting if necessary a constant from ξ, one may assume without loss of generality that $\xi \le \varphi^\alpha$ on $\partial\Omega$. Then, for all $\theta \in (0,1)$, we define the function $u^{\alpha,\theta}$ by

$$\begin{aligned} u^{\alpha,\theta}(x,t) := \theta\big(u^\alpha(x,t) - \frac{4B}{K} e^{\frac{Kt}{4}} \sqrt[4]{\alpha}\big) \\ + (1-\theta)\xi(x) + (1-\theta)\nu(t - T_\theta) \quad \text{on } \overline{\Omega} \times [0, T_\theta], \end{aligned}$$

where T_θ is chosen large enough in order to have

$$u^{\alpha,\theta} \;\le\; v \quad \text{ on } \overline{\Omega} \times \{0\}\,. \tag{60}$$

Indeed, on $\overline{\Omega} \times \{0\}$, we have

$$u^{\alpha,\theta}(x,0) = \theta\Big(u^\alpha(x,0) - \frac{4B}{K}\sqrt[4]{\alpha}\Big) + (1-\theta)\xi(x) - (1-\theta)\nu T_\theta$$

and since the functions u^α, ξ and v are bounded, we may choose the parameter T_θ large enough to obtain the inequality (60). Moreover, since $\xi \le \varphi^\alpha$ on $\partial\Omega$, it is clear enough that $u^{\alpha,\theta} \le \varphi^\alpha$ on $\partial\Omega \times [0, T_\theta]$. Finally, because of (59) and since the Hamiltonian is convex in (56), an easy computation shows that $u^{\alpha,\theta}$ satisfies

$$\frac{\partial u^{\alpha,\theta}}{\partial t} + H(x, Du^{\alpha,\theta}) \;\le\; 0 \quad \text{ in } \Omega \times (0, T_\theta]\,.$$

Therefore, $u^{\alpha,\theta}$ is a viscosity subsolution of (58).

Finally, since $u^{\alpha,\theta}$ and φ^α are Lipschitz continuous, an easy adaptation of the comparison result of [6] for the Cauchy problem (58) (see also [18] and [20]) yields $u^{\alpha,\theta} \le v$ in $\Omega \times [0, T_\theta]$, because of the inequality (60) on $\overline{\Omega} \times \{0\}$. We consider this inequality at time T_θ. The last term in the definition of $u^{\alpha,\theta}$ vanishes and, letting α tend to 0, we get

$$\theta\mu u_* + (1-\theta)\xi \;\le\; v \quad \text{ in } \Omega\,.$$

It remains to let $\theta \to 1$ and $\mu \to 1$. And the proof is complete. $\square$

We turn to the proof of Lemma 3.2.

Proof of Lemma 3.2. It is easy to check that u^α is Lipschitz continuous in space and time variables on $\overline{\Omega} \times [0,T]$. In fact, this regularization of μu_* is one of the main reasons for performing the inf-convolution procedure (cf. [9] and [17]). Other classical facts are the following: since u_* is l.s.c. , for any $x \in \Omega$ and $t \in [0,T]$, there exists a point $y_\alpha \in \overline{\Omega}$ such that

$$u^\alpha(x,t) = \mu u_*(y_\alpha) + e^{-Kt}\phi_\alpha(x,y_\alpha) + L(d(x) - d(y_\alpha)). \tag{61}$$

Since $\phi_\alpha(x,x) = 0$ for $x \in \overline{\Omega}$, we have from the definition of u^α $u^\alpha \le \mu u$ on $\overline{\Omega} \times [0,T_\theta]$. Combining this with (61), we obtain

$$e^{-Kt}\phi_\alpha(x,y_\alpha) \ \le\ 2\|u\|_\infty + 2L\|d\|_\infty\,. \tag{62}$$

Moreover, by using Young's inequality

$$R\frac{|x-y|^3}{\alpha}|d(x)-d(y)| \ \le\ \frac{3}{4}\frac{|x-y|^4}{\alpha} + \frac{1}{4}R^4\frac{|d(x)-d(y)|^4}{\alpha}\,,$$

and taking $S > \frac{1}{4}R^4$, we get $\phi_\alpha(x,y) \ge \frac{1}{4}\frac{|x-y|^4}{\alpha}$. Then this implies with (62) that

$$|x-y_\alpha| \ \le\ \bar{B}e^{\frac{Kt}{4}}\sqrt{[}4]{\alpha} \tag{63}$$

with $\bar{B} := 2\sqrt{[}4]{\|u\|_\infty + L\|d\|_\infty}$.

Since the function u^α is Lipschitz continuous in $\Omega \times (0,T)$, it is differentiable almost everywhere by Rademacher's Theorem and, as in the proof of Theorem 3.1, in order to prove that u^α is a viscosity subsolution of (56) in $\Omega \times (0,T)$, it suffices to show that u^α satisfies the equation (56) in the almost everywhere sense, since the Hamiltonian is convex (cf. [18]). And then, if u^α is a subsolution of (56) in $\Omega \times (0,T)$, u^α is also a viscosity subsolution of (56) in $\Omega \times (0,T]$, by a classical property of the Cauchy problem (cf. [7]).

If $(x_0,t_0) \in \Omega \times (0,T)$ is a point where u^α is differentiable then, by nonsmooth analysis arguments analogous to the ones we already use in the proof of Theorem 3.1, we have

$$Du^\alpha(x_0,t_0) = e^{-Kt_0}D_x\phi_\alpha(x_0,y_\alpha) - Ln(x_0)\,,$$

and

$$\frac{\partial u^{\alpha,\beta}}{\partial t}(x_0,t_0) = -Ke^{-Kt_0}\phi_\alpha(x_0,y_\alpha)\,,$$

where y_α is a point of $\overline{\Omega}$ such that

$$u^\alpha(x_0,t_0) = \mu u_*(y_\alpha) + e^{-Kt_0}\phi_\alpha(x_0,y_\alpha) + L(d(x_0) - d(y_\alpha))\,.$$

By the definition of u^α, y_α is a minimum point of $u_* - \frac{1}{\mu}\psi$ with

$$\psi(y) \ :=\ -e^{-Kt_0}\phi_\alpha(x_0,y) - L(d(x_0) - d(y))\,.$$

Then, in order to apply Theorem 3.1, we need the following lemma:

Lemma 3.3. *There exist constants L and R depending only on the functions b, σ and d such that, for ε small enough and for all $(x_0,t_0)\in\Omega\times(0,T)$, we have $y_\alpha\in\Omega$.*

We postpone the proof of this lemma. Then Theorem 3.1 implies

$$-\frac{1}{2\mu}|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2+b(y_\alpha)\cdot D\psi(y_\alpha)\ \geq\ 0\,, \tag{64}$$

where $D\psi(y_\alpha)=p_\alpha-\lambda_\alpha n(y_\alpha)$ with

$$p_\alpha\ :=\ e^{-Kt_0}\Big[4(x_0-y_\alpha)\frac{|x_0-y_\alpha|^2}{\alpha}+3R(x_0-y_\alpha)\frac{|x_0-y_\alpha|}{\alpha}(d(x_0)-d(y_\alpha))\Big]$$

and

$$\lambda_\alpha\ :=\ e^{-Kt_0}\Big[R\frac{|x_0-y_\alpha|^3}{\alpha}+4S(d(x_0)-d(y_\alpha))\frac{|d(x_0)-d(y_\alpha)|^2}{\alpha}\Big]+L\,.$$

Then, with these notations, we have $Du^\alpha(x_0,t_0)=p_\alpha-\lambda_\alpha n(x_0)$. Using the inequality (64), we get

$$\begin{aligned}&\frac{1}{2}|\sigma^T(x_0)Du^\alpha(x_0,t_0)|^2-b(x_0)\cdot Du^\alpha(x_0,t_0)\\ &\leq\frac{1}{2}(1-\frac{1}{\mu})|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2+\frac{1}{2}\Big(|\sigma^T(x_0)Du^\alpha(x_0,t_0)|^2-|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2\Big)\\ &\quad+\Big(b(y_\alpha)\cdot D\psi(y_\alpha)-b(x_0)\cdot Du^\alpha(x_0,t_0)\Big).\end{aligned} \tag{65}$$

Because of the quadratic term of the Hamiltonian, we proceed as in Theorem 3.1. First we estimate $|Du^\alpha(x_0,t_0)-D\psi(y_\alpha)|$:

$$\begin{aligned}&|Du^\alpha(x_0,t_0)-D\psi(y_\alpha)|\leq|\lambda_\alpha||n(x_0)-n(y_\alpha)|\leq|\lambda_\alpha|\|Dn\|_\infty|x_0-y_\alpha|\\ &\leq\|Dn\|_\infty\Big(e^{-Kt_0}\Big[R\frac{|x_0-y_\alpha|^4}{\alpha}+4S\frac{|d(x_0)-d(y_\alpha)|^3}{\alpha}|x_0-y_\alpha|\Big]+L|x_0-y_\alpha|\Big).\end{aligned}$$

Using the estimate (63) and Young's inequality, we obtain after tedious but straightforward computations

$$|Du^\alpha(x_0,t_0)-D\psi(y_\alpha)|\ \leq\ K_1(e^{-Kt_0}\phi_\alpha(x_0,y_\alpha)+\bar{B}e^{\frac{Kt_0}{4}}\sqrt{[}4]\alpha)$$

for some constant K_1, independent of K.

Next, using the Lipschitz continuity of σ, we compute

$$|\sigma^T(x_0)-\sigma^T(y_\alpha)||D\psi(y_\alpha)|\ \leq\ C|x_0-y_\alpha|(|p_\alpha|+|\lambda_\alpha|)\,.$$

Again, using (63) and Young's inequality, we deduce

$$|\sigma^T(x_0) - \sigma^T(y_\alpha)||D\psi(y_\alpha)| \;\le\; K_2(e^{-Kt_0}\phi_\alpha(x_0,y_\alpha) + \bar{B}e^{\frac{Kt_0}{4}}\sqrt[4]{\alpha})\,,$$

for some constant K_2, independent of K.

Finally, we also get

$$|b(x_0) - b(y_\alpha)||D\psi(y_\alpha)| \;\le\; K_3(e^{-Kt_0}\phi_\alpha(x_0,y_\alpha) + \bar{B}e^{\frac{Kt_0}{4}}\sqrt[4]{\alpha})\,,$$

for some constant K_3, independent of K. Then we introduce

$$M \;:=\; \bar{K}(e^{-Kt_0}\phi_\alpha(x_0,y_\alpha) + \bar{B}e^{\frac{Kt_0}{4}}\sqrt[4]{\alpha})$$

where

$$\bar{K} \;:=\; \max\Big\{(\|\sigma\|_\infty + \|b\|_\infty)K_1,\; K_2,\; K_3\Big\}\,.$$

And, using the same arguments as in the proof of Theorem 3.1, we deduce from (65)

$$\begin{aligned} &H(x_0, Du^\alpha(x_0,t_0)) \\ &\quad\le \frac{1}{2}(1-\frac{1}{\mu})|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2 + 2M|\sigma^T(y_\alpha)D\psi(y_\alpha)| + 2M^2 + 2M\,. \end{aligned} \tag{66}$$

It remains to introduce the derivative of u^α in time; i.e.,

$$\frac{\partial u^{\alpha,\beta}}{\partial t}(x_0,t_0) = -Ke^{-Kt_0}\phi_\alpha(x_0,y_\alpha)\,.$$

Combining this with (66), we get

$$\begin{aligned} \frac{\partial u^\alpha}{\partial t}(x_0,t_0) + H(x_0, Du^\alpha(x_0,t_0)) &\le \frac{1}{2}(1-\frac{1}{\mu})|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2 \\ &\quad + 2M|\sigma^T(y_\alpha)D\psi(y_\alpha)| + 2M^2 + 2M - Ke^{-Kt_0}\phi_\alpha(x_0,y_\alpha). \end{aligned}$$

Now we use the Cauchy-Schwarz inequality,

$$2M|\sigma^T(y_\alpha)D\psi(y_\alpha)| \le \frac{1}{2}(\frac{1}{\mu}-1)|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2 + \frac{2\mu}{1-\mu}M^2\,,$$

to obtain

$$\frac{\partial u^\alpha}{\partial t}(x_0,t_0) + H(x_0, Du^\alpha(x_0,t_0)) \le \frac{2}{1-\mu}M^2 + 2M - Ke^{-Kt_0}\phi_\alpha(x_0,y_\alpha)\,.$$

But, by (62), $M \leq \tilde{K}$ for some constant $\tilde{K}$ depending only on u and d and therefore taking $K = \frac{2\tilde{K}}{1-\mu} + 2$, yields

$$\frac{\partial u^\alpha}{\partial t}(x_0, t_0) + H(x_0, Du^\alpha(x_0, t_0)) \leq Be^{\frac{Kt_0}{4}} \sqrt{[4]}\alpha\,,$$

with $B = K\bar{K}\bar{B}$. and the proof is complete. □

Before proving the property of u^α on the boundary, we give the

Proof of Lemma 3.3. Since the functions b, σ and n are Lipschitz continuous on $\overline{\Omega}$, we may replace the assumption (7) by

$$\forall x \in \Omega^{\delta'}, \quad |\sigma^T(x)n(x)| \;\geq\; \frac{1}{2}\zeta > 0 \quad \text{ or } \quad b(x) \cdot n(x) \geq \frac{1}{2}\zeta > 0\,, \tag{67}$$

for some constant δ' small enough where $\Omega^{\delta'} := \{x \in \Omega;\ d(x) < \delta'\}$. To simplify matters and without loss of generality we may assume that $\delta = \delta'$ (otherwise we argue with $\delta'' = \min\{\delta, \delta'\}$).

If $x_0 \notin \Omega^\delta$, by using the inequality (63), we get

$$d(y_\alpha) \geq d(x_0) - |x_0 - y_\alpha| \geq \delta - \bar{B}e^{\frac{Kt_0}{4}} \sqrt{[4]}\alpha.$$

Then, if α is small enough, we have $d(y_\alpha) > 0$.

If $x_0 \in \Omega^\delta$, we claim that

$$d(x_0) \;\leq\; d(y_\alpha)\,. \tag{68}$$

This inequality gives the answer since $x_0 \in \Omega$.

In order to prove (68), we argue by contradiction assuming that the point y_α is such that

$$d(x_0) \;>\; d(y_\alpha)\,, \tag{69}$$

and we first consider the case when $d(y_\alpha) > 0$, i.e., when $y_\alpha \in \Omega$. By Theorem 3.1, we get

$$-\frac{1}{2\mu}|\sigma^T(y_\alpha)D\psi(y_\alpha)|^2 + b(y_\alpha) \cdot D\psi(y_\alpha) \;\geq\; 0\,, \tag{70}$$

where $D\psi(y_\alpha) = p_\alpha - \lambda_\alpha n(y_\alpha)$ with the notations of Lemma 3.2.

Now we are going to use the assumption (7). To do so, we need the

Lemma 3.4. *Let H be the Hamiltonian given by*

$$H(x,p) = \frac{1}{2}|\sigma^T(x)p|^2 - b(x) \cdot p \quad \text{on } \overline{\Omega} \times \mathbb{R}^N\,.$$

If σ and b are Lipschitz continuous on $\overline{\Omega}$ and if (7) holds then there exists a constant E such that, for $x \in \Omega^\delta$ and for $p \in \mathbb{R}^N$ such that, if $H(x, p - \lambda n(x)) \leq 0$ for some $\lambda \in \mathbb{R}^+$, we have $\lambda \leq E(1 + |p|)$.

Proof. By definition of Ω^δ, the assumption (7) holds at the point x. Thus we have two cases to consider. If $b(x)\cdot n(x) \geq \frac{1}{2}\zeta$, then the inequality $H(x, p-\lambda n(x)) \leq 0$ gives

$$0 \leq b(x)\cdot(p-\lambda n(x)) \leq \|b\|_\infty |p| - \frac{1}{2}\zeta\lambda. \tag{71}$$

And thus $\lambda \leq \frac{2\|b\|_\infty}{\zeta}|p|$. Otherwise, if $|\sigma^T(x)n(x)| \geq \frac{1}{2}\zeta$, then the inequality $H(x, p-\lambda n(x)) \leq 0$ implies

$$\begin{aligned}
&\frac{1}{2}|\sigma^T(x)(p-\lambda n(x))|^2 \leq b(x)\cdot(p-\lambda n(x))\\
&\frac{1}{2}|\sigma^T(x)(p-\lambda n(x))|^2 \leq \|b\|_\infty(|p|+\lambda)\\
&\frac{1}{\sqrt{2}}\Big(\frac{1}{2}\zeta\lambda - \|\sigma\|_\infty|p|\Big) \leq \sqrt{\|b\|_\infty(|p|+\lambda)}\,.
\end{aligned}$$

But for any $\gamma > 0$ the right-hand side is estimated by

$$\sqrt{\|b\|_\infty(|p|+\lambda)} \leq \gamma + \frac{\|b\|_\infty(|p|+\lambda)}{4\gamma}\,.$$

We choose $\gamma = \frac{4\sqrt{2}\|b\|_\infty}{\zeta}$ and we obtain $\frac{1}{2\sqrt{2}}\zeta\lambda \leq \tilde{C}(|p|+1)$, for some constant $\tilde{C}$ depending only on b, σ and ζ. And the proof is complete. □

Then this lemma implies a contradiction with the inequality (70) if we have $\lambda_\alpha > E(1+|p_\alpha|)$ and we are now going to show that this last inequality holds if $d(x_0)-d(y_\alpha) > 0$. Since $d(x_0)-d(y_\alpha) > 0$, using Young's inequality, we obtain

$$3R\frac{|x_0-y_\alpha|^2}{\alpha}|d(x_0)-d(y_\alpha)| \;\leq\; 2\frac{|x_0-y_\alpha|^3}{\alpha} + R^3\frac{|d(x_0)-d(y_\alpha)|^3}{\alpha}\,.$$

Now we estimate the quantity

$$\begin{aligned}
&E(1+|p_\alpha|)-\lambda_\alpha\\
&\quad\leq E-L+e^{-Kt_0}\big[(6E-R)\frac{|x_0-y_\alpha|^3}{\alpha} + (R^3E-4S)\frac{|d(x_0)-d(y_\alpha)|^3}{\alpha}\big].
\end{aligned}$$

And the right-hand side of this equality is negative if $R > 6E$ and $L > E$ since we already know that $S > \frac{1}{4}R^4$. Hence we get a contradiction with (70).

It remains to consider the case when the minimum point y_α is on the boundary $\partial\Omega$. We set

$$\chi_\varepsilon(y) \;:=\; u_*(y) - \frac{1}{\mu}\psi(y) + |y-y_\alpha|^2 + \frac{\varepsilon}{d(y)}\,,$$

for every $y \in \Omega$ and $\varepsilon > 0$. We add the term $|\cdot - y_\alpha|^2$ in order to be sure that the point y_α is a strict local minimum point of $u_*(\cdot) - \frac{1}{\mu}\psi(\cdot) + |\cdot - y_\alpha|^2$. Then, there exists a sequence $(y_\varepsilon)_\varepsilon$ of local minimum points of χ_ε such that

$$\lim_{\varepsilon\to 0} y_\varepsilon = y_\alpha \quad \text{ and } \quad \lim_{\varepsilon\to 0} \chi_\varepsilon(y_\varepsilon) = u_*(y_\alpha) - \frac{1}{\mu}\psi(y_\alpha)\,.$$

Moreover, since the point y_ε is in Ω, Theorem 3.1 implies

$$-H\big(y_\varepsilon, \frac{1}{\mu}D\psi(y_\alpha) - 2(y_\varepsilon - y_\alpha) - \frac{\varepsilon}{d(y_\varepsilon)^2}n(y_\varepsilon)\big) \geq 0. \tag{72}$$

But, since $d(y_\varepsilon) \to 0$ as $\varepsilon \to 0$, we have $d(y_\varepsilon) < d(x_0)$ for ε small enough. Then using the same arguments as in the preceding case, we get a contradiction with the equation (72) since the term $|y_\varepsilon - y_\alpha|$ tends to 0 and since the additional term in the normal direction is negative. Hence the proof is complete. □

In order to conclude the proof of Lemma 3.2, it remains to prove that $u^\alpha \leq \varphi^\alpha$ on $\partial\Omega \times [0,T]$. For any $y \in \partial\Omega$, there exists a sequence $(y_p)_p$ of points in Ω such that $y_p \to y$ and

$$\lim_p u(y_p) = \liminf_{z\to y, z\in\Omega} u(z)\,.$$

By the property (19),

$$\lim_p u(y_p) \;\leq\; \varphi_*(y)\,.$$

For any other point $x \in \partial\Omega$, we may construct a sequence $(x_p)_p$ in Ω defined, for every p, by $x_p := x - d(y_p)n(x)$. Since we have $d(y_p) \to 0$ as $n \to \infty$, the sequence $(x_p)_p$ converges to the point x. Notice also that, for p large enough, $d(x_p) = d(y_p)$.

Thus, by the definition of u^α, we have

$$\begin{aligned} u^\alpha(x_p,t) &\leq \mu u_*(y_p) + e^{-Kt}\phi_\alpha(x_p,y_p) + L(d(x_p) - d(y_p)) \\ &\leq \mu u_*(y_p) + e^{-Kt}\Big[\frac{|x_p - y_p|^4}{\alpha} + R\frac{|x_p-y_p|^3}{\alpha}(d(x_p) - d(y_p)) \\ &\quad + S\frac{|d(x_p) - d(y_p)|^4}{\alpha}\Big] + L(d(x_p) - d(y_p)). \end{aligned}$$

And letting $p \to \infty$, this implies

$$u^\alpha(x,t) \leq \mu\varphi_*(y) + e^{-Kt}\frac{|x-y|^4}{\alpha}\,.$$

We may assume without loss of generality that φ_* is positive since, if this is not the case, we may add a positive constant to each functions u, v and φ since the Hamiltonian does not depend on u. Therefore, we obtain

$$u^\alpha(x,t) \;\leq\; \varphi_*(y) + e^{-Kt}\frac{|x-y|^4}{\alpha}\,.$$

Hence, since x and y are arbitrary, we get $u^\alpha \le \varphi^\alpha$ on $\partial\Omega \times [0,T]$. $\square$

4. Appendix: the gradient estimate. The goal of this appendix is to present a pointwise estimate on the first derivatives of a solution v of

$$-\frac{\varepsilon^2}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} - \sum_i b_i(x)\frac{\partial v}{\partial x_i} = 0 \quad \text{in } \Omega \tag{73}$$

where we emphasize the dependence on the parameter ε and on the distance to the boundary.

Theorem 4.1. *We assume that the functions a and b are Lipschitz-continuous functions on $\overline{\Omega}$ and that*

$$\exists \gamma > 0 \quad \textit{such that} \quad \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \ \ge\ \gamma|\xi|^2 \quad \forall \xi \in \mathbb{R}^N,\ \forall x \in \Omega\,. \tag{74}$$

If $v \in C^2(\Omega)$ is a solution of (73) and if $0 < \varepsilon \le 1$, then for $x \in \Omega$, one has

$$|Dv(x)| \ \le\ C_1 + C_2\frac{\varepsilon^2}{d(x)},$$

where C_1, C_2 are positive constants and $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Proof. This proof consists in reproducing the arguments of the related proof given in [14]; we just detail the arguments to get the dependence on ε and of d. We give the proof for the convenience of the reader.

To simplify the exposition, we only give the proof in the case when a, b and v are smooth functions in Ω; the general case can then be obtained by standard approximation arguments. We are going to apply the classical Bernstein's method.

Let Ω' be a strict subset of Ω ($\Omega' \subset\subset \Omega$). We write $\delta = \operatorname{dist}(\Omega', \partial\Omega)$ (> 0). We consider a regular function η which is equal to 1 in Ω' and 0 outside a neighborhood of Ω'.

We define $z \equiv \eta^4|Dv|^2$. Let x_0 be a maximum point of z on $\overline{\Omega}$. We may assume that $\max_{\overline{\Omega}} z > 0$. Because of the function η, x_0 is in Ω. Therefore,

$$\frac{\partial z}{\partial x_i}(x_0) = 0\,.$$

In order to simplify the notation, we drop the reference to the point x_0. We compute

$$\frac{\partial z}{\partial x_i} = 4\eta^3\frac{\partial \eta}{\partial x_i}|Dv|^2 + 2\eta^4\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial x_i \partial x_j} &= 12\eta^2\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}|Dv|^2 + 4\eta^3\frac{\partial^2 \eta}{\partial x_i \partial x_j}|Dv|^2 + 8\eta^3\frac{\partial \eta}{\partial x_i}\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_j \partial x_k}\\ &\quad + 8\eta^3\frac{\partial \eta}{\partial x_j}\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k} + 2\eta^4\sum_k \frac{\partial^2 v}{\partial x_j \partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k} + 2\eta^4\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}.\end{aligned}$$

Using $\frac{\partial z}{\partial x_i} = 0$, we obtain

$$\eta^3\frac{\partial \eta}{\partial x_i}\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_j \partial x_k} = -2\eta^2\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}|Dv|^2.$$

And then, we do some simplifications:

$$\frac{\partial^2 z}{\partial x_i \partial x_j} = -20\eta^2 \frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j}|Dv|^2 + 4\eta^3 \frac{\partial^2 \eta}{\partial x_i \partial x_j}|Dv|^2 + 2\eta^4 \sum_k \frac{\partial^2 v}{\partial x_j \partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k} + 2\eta^4 \sum_k \frac{\partial v}{\partial x_k}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}. \tag{75}$$

Moreover, at the maximum point x_0, we have

$$0 \le -\frac{\varepsilon^2}{2}\sum_{i,j} a_{i,j}\frac{\partial^2 z}{\partial x_i \partial x_j}.$$

Combining this with (75), we get

$$0 \le 10\varepsilon^2\eta^2|Dv|^2 \sum_{i,j} a_{i,j}\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j} - 2\varepsilon^2\eta^3|Dv|^2 \sum_{i,j} a_{i,j}\frac{\partial^2 \eta}{\partial x_i \partial x_j} - \varepsilon^2\eta^4 \sum_{i,j,k} a_{i,j}\frac{\partial^2 v}{\partial x_j \partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k} + 2\eta^4 \sum_{i,j,k}\Big(-\frac{\varepsilon^2}{2}\frac{\partial v}{\partial x_k} a_{i,j}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}\Big).$$

We note that

$$\frac{\partial}{\partial x_k}\Big(-\frac{\varepsilon^2}{2}\sum_{i,j} a_{i,j}\frac{\partial^2 v}{\partial x_i \partial x_j}\Big) = -\frac{\varepsilon^2}{2}\sum_{i,j}\frac{\partial a_{i,j}}{\partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i,j}\Big(-\frac{\varepsilon^2}{2} a_{i,j}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k}\Big),$$

and we use the ellipticity condition (74); we obtain

$$0 \le 10\varepsilon^2\eta^2|Dv|^2 \sum_{i,j} a_{i,j}\frac{\partial \eta}{\partial x_i}\frac{\partial \eta}{\partial x_j} - 2\varepsilon^2\eta^3|Dv|^2 \sum_{i,j} a_{i,j}\frac{\partial^2 \eta}{\partial x_i \partial x_j} - \varepsilon^2\eta^4\gamma \sum_k\Big|\frac{\partial}{\partial x_k}Dv\Big|^2 + \varepsilon^2\eta^4 \sum_{i,j,k}\frac{\partial v}{\partial x_k}\frac{\partial a_{i,j}}{\partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_j} + 2\eta^4 \sum_{i,j,k}\frac{\partial v}{\partial x_k}\frac{\partial}{\partial x_k}\Big(-\frac{\varepsilon^2}{2} a_{i,j}\frac{\partial^2 v}{\partial x_i \partial x_j}\Big). \tag{76}$$

Since v satisfies (73), we get

$$\frac{\partial}{\partial x_k}\Big(-\frac{\varepsilon^2}{2} a_{i,j}\frac{\partial^2 v}{\partial x_i \partial x_j}\Big) = \frac{\partial}{\partial x_k}\Big(-\frac{1}{2}\sum_{i,j} a_{i,j}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_i b_i\frac{\partial v}{\partial x_i}\Big) \tag{77}$$

$$= -\frac{1}{2}\sum_{i,j}\frac{\partial a_{i,j}}{\partial x_k}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} - \sum_{i,j} a_{i,j}\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_j \partial x_k} + \sum_i \frac{\partial b_i}{\partial x_k}\frac{\partial v}{\partial x_i} + \sum_i b_i \frac{\partial^2 v}{\partial x_i \partial x_k}.$$

Then we may assume that

$$\Big\|\frac{\partial \eta}{\partial x_i}\Big\|_{L^\infty} \le C\frac{\|\eta\|_\infty}{\delta} \quad \text{and} \quad \Big\|\frac{\partial^2 \eta}{\partial x_i \partial x_j}\Big\|_{L^\infty} \le C\frac{\|\eta\|_\infty}{\delta^2},$$

where C is a generic positive constant.

Combining this with (76) and (77), we find

$$0 \le C\frac{\varepsilon^2}{\delta^2}(\eta^2|Dv|^2 + \eta^3|Dv|^2) - \varepsilon^2\eta^4\gamma|D^2 v|^2 + C\varepsilon^2\eta^4|Dv||D^2 v| - \eta^4 \sum_{i,j,k}\frac{\partial a_{i,j}}{\partial x_k}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k} - 2\eta^4 \sum_{i,j}\Big(a_{i,j}\frac{\partial v}{\partial x_i}\sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_j \partial x_k}\Big) + 2\eta^4 \sum_{i,k}\frac{\partial b_i}{\partial x_k}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_k} + 2\eta^4 \sum_i \Big(b_i \sum_k \frac{\partial v}{\partial x_k}\frac{\partial^2 v}{\partial x_i \partial x_k}\Big). \tag{78}$$

Using $\frac{\partial z}{\partial x_i} = 0$ again, we obtain

$$
\begin{aligned}
0 \leq C\frac{\varepsilon^2}{\delta^2}(1+\eta)\eta^2|Dv|^2 - \varepsilon^2\eta^4\gamma|D^2v|^2 + C\varepsilon^2\eta^4|Dv||D^2v| + C\eta^4|Dv|^3 \\
+ 4\eta^3\sum_{i,j} a_{i,j}\frac{\partial v}{\partial x_i}\frac{\partial \eta}{\partial x_j}|Dv|^2 + C\eta^4|Dv|^2 - 4\eta^3\sum_i b_i\frac{\partial \eta}{\partial x_i}|Dv|^2\,.
\end{aligned}
\tag{79}
$$

We apply Young's inequality to get

$$
C\varepsilon^2\eta^4|Dv||D^2v| \ \leq\ \frac{\varepsilon^2\eta^4\gamma}{2}|D^2v|^2 + \frac{\varepsilon^2\eta^4C^2}{2\gamma}|Dv|^2\,.
$$

Plugging this in (79), we obtain

$$
\begin{aligned}
0 \leq C\frac{\varepsilon^2}{\delta^2}(1+\eta)\eta^2|Dv|^2 - \varepsilon^2\eta^4\gamma|D^2v|^2 + \frac{\varepsilon^2\eta^4\gamma}{2}|D^2v|^2 + \frac{\varepsilon^2\eta^4C^2}{2\gamma}|Dv|^2 \\
+ C\eta^4|Dv|^3 + C\eta^3\frac{1}{\delta}|Dv|^3 + C\eta^4|Dv|^2 + C\eta^3\frac{1}{\delta}|Dv|^2\,.
\end{aligned}
$$

Finally

$$
\frac{\gamma}{2}\varepsilon^2\eta^4|D^2v|^2 \ \leq\ C\Big(\frac{\varepsilon^2}{\delta^2} + \frac{1}{\delta} + 1\Big)\eta^2|Dv|^2 + C\Big(\frac{1}{\delta} + 1\Big)\eta^3|Dv|^3\,. \tag{80}
$$

Furthermore, we deduce from (73)

$$
\Big|\frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} - \sum_i b_i(x)\frac{\partial v}{\partial x_i}\Big|^2 = \Big|\frac{\varepsilon^2}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}\Big|^2 \leq C\varepsilon^4|D^2v|^2 \quad \text{in } \Omega.
$$

And using (74), we obtain after straightforward computations

$$
|Dv|^4 \ \leq\ C(1+\varepsilon^4|D^2v|^2) \quad \text{in } \Omega\,.
$$

Combining this with the estimate (80), we obtain

$$
|Dv|^4 \ \leq\ C + \Big(\frac{\varepsilon^2}{\delta} + 1\Big)^2|Dv|^2 + \Big(\frac{\varepsilon^2}{\delta} + 1\Big)|Dv|^3 \quad \text{in } \Omega'\,.
$$

Then it is easy to show

$$
|Dv| \ \leq\ C + C\frac{\varepsilon^2}{\delta} \quad \text{in } \Omega\,,
$$

and the result follows.

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