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CLASSICAL SOLUTIONS FOR HELE-SHAW MODELS WITH SURFACE TENSION

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Abstract. It is shown that surface tension effects on the free boundary are regularizing for Hele-Shaw models. This implies, in particular, existence and uniqueness of classical solutions for a large class of initial data. As a consequence, we give a rigorous proof of the fact that homogeneous Hele-Shaw flows with positive surface tension are volume preserving and area shrinking.

1. Introduction. Recently, N. Alikakos, P. Bates, and X. Chen ([1]) proved that level surfaces of solutions to the Cahn-Hilliard equation tend to solutions of the two-phase Hele-Shaw problem with surface tension under the assumption that classical solutions of the latter exist. In the present paper we show, in particular, that the above assumption is in fact satisfied; i.e., we prove existence and uniqueness of classical solutions to one- and two-phase Hele-Shaw problems with surface tension.

It should be mentioned that even weak solutions to Hele-Shaw problems with surface tension were not known to exist in higher space dimensions. For the two-phase Hele-Shaw problem in two dimensions, X. Chen ([7]) recently proved the local existence of a weak solution for an arbitrary (smooth) initial curve, and global existence of a weak solution when the initial curve is nearly circular. It should be emphasized that there are no uniqueness results in [7]. Also in the two-dimensional case, P. Constantin and M. Pugh ([9]) established global analytic solutions for the one-phase problem, provided the initial curves are small analytic perturbations of circles. Finally, still in two space dimensions and for a particular geometry, i.e., for strip-like domains, J. Duchon and R. Robert ([12]) established the existence of local solutions for the one-phase problem.

Our approach works for one- and two-phase problems in any space dimension and we obtain classical solutions and uniqueness for rather general initial data.

We first consider the one-phase problem. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and assume that its boundary $\partial\Omega$ consists of two disjoint nonempty components J and Γ. Later on, we will model over the exterior component Γ a moving interface,

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whereas the interior component J describes a fixed portion of the boundary. Let ν denote the outer unit normal field over Γ and fix $\alpha \in (0,1)$. Given $a > 0$, let

$$\mathfrak{A} := \{\rho \in C^{2+\alpha}(\Gamma) : \|\rho\|_{C^1(\Gamma)} < a\}.$$

For each $\rho \in \mathfrak{A}$ define the map

$$\theta_\rho := id_\Gamma + \rho\nu$$

and let $\Gamma_\rho := im(\theta_\rho)$ denote its image. Obviously, θ_ρ is a $C^{2+\alpha}$ diffeomorphism mapping Γ onto Γ_ρ, provided $a > 0$ is chosen sufficiently small. In addition, we assume that $a > 0$ is small enough such that Γ_ρ and J are disjoint for each $\rho \in \mathfrak{A}$. Let Ω_ρ denote the domain in $\mathbb{R}^n$ being diffeomorphic to Ω and whose boundary is given by J and Γ_ρ. To describe the evolution of the hypersurface Γ_ρ, fix some $T > 0$. Then each map $\rho : [0,T] \to \mathfrak{A}$ defines a collection of hypersurfaces $\Gamma_{\rho(t)}$ and domains $\Omega_{\rho(t)},\ t \in [0,T]$. Let us also introduce the following generalized parabolic cylinder

$$\Omega_{\rho,T} := \big\{(x,t) \in \mathbb{R}^n \times (0,T] : x \in \Omega_{\rho(t)}\big\} = \bigcup_{t\in(0,T]} \big(\Omega_{\rho(t)} \times \{t\}\big)$$

and, correspondingly,

$$\Gamma_{\rho,T} := \big\{(x,t) \in \mathbb{R}^n \times (0,T] : x \in \Gamma_{\rho(t)}\big\} = \bigcup_{t\in(0,T]} \big(\Gamma_{\rho(t)} \times \{t\}\big).$$

Observe that $\Omega_{0,T}$ is just the standard parabolic cylinder $\Omega \times (0,T]$. Similarly, $\Gamma_{0,T} = \Gamma \times (0,T]$. For the sake of completeness, we write $J_T := J \times (0,T]$. Then, given any initial value $\rho_0 \in \mathfrak{A}$, consider the *moving boundary problem* of determining a pair (u,ρ) satisfying the following set of equations:

$$\begin{aligned} \Delta u = 0 &\quad \text{in } \ \Omega_{\rho,T} \\ u = \sigma\kappa_\rho &\quad \text{on } \ \Gamma_{\rho,T} \\ (1-\delta)u + \delta(\nabla u|\nu_J) = b &\quad \text{on } \ J_T \\ \partial_t S_\rho - (\nabla u|\nabla S_\rho) = 0 &\quad \text{on } \ \Gamma_{\rho,T} \\ \rho(0,\cdot) = \rho_0 &\quad \text{on } \ \Gamma. \end{aligned} \tag{1.1}$$

Here, $\sigma > 0$ is a positive constant, called *surface tension*, and $\kappa_{\rho(t)}(x)$ denotes the mean curvature of $\Gamma_{\rho(t)}$ at $x \in \Gamma_{\rho(t)},\ t \in [0,T]$. We use the sign convention that convex hypersurfaces have positive mean curvature. In particular, we have $\kappa_0 \equiv 1$ if Γ is the unit sphere. Moreover, Δ and ∇ stand for the Laplacian and the gradient, respectively, in the Euclidean metric. The outer unit normal field over J is denoted by ν_J and S_ρ is a defining function for Γ_ρ; i.e., $S_\rho^{-1}(0) = \Gamma_\rho,\ \rho \in \mathfrak{A}$. A precise definition of S_ρ is given below. Finally, the function b is given and the integer $\delta \in \{0,1\}$ is introduced to label the boundary condition on the fixed boundary J (where $\delta = 0$ corresponds to a Dirichlet boundary condition and $\delta = 1$ corresponds to a Neumann condition). The entire system (1.1) is called the classical formulation of the *one-phase Hele-Shaw model with surface tension*.

The equations in (1.1) express that the free boundary moves with normal velocity V given as a nonlocal functional $\mathcal{F}(\kappa)$ of the mean curvature κ. More precisely, let V be the normal velocity taken to be positive for expanding hypersurfaces and let $\mathcal{F}(\kappa) := -(\nabla u_\kappa | N)$, where u_κ satisfies the first three equations in (1.1) and where N is the outer unit normal field on the moving boundary. Then the fourth equation of (1.1) implies the relation

$$V = \mathcal{F}(\kappa) \quad \text{on} \quad \Gamma_{\rho,T},$$

which can be considered as a nonlocal generalization of the usual motion by mean curvature; see [23], [26], [19], [21].

To state our results clearly, we need some notations. Let us first give the definition of S_ρ. For this, pick $a_0 \in (0, \operatorname{dist}(\Gamma, J))$ and let

$$\mathcal{N} : \Gamma \times (-a_0, a_0) \to \mathbb{R}^n, \qquad \mathcal{N}(x,\lambda) := x + \lambda\nu(x).$$

Then $\mathcal{N}$ is a smooth diffeomorphism onto its image $\mathcal{R} := im(\mathcal{N})$; i.e.,

$$\mathcal{N} \in \mathit{Diff}^\infty(\Gamma \times (-a_0, a_0), \mathcal{R}),$$

provided $a_0 > 0$ is small enough. It is convenient to decompose the inverse of $\mathcal{N}$ into $\mathcal{N}^{-1} = (X, \Lambda)$, where

$$X \in C^\infty(\mathcal{R}, \Gamma) \qquad \text{and} \qquad \Lambda \in C^\infty(\mathcal{R}, (-a_0, a_0)).$$

Note that $X(y)$ is the nearest point on Γ to y, and that $\Lambda(y)$ is the signed distance from y to Γ (that is, to $X(y)$). The neighborhood $\mathcal{R}$ consists of those points with distance less than a_0 to Γ. Given $\rho \in \mathfrak{A}$, define now

$$S_\rho : \mathcal{R} \to \mathbb{R}, \qquad S_\rho(y) := \Lambda(y) - \rho(X(y)).$$

Then it is not difficult to verify that $\Gamma_\rho = S_\rho^{-1}(0)$. Finally, we write $S_\rho(y,t) := \Lambda(y) - \rho(X(y),t)$ for any function $\rho : [0,T] \to \mathfrak{A}$ and $(y,t) \in \mathcal{R} \times [0,T]$.

Next we introduce some function spaces. Given an open subset U of $\mathbb{R}^n$, let $h^s(U)$ denote the little Hölder space of order $s > 0$, i.e., the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the Banach space of all bounded and uniformly Hölder-continuous functions of order s. If M is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ the spaces $h^s(M)$ are defined by means of a smooth atlas for M. Finally, we fix $\alpha_0, \beta \in (\alpha, 1)$ such that $\beta < \alpha_0$ and we set

$$\mathcal{V} := h^{2+\alpha_0}(\Gamma) \cap \mathfrak{A}, \qquad \mathcal{U} := h^{2+\beta}(\Gamma) \cap \mathfrak{A}.$$

A pair (u,ρ) is called a *classical* (*smooth*) *solution* of (1.1), if

$$u(\cdot,t)\in C^\infty(\bar\Omega_{\rho(t)}),\quad t\in(0,T],\qquad \rho\in C([0,T],\mathcal{V})\cap C^\infty(\Gamma\times(0,T))$$

and if (u,ρ) satisfies the equations in (1.1) point-wise. Recall that the function b and the surface tension σ are known quantities, i.e., we assume that

$$b\in C^\infty(J)\qquad\text{and}\qquad \sigma>0$$

are given. Our main result for problem (1.1) now reads as follows:

Theorem 1. *Given any initial value $\rho_0\in\mathcal{V}$, there exists a unique classical solution (u,ρ) of (1.1) on a sufficiently small interval of existence $(0,T]$. Moreover, the moving boundary ρ is analytic in the time variable.*

Of course, we can choose $\rho_0\equiv 0$ above. Then Theorem 1 guarantees a classical solution to (1.1) starting from the initial hypersurface Γ. Observe, however, that we also get a classical solution to problem (1.1) for any $C^{2+\alpha_0}$ initial hypersurface Γ_{ρ_0} which is close to Γ in the sense that ρ_0 belongs to $\mathcal{V}$. If we content ourselves with solutions in $h^{3+\alpha}(\Gamma)$ rather than with smooth solutions, then Ω can be chosen to be of class $h^{3+\alpha}$.

Consider problem (1.1) with positive surface tension and with a homogeneous Neumann boundary condition on J, i.e., assume that $\sigma>0$, $\delta=1$, $b=0$, and let (u,ρ) be the solution of (1.1) starting from the initial curve Γ_0. In the following it is convenient to use the notation $\Gamma_t:=\Gamma_{\rho(t)}$ and $\Omega_t:=\Omega_{\rho(t)}$ for $t\in[0,T]$. Furthermore, let $\mathrm{Vol}(t)$ and $A(t)$ denote the volume of Ω_t and the area of Γ_t, respectively. Thanks to Theorem 1 it is easy to prove that the Hele-Shaw model is volume preserving and area shrinking. More precisely, we have

Theorem 2. *The functions $Vol(\cdot)$ and $A(\cdot)$ belong to $C^\infty((0,T),\mathbb{R})$ and satisfy*

$$\frac{d}{dt}\,Vol(t)=0,\qquad \sigma\frac{d}{dt}A(t)=-(n-1)\int_{\Omega_t}|\nabla u(\cdot,t)|^2\,dx,\quad t\in(0,T).$$

Proof. (i) From Theorem 1 we know that $\rho\in C^\infty(\Gamma\times(0,T))$. Hence, setting $\theta(x,t):=\theta_{\rho(t)}(x)$ for $(x,t)\in\Gamma\times(0,T)$, we have that $\theta\in C^\infty(\Gamma\times(0,T),\mathbb{R}^n)$. We will show in Section 2 that $\theta(\cdot,t)$ admits an extension, again denoted by the same symbol, satisfying $\theta\in C^\infty(\Omega\times(0,T),\mathbb{R}^n)$ and $\theta(\cdot,t)\in Diff^\infty(\Omega,\Omega_t)$ for each $t\in(0,T)$. Hence the first assertion follows from the relations

$$\mathrm{Vol}(t)=\int\limits_\Omega|\det[D_1\theta(\cdot,t)]|\,dx,\quad A(t)=\int\limits_\Gamma\sqrt{\det[D_1\theta(\cdot,t)^TD_1\theta(\cdot,t)]}\,d\sigma,$$

where $d\sigma$ denotes the volume element of Γ.

(ii) Let $N(\cdot,t)$ be the outer unit normal field of Γ_t and let V denote the normal velocity of $[t \mapsto \Gamma_t]$ in direction of N; i.e., let

$$V(\theta(x,t),t) := \big(D_2\theta(x,t) | N(\theta(x,t),t)\big)$$

for $(x,t) \in \Gamma \times (0,T)$. Recall that $S_\rho(\theta(x,t),t) = 0$ for $(x,t) \in \Gamma \times (0,T)$. Hence we conclude that

$$V(\theta(x,t),t) = -\frac{\partial_t S_\rho(\theta(x,t),t)}{|\nabla S_\rho(\theta(x,t),t)|}, \quad (x,t) \in \Gamma \times (0,T),$$

since the outer unit normal field N on Γ_t is given by $\nabla S_\rho |\nabla S_\rho|^{-1}$. Consequently, the boundary condition

$$\partial_t S_\rho - (\nabla u | \nabla S_\rho) = 0 \quad \text{on} \quad \Gamma_{\rho,T}$$

implies the relation

$$V = -(\nabla u | N) \quad \text{on} \quad \Gamma_{\rho,T}. \tag{1.2}$$

In the following, $d\sigma_t$ denotes the volume element of Γ_t. In terms of V, the temporal changes of Vol and A are given by the formulas

$$\frac{d}{dt}\mathrm{Vol}(t) = \int_{\Gamma_t} V \, d\sigma_t, \qquad \frac{d}{dt}A(t) = (n-1)\int_{\Gamma_t} \kappa V \, d\sigma_t;$$

see, e.g., Theorem 2E in [22], Theorem 4 in [31], or page 462 in [5]. Since u satisfies the equations in (1.1) pointwise, Gauss's theorem and (1.2) yield

$$\frac{d}{dt}\mathrm{Vol}(t) = \int_{\Gamma_t} V \, d\sigma_t = -\int_{\Gamma_t} (\nabla u | N) \, d\sigma_t = -\int_{\Omega_t} \Delta u \, dx = 0$$

and

$$\sigma \frac{1}{n-1}\frac{d}{dt}A(t) = \sigma \int_{\Gamma_t} \kappa V \, d\sigma_t = -\int_{\Gamma_t} (u\nabla u | N) \, d\sigma_t = -\int_{\Omega_t} |\nabla u|^2 dx,$$

which completes the proof. $\square$

An analogous result of Theorem 2 for the two-dimensional Hele-Shaw flow was proved in [7]. However, Chen's result holds only almost everywhere on $(0,T)$ and its proof is considerably more involved than the proof of Theorem 2. This is of course due to fact that in [7] too few regularity properties of the solutions are established in order to satisfy the equation point-wise.

Let us also discuss the above result comparing it to the one-phase problem without surface tension, given by $\sigma = 0$. In the latter, the sign of the function b becomes significant. Recently, it was shown in [17] that the one-phase Hele-Shaw problem is well-posed if $b \succeq 0$, i.e., if $b(x) \geq 0$ for $x \in J$ but $b \not\equiv 0$. This is in clear contrast to the case $\sigma > 0$ where the sign of b has no influence on the problem being well-posed, as Theorem 1 shows. Hence surface tension has a regularizing effect on Hele-Shaw models; see also [13], [30], [25], and [22].

We call problem (1.1) *linearly ill-posed* on $\mathcal{V}$ if the linearized equation on a fixed reference domain is ill-posed in the sense of Hadamard. Then we have the following sharp alternative:

Theorem 3. *The Hele-Shaw problem* (1.1) *is well-posed on* $\mathcal{V}$ *if* $\sigma > 0$ *or if* $\sigma = 0$ *and* $b \succeq 0$. *It is linearly ill-posed on* $\mathcal{V}$ *if* $\sigma < 0$ *or if* $\sigma = 0$ *and* $b \preceq 0$.

Proof. This follows from Theorem 1, Remark 4.3 below, Theorem 1 in [17], and Remark 5.3 in [17]. □

The previous result indicates that problem (1.1) should also make sense when the fixed part J of the boundary is empty, provided the surface tension does not vanish. In fact, an inspection of the proof of Theorem 1 shows that in the case $J = \emptyset$, $\sigma \neq 0$ the results of Theorem 1 remain valid.

Let us now turn to the two-phase Hele-Shaw model. To begin with, assume again that Ω^1 is a bounded smooth domain in $\mathbb{R}^n$ such that its boundary $\partial\Omega^1$ consists of two disjoint components, the interior part J^1 and the exterior part Γ. In addition, let also Ω be a bounded smooth domain in $\mathbb{R}^n$ containing Ω^1 and possessing a boundary with two disjoint components. The interior part of $\partial\Omega$ is assumed to coincide with J^1 and the exterior part is called J^2. Finally, we let $\Omega^2 := \Omega \setminus \overline{\Omega^1}$ and we use the same notation as above for $\mathcal{V}$, Γ_ρ, $\Gamma_{\rho,T}$, S_ρ, J^i_T, and $\Omega^i_{\rho,T}$, $i \in \{1, 2\}$ and $T > 0$. Of course, in this situation we assume that the positive constant a in $\mathcal{V}$ is chosen small enough so that Γ_ρ intersects neither J^1 nor J^2. Then we consider the following two-phase problem: Given an initial value $\rho_0 \in \mathcal{V}$, find a triple (u^1, u^2, ρ) such that

$$\begin{aligned}
&(u_1(\cdot,t), u_2(\cdot,t)) \in C^\infty(\bar\Omega^1_{\rho(t)}) \times C^\infty(\bar\Omega^2_{\rho(t)}), \quad t \in (0,T] \\
&\rho \in C([0,T],\mathcal{V}) \cap C^\infty(\Gamma \times (0,T))
\end{aligned} \tag{1.3}$$

and such that

$$\begin{aligned}
\Delta u^i &= 0 && \text{in } \ \Omega^i_{\rho,T} \\
(\nabla u^i | \nu_J) &= 0 && \text{on } \ J^i_T \\
u^i &= \sigma\kappa_\rho && \text{on } \ \Gamma_{\rho,T} \\
\partial_t S_\rho - (\nabla u^1 - \nabla u^2 | \nabla S_\rho) &= 0 && \text{on } \ \Gamma_{\rho,T} \\
\rho(0,\cdot) &= \rho_0 && \text{on } \ \Gamma.
\end{aligned} \tag{1.4}$$

Here we use the sign convention that κ_ρ denotes the mean curvature of Γ_ρ with respect to Ω^1_ρ. System (1.4) is the classical formulation of the *two-phase Hele-Shaw model with surface tension*. For problem (1.4) we have the following general existence, uniqueness, and regularity result:

Theorem 4. *Assume that* $\sigma > 0$. *Then, given any initial value* $\rho_0 \in \mathcal{V}$, *problem* (1.4) *possesses a unique classical solution* (u^1, u^2, ρ) *in the class* (1.3) *for a sufficiently small* $T > 0$. *Moreover, the interface depends analytically on the time variable. If* $\sigma < 0$, *then problem* (1.4) *is linearly ill-posed on* $\mathcal{V}$.

We mention that, besides Hele-Shaw flows, problems (1.1) and (1.4) also encompass quasi-static one- and two-phase Stefan problems, respectively, with an interfacial free energy condition on the moving boundary. In the latter case, system (1.4) is also called Mullins-Sekerka model, (cf. [22, page 136]). In addition, it is worth noting also that the homogeneous two-phase Hele-Shaw model is volume preserving and area shrinking, and that Theorem 4 remains true if the fixed part J of the boundary is empty. This follows similarly as in the proof of Theorem 2.

To prove the above results we use the techniques developed in [15], [16], and [17]; i.e., we transform the original problem to a system of equations on a fixed reference domain. After a natural reduction of the transformed problem we are led to a nonlinear evolution equation for the distance function ρ only. In the case of positive surface tension the propagator of this evolution equation turns out to be a nonlinear, nonlocal pseudo-differential operator of *third* order. A careful analysis of that operator discloses that it carries in addition a quasilinear structure of parabolic type. This enables us to use H. Amann's theory of abstract parabolic evolution equations ([2]), or the results in [6], to find the moving boundary ρ. This method has been successfully used to study problems arising in gravity flows of incompressible fluids through porous media and to Hele-Shaw problems without surface tension; see [15], [16], and [17]. Finally, we mention that the above results have been announced in [18].

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Note. After finishing this paper, N. Alikakos drew our attention to a manuscript by X. Chen, J. Hong, and F. Yi ([8]) in which also existence and uniqueness of classical solutions for the two-phase problem (1.4) is proved. However, it is noteworthy to mention that Chen et al. assume that the initial hypersurface belongs to $\mathcal{V} \cap C^{3+\alpha}(\Gamma)$ which is a true subspace of $\mathcal{V}$.

2. The equations on a fixed domain.

We first focus our attention on problem (1.1). Throughout this paper we choose either a Dirichlet or a Neumann boundary condition on the fixed component J of the boundary; i.e., we fix $\delta \in \{0, 1\}$. Also the rate b of injection (or suction) and the surface tension σ are known quantities. We assume in the following that

$$b \in C^\infty(J) \qquad \text{and} \qquad \sigma > 0.$$

In addition, let us introduce some function spaces which we will need in what follows. Assume that U is an open subset of $\mathbb{R}^m$. Given $k \in \mathbb{N} \cup \{\infty\}$, let $C^k(U)$ denote the space of all $f : U \to \mathbb{R}$ having continuous derivatives up to order k. The closed subspace of $C^k(U)$, consisting of all maps from U into $\mathbb{R}$ which have bounded and uniformly continuous derivatives up to order k is denoted by $BUC^k(U)$. Given $\alpha \in (0, 1)$, the space $C^{k+\alpha}(U)$ stands for all $f \in C^k(U)$ having locally α-Hölder-continuous derivatives of order k and $BUC^{k+\alpha}(U)$ stands for all $f \in BUC^k(U)$ having uniformly α-Hölder-continuous derivatives of order k. In addition, $C^\omega(U)$ denotes the space of all real analytic functions on U.

Furthermore, we write $\mathcal{S}(\mathbb{R}^m)$ for the Schwartz space, i.e., the Fréchet space of all rapidly decreasing smooth functions on $\mathbb{R}^m$. Let r_U denote the restriction operator with respect to U. Then the **little Hölder spaces** $h^r(U),\ r \in \mathbb{R}$, are defined as

$$h^r(U) := \text{closure of} \quad r_U\big(\mathcal{S}(\mathbb{R}^m)\big) \quad \text{in} \quad B^r_{\infty\infty}(U),$$

where $B^r_{\infty\infty}$ stands for a class of Besov spaces; see [34]. Observe that $B^s_{\infty\infty}(U) = BUC^s(U)$ for $s \in \mathbb{R}^+ \setminus \mathbb{N}$, so that this definition coincides with the previous one when U is bounded and $r > 0$. Finally, assume that M is an m-dimensional smooth submanifold of $\mathbb{R}^n$. Then the spaces $BUC^s(M)$, $s \geq 0$, and $h^r(M)$, $r \in \mathbb{R}$, are defined as usual by means of a smooth atlas for M; see [34].

Next, let us introduce an appropriate extension of the diffeomorphism θ_ρ to $\mathbb{R}^n$. For this we assume that $a \in (0, a_0/4)$ and we fix a $\varphi \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\varphi(\lambda) = \begin{cases} 1 & \text{if} \quad |\lambda| \leq a \\ 0 & \text{if} \quad |\lambda| \geq 3a \end{cases}$$

and such that $\sup |\partial\varphi(\lambda)| < 1/a$. Then we define for each $\rho \in \mathcal{U}$ the map

$$\Theta_\rho(y) := \begin{cases} \mathcal{N}\big(X(y), \Lambda(y) + \varphi(\Lambda(y))\rho(X(y))\big) & \text{if} \quad y \in \mathcal{R}, \\ y & \text{if} \quad y \notin \mathcal{R}. \end{cases} \tag{2.1}$$

Note that $[\lambda \mapsto \lambda + \varphi(\lambda)\rho]$ is strictly increasing since $|\partial\varphi(\lambda)\rho| < 1$. Then it is not difficult to verify that

$$\Theta_\rho \in \mathit{Diff}^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n) \cap \mathit{Diff}^{2+\alpha}(\Omega, \Omega_\rho) \quad \text{and} \quad \Theta_\rho | \Gamma = \theta_\rho.$$

Moreover, we observe that there exists an open neighborhood U of J such that

$$\Theta_\rho | U = id_U. \tag{2.2}$$

It should be mentioned that the above diffeomorphism was first introduced by E.I. Hanzawa ([24]) to transform multi-dimensional Stefan problems to fixed domains. In the following we use the same symbol θ_ρ for both diffeomorphisms θ_ρ and Θ_ρ. The pull-back operator induced by θ_ρ is given as

$$\theta^* u := \theta^*_\rho u := u \circ \theta_\rho \quad \text{for} \quad u \in BUC(\Omega_\rho).$$

Similarly, the corresponding push-forward operator is defined as

$$\theta_* v := \theta^\rho_* v := v \circ \theta^{-1}_\rho \quad \text{for} \quad v \in BUC(\Omega).$$

In a similar way as in Lemma 2.1 in [17] one proves that, given $\rho \in \mathfrak{A} \cap h^{k+\alpha}(\Gamma)$ with $k \in \mathbb{N}$ and $k \geq 2$, we have

$$\theta^*_\rho \in \text{Isom}\,(h^{k+\alpha}(\Omega_\rho), h^{k+\alpha}(\Omega)) \cap \text{Isom}(h^{k+\alpha}(\Gamma_\rho), h^{k+\alpha}(\Gamma)) \tag{2.3}$$

with

$$[\theta_\rho^*]^{-1} = \theta_*^\rho. \tag{2.4}$$

Based on the above transformation operators, we are now able to associate to the original moving boundary problem (1.1) a changed version on the (fixed) reference domain Ω. To do this, let

$$\begin{aligned} &A(\rho)v := -\theta_\rho^*\big(\Delta(\theta_*^\rho v)\big), \qquad B(\rho)v := \gamma\theta_\rho^*(\nabla(\theta_*^\rho v)|\nabla S_\rho) \\ &Cv := (1-\delta)\gamma_J v + \delta(\gamma_J \nabla v|\nu_J), \end{aligned}$$

for $v \in C^2(\Omega) \cap BUC^1(\Omega)$ and $\rho \in \mathcal{U}$. Here γ and γ_J denote the trace operators with respect to Γ and J, respectively, and ν_J stands for the outer unit normal field over J. It should be observed that $A(\rho)$ is the Laplace-Beltrami operator on Ω with respect to the metric induced by θ_ρ. Similarly, $B(\rho)$ and C (in the case $\delta = 1$, of course) are the corresponding derivatives with respect to the outer normal; see also Lemma 2.3 in [17]. Hence these operators act linearly on the space $C^2(\Omega) \cap BUC^1(\Omega)$. We also introduce the transformed mean curvature operator

$$H(\rho) := \theta_\rho^* \kappa_\rho, \qquad \rho \in \mathcal{U}. \tag{2.5}$$

Then we consider the following problem: Given $\rho_0 \in \mathcal{V}$, find a pair (v, ρ) such that

$$\begin{array}{rll} A(\rho)v = 0 & \text{in} & \Omega_{0,T} \\ v = \sigma H(\rho) & \text{on} & \Gamma_{0,T} \\ Cv = b & \text{on} & J_T \quad 2 \\ \partial_t \rho + B(\rho)v = 0 & \text{on} & \Gamma_{0,T} \\ \rho(0,\cdot) = \rho_0 & \text{on} & \Gamma. \end{array} \qquad (2.6)_{\rho_0}$$

We recall that σ and b are known quantities. A pair (v, ρ) is called a **classical solution** of $(2.6)_{\rho_0}$, if

$$v(\cdot, t) \in C^\infty(\bar{\Omega}), \quad t \in (0, T], \quad \rho \in C([0, T], \mathcal{V}) \cap C^\infty(\Gamma \times (0, T))$$

and if (v, ρ) satisfies the equations in $(2.6)_{\rho_0}$ point-wise. The following Lemma is a consequence of (2.3) and (2.4):

Lemma 2.1. *Let $\rho_0 \in \mathcal{V}$ be given.*

a) *If (u, ρ) is a classical solution of $(1.1)_{\rho_0}$ then $(\theta_\rho^* u, \rho)$ is a classical solution of $(2.6)_{\rho_0}$.*
b) *If (v, ρ) is a classical solution of $(2.6)_{\rho_0}$ then $(\theta_*^\rho v, \rho)$ is a classical solution of $(1.1)_{\rho_0}$.*

We close this section by proving the following results for elliptic boundary value problems in little Hölder spaces. We shall use these results in Sections 4 and 5.

Lemma 2.2. *Let $\sigma \in [\alpha, \beta]$ be fixed. Then*

a)
$$(A, B) \in C^\omega\big(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\Omega), h^{\sigma-1}(\Omega) \times h^\sigma(\Gamma))\big).$$

b) Let $\rho \in \mathcal{U}$ be given and let γ denote the trace operator with respect to Γ. Then

$$(A(\rho), \gamma, C) \in Isom\big(h^{1+\sigma}(\Omega), h^{\sigma-1}(\Omega) \times h^{1+\sigma}(\Gamma) \times h^{1+\sigma-\delta}(J)\big). \tag{2.7}$$

Proof. a) Given $\rho \in \mathcal{U}$, let $[g^{jk}(\rho)] := [g_{jk}(\rho)]^{-1}$, where $g_{jk}(\rho) := (\partial_j\theta_\rho|\partial_k\theta_\rho)$, $1 \le j, k \le n$, denote the components of the metric tensor induced by θ_ρ. Observe that

$$[\rho \mapsto \theta_\rho] \in C^\omega(\mathcal{U}, h^{2+\beta}(U, \mathbb{R}^n)),$$

where U is an open and bounded set containing Ω. Hence the map $[\rho \mapsto g^{jk}(\rho)] : \mathcal{U} \to h^{1+\beta}(U))$ is analytic for $1 \le j, k \le n$ and we can conclude that

$$A(\rho) = a_{jk}(\rho)\partial_j\partial_k + a_j(\rho)\partial_j, \qquad B(\rho) = b_j(\rho)\gamma\partial_j\,,$$

where the coefficients satisfy

$$[\rho \mapsto \big(a_{jk}(\rho), a_j(\rho), b_j(\rho)\big)] \in C^\omega\big(\mathcal{U}, h^{1+\beta}(\Omega) \times h^\beta(\Omega) \times h^{1+\beta}(\Gamma)\big)$$

for $1 \le j, k \le n$. Thus the assertion follows from the fact that the mappings

$$\begin{aligned} &[(a, v) \mapsto av] : h^{1+\beta}(\Omega) \times h^{\sigma-1}(\Omega) \to h^{\sigma-1}(\Omega), \\ &[(a, v) \mapsto av] : h^\beta(\Omega) \times h^\sigma(\Omega) \to h^{\sigma-1}(\Omega), \\ &[(a, v) \mapsto av] : h^{1+\beta}(\Gamma) \times h^\sigma(\Gamma) \to h^\sigma(\Gamma) \end{aligned} \tag{2.8}$$

are bilinear and continuous; see Theorem 2.8.2 in [34].

b) For uniformly elliptic operators of the form $A = a_{jk}\partial_j\partial_k + a_j\partial_j$ with smooth coefficients a_{jk} and a_j, (2.7) is shown in [34], Theorem 4.3.4 and Corollary 4.3.2, since the maximum principle implies that the Hypothesis of Section 4.3.1 is satisfied. Moreover, an inspection of the proofs in [34] and the continuity of the first two multiplier operators in (2.8) show that the same assertions also hold true for coefficients $a_{jk} \in h^{1+\beta}(\Omega)$ and $a_j \in h^\beta(\Omega)$. $\square$

Let us introduce the following natural decomposition $(A(\rho), \gamma, C)^{-1} = S \oplus T \oplus R$ by setting

$$\begin{aligned} S(\rho) &:= \big(A(\rho), \gamma, C\big)^{-1}(\cdot, 0, 0) \in \mathcal{L}(h^{\sigma-1}(\Omega), h^{1+\sigma}(\Omega)), \\ T(\rho) &:= \big(A(\rho), \gamma, C\big)^{-1}(0, \cdot, 0) \in \mathcal{L}(h^{1+\sigma}(\Gamma), h^{1+\sigma}(\Omega)), \\ R(\rho) &:= \big(A(\rho), \gamma, C\big)^{-1}(0, 0, \cdot) \in \mathcal{L}(h^{1+\sigma-\delta}(J), h^{1+\sigma}(\Omega)), \end{aligned} \tag{2.9}$$

for each $\sigma \in [\alpha, \beta]$. These operators will be used in Section 4 to reduce the system (2.6) to an evolution equation for the free boundary only. In the next Lemma we investigate the dependence of these operators on ρ.

Lemma 2.3. *Let $\sigma \in [\alpha, \beta]$ be fixed. Then*

$$[\rho \mapsto T(\rho)] \in C^\omega\big(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\Gamma), h^{1+\sigma}(\Omega))\big),$$
$$[\rho \mapsto R(\rho)] \in C^\omega\big(\mathcal{U}, \mathcal{L}(h^{1+\sigma-\delta}(J), h^{1+\sigma}(\Omega))\big).$$

Proof To shorten the notation let us introduce the spaces $F_0 := h^{\sigma-1}(\Omega)$, $F_1 := h^{1+\sigma}(\Omega)$, $E_1 := h^{1+\sigma}(\Gamma)$, and $E_\delta := h^{1+\sigma-\delta}(\Gamma)$. It follows from Lemma 2.2 that $[\rho \mapsto (A(\rho), \gamma, C)] \in C^\omega(\mathcal{U}, Isom(F_1, F_0 \times E_1 \times E_\delta))$. Moreover, observe that the inversion

$$[G \mapsto G^{-1}] \,:\, Isom(F_1, F_0 \times E_1 \times E_\delta) \to \mathcal{L}(F_0 \times E_1 \times E_\delta, F_1)$$

is analytic as well. Next, let $V \in \mathcal{L}(F_0 \times E_1 \times E_\delta, F_1)$ be given and define $e(V) \in \mathcal{L}(E_1, F_1)$ by $e(V)g := V(0, g, 0)$, $g \in E_1$. It is easily verified that the evaluation map e satisfies $e \in \mathcal{L}\big(\mathcal{L}(F_0 \times E_1 \times E_\delta, F_1), \mathcal{L}(E_1, F_1)\big)$ and consequently

$$e \in C^\omega\big(\mathcal{L}(F_0 \times E_1 \times E_\delta, F_1), \mathcal{L}(E_1, F_1)\big).$$

Now the first assertion follows from the identity $T = e \circ (A(\cdot), \gamma, C)^{-1}$ and the fact that the composition of analytic maps is analytic as well, and the second statement follows by a similar argument. ☐

3. The mean curvature operator. Let η denote the standard Euclidean metric on $\mathbb{R}^n$ and let $\theta^*\eta$ denote the Riemannian metric on $\mathbb{R}^n$ and on Γ, respectively, induced by the diffeomorphisms

$$\theta_\rho \in Diff^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n) \cap Diff^{2+\alpha}(\Gamma, \Gamma_\rho);$$

i.e.,

$$\theta^*\eta(V, W) := \eta(\theta_* V, \theta_* W)$$

for V, $W \in \mathfrak{X}(\mathbb{R}^n)$ or for V, $W \in \mathfrak{X}(\Gamma)$. Here, of course, $\theta_* V$ stands for the push forward of the vector field V; see [27, page 10]. As a first result we now prove that the mean curvature operator H defined in (2.5) depends analytically on $\rho \in \mathcal{U}$ and that it carries a quasilinear structure in the following sense.

Lemma 3.1. *There exist*

$$P \in C^\omega(\mathcal{U}, \mathcal{L}(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma))) \quad \textit{and} \quad K \in C^\omega(\mathcal{U}, h^{1+\beta}(\Gamma))$$

such that

$$H(\rho) = P(\rho)\rho + K(\rho) \quad \textit{for} \quad \rho \in \mathcal{U} \cap h^{3+\alpha}(\Gamma).$$

Proof. Let S_Γ denote the second fundamental form of $(\Gamma, \theta_\rho^*\eta)$ in $(\mathbb{R}^n, \theta_\rho^*\eta)$ with respect to the outer unit normal field ν; cf. Chapter VII in [28]. Then it can be shown that

$$H(\rho) = \theta_\rho^* \kappa_\rho = \frac{1}{n-1}\,\mathrm{trace}\,(S_\Gamma). \tag{3.1}$$

Furthermore, observe that

$$[\rho \mapsto \theta_\rho] \in C^\omega(\mathcal{U}, h^{2+\beta}(U, \mathbb{R}^n)),$$

where U is an open and bounded set containing Ω. This shows that the metric $\theta_\rho^*\eta$ depends analytically on $\rho \in \mathcal{U}$. Now the assertion follows from (3.1), formula (4.11) in [11] and the fact that ρ and $\partial_j\rho$, $1 \le j \le n-1$, induce point-wise multiplication operators on $h^{1+\alpha}(\Gamma)$ and on $h^{1+\beta}(\Gamma)$ for each $\rho \in \mathcal{U}$. $\square$

Given $\rho \in \mathcal{U}$, it is convenient to express the linear operator $P(\rho)$ by means of local coordinates. To make this precise we need a few notations. Given $\kappa \in (0, a]$, let $\mathcal{R}_\kappa := \mathcal{N}(\Gamma \times (-\kappa, 0])$. Then there exists $m := m_\kappa \in \mathbb{N}$ and an atlas $\{(U_l, \varphi_l) : 1 \le l \le m\}$ of $\mathcal{R}_\kappa$ such that $\operatorname{diam}(U_l) < 2\kappa$ for all $l \in \{1, \dots, m\}$. Let

$$s_l \in C^\infty((-\delta,\delta)^{n-1}, U_l), \quad l \in \{1, \dots, m\},$$

be a parametrization of $U_l \cap \Gamma$. Furthermore, let $X := (-\delta,\delta)^{n-1} \times (-\delta, 0]$, $Y := (-\delta,\delta)^{n-1} \times \{0\} \equiv (-\delta,\delta)^{n-1}$, and define

$$\mu_l : X \to U_l, \quad (\omega, r) \mapsto s_l(\omega) + r\nu(s_l(\omega)), \quad 1 \le l \le m.$$

Without loss of generality, we may assume that $\delta = \kappa$ and that $\mu_l = \varphi_l^{-1}$ for $1 \le l \le m$. The additional parameter κ is introduced to control the size of the chart domain U_l. This fact will be used in Section 5 to employ a perturbation result; cf. estimate (5.7). Finally, to further economize our notation, we set $\mu := \mu_l$, $U := U_l$ and we let

$$\hat\rho := \mu_l^*\rho, \quad \rho \in \mathcal{U}.$$

Moreover, we use the notation

$$\partial_j := \partial_{\omega_j}, \quad 1 \le j \le n-1, \qquad \partial_n := \partial_r.$$

We now define

$$\widetilde{w}_{jk} := (\partial_j s|\partial_k s) + \pi\big((\partial_j\mu^*\nu|\partial_k s) + (\partial_k\mu^*\nu|\partial_j s)\big) + \pi^2(\partial_j\mu^*\nu|\partial_k\mu^*\nu),$$

for $1 \le j,k \le n-1$ and where $\pi(\omega, r) := r$ for $(\omega, r) \in X$. Clearly, $[\widetilde{w}_{jk}]$ is symmetric. In addition, observe that $[(\partial_j s|\partial_k s)]$ is uniformly positive definite on Y. Hence we may assume that also $[\widetilde{w}_{jk}]$ is uniformly positive definite on X, provided $a > 0$ is small enough.

Next, let $w_{jk}(\rho)$ be the Nemitskii operator induced by $\widetilde{w}_{jk}$; i.e.,

$$w_{jk}(\rho)(\omega) := \widetilde{w}_{jk}(\omega, \hat\rho(\omega)), \quad \omega \in Y, \quad 1 \le j,k \le n-1. \tag{3.2}$$

Similarly, we set

$$w^{jk}(\rho)(\omega) := \widetilde{w}^{jk}(\omega, \hat\rho(\omega)), \quad \omega \in Y, \quad 1 \le j,k \le n-1,$$

where $\widetilde{w}^{jk}$ are the components of the inverse of $[\widetilde{w}_{jk}]$. Finally, given any Riemannian manifold (M, g), let ∇^M, Δ^M, and hess^M, respectively, denote the gradient, the Laplace-Beltrami operator, and the Hessian of (M, g). Our next goal is to show that the principal part of $H(\rho)$, i.e., the linear operator $P(\rho)$, is a uniformly elliptic operator. For this, let

$$\mathcal{P}(\rho)\mu^* := \mu^* P(\rho), \qquad \rho \in \mathcal{U},$$

be the localized version of $P(\rho)$. Then we have

Lemma 3.2. *Given $\rho \in \mathcal{U}$, the operator $\mathcal{P}(\rho)$ is uniformly elliptic on Y, i.e., there exist*

$$\widetilde{p}_{jk} \in C^\infty(Y \times (-a, a) \times \mathbb{R}^{n-1}, \mathbb{R}), \qquad 1 \leq j,\, k \leq n-1,$$

such that

$$[p_{jk}(\rho)] \quad \textit{is symmetric and uniformly positive definite on } Y \tag{3.3}$$

and such that

$$\mathcal{P}(\rho) = -\sum_{j,k=1}^{n-1} p_{jk}(\rho)\partial_j\partial_k.$$

Proof. a) Given any differential operator $\mathcal{D}$, we have that

$$\mathcal{D}^{\mathbb{R}^n}\mu_* = \mu_*\mathcal{D}^X,$$

where $\mathbb{R}^n = (\mathbb{R}^n, \eta)$ and where $X := (X, g_X)$ with $g_X := \mu^*\eta$. Hence we find from Theorem 4.5 in [11] (see also (A13) on page 384 in [20]) for each $\omega \in Y$ the following formula:

$$\begin{aligned}\mathcal{H}(\hat{\rho})(\omega) &= \mu^*\theta^*\kappa_\rho(\omega)\\ &= \frac{1}{(n-1)|\nabla^{\mathbb{R}^n} S_\rho|}\Big(\Delta^{\mathbb{R}^n} S_\rho - \frac{\text{hess}^{\mathbb{R}^n} S_\rho(\nabla^{\mathbb{R}^n} S_\rho, \nabla^{\mathbb{R}^n} S_\rho)}{|\nabla^{\mathbb{R}^n} S_\rho|^2}\Big)\Big|_{\theta(\mu(\omega))}\\ &= \frac{1}{(n-1)\|\nabla^X \hat{S}_\rho\|_X}\Big(\Delta^X \hat{S}_\rho - \frac{\text{hess}^X \hat{S}_\rho(\nabla^X \hat{S}_\rho, \nabla^X \hat{S}_\rho)}{\|\nabla^X \hat{S}_\rho\|_X^2}\Big)\Big|_{(\omega, \hat{\rho}(\omega))},\end{aligned}$$

where we used the notation $\hat{S}_\rho := \mu^* S_\rho$, $\rho \in \mathcal{U}$.

b) Observe that

$$\hat{S}_\rho(\omega, r) = r - \hat{\rho}(\omega), \qquad (\omega, r) \in X \tag{3.4}$$

and that

$$[g_X^{jk}]_{1\leq j,k\leq n} = \begin{bmatrix} [\widetilde{w}^{lm}]_{1\leq l,m\leq n-1} & 0\\ 0 & 1\end{bmatrix}, \tag{3.5}$$

since $(\partial_j s|\nu) = (\partial_j \nu|\nu) = 0$. Hence, letting

$$l_\rho(\omega) := \sqrt{g_X(\nabla^X \hat{S}_\rho, \nabla^X \hat{S}_\rho)}\,\big|_{(\omega,\hat{\rho}(\omega))}, \qquad \rho \in \mathcal{U}, \quad \omega \in Y,$$

one easily finds that

$$l_\rho = \sqrt{1 + w^{jk}(\rho)\partial_j\hat{\rho}\partial_k\hat{\rho}}\,. \tag{3.6}$$

Moreover, using local representations of Δ^X and hess^X, the last equation in step a) yields the formula

$$\mathcal{P}(\rho) = \frac{1}{(n-1)l_\rho^3}\big(-l_\rho^2 w^{jk}(\rho) + w^{jl}(\rho)\partial_l\hat{\rho}w^{km}(\rho)\partial_m\hat{\rho}\big)\partial_j\partial_k. \tag{3.7}$$

c) Finally, given $\xi \in T^*(Y)$, let $p(\rho)(\xi)$ denote the symbol of $\mathcal{P}(\rho)$. Then it follows from (3.6), (3.7), and the Cauchy-Schwarz inequality that

$$\begin{aligned} p(\rho)(\xi) &= \frac{1}{(n-1)l_\rho^3}\big[w_Y^*(\xi,\xi) + w_Y^*(d\hat{\rho},d\hat{\rho})w_Y^*(\xi,\xi) - (w_Y^*(d\hat{\rho},\xi))^2\big] \\ &\geq \frac{w_Y^*(\xi,\xi)}{(n-1)l_\rho^3}, \qquad \xi \in T^*(Y), \end{aligned}$$

where w_Y^* denotes the metric on $T^*(Y)$ induced by $[w^{jk}]$; i.e., $w_Y^*(\xi,\zeta) := w^{jk}\xi_j\zeta_k$ for $\xi,\zeta \in T^*(Y)$, and where $d\tau := \partial_j\tau dx^j \in T^*(Y)$ denotes the exterior differential of $\tau \in C^1(Y)$. This proves the assertion. □

Remark 3.3. We have that

$$P(0) = -\frac{1}{n-1}\Delta_\pi^\Gamma,$$

where Δ_π^Γ denotes the principal part of the Laplace-Beltrami operator on $\Gamma = (\Gamma,\eta)$.

Proof. This follows from formula (3.7) and the fact that $w_{jk}(0) = (\partial_j s|\partial_k s)$ for $1 \leq j,k \leq n-1$.

4. The reduced equation. In this section we reduce the system (2.6) for the pair (v,ρ) to a single evolution equation for the distance function ρ only. It turns out that the principal part Φ of the operator appearing in that evolution equation is a quasilinear pseudo-differential operator of *third* order. More precisely, let

$$\Phi(\rho) := \sigma B(\rho)T(\rho)P(\rho), \qquad F(\rho) := -B(\rho)[\sigma T(\rho)K(\rho) + R(\rho)b],$$

for $\rho \in \mathcal{U}$. It follows from Lemmas 2.3 and 3.1 that the mappings

$$\Phi : \mathcal{U} \to \mathcal{L}(h^{3+\alpha}(\Gamma), h^\alpha(\Gamma)) \qquad \text{and} \qquad F : \mathcal{U} \to h^\beta(\Gamma)$$

are well-defined. Given $\rho_0 \in \mathcal{V}$, we may therefore consider the nonlinear evolution equation on $h^\alpha(\Gamma)$ for the above operators, i.e.,

$$\partial_t \rho + \Phi(\rho)\rho = F(\rho), \qquad \rho(0) = \rho_0. \tag{4.1}$$

To investigate equation (4.1) we shall use H. Amann's theory of abstract quasilinear evolution equations of parabolic type; see [2]. A thorough knowledge of the linear part $\Phi(\rho)$ is fundamental in order to apply this theory. For this, let E_0 and E_1 be Banach spaces such that E_1 is densely injected in E_0 and let $\mathcal{H}(E_1, E_0)$ denote the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the generator of a strongly continuous analytic semigroup on E_0. The basic result for the operator $\Phi(\rho) = \sigma B(\rho)T(\rho)P(\rho)$ is contained in the following theorem. Its proof is postponed to Section 5.

Theorem 4.1. *Given $\rho \in \mathcal{U}$, the following generation property holds*:

$$B(\rho)T(\rho)P(\rho) \in \mathcal{H}(h^{3+\alpha}(\Gamma), h^\alpha(\Gamma)).$$

Remarks 4.2. a) Observe that among the given quantities σ and b only the surface tension σ appears in $\Phi(\rho)$ and that all terms involving the rate of injection b are contained in the lower-order term $F(\rho)$. This shows that the surface tension has a regularizing effect and that the sign of b has no influence on the problem to be well-posed. This is in clear contrast to the problem without surface tension, which is only well-posed for nonnegative b.

b) If $\sigma = 0$, the evolution equation (4.1) reduces to the problem

$$\partial_t \rho = F(\rho), \qquad \rho(0) = \rho_0, \tag{4.2}$$

which is no longer quasilinear but of fully nonlinear type. To solve (4.2) one can use the theory of maximal regularity ([10], [6], [32]), provided the Fréchet derivative ∂F of F generates an analytic semigroup on appropriate Hölder spaces; see [17].

c) The operator $\Phi(\rho)$ consists of the principal (elliptic) part of the mean curvature operator H and of the operator BT, which is sometimes called Dirichlet-Neumann operator; see [14].

Proof of Theorem 1. Let $\rho_0 \in \mathcal{V}$ be given.

a) In a first step we prove that the nonlinear evolution equation (4.1) has a unique solution ρ which satisfies

$$\rho \in C([0,T], \mathcal{V}) \cap C((0,T], h^{3+\alpha}(\Gamma)) \cap C^1((0,T], h^\alpha(\Gamma)). \tag{4.3}$$

To do so, let $E_0 := h^\alpha(\Gamma)$ and $E_1 := h^{3+\alpha}(\Gamma)$ and set

$$E_\theta := (E_0, E_1)^0_{\theta,\infty}, \qquad \theta \in (0,1),$$

where $(\cdot,\cdot)^0_{\theta,\infty}$ denotes the continuous interpolation method. Next we fix

$$\theta_1 := \frac{2+\alpha_0-\alpha}{3}, \qquad \theta_0 := \frac{2+\beta-\alpha}{3}, \qquad \theta := \frac{\beta-\alpha}{3}.$$

Since the little Hölder spaces are stable under continuous interpolation we get

$$E_{\theta_1} = h^{2+\alpha_0}(\Gamma), \qquad E_{\theta_0} = h^{2+\beta}(\Gamma), \qquad E_\theta = h^\beta(\Gamma).$$

Now Lemma 2.3 and Lemma 3.1 yield $(\Phi, F) \in C^\omega(\mathcal{U}, \mathcal{L}(E_1, E_0) \times E_\theta)$ and the assertion follows from Theorem 4.1 above and from Theorem 12.1 in [2]. We mention that this result can also be obtained by using Theorem 2.11 in [6].

b) We claim that the solution of (4.1) is analytic in the time variable; that is,

$$\rho \in C^\omega((0,T), h^{3+\alpha}(\Gamma)).$$

In fact, this follows from Corollary 2.13 in [6], since it is not difficult to verify that the particular assumption of maximal regularity needed in [6] is satisfied.

c) In a next step we use a bootstrapping argument to establish that

$$\rho \in C^\infty(\Gamma \times (0,T)).$$

For this we note that the quasilinear evolution equation (4.1) admits a smoothing property. We already know that solutions with initial values in $\mathcal{V}$ are in $h^{3+\alpha}(\Gamma)$ for any positive time. Let $\tau \in (0,T)$ be arbitrary. Then $\rho_1 := \rho(\tau) \in h^{3+\alpha}(\Gamma)$, and we take ρ_1 as initial value for the evolution equation

$$\partial_t \rho + \Phi(\rho)\rho = F(\rho), \qquad \rho(\tau) = \rho_1. \tag{4.4}$$

Let $\alpha_1, \beta_1 \in (0,\alpha)$ be fixed such that $\alpha_1 < \beta_1$. An inspection of the proofs in Lemmas 2.2, 2.3, and 3.1 shows that the mappings Φ and F do also satisfy

$$(\Phi, F) \in C^\omega\big(\mathfrak{A} \cap h^{3+\beta_1}(\Gamma), \mathcal{L}(h^{4+\alpha_1}(\Gamma), h^{1+\alpha_1}(\Gamma)) \times h^{1+\beta_1}(\Gamma)\big).$$

Moreover, the same techniques as in the proof of Theorem 4.1 can be used to prove that

$$B(\rho)T(\rho)P(\rho) \in \mathcal{H}(h^{4+\alpha_1}(\Gamma), h^{1+\alpha_1}(\Gamma)), \quad \rho \in \mathfrak{A} \cap h^{3+\beta_1}(\Gamma).$$

We can therefore apply Theorem 12.1 in [2] again and we see that problem (4.4) has a unique maximal solution $\rho(\cdot, \rho_1)$ with

$$\rho(\cdot,\rho_1) \in C([\tau, t_1^+), \mathcal{V}_1) \cap C((\tau, t_1^+), h^{4+\alpha_1}(\Gamma)) \cap C^1((\tau, t_1^+), h^{1+\alpha_1}(\Gamma)),$$

where $\mathcal{V}_1 := \mathcal{V} \cap h^{3+\alpha}(\Gamma)$ and where $[\tau, t_1^+)$ is the maximal interval of existence. It follows that

$$\rho(\cdot,\rho_1) \in C([\tau, t_1^+), \mathcal{V}_1) \cap C^1([\tau, t_1^+), h^\alpha(\Gamma)),$$

and we conclude that $\rho(\cdot,\rho_1) = \rho$ on the common interval of existence. Next we show that $t_1^+ > T$. Let us assume that $t_1^+ \le T$. Since $h^{4+\alpha_1}(\Gamma)$ is compactly embedded in $h^{1+\alpha_1}(\Gamma)$, it follows from Theorem 12.5 in [2] that $\rho(t,\rho_1)$ either approaches the boundary of $\mathcal{V} \cap h^{3+\beta_1}(\Gamma)$ or that $\|\rho(t,\rho_1)\|_{3+\alpha}$ converges to infinity as t converges to t_1^+. Since $\rho(\cdot,\rho_1)=\rho$ on $[\tau,t_1^+)$ we conclude that ρ has the same property. But this cannot occur because we already know that ρ satisfies (4.3). Therefore, the assumption leads to a contradiction. Since τ can be chosen arbitrarily we obtain that

$$\rho \in C((0,T],h^{4+\alpha_1}(\Gamma)) \cap C^1((0,T],h^{1+\alpha_1}(\Gamma)).$$

In a next step we choose $\alpha_2,\beta_2 \in (0,\alpha_1)$ such that $\alpha_2<\beta_2$ and repeat the steps above. By induction we can then prove that

$$\rho \in C((0,T],h^{k+\sigma}(\Gamma)), \quad k\in\mathbb{N}, \quad \sigma\in(0,1).$$

By combining this result with the result in part b) we get the assertion.

d) Next we set

$$v(\cdot,t) := \sigma T(\rho(t))H(\rho(t)) + R(\rho(t))b, \quad t\in(0,T].$$

We recall that the solution operators $T(\rho)$ and $R(\rho)$ were introduced in (2.9). In the following we fix $t\in(0,T]$ and we simply write ρ for $\rho(t)$. Thanks to part c) of the proof, we know that $\rho\in C^\infty(\Gamma)$. This implies that the transformation θ_ρ introduced in Section 2 is a C^∞-diffeomorphism. As a consequence, the transformed mean curvature satisfies $H(\rho)\in C^\infty(\Gamma)$ and the transformed differential operators $\big(A(\rho),B(\rho)\big)$ have C^∞-coefficients. Now Theorem 4.3.1 in [34], see in particular formula 4.3.1/2, shows that $v:=v(\cdot,t)\in C^\infty(\bar\Omega)$ is the unique solution of the first three equations in (2.6).

e) Finally, we infer from Lemma 2.1 that $(\theta_\rho^* v,\rho)$ is the unique solution of (1.1), and so the proof of Theorem 1 is now completed. □

Lemma 2.1 suggests calling problem (1.1) *linearly ill-posed* at $\rho\in\mathcal{V}$ if the linear equation

$$\partial_t\tau + \Phi(\rho)\tau = 0, \qquad \tau(0)=\tau_0$$

is ill-posed in the sense of Hadamard, i.e., if $-\Phi(\rho)$ does not generate a strongly continuous semigroup on $h^\alpha(\Gamma)$.

Consequently, Theorem 4.1 and a well-known characterization of generators of analytic semigroups immediately imply the following

Remark 4.3. Problem (1.1) is linearly ill-posed if $\sigma<0$.

5. Proof of Theorem 4.1. Throughout this section we fix $\rho\in\mathcal{U}$. Let us first introduce local representations of the operators $A(\rho)$ and $B(\rho)$ by setting

$$\mathcal{A}(\rho)\mu^* := \mu^* A(\rho) \quad \text{and} \quad \mathcal{B}(\rho)\mu^* := \mu^* B(\rho), \quad \rho\in\mathcal{U},$$

respectively.

In the following we determine the structure of the coefficients of $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$. For this we use the following notation: Let

$$D := X\times(-a,a)\times\mathbb{R}^{n-1}\times\mathbb{R}^{2n-2},$$

and assume that $\widetilde{a} \in C^\infty(D,\mathbb{R})$. Given $\tau \in C^2(\Gamma)$, we then let a denote the Nemitskii operator induced by $\widetilde{a}$; i.e.,

$$a(\tau)(\omega,r) := \widetilde{a}((\omega,r),\mu^*\tau(\omega),\partial\mu^*\tau(\omega),\partial^2\mu^*\tau(\omega)), \quad (\omega,r)\in X.$$

Recall that $A(\rho)=\Delta^\Omega$ and that $B(\rho)=\partial_\nu^\Gamma$; i.e., $A(\rho)$ is the Laplace-Beltrami operator with respect to $(\Omega,\theta_\rho^*\eta)$, and $B(\rho)$ is the directional derivative with respect to the outer normal field on $(\Gamma,\theta_\rho^*\eta)$. Hence there exist

$$\widetilde{a}_{jk},\ \widetilde{a}_j \in C^\infty(D,\mathbb{R}), \qquad 1\le j,\ k\le n,$$

and

$$\widetilde{b}_j \in C^\infty(E,\mathbb{R}), \qquad 1\le j\le n,$$

where $E := Y\times(-a,a)\times\mathbb{R}^{n-1}$, such that

$$\begin{aligned} &[a_{jk}(\rho)] \quad \text{is symmetric and uniformly positive definite on } X,\\ &b_n(\rho) \quad \text{is uniformly positive on } Y, \end{aligned} \tag{5.1}$$

and such that

$$\begin{aligned} \mathcal{A}(\rho) &= -\sum_{j,k=1}^n a_{jk}(\rho)\partial_j\partial_k + \sum_{j=1}^n a_j(\rho)\partial_j\\ \mathcal{B}(\rho) &= -\sum_{j=1}^n b_j(\rho)\partial_j. \end{aligned} \tag{5.2}$$

In the next step we introduce linear differential operators with constant coefficients by freezing the localizations at ρ and at 0. More precisely, let

$$\begin{aligned} &a_{jk}^0 := a_{jk}(\rho)(0,0), \qquad b_j^0 := b_j(\rho)(0), \qquad 1\le j,\ k\le n,\\ &p_{jk}^0 := p_{jk}(\rho)(0), \qquad 1\le j,\ k\le n-1, \end{aligned} \tag{5.3}$$

where $[p_{jk}]$ are the coefficients of $\mathcal{P}$; see Lemma 3.2. Now define the following linear differential operators having constant coefficients:

$$\mathcal{A}_0 := 1-\sum_{j,k=1}^n a_{jk}^0\partial_j\partial_k, \qquad \mathcal{B}_0 := -\sum_{j=1}^n b_j^0\gamma_0\partial_j, \qquad \mathcal{P}_0 := 1-\sum_{j,k=1}^{n-1} p_{jk}^0\partial_j\partial_k.$$

From (5.1) we know that $\mathcal{A}_0$ is an elliptic operator. Hence the following boundary value problem for the half space $\mathbb{H}^n := \{x\in\mathbb{R}^n : x_n>0\}$ is well-posed in $h^{1+\alpha}(\mathbb{H}^n)$:

$$\mathcal{A}_0 u = 0 \quad \text{in} \quad \mathbb{H}^n, \qquad \gamma_0 u = g \quad \text{on} \quad \mathbb{R}^{n-1}, \tag{5.4}$$

where γ_0 denotes the corresponding trace operator, i.e., the restriction operator to $\partial\mathbb{H}^n \equiv \mathbb{R}^{n-1}$. Let $\mathcal{T}_0$ denote the corresponding solution operator; i.e., given $g \in h^{1+\alpha}(\mathbb{R}^{n-1})$, let $\mathcal{T}_0 g$ be the unique solution of (5.4). It can be shown that

$$\mathcal{T}_0 \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{n-1}); h^{1+\alpha}(\mathbb{H}^n)),$$

see Appendix B in [15]. The operator $\mathcal{B}_0\mathcal{T}_0\mathcal{P}_0$ should be regarded as the principal part of $\Phi(\rho)$ with coefficients fixed at $\mu_l(0,0)$. Our next goal is to show that $\mathcal{B}_0\mathcal{T}_0\mathcal{P}_0$ is a Fourier multiplier operator. To determine its symbol we need some preparation. Let

$$\vec{a} := (a^0_{1n}, \ldots, a^0_{(n-1)n}), \qquad a_0(\xi) := \sum_{j,k=1}^{n-1} a^0_{jk}\xi^j\xi^k, \quad \xi \in \mathbb{R}^{n-1},$$

and, for fixed $(\xi, \gamma) \in \mathbb{R}^{n-1} \times (0, \infty)$, define the following parameter-dependent quadratic polynomial:

$$q_{\xi,\gamma}(z) := \gamma^2 + a_0(\xi) + 2i(\vec{a}|\xi)z - a^0_{nn}z^2, \quad z \in \mathbf{C}.$$

Observe that the matrix $[a^0_{jk}]$ is positive definite; see (5.1) and (5.3). Hence it follows that, given $(\xi, \gamma) \in \mathbb{R}^{n-1} \times (0, \infty)$, there exists exactly one root $\lambda(\xi, \gamma)$ of $q_{\xi,\gamma}(\cdot)$ with positive real part, which is given by

$$\lambda(\xi, \gamma) = \frac{i(\vec{a}|\xi)}{a^0_{nn}} + \frac{1}{a^0_{nn}}\sqrt{a^0_{nn}(\gamma^2 + a_0(\xi)) - (\vec{a}|\xi)^2}. \tag{5.5}$$

Moreover, we set

$$\vec{b} := (b^0_1, \ldots, b^0_{n-1}), \qquad p_0(\xi, \gamma) := \gamma^2 + \sum_{j,k=1}^{n-1} p^0_{jk}\xi^j\xi^k$$

for $(\xi, \gamma) \in \mathbb{R}^{n-1} \times (0, \infty)$. Finally, let

$$a(\xi, \gamma) := \{b^0_n\lambda(\xi, \gamma) - i(\vec{b}|\xi)\}p_0(\xi, \gamma)$$

for $(\xi, \gamma) \in \mathbb{R}^{n-1} \times (0, \infty)$. The function a will serve as a parameter-dependent Fourier multiplier. More precisely, let $\gamma \in (0, \infty)$ be given and let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^{n-1}$. Then we shall see that the Fourier multiplier operator $\mathcal{F}^{-1}a(\cdot, \gamma)\mathcal{F}$ is a nicely behaved operator. The additional parameter γ is introduced to homogenize these symbols. Observe that the function a is positively homogeneous of degree 3.

Lemma 5.1. $\mathcal{B}_0\mathcal{T}_0\mathcal{P}_0$ *is a Fourier multiplier operator with symbol* $a(\cdot,1)$*; i.e.,*

$$\mathcal{B}_0\mathcal{T}_0\mathcal{P}_0 = \mathcal{F}^{-1}a(\cdot,1)\mathcal{F}.$$

Proof. Obviously, $p_0(\cdot,1)$ is the symbol of the operator $\mathcal{P}_0$, which means that $\mathcal{P}_0 = \mathcal{F}^{-1}p_0(\cdot,1)\mathcal{F}$. In addition, it follows from the proof of Lemma B.3 in [15], that the following representation holds:

$$\mathcal{B}_0\mathcal{T}_0 = \mathcal{F}^{-1}\big\{b_n^0\lambda(\cdot,1) - i(\vec{b}|\cdot)\big\}\mathcal{F},$$

which completes the proof. □

In [5], H. Amann developed a theory of parameter-dependent Fourier multiplier operators on general function spaces. It is shown in [4] that parameter-dependent Fourier multipliers are very useful in studying generation properties of Fourier multiplier operators on Besov spaces, thus in particular on Hölder spaces. To be more precise, let $\alpha_* > 0$, $r > 0$ be given and define

$$\begin{aligned}\mathcal{E}ll\mathcal{S}_r^\infty(\alpha_*) := \Big\{&a \in C^\infty(\mathbb{R}^{n-1}\times(0,\infty))\,;\, a \text{ is positively homogeneous}\\ &\text{of degree } r,\ \text{all derivatives of } a \text{ are bounded on } |\xi|^2+\mu^2=1,\\ &\text{and } Re\,a(\xi,\mu) \ge \alpha_*(|\xi|^2+\mu^2)^{r/2},\ (\xi,\mu)\in\mathbb{R}^{n-1}\times(0,\infty)\Big\}.\end{aligned}$$

Then, given $b \in \mathcal{E}ll\mathcal{S}_r^\infty(\alpha_*)$, it can be proved that

$$\mathcal{F}^{-1}b(\cdot,\gamma)\mathcal{F} \in \mathcal{H}(h^{s+r}(\mathbb{R}^{n-1}), h^s(\mathbb{R}^{n-1}))$$

for all $\gamma > 0$ and all $s \in (0,\infty)$. For a proof of the above result we refer to [4]. Using this general result it is now easy to establish the following

Corollary 5.2. $\mathcal{B}_0\mathcal{T}_0\mathcal{P}_0 \in \mathcal{H}(h^{3+\alpha}(\mathbb{R}^{n-1}), h^\alpha(\mathbb{R}^{n-1}))$.

Proof. (i) It suffices to verify that a belongs to the class $\mathcal{E}ll\mathcal{S}_3^\infty(\alpha_*)$. To see this, observe that $a \in C^\infty(\mathbb{R}^{n-1}\times(0,\infty))$ and that a is positively homogeneous of degree 3. Moreover, it is easily verified that all derivatives of a are bounded on $[\gamma^2+|\xi|^2=1]$.

(ii) Recall that thanks to (3.3) we know that $[p_{jk}^0]$ is positive definite. Hence we find a positive constant p_* such that

$$p_0(\xi,\gamma) = \gamma^2 + \sum_{jk}^{n-1} p_{jk}^0\xi^j\xi^k \ge p_*(\gamma^2+|\xi|^2),$$

for all $(\xi,\gamma)\in\mathbb{R}^{n-1}\times(0,\infty)$. Similarly, it follows from (5.1) that there exists a positive constant r_* such that

$$Re\,\lambda(\xi,\gamma) \ge r_*\sqrt{\gamma^2+|\xi|^2}, \quad (\xi,\gamma)\in\mathbb{R}^{n-1}\times(0,\infty).$$

Hence letting $\alpha_* := b_n^0 r_* p_* > 0$ we see, again due to (5.1), that α_* is positive. Moreover, it follows that

$$Re\, a(\xi,\gamma) = b_n^0 p_0(\xi,\gamma) Re\, \lambda(\xi,\gamma) \geq \alpha_*(\gamma^2 + |\gamma|^2)^{3/2}$$

for all $(\xi,\gamma) \in \mathbb{R}^{n-1} \times (0,\infty)$. This shows that a belongs to the class $\mathcal{E}ll\mathcal{S}_3^\infty(\alpha_*)$ and completes the proof. □

Proof of Theorem 4.1. (i) Assume that E_1 and E_0 are two Banach spaces such that $E_1 \hookrightarrow E_0$ and such that E_1 is dense in E_0. Given $A \in \mathcal{L}(E_1, E_0)$, it follows from Remark I.1.2.1a) in [3] that A belongs to $\mathcal{H}(E_1, E_0)$ if and only if there exist positive constants C and λ_* such that

$$\begin{aligned}&\lambda_* + A \in Isom(E_1, E_0),\\ &|\lambda|\, \|x\|_{E_0} + \|x\|_{E_1} \leq C\|(\lambda + A)x\|_{E_0}, \quad x \in E_1,\ \lambda \in [Re\, z \geq \lambda_*].\end{aligned}$$

(ii) To simplify our notation we let

$$\mathcal{G}_0(\rho) := \mathcal{B}_0\mathcal{T}_0\mathcal{P}_0 \quad \text{and} \quad G(\rho) := B(\rho)T(\rho)P(\rho), \quad \rho \in \mathcal{U},$$

and we use the symbols $|\cdot|_s$ and $\|\cdot\|_s$ exclusively for the norms in $h^s(\mathbb{R}^{n-1})$ and $h^s(\Gamma)$, respectively. Moreover we fix a compact subset K of $\mathcal{U}$. It follows from Corollary 5.2 and (i) that there exist positive constants λ_1 and C_1, independent of $\kappa \in (0, a]$, such that

$$|g|_{3+\alpha} + |\lambda||g|_\alpha \leq C_1|(\lambda + \mathcal{G}_0(\rho))g|_\alpha \tag{5.6}$$

for all $g \in h^{3+\alpha}(\mathbb{R}^{n-1})$, $\lambda \in [Re\, z \geq \lambda_1]$, and $\rho \in K$.

(iii) Given $\gamma \in (0,\alpha)$, there exist $\kappa \in (0, a]$, a partition of unity $\{(U_l, \psi_l) : 1 \leq l \leq m_\kappa\}$ for Γ, and a positive constant $C_2 := C_2(C_1, \gamma, K, \kappa)$ such that

$$|\mu_l^*\big(\psi_l G(\rho)h\big) - \mathcal{G}_0(\rho)\mu_l^*(\psi_l h)|_\alpha \leq \frac{1}{2C_1}|\mu_l^*(\psi_l h)|_{3+\alpha} + C_2\|h\|_{3+\gamma} \tag{5.7}$$

for all $h \in h^{3+\alpha}(\Gamma)$, $l \in \{1,\ldots,m_\kappa\}$, and $\rho \in K$. We omit an explicit proof of the above estimate, but we refer to the techniques used in Lemma 5.1 in [17] or in Lemma 6.1 in [15], where similar results are shown.

(iv) From (5.6) and (5.7) we now conclude that

$$\begin{aligned}&|\mu_l^*(\psi_l h)|_{3+\alpha} + |\lambda||\mu_l^*(\psi_l h)|_\alpha\\ &\qquad \leq 2C_1\big\{|\mu_l^*\big(\psi_l(\lambda + G(\rho))h\big)|_\alpha + C_2\|h\|_{3+\gamma}\big\}\end{aligned} \tag{5.8}$$

for all $h \in h^{3+\alpha}(\Gamma)$, $\lambda \in [Re\, z \geq \lambda_1]$, $l \in \{1,\ldots,m_\kappa\}$, and $\rho \in K$. Next observe that

$$\big[h \mapsto \max_{1\leq l\leq m_\kappa} |\mu_l^*(\psi_l h)|_{k+\alpha}\big]$$

defines an equivalent norm on $h^{k+\alpha}(\Gamma)$, $k \in \mathbb{N}$, due to the fact that the family $\{(U_l, \psi_l) : 1 \le l \le m_\kappa\}$ is a localization sequence for Γ; see [34]. Hence (5.8) implies the existence of a positive constant C such that

$$\|h\|_{3+\alpha} + |\lambda|\,\|h\|_\alpha \le \frac{C}{2}\|(\lambda + G(\rho))h\|_\alpha + C\|h\|_{3+\gamma} \tag{5.9}$$

for all $h \in h^{3+\alpha}(\Gamma)$, $\lambda \in [Re\, z \ge \lambda_1]$, and $\rho \in K$.

Finally, it follows from the fact that the little Hölder spaces are stable under continuous interpolation that there exists a positive constant C_3 such that

$$\|h\|_{3+\gamma} \le \frac{1}{2C}\|h\|_{3+\alpha} + C_3\|h\|_\alpha, \qquad h \in h^{3+\alpha}(\Gamma).$$

Now we conclude from (5.9) that

$$\|h\|_{3+\alpha} + |\lambda|\,\|h\|_\alpha \le C\|(\lambda + G(\rho))h\|_\alpha \tag{5.10}$$

for all $h \in h^{3+\alpha}(\Gamma)$, $\lambda \in [Re\, z \ge \lambda_*]$, and $\rho \in K$, and where we have set $\lambda_* := 2\max\{\lambda_1, CC_3\}$.

(v) Thanks to (5.10) it suffices to prove that, given $\rho \in \mathcal{U}$, the operator $\lambda_* + G(\rho)$ is surjective. Moreover observe that $\mathcal{U}$ is star-shaped with respect to 0 and that $K := \{t\rho : t \in [0,1]\}$ is a compact subset of $\mathcal{U}$. Since the estimate in (5.10) is uniform with respect to $\rho \in K$, a standard homotopy argument shows that it suffices to verify that $\lambda_* + G(0) = \lambda_* + B(0)T(0)P(0)$ is surjective.

It is known that the Dirichlet-Neumann operator $B(0)T(0)$ belongs to the space $\mathcal{H}(h^{1+\alpha}(\Gamma), h^\alpha(\Gamma))$; see the proof of Corollary 6.3 in [15]. Hence we may assume that

$$\lambda_* + B(0)T(0) \in Isom(h^{1+\alpha}(\Gamma), h^\alpha(\Gamma)).$$

Since also

$$(1 - \frac{1}{n-1}\Delta_\pi^\Gamma) \in Isom(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)),$$

we find that

$$\big(\lambda_* + B(0)T(0)\big)(1 - \frac{1}{n-1}\Delta_\pi^\Gamma) \in Isom(h^{3+\alpha}(\Gamma), h^\alpha(\Gamma)). \tag{5.11}$$

Moreover, recall that $P(0) = -\frac{1}{n-1}\Delta_\pi^\Gamma$; see Remark 3.3. Hence we obtain that

$$\lambda_* + G(0) - \big(\lambda_* + B(0)T(0)\big)(1 - \frac{1}{n-1}\Delta_\pi^\Gamma) = \frac{\lambda_*}{n-1}\Delta_\pi^\Gamma - B(0)T(0). \tag{5.12}$$

Finally, it follows from standard elliptic regularity theory that the solution operator $T(0)$ belongs to $\mathcal{L}(h^{3+\alpha}(\Gamma), h^{3+\alpha}(\Omega))$. Hence we see that

$$\frac{\lambda_*}{n-1}\Delta_\pi^\Gamma - B(0)T(0) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)).$$

Since $h^{1+\alpha}(\Gamma)$ is compactly embedded in $h^\alpha(\Gamma)$, we conclude from (5.11) and (5.12) that $\lambda_* + G(0)$ is a compactly perturbed isomorphism, hence a Fredholm operator of index 0. Since it is injective too, see (5.10), we get the assertion. □

Proof of Theorem 4. We briefly sketch the proof of Theorem 4. As in Section 2, given $\rho \in \mathcal{U}$, we find transformations θ^i_ρ mapping Ω^i_ρ onto the reference domains Ω^i. Let $H(\rho)$ be the mean curvature operator of Γ with respect to Ω^1 and perform, similarly as in Lemma 3.1, a quasilinear decomposition of the form $H(\rho) = P(\rho)\rho + K(\rho)$. In addition, let $T_i(\rho)$ denote the solution operator of the corresponding transformed elliptic boundary value problems on Ω^i and put

$$\begin{aligned}\Phi(\rho) &:= \sigma\big(B_1(\rho)T_1(\rho) + B_2(\rho)T_2(\rho)\big)P(\rho),\\ F(\rho) &:= -\sigma\big(B_1(\rho)T_1(\rho) + B_2(\rho)T_2(\rho)\big)K(\rho).\end{aligned}$$

Here $B_i(\rho)$ denote the directional derivatives on Γ with respect to the outer normals on Ω^i, respectively. Then Lemma 2.1 is easily adapted to the situation of problem (1.4). Finally, the symbol of the principal part $\Phi(\rho)$ with coefficients fixed at $\rho \in \mathcal{U}$ and at $\mu_l(0)$ is given by

$$\sigma\big\{b^0_{n,1}\lambda_1(\cdot,1) + b^0_{n,2}\lambda_2(\cdot,1) - i(\vec{b}_1 + \vec{b}_2|\cdot)\big\}p_0,$$

where λ_i denote the eigenvalues with positive real part of the elliptic boundary value problems on Ω^i according to (5.5). After an analogous perturbation procedure as for problem (1.1) one then gets that

$$\sigma\big(B_1(\rho)T_1(\rho) + B_2(\rho)T_2(\rho)\big)P(\rho) \in \mathcal{H}(h^{3+\alpha}(\Gamma), h^\alpha(\Gamma)),$$

which implies the assertions of Theorem 4.

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