Advances in Differential Equations Volume 2, Number 5, September 1997, pp. 831–866.

THE DIRICHLET PROBLEM FOR A CLASS OF ULTRAPARABOLIC EQUATIONS

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(Submitted by: G. Da Prato)

Abstract. In this paper we study the Dirichlet problem for a class of ultraparabolic equations. More precisely, we prove the existence of a generalized Perron-Wiener solution and we provide a geometric condition for the regularity of the boundary points which extends the classical Zaremba exterior cone criterion to our setting. The main steps for deriving our results are: i) the introduction in $\mathbf{R}^{N+1}$ of a homogeneous structure; ii) the proof of some interior estimates in a suitable space of Hölder-continuous functions; iii) the construction of a basis of open subsets of $\mathbf{R}^{N+1}$ for which the Dirichlet problem is univocally solvable.

1. Introduction and main results. We consider in $\mathbf{R}^{N+1}$ the second-order linear operators

$$L = \sum_{i,j=1}^{q} a_{ij}(z)\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t, \tag{1.1}$$

where $z = (x,t) \in \mathbf{R}^{N+1}$, $1 \le q \le N$ and $b_{ij} \in \mathbf{R}$ for every $i,j = 1,\dots,N$.

$A_0(z) = (a_{ij}(z))_{i,j=1,\dots,q}$ is a symmetric matrix, which is positive definite in $\mathbf{R}^q$, $B = (b_{ij})_{i,j=1,\dots,N}$ is a constant matrix of the form

$$B = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{1.2}$$

where each B_i is a $p_{i-1} \times p_i$ block matrix of rank p_i, $i = 1,\dots,r$ with $q = p_0 \ge p_1 \ge \dots \ge p_r$ and $p_0 + p_1 + \dots + p_r = N$.

Operators like (1.1) naturally arise in the stochastic theory of diffusion processes. For example, if we choose

$$A_0 = I_n \quad \text{and} \quad B = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$$

Accepted for publication September 1996.
AMS Subject Classifications: 35K70, 35K22, 35G15.

we obtain in $\mathbf{R}^{2n+1}$

$$L = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i=1}^{n} x_i \partial_{x_{n+i}} - \partial_t, \tag{1.3}$$

which is the simplest prototype of the Kolmogorov operator and describes the probability of a physical system with $2n$ degrees of freedom (cf. [20, page 167]).

As in the classical theory of elliptic and parabolic operators, we will study the operators (1.1) as an Hölder perturbation of the *frozen* operators L_{z_0}:

$$L_{z_0} = \sum_{i,j=1}^{q} a_{ij}(z_0)\partial_{x_i x_j}^2 + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t, \quad z_0 \in \mathbf{R}^{N+1}. \tag{1.4}$$

It is well known (see for example [8]) that the operators (1.4) are hypoelliptic since they satisfy the Hörmander condition; see [6]. Moreover, they are invariant with respect to a dilation group $(D(\lambda))_{\lambda>0}$ and left invariant with respect to a translation group $(\mathbf{R}^{N+1}, \circ)$. The dilation $D(\lambda)$ is defined as

$$D(\lambda) = \text{diag}\,(\lambda I_{p_0}, \lambda^3 I_{p_1}, \lambda^5 I_{p_2}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2), \tag{1.5}$$

where I_k denotes the $k \times k$ identity matrix and $p_1, \dots, p_r$ are introduced in (1.2). Hence the homogeneous dimension of $\mathbf{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$ is

$$Q + 2 = p_0 + 3p_1 + 5p_2 + \cdots + (2r+1)p_r + 2. \tag{1.6}$$

The composition law of the group $(\mathbf{R}^{N+1}, \circ)$ is defined as

$$(x,t) \circ (y,\tau) = (y + E(\tau)x, t + \tau) \quad \forall (x,t), (y,\tau) \in \mathbf{R}^{N+1}, \tag{1.7}$$

where

$$E(\tau) = \exp(-\tau B^T). \tag{1.8}$$

The identity element of the group is $(0,0)$ and the inverse element $(x,t)^{-1} = (-E(-t)x, -t)$ for every $(x,t) \in \mathbf{R}^{N+1}$.

Note that neither $D(\lambda)$ nor the translation group depend on the point z_0, since they depend on the matrix B. Hence the introduction in $\mathbf{R}^{N+1}$ of a homogeneous seminorm of degree 1 with respect to $D(\lambda)$ seems natural.

Definition 1.1. For any $z = (x_1, \dots, x_N, t) \in \mathbf{R}^{N+1}$ we define

$$||z|| \equiv |||x||| + |t|^{\frac{1}{2}} \equiv \sum_{i=1}^{N} |x_i|^{\frac{1}{\alpha_i}} + |t|^{\frac{1}{2}},$$

where $\alpha_1 = \cdots = \alpha_{p_0} = 1$, $\alpha_{p_0+1} = \cdots = \alpha_{p_0+p_1} = 3$, $\dots$, $\alpha_{p_0+\cdots+p_{r-1}+1} = \cdots = \alpha_N = 2r+1$.

(See Section 2 below for the proof of some properties of $||\cdot||$.) We also define a *distance* which is invariant with respect to the left translations of $(\mathbf{R}^{N+1},\circ)$

$$d(z,\zeta)=||\zeta^{-1}\circ z||.$$

We will always work in the following space of Hölder- continuous functions.

Definition 1.2. Let α be a positive constant, $\alpha<1$, and let Ω be an open subset of $\mathbf{R}^{N+1}$. We will say that a function $f:\Omega\to\mathbf{R}$ is Hölder continuous with exponent α in Ω with respect to the groups $(\mathbf{R}^{N+1},\circ)$ and $(D(\lambda))_{\lambda>0}$ (in short: B-Hölder continuous with exponent α) if there exists a positive constant k such that

$$|f(z)-f(\zeta)|\le k\,||\zeta^{-1}\circ z||^{\alpha}\quad \forall z,\zeta\in\Omega.$$

We will denote by $C^{\alpha}(\Omega;B)$ the space of the B- Hölder-continuous functions and by $|\cdot|_{\alpha,\Omega}$ the norm

$$|f|_{\alpha,\Omega}=\sup_{\Omega}|f|+\sup_{\substack{z,\zeta\in\Omega\\ z\ne\zeta}}\frac{|f(z)-f(\zeta)|}{||\zeta^{-1}\circ z||^{\alpha}}.$$

We will say that $f\in C^{2+\alpha}(\Omega;B)$ if

$$|f|_{2+\alpha,\Omega}=\sup_{\Omega}|f|+\sum_{i=1}^{q}\sup_{z\in\Omega}|f_i(z)|+\sum_{i,j=1}^{q}\sup_{z\in\Omega}|f_{ij}(z)|$$
$$+\sum_{i,j=1}^{q}\sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}}\frac{|f_{ij}(z)-f_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}+\sup_{z\in\Omega}|Yf(z)|+\sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}}\frac{|Yf(z)-Yf(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}<\infty,$$

where $f_i=\partial_{x_i}f$, $f_{ij}=\partial^2_{x_ix_j}f$ and Y is the first-order differential operator:

$$Y=\langle x,BD\rangle-\partial_t\equiv\sum_{i,j=1}^{N}b_{ij}x_i\partial_{x_j}-\partial_t,$$

where $D=(\partial_{x_1},\dots,\partial_{x_N})$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the gradient and the inner product in $\mathbf{R}^N$.

Moreover, we will say that f is a locally B-Hölder-continuous function, and we will write $f\in C^{\alpha}_{loc}(\Omega;B)$ (respectively $f\in C^{2+\alpha}_{loc}(\Omega;B)$) if $f\in C^{\alpha}(\Omega';B)$ ($f\in C^{2+\alpha}(\Omega';B)$) for every subset $\Omega'\subset\subset\Omega$.

We observe that, if f is an Hölder-continuous function of exponent α in the usual sense, then f is B-Hölder continuous of exponent α. Vice versa, if $f\in C^{\alpha}(\Omega;B)$ then f is $\alpha\beta$-Hölder continuous in usual sense, where β is the real constant defined by $\beta=\frac{1}{2r+1}$. Here r denotes the number of nonzero blocks of the matrix B (see

(1.2)). This remark follows immediately from inequalities (iii) of Proposition 2.1 in the next section.

Definition 1.3. Let $0 < \alpha < 1$ and let Ω be an open subset of $\mathbf{R}^{N+1}$. We will say that a bounded function $f : \Omega \to \mathbf{R}$ belongs to $C_d^\alpha(\Omega; B)$ if

$$\sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^\alpha \frac{|f(z) - f(\bar{z})|}{||\bar{z}^{-1} \circ z||^\alpha} < \infty,$$

where

$$d_{z,\bar{z}} = \min\{d_z, d_{\bar{z}}\}, \quad \text{and} \quad d_z = \text{dist } (z, \partial\Omega) = \inf_{\zeta\in\partial\Omega} ||\zeta^{-1} \circ z||.$$

We define

$$|f|_{\alpha,d;\Omega} = \sup_\Omega |f| + \sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^\alpha \frac{|f(z) - f(\bar{z})|}{||\bar{z}^{-1} \circ z||^\alpha},$$

and we will say that $f \in C_d^{2+\alpha}(\Omega; B)$ if

$$\begin{aligned} |f|_{2+\alpha,d;\Omega} &= \sup_\Omega |f| + \sum_{i=1}^q \sup_{z\in\Omega} d_z|f_i(z)| + \sum_{i,j=1}^q \sup_{z\in\Omega} d_z^2|f_{ij}(z)| \\ &+ \sum_{i,j=1}^q \sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^{2+\alpha} \frac{|f_{ij}(z) - f_{ij}(\bar{z})|}{||\bar{z}^{-1} \circ z||^\alpha} \\ &+ \sup_{z\in\Omega} d_z^2|Yf(z)| + \sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^{2+\alpha} \frac{|Yf(z) - Yf(\bar{z})|}{||\bar{z}^{-1} \circ z||^\alpha} \\ &\equiv \sup_\Omega |f| + \sum_{i=1}^q \sup_\Omega d|f_i| + |d^2Yf|_{\alpha,d;\Omega} + \sum_{i,j=1}^q |d^2 f_{ij}|_{\alpha,d;\Omega} < \infty. \end{aligned}$$

We next introduce further hypotheses on the operators L in (1.1).

Hypothesis (H). *For every $i, j = 1, \dots, q$ the coefficients a_{ij} belong to $C_{loc}^\alpha(\mathbf{R}^{N+1};\, B)$, where $\alpha \in (0, 1)$. Moreover, there exists a positive constant μ such that*

$$\frac{1}{\mu} \sum_{i=1}^q |\xi_i|^2 \leq \sum_{i,j=1}^q a_{ij}(z)\xi_i\xi_j \leq \mu \sum_{i=1}^q |\xi_i|^2$$

for all $(\xi_1, \dots, \xi_q) \in \mathbf{R}^q$ and for all $z \in \mathbf{R}^{N+1}$.

We remark that if the coefficients a_{ij} of (1.1) are smooth, the operators L belong to the class first introduced by Hörmander in [6], Oleĭnik-Radkevič in [13], and later studied by Rothschild-Stein in [16].

Whereas the operators of "parabolic" type $\sum_{i=1}^{q} X_i^2 - \partial_t$ with C^∞ coefficients have been widely studied, only few results are known for the operators (1.1) when the coefficients are not smooth (see Weber, [22]; Il'In, [7]; Sonin, [21]; Genčev, [5]). In particular, in [18] the author proves the existence of a solution of the Cauchy-Dirichlet problem for the operator (1.3) in $\mathbf{R}^3$ in the domain $\{(x_1, x_2, t) \in \mathbf{R}^3 | \, x_1 \geq 0, x_2 \in \mathbf{R}, 0 < t < T\}$. Hölder estimates of the solutions are proved only for the operators with constant coefficients: in [19] a local Hölder estimate for the solutions of Kolmogorov operator (1.3) is proved.

In [8] Lanconelli and Polidoro have carried out a systematic study for the operators (1.1) with constant coefficients a_{ij}. In [14] the author shows the existence of the fundamental solution Γ of (1.1) with the Levi's parametrix method, provides a precise local estimate of Γ in terms of the fundamental solution of the frozen operators L_{z_0} and shows the Harnack inequality for nonnegative solutions. The Harnack inequality for a particular subclass of operators like (1.1) has been proved by Qi Zhang ([23]). A global lower bound for the fundamental solution of the operators in divergence form can be found in [15].

Finally, we also quote the recent paper [9] where the author studies the Cauchy problem for the operators (1.1) with constant coefficients a_{ij}, and proves a global Hölder estimate with respect to the spatial variable x.

In this paper we first prove the following Schauder interior estimates.

Theorem 1.4 (Schauder estimates)**.** *Let L be as in* (1.1) *satisfying hypothesis* (H). *Let $f \in C_d^\alpha(\Omega; B)$ and let u be a bounded function belonging to $C_{loc}^{2+\alpha}(\Omega; B)$ such that $Lu = f$ in Ω. Then $u \in C_d^{2+\alpha}(\Omega; B)$ and there exists a constant $c > 0$, independent of u, such that*

$$|u|_{2+\alpha,d;\Omega} \leq c\,(\sup_{\Omega} |u| + |d^2 f|_{\alpha,d;\Omega}). \tag{1.9}$$

(*The constant c depends on the constant μ in* (H) *and on the Hölder norm of the coefficients a_{ij}*).

Afterward, we prove the existence of a *generalized solution*, in the sense of Perron-Wiener, of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ \;\; u = \phi & \text{on} \quad \partial\Omega. \end{cases} \tag{1.10}$$

Precisely, our main result can be stated as follows:

Theorem 1.5 (Existence of a generalized solution)**.** *Let L be as in* (1.1) *satisfying hypothesis* (H). *Let Ω be an open bounded subset of $\mathbf{R}^{N+1}$, $f \in C^\alpha(\Omega; B)$ and $\phi \in C(\partial\Omega)$. Then, there exists a solution $u \in C_{loc}^{2+\alpha}(\Omega; B)$ of $Lu = f$ in Ω such that $\lim_{z \to z_0} u(z) = \phi(z_0)$ for every L-regular point $z_0 \in \partial\Omega$.*

A boundary point z_0 of Ω is *L-regular* if there exists a local barrier in z_0, that is, there exist a neighborhood V of z_0 and a function $w \in C^{2+\alpha}(V; B)$ such that $w(z_0) = 0$, $w(z) > 0$ for $z \in \overline{\Omega \cap V} \setminus \{z_0\}$ and $Lw < 0$ in $\Omega \cap V$.

The proof is organized as follows. In Section 2 we show some properties of the homogeneous norm $||\cdot||$ and of the fundamental solution of L_{z_0}, and we prove a representation formula for functions $u \in C_0^\infty(\mathbf{R}^{N+1})$. Some interpolation inequalities enable us to deduce Theorem 1.4 (see Section 3).

We next apply this result to the Dirichlet problem (1.10): in Section 4 we prove the existence of a solution on a suitable class of L-regular subsets (see Definition 4.1 below), and in Section 5 we prove Theorem 1.5.

Finally, in Section 6, we give some geometric criteria for the L-regularity of the boundary points of arbitrary open sets. In particular we extend the classical Zaremba exterior cone criterion for elliptic operators to our setting. Our result also extends an Effros-Kazdan criterion for the heat operator ([4]) and a result proved in [17] for the Kolmogorov operator in $\mathbf{R}^3$.

2. Some preliminary results. In this section we will prove some preliminary results: we first prove some properties of the homogeneous norm $||\cdot||$ introduced in Definition 1.1, then we study the properties of the fundamental solution Γ^0 of L_{z_0}. Finally, we write a representation formula for functions $u \in C_0^\infty(\mathbf{R}^{N+1})$ in terms of Γ^0.

Proposition 2.1. *The function $z \mapsto ||z||$ satisfies*

(i) $||D(\lambda)z|| = \lambda||z||$ *for every $z \in \mathbf{R}^{N+1}$ and for every $\lambda > 0$.*

(ii) *For every $z, \zeta \in \mathbf{R}^{N+1}$*

$$||z+\zeta|| \le c\,(||z|| + ||\zeta||), \qquad ||z \circ \zeta|| \le c\,(||z|| + ||\zeta||),$$

and

$$\frac{1}{c}||z|| \le ||z^{-1}|| \le c||z||,$$

for some constant $c = c(B) \ge 1$.

(iii)

$$\tfrac{1}{c}\,|z-\zeta| \le ||\zeta^{-1} \circ z|| \le c\,|z-\zeta|^\beta \quad \text{if} \quad |z-\zeta|, ||\zeta^{-1}\circ z|| \le 1,$$

for some constant $c = c(B) > 0$. Here $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^{N+1}$ and $\beta = 1/(2r+1)$ (r is the number of nonzero blocks of the matrix B, so that β depends on the height of commutators which generate the Lie algebra Lie$\{\partial_1, \dots, \partial_q, Y\}$).

Proof. The proof of (i) follows from Definition 1.1, whereas for the proof of (ii) we refer to [10] or [2]. Let us now prove the first inequality in (iii). It is sufficient to show that

$$|z \circ \zeta - z| \le c\,||\zeta|| \quad \text{for all } z, \zeta \in K \text{ and } ||\zeta|| \le 1,$$

where K is a bounded subset of $\mathbf{R}^{N+1}$. If $z = (x,t)$ and $\zeta = (\xi, \tau)$, then from the definition of the translation $\circ$ in (1.7) we have

$$|z \circ \zeta - z| = |(\xi + E(\tau)x, t+\tau) - (x,t)| = |\xi + (E(\tau) - I)x| + |\tau|,$$

where I indicates the $N \times N$ identity matrix. On the other hand, since

$$E(t) = \sum_{k=0}^{r} (-1)^k \frac{t^k}{k!} (B^T)^k, \tag{2.1}$$

we get

$$|(E(\tau) - I)x| = \Big|\sum_{k=1}^{r} \frac{(-\tau B^T)^k}{k!}\, x\Big| \le (|\tau| \le 1) \le |\tau| \sum_{k=1}^{r} \frac{|(B^T)^k\, x|}{k!} \le c(K, B)\, |\tau|.$$

Then

$$|z \circ \zeta - z| \le c \,\Big(\sum_{i=1}^{N} |\xi_i| + |\tau|\Big) \le (||\zeta|| \le 1) \le c\, ||\zeta||.$$

Now we prove the second inequality in (iii). If $\zeta = (\xi, \tau)$ and $z = (x, t)$, we have from (2.1)

$$\begin{aligned}
||\zeta^{-1} \circ z|| &= \sum_{i=1}^{N} |(x - E(t-\tau)\xi)_i|^{\frac{1}{\alpha_i}} + |t - \tau|^{\frac{1}{2}} \\
&= \sum_{i=1}^{q} |x_i - \xi| + \sum_{i=q+1}^{q+p_1} |(x_i - (t-\tau)\ l.c.\{\xi_1, \dots, \xi_q\} - \xi_i|^{\frac{1}{\alpha_i}} \\
&+ \sum_{i=q+p_1}^{q+p_1+p_2} \big|x_i - (t-\tau)^2\, c.l.\{\xi_1, \dots, \xi_q\} + (t-\tau)\, l.c.\{\xi_{q+1}, \dots, \xi_{q+p_1}\} - \xi_i\big|^{\frac{1}{\alpha_i}} \\
&+ \cdots + |t - \tau|^{\frac{1}{2}},
\end{aligned}$$

where $l.c.\{a_1, \dots, a_s\}$ indicates a linear combination, with constant coefficients, of $a_1, \dots, a_s$. Since $\beta \le \frac{1}{\alpha_i}$ for $i = 1, \dots, N$ and $|x_i - \xi_i| \le 1$, $|t - \tau| \le 1$ we obtain

$$||\zeta^{-1} \circ z|| \le c\,(\sum_{i=1}^{N} |x_i - \xi_i|^{\frac{\beta}{2}} + |t - \tau|^{\frac{\beta}{2}}) \le c\, |\zeta - z|^{\beta}.$$

This inequality completes the proof of Proposition 2.1.

Let $z_0 \in \mathbf{R}^{N+1}$ and L_{z_0} be the frozen operator defined in (1.4). If we denote by $\Gamma(z_0; \cdot) = \Gamma^0(\cdot)$ the fundamental solution of L_{z_0} with pole at zero, then

$$\Gamma^0(x, t) = \frac{t^{-\frac{Q}{2}}}{(4\pi)^{N/2} (\det C(z_0))^{1/2}} \exp\big(-\tfrac{1}{4} \langle C^{-1}(z_0) D_0(t^{-\frac{1}{2}}) x, D_0(t^{-\frac{1}{2}}) x\rangle\big), \tag{2.2}$$

if $t > 0$, $\Gamma^0(x, t) = 0$ if $t \le 0$ (see [8]). Here $D_0(\lambda)$ denotes the restriction to $\mathbf{R}^N$ of the dilation $D(\lambda)$ defined in (1.5) and $C(z_0)$ denotes the positive definite matrix $\int_0^1 E(s) A(z_0) E^T(s)\, ds$ where $A(z_0)$ is the $N \times N$ constant matrix

$$A(z_0) = \begin{pmatrix} A_0(z_0) & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0(z_0) = (a_{ij}(z_0))_{i,j=1,\dots,q}. \tag{2.3}$$

By the left invariance of L_{z_0}, its fundamental solution with pole at ζ simply is:

$$\Gamma^0(z; \zeta) = \Gamma^0(\zeta^{-1} \circ z; 0) = \Gamma^0(\zeta^{-1} \circ z).$$

Proposition 2.2. *Let* $z_0 \in \mathbf{R}^{N+1}$ *and* $\Gamma^0(\cdot) = \Gamma(z_0; \cdot)$. *Then*

(i) $\Gamma^0 \in C^\infty(\mathbf{R}^{N+1} \setminus \{0\})$;

(ii) Γ^0 *is* $D(\lambda)$*-homogeneous of degree* $-Q$, $\Gamma^0_i = \partial_{x_i}\Gamma^0$ *and* $\Gamma^0_{ij} = \partial^2_{x_i x_j}\Gamma^0$ *are respectively* $D(\lambda)$*-homogeneous of degree* $-Q-\alpha_i$ *and* $-Q-\alpha_i-\alpha_j$ *for every* $i, j = 1, \dots, N$.

(iii) (*Vanishing property*). *For all* $0 < a < b$ *and for all* $i, j = 1, \dots, q$ *we have*

$$\int_{a \le ||\zeta|| \le b} \Gamma^0_{ij}(\zeta)\, d\zeta = 0.$$

Proof. (i) and (ii) follow from (2.2). In order to prove (iii) we only have to show that the function

$$\epsilon \mapsto \int_{||\zeta||=\epsilon} \Gamma^0_j(\zeta)\nu_i(\zeta)\, d\sigma_\zeta,$$

is constant, where ν is the outer normal to the surface $\{\zeta \in \mathbf{R}^{N+1} : ||\zeta|| = \epsilon\}$. From Definition 1.1, it immediately follows that $||(x,t)|| = \epsilon$, $t > 0$ if and only if

$$t = \big(\epsilon - \sum_{k=1}^N |x_k|^{\frac{1}{\alpha_k}}\big)^2;$$

then, since $\Gamma = 0$ if $t < 0$ and Γ^0_j is $D(\lambda)$-homogeneous of degree $-Q-1$, we get

$$\int_{||\zeta||=\epsilon} \Gamma^0_j(\zeta)\nu_i(\zeta)\, d\sigma_\zeta = \int_{||\zeta||=\epsilon, t>0} \Gamma^0_j(\zeta)\nu_i(\zeta)\, d\sigma_\zeta$$

$$= \int_{||D_0(\epsilon^{-1})x|| \le 1} \Gamma^0_j\big(x, \big(\epsilon - \sum_{k=1}^N |x_k|^{\frac{1}{\alpha_k}}\big)^2\big) \frac{2x_i}{\alpha_i}\big(\epsilon - \sum_{k=1}^N |x_k|^{\frac{1}{\alpha_k}}\big)^2 |x_i|^{\frac{1}{\alpha_i}-2}\, dx$$

$$= (y = D_0\left(\epsilon^{-1}\right) x)$$

$$= \int_{||y|| \le 1} \Gamma^0_j\big(D_0(\epsilon)y, \epsilon^2\big(1 - \sum_{k=1}^N |y_k|^{\frac{1}{\alpha_k}}\big)^2\big) \frac{2y_i}{\alpha_i} \epsilon^{Q+2-\alpha_i}\big(1 - \sum_{k=1}^N |y_k|^{\frac{1}{\alpha_k}}\big) |y_i|^{\frac{1}{\alpha_i}-2}\, dy$$

$$= (\alpha_i = 1) = \int_{||y|| \le 1} \Gamma^0_j\big(y, \big(1 - \sum_{k=1}^N |y_k|^{\frac{1}{\alpha_k}}\big)^2\big) \frac{2y_i}{\alpha_i}\big(1 - \sum_{k=1}^N |y_k|^{\frac{1}{\alpha_k}}\big) |y_i|^{\frac{1}{\alpha_i}-2}\, dy.$$

So that

$$\int_{||\zeta||=\epsilon} \Gamma^0_j(\zeta)\nu_i(\zeta)\, d\sigma_\zeta = \int_{||\zeta||=1} \Gamma^0_j(\zeta)\nu_i(\zeta)\, d\sigma_\zeta \quad \forall \epsilon > 0, \tag{2.4}$$

which is our assertion.

Proposition 2.3 (Hörmander-type inequality)**.** *Let* $z_0 \in \mathbf{R}^{N+1}$ *and* $\Gamma^0(\cdot) = \Gamma(z_0; \cdot)$. *Then for every* $i, j = 1, \ldots, q$ *there exist two positive constants* c *and* $M > 1$ *such that*

$$|\Gamma^0_{ij}(\eta^{-1} \circ \zeta) - \Gamma^0_{ij}(\eta^{-1} \circ z)| \leq c \, \frac{||\zeta^{-1} \circ z||}{||z^{-1} \circ \eta||^{Q+3}}, \tag{2.5}$$

if $||z^{-1} \circ \eta|| \geq M \, ||\zeta^{-1} \circ z||$.

Proposition 2.3 will be derived from the following general lemma.*

Lemma 2.4. *If* $f \in C^1(\mathbf{R}^{N+1} \setminus \{0\})$, f *is* $D(\lambda)$*-homogeneous of degree* $\alpha \in \mathbf{R}$, *then there are two constants* $c > 0$ *and* $M > 1$ *such that*

$$|f(u) - f(v)| \leq c \, ||v^{-1} \circ u|| \; ||u||^{\alpha - 1}$$

provided $||u|| \geq M ||v^{-1} \circ u||$.

Proof. At first, we suppose that $||u|| = 1$. Since $f \in C^1(\mathbf{R}^{N+1} \setminus \{0\})$, we have

$$\begin{aligned} |f(u) - f(v)| &\leq \sup_{m_0 \leq ||w|| \leq m_1} |Df(w)| \; |u - v| \leq \text{(Proposition 2.1)} \\ &\leq c \sup_{m_0 \leq ||w|| \leq m_1} |Df(w)| \; ||v^{-1} \circ u||, \end{aligned}$$

where m_0 and m_1 are two positive constants chosen as follows. Since

$$1 = ||u|| \leq c \, (\, ||v^{-1} \circ u|| + ||u||) \leq c \Big(\frac{1}{M} + ||v|| \Big),$$

then by choosing $M > c$, we get $||v|| \geq \frac{1}{c} - \frac{1}{M} \equiv m_0 > 0$. Moreover,

$$||v|| \leq c \, (\, ||v \circ u^{-1}|| + ||u||) \leq c \, (c_0 \, ||u^{-1} \circ v|| + ||u||) \leq c \Big(\frac{c_0}{M} + 1 \Big) \equiv m_1.$$

If $u, v \in \mathbf{R}^{N+1} \setminus \{0\}$, we obtain

$$\begin{aligned} |f(u) - f(v)| &= ||u||^{\alpha} |f(D(||u||^{-1})u) - f(D(||u||^{-1})v)| \\ &\leq c \, ||u||^{\alpha} ||(D(||u||^{-1})v)^{-1} \circ (D(||u||^{-1})u)|| = c \, ||v^{-1} \circ u|| \; ||u||^{\alpha - 1}. \end{aligned}$$

Here, we have used the following equalities:

$$\begin{aligned} (D(||u||^{-1})v)^{-1} \circ (D(||u||^{-1})u) &= \text{(from the definitions of } \circ \text{ and } D(\lambda)) \\ &= (D(||u||^{-1})v^{-1}) \circ (D(||u||^{-1})u) = D(||u^{-1}||)(v^{-1} \circ u). \end{aligned}$$

*We thank S. Polidoro for providing us with the proof of lemma.

Theorem 2.5 (Representation formula)**.** *Let $u \in C_0^\infty(\mathbf{R}^{N+1})$ and $z_0 \in \mathbf{R}^{N+1}$. Then, for every $z \in \mathbf{R}^{N+1}$, we have*

$$\begin{aligned}\partial^2_{x_i x_j} u(z) = &- \lim_{\epsilon \to 0} \int_{||\zeta^{-1}\circ z|| \geq \sqrt{\epsilon}} \Gamma^0_{ij}(\zeta^{-1}\circ z) L_{z_0} u(\zeta)\, d\zeta \\ &- L_{z_0} u(z) \int_{||\zeta||=1} \Gamma^0_i(\zeta)\nu_j(\zeta)\, d\sigma, \quad i,j = 1,\dots,q.\end{aligned} \tag{2.6}$$

Here, ν_j is the j-th component of the outer normal to the surface $\{\zeta \in \mathbf{R}^{N+1} : ||\zeta|| = 1\}$.

Proof. See [10] or [2].

3. Schauder estimates. In this section we shall give the proof of Theorem 1.4. For every $z \in \mathbf{R}^{N+1}$ and $r > 0$ we indicate by $B(z,r)$ or by $B_r(z)$ the set

$$B(z,r) = \{\zeta \in \mathbf{R}^{N+1} : ||\zeta^{-1}\circ z|| < r\}.$$

We explicitly remark that

$$|B(z,r)| = |B(0,r)| = |B(0,1)| r^{Q+2}, \tag{3.1}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbf{R}^{N+1}$ and $Q+2$ is the homogeneous dimension of $\mathbf{R}^{N+1}$ with respect to $(D(\lambda))_{\lambda>0}$ as defined in (1.6). Indeed, the matrix E as defined in (1.8) verifies the identity

$$E(\lambda^2 t) = D_0(\lambda) E(t) D_0(\lambda^{-1}), \quad \text{for all } \lambda > 0 \text{ and for all } t \in \mathbf{R} \tag{3.2}$$

(see (2.20) in [8]). Therefore $\det E(\lambda^2 t) = \det E(t)$ for all $\lambda > 0$. Since $\det E(0) = 1$, by letting λ tend to 0 we get $\det E(\tau) = 1$ for all $\tau \in \mathbf{R}$. From this the first equality in (3.1) follows. Moreover, from (1.6), we have $|B(0,r)| = |B(0,1)| r^{Q+2}$.

In particular $|B(z,2r)| = 2^{Q+2}|B(z,r)|$, which corresponds to the so-called doubling condition.

Proof of Theorem 1.4. We remark that it suffices to prove inequality (1.9) for compact subsets of Ω. Let $(\Omega_k)_{k\in N}$ be a sequence of open subsets of Ω such that $\Omega_k \subset \Omega_{k+1} \subset\subset \Omega$ for all k and $\cup_{k\in N}\Omega_k = \Omega$. From (1.9) we have that $|u|_{2+\alpha,d;\Omega_k}$ is finite, and for every $z, \bar{z} \in \Omega$, $z \neq \bar{z}$ and for sufficiently large k, we infer that

$$\begin{aligned}|u(z)| + d_z(\Omega_k)|u_i(z)| &+ d_z^2(\Omega_k)|u_{ij}(z)| + d_{z,\bar{z}}^{2+\alpha}(\Omega_k)\frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1}\circ z||^\alpha} \\ &+ d_z^2(\Omega_k)|Yu(z)| + d_{z,\bar{z}}^{2+\alpha}(\Omega_k)\frac{|Yu(z) - Yu(\bar{z})|}{||\bar{z}^{-1}\circ z||^\alpha} \\ &\leq c\,(\sup_{\Omega_k}|u| + |d^2 f|_{\alpha,d;\Omega_k}) \leq c\,(\sup_{\Omega}|u| + |d^2 f|_{\alpha,d;\Omega}),\end{aligned}$$

where $d_{z,\bar z}(\Omega_k) = \min\{d_z(\Omega_k), d_{\bar z}(\Omega_k)\}$ and $d_z(\Omega_k) = \operatorname{dist}(z, \partial\Omega_k)$. Hence, on letting i tend to infinity, we obtain the inequality

$$|u(z)| + d_z|u_i(z)| + d_z^2|u_{ij}(z)| + d_{z,\bar z}^{2+\alpha}\frac{|u_{ij}(z) - u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^\alpha} + d_z^2|Yu(z)| + d_{z,\bar z}^{2+\alpha}\frac{|Yu(z) - Yu(\bar z)|}{||\bar z^{-1}\circ z||^\alpha} \le c\,(\sup_\Omega |u| + |d^2 f|_{\alpha,d;\Omega}),$$

for every $z, \bar z \in \Omega$, which implies the result. We may therefore assume that $u \in C_d^{2+\alpha}(\Omega)$.

It is convenient to split our proof into three stages. We first prove an estimate in the B-Hölder spaces for the solutions of $L_{z_0}u = g \in C_d^\alpha(\Omega; B)$, where L_{z_0} is defined in (1.4) and u is a function with compact support. In the second part, we extend this estimate for a general u. Finally, by using interpolation inequalities, we obtain the thesis.

(a) *First step.* Let $z_0 \in \Omega$ and let L_{z_0} be the corresponding frozen operator. We indicate by Γ^0 the fundamental solution of L_{z_0} and set $g \equiv L_{z_0}u$. We suppose that u has compact support, then from the representation formula (2.6), we can directly estimate the difference $|u_{ij}(z) - u_{ij}(\bar z)|$ for all $z, \bar z \in \Omega$. By the vanishing property (Proposition 2.2), we have

$$\lim_{\epsilon\to 0}\int_{||\zeta^{-1}\circ z||\ge\sqrt\epsilon}\Gamma_{ij}^0(\zeta^{-1}\circ z)g(\zeta)\,d\zeta - \lim_{\epsilon\to 0}\int_{||\zeta^{-1}\circ \bar z||\ge\sqrt\epsilon}\Gamma_{ij}^0(\zeta^{-1}\circ \bar z)g(\zeta)\,d\zeta = \int\Gamma_{ij}^0(\zeta^{-1}\circ z)(g(\zeta) - g(z))\,d\zeta - \int\Gamma_{ij}^0(\zeta^{-1}\circ \bar z)(g(\zeta) - g(\bar z))\,d\zeta.$$

Let $\delta = ||\bar z^{-1}\circ z|| > 0$ and let M be the positive constant in the Hörmander inequality (Proposition 2.3). We write

$$\begin{aligned}&\int\Gamma_{ij}^0(\zeta^{-1}\circ z)(g(\zeta) - g(z))\,d\zeta - \int\Gamma_{ij}^0(\zeta^{-1}\circ \bar z)(g(\zeta) - g(\bar z))\,d\zeta\\ &\quad= \int_{M_1}\Gamma_{ij}^0(\zeta^{-1}\circ z)(g(\zeta) - g(z))\,d\zeta + \int_{M_2}\Gamma_{ij}^0(\zeta^{-1}\circ z)(g(\zeta) - g(z))\,d\zeta\\ &\quad\quad- \int_{M_1}\Gamma_{ij}^0(\zeta^{-1}\circ \bar z)(g(\zeta) - g(\bar z))\,d\zeta - \int_{M_2}\Gamma_{ij}^0(\zeta^{-1}\circ \bar z)(g(\zeta) - g(\bar z))\,d\zeta\\ &\quad\equiv I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where $M_1 = \{\zeta \in \mathbf{R}^{N+1} : ||\zeta^{-1}\circ z|| \le M\delta\}$, $M_2 = \{\zeta \in \mathbf{R}^{N+1} : ||\zeta^{-1}\circ z|| \ge M\delta\}$, and we estimate separately I_1, I_2, I_3 and I_4.

Let Ω' be the support of u; then by (ii) in Proposition 2.2 we obtain

$$|I_1| \le |g|_{\alpha;\Omega'}\int_{M_1}\frac{||\zeta^{-1}\circ z||^\alpha}{||\zeta^{-1}\circ z||^{Q+2}}\,d\zeta = c\,|g|_{\alpha;\Omega'}\int_0^{M\delta}\rho^{\alpha-1}d\rho = |g|_{\alpha;\Omega'}\,M^\alpha\,\delta^\alpha. \quad (3.3)$$

An analogous procedure can be used for estimating $|I_3|$. From Proposition 2.3, we get

$$|I_2 + I_4| = \Big|\int_{M_2} (\Gamma^0_{ij}(\zeta^{-1} \circ z) - \Gamma^0_{ij}(\zeta^{-1} \circ \bar{z}))(g(\zeta) - g(z))d\zeta\Big| \tag{3.4}$$
$$\leq c\,|g|_{\alpha;\Omega'}\,\delta \int_{M_2} \frac{d\zeta}{||\zeta^{-1} \circ z||^{Q+3-\alpha}} = c\,|g|_{\alpha;\Omega'}\,\delta \int_{M\delta}^{+\infty} \rho^{\alpha-2} = c\,|g|_{\alpha;\Omega'}\,\delta^{\alpha}.$$

Then from the representation formula (2.6) and inequalities (3.3) and (3.4) we infer that

$$|u_{ij}(z) - u_{ij}(\bar{z})| \leq c\,|g|_{\alpha;\Omega'} ||\bar{z}^{-1} \circ z||^{\alpha}, \quad \forall z, \bar{z} \in \Omega,$$

so that there exists a positive constant c such that

$$d^{2+\alpha}_{z,\bar{z}} \frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1} \circ z||^{\alpha}} \leq c\,|g|_{\alpha,d;\Omega} \quad \forall z, \bar{z} \in \Omega,\ z \neq \bar{z}, \tag{3.5}$$

where $d_{z,\bar{z}}$ is as in Definition 1.3.

(b) *Second step.* For every fixed $r > 0$ we shall prove that if $z, \bar{z} \in \Omega$, $z \neq \bar{z}$ and $||\bar{z}^{-1} \circ z|| \leq \frac{r}{2a}$, where a is a positive constant to be specified later, then

$$r^{2+\alpha} \frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1} \circ z||^{\alpha}} \leq c\,(|r^2 g|_{\alpha,d;B_r(\bar{z})} + \sup_{B_r(\bar{z})} |u|), \tag{3.6}$$

where $g \equiv L_{z_0} u$.

In order to prove (3.6), we indicate by L_0^* the adjoint operator of L_{z_0}; that is,

$$L_0^* = \sum_{i,j=1}^{q} a_{ij}(z_0)\partial^2_{x_i x_j} - Y.$$

The function Γ^* defined by $\Gamma^*(z_0; z) = \Gamma^0(z; z_0)$ for all $z, z_0 \in \mathbf{R}^{N+1}$, $z \neq z_0$ is the fundamental solution of L_0^*. In the sequel, we use Green's identity

$$vL_{z_0}u - uL_0^*v = \sum_{i=1}^{q} \partial_{x_i}\Big(\sum_{j=1}^{q} va_{ij}(z_0)u_j - \sum_{j=1}^{q} ua_{ij}(z_0)v_j\Big) + Y(uv), \tag{3.7}$$

with $v(z) = \phi(z)\Gamma^*(z;\zeta)$, $\bar{z} \in \Omega$ and ϕ a function of class $C_0^{\infty}(B_r(\bar{z}))$, such that $\phi(z) \equiv 1$ in $B_{r/2}(\bar{z})$ and $\phi(z) \equiv 0$ in $B_r(\bar{z}) \setminus B_{\frac{r}{2}+\frac{r}{2a}}(\bar{z})$. Besides, $|\phi_i(z)| \leq \frac{c}{r}$, $|\phi_{ij}(z)| \leq \frac{c}{r^2}$ and $|Y\phi(z)| \leq \frac{c}{r^2}$ for all $z \in B_r(\bar{z})$ and for all $i, j = 1, \ldots, q$.

Then if $\zeta \in B_{\frac{r}{2}}(\bar{z})$, upon integration on $B_r(\bar{z})$ of (3.7), we obtain

$$u(\zeta) = -\int_{B_r(\bar{z})} \phi(z)g(z)\Gamma^*(z;\zeta)\,dz + \int_{B_r(\bar{z})} u(z)L_0^*(\phi(z)\Gamma^*(z;\zeta))\,dz, \quad \zeta \in B_{\frac{r}{2}}(\bar{z}).$$

Consequently, we have

$$\begin{aligned} u_{ij}(\zeta) =& - \partial^2_{\xi_i\xi_j} \int_{B_r(\bar z)} \phi(z) g(z)\Gamma^*(z;\zeta)\,dz + \partial^2_{\xi_i\xi_j} \int_{B_r(\bar z)} u(z) L_0^*(\phi(z)\Gamma^*(z;\zeta))\,dz \\ \equiv& v_{ij}(\zeta) + w_{ij}(\zeta), \quad i,j = 1,\dots,q. \end{aligned} \tag{3.8}$$

The estimate (3.6) for the first term v_{ij} in (3.8) can be obtained as in step (a) of the proof, then, we consider the second term w_{ij}. From the definition of ϕ and the fact that $\partial_{x_i x_j}(L_0^*\Gamma^*) = L_0^*\Gamma^*_{ij} = 0$, we have

$$\begin{aligned} w_{ij}(\zeta) =& \partial^2_{\xi_i\xi_j} \int_{B_r(\bar z)} u(z)\, L_0^*(\phi(z)\Gamma^*(z;\zeta))\,dz \\ =& \int_{B_{\frac{r}{2}+\frac{r}{2a}}(\bar z)\setminus B_{\frac{r}{2}}(\bar z)} u(z)\, L_0^*(\phi(z)\Gamma^*_{ij}(z;\zeta))\,dz, \quad \zeta \in B_{\frac{r}{2}}(\bar z). \end{aligned} \tag{3.9}$$

Let $z \in \Omega$ be such that $||\bar z^{-1} \circ z|| \le \frac{r}{2a}$; then

$$\begin{aligned} r^\alpha \frac{|w_{ij}(z) - w_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^\alpha} \le& \frac{c\, r^\alpha}{||\bar z^{-1}\circ z||^\alpha} \sup_{B_r(\bar z)} |u| \\ & \Big(\sup_{B_r(\bar z)} |\phi_{ij}| \int_{B_{\frac{r}{2}+\frac{r}{2a}}(\bar z)\setminus B_{\frac{r}{2}}(\bar z)} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar z^{-1}\circ\zeta)|\,d\zeta \\ & + \sup_{B_r(\bar z)} |\phi_k| \int_{B_{\frac{r}{2}+\frac{r}{2a}}(\bar z)\setminus B_{\frac{r}{2}}(\bar z)} |\Gamma_{ijk}(z^{-1}\circ\zeta) - \Gamma_{ijk}(\bar z^{-1}\circ\zeta)|\,d\zeta \\ & + \sup_{B_r(\bar z)} |Y\phi| \int_{B_{\frac{r}{2}+\frac{r}{2a}}(\bar z)\setminus B_{\frac{r}{2}}(\bar z)} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar z^{-1}\circ\zeta)|\,d\zeta\Big) \\ \equiv& \frac{r^\alpha}{||\bar z^{-1}\circ z||^\alpha} \sup_{B_r(\bar z)} |u|\; (I_1 + I_2 + I_3). \end{aligned} \tag{3.10}$$

We next determine an estimate for I_1, I_2 and I_3.

Let us write

$$\begin{aligned} I_1 = \sup_{B_r(\bar z)} |\phi_{ij}|\, \Big(& \int_{A_1} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar z^{-1}\circ\zeta)|\,d\zeta \\ & + \int_{A_2} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar z^{-1}\circ\zeta)|\,d\zeta\Big), \end{aligned} \tag{3.11}$$

where A_1 and A_2 are the sets

$$A_1 = (B_{\frac{r}{2}+\frac{r}{2a}}(\bar z) \setminus B_{\frac{r}{2}}(\bar z)) \cap \{\zeta \in \mathbf{R}^{N+1} : ||\bar z^{-1}\circ\zeta|| \ge M\,||\bar z^{-1}\circ z||\},$$

$$A_2 = (B_{\frac{r}{2}+\frac{r}{2a}}(\bar z) \setminus B_{\frac{r}{2}}(\bar z)) \cap \{\zeta \in \mathbf{R}^{N+1} : ||\bar z^{-1}\circ\zeta|| \le M\,||\bar z^{-1}\circ z||\},$$

and M is the positive constant in Proposition 2.3. We thus have

$$\int_{A_1} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar{z}^{-1}\circ\zeta)|\, d\zeta \\ \leq c\,||\bar{z}^{-1}\circ z||\int_{A_1} \frac{d\zeta}{||\bar{z}^{-1}\circ\zeta||^{Q+3}} \leq c\Big(\frac{||\bar{z}^{-1}\circ z||}{r}\Big)^{\alpha}. \tag{3.12}$$

On the other hand, from (ii) in Proposition 2.2 we have

$$\int_{A_2} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar{z}^{-1}\circ\zeta)|\, d\zeta \leq c\int_{A_2}\Big(\frac{1}{||z^{-1}\circ\zeta||^{Q+2}} + \frac{1}{||\bar{z}^{-1}\circ\zeta||^{Q+2}}\Big)\, d\zeta,$$

where

$$\int_{A_2} \frac{d\zeta}{||z^{-1}\circ\zeta||^{Q+2}} \leq c\int_{c_2 r}^{c_1||\bar{z}^{-1}\circ z||} \frac{d\rho}{\rho} \leq c\,\frac{||\bar{z}^{-1}\circ z||}{r}.$$

Here c_1 and c_2 are two positive constants given by the following two inequalities:

$$||z^{-1}\circ\zeta|| \leq (\text{Proposition 2.1}) \leq c(||z^{-1}\circ\bar{z}|| + ||\bar{z}^{-1}\circ\zeta||) \leq c_2||\bar{z}^{-1}\circ z||,$$

$$||z^{-1}\circ\zeta|| \geq \frac{1}{c}||\zeta^{-1}\circ\bar{z}|| - c||\bar{z}^{-1}\circ z|| \geq \frac{1}{c^2}||\bar{z}^{-1}\circ\zeta|| - \frac{cr}{2a} \geq \frac{r}{2}\,\frac{2a - c^3}{2ac^2} \equiv c_1 r$$

and a is a positive constant such that c_1 is positive. We then deduce

$$\int_{A_2} |\Gamma_{ij}(z^{-1}\circ\zeta) - \Gamma_{ij}(\bar{z}^{-1}\circ\zeta)|\, d\zeta \leq c\Big(\frac{||\bar{z}^{-1}\circ z||}{r}\Big)^{\alpha}, \tag{3.13}$$

and, from (3.11)–(3.13), we obtain

$$I_1 \leq c\sup_{B_r(\bar{z})}|\phi_{ij}|\,\Big(\frac{||\bar{z}^{-1}\circ z||}{r}\Big)^{\alpha}. \tag{3.14}$$

The estimate of I_2 in (3.10) is similar to that of I_1. Indeed, from Lemma 2.4, the function Γ_{ijk} satisfies the Hörmander inequality

$$|\Gamma_{ijk}(z^{-1}\circ\zeta) - \Gamma_{ijk}(\bar{z}^{-1}\circ\zeta)| \leq c\,\frac{||\bar{z}^{-1}\circ z||}{||\bar{z}^{-1}\circ\zeta||^{Q+4}} \quad i,j,k = 1,\dots,q,$$

provided $||\bar{z}^{-1}\circ\zeta|| \geq M\,||\bar{z}^{-1}\circ z||$, c and M being suitable positive constants. Hence, arguing as before, we get

$$I_2 \leq c\frac{\sup_{B_r(\bar{z})}|\phi_k|}{r}\,\Big(\frac{||\bar{z}^{-1}\circ z||}{r}\Big)^{\alpha}. \tag{3.15}$$

The integral I_3 can be estimated like I_1; that is,

$$I_3 \le c \sup_{B_r(\bar z)} |Y\phi| \, \big(\frac{||\bar z^{-1} \circ z||}{r}\big)^{\alpha}. \tag{3.16}$$

Then, from (3.10), (3.14), (3.15), (3.16) and from the conditions on ϕ we have

$$r^{\alpha} \frac{|w_{ij}(z) - w_{ij}(\bar z)|}{||\bar z^{-1} \circ z||^{\alpha}} \le c\, r^{-2} \sup_{B_r(\bar z)} |u|. \tag{3.17}$$

Combining (3.8), (3.5) and (3.17) we thus have

$$\begin{aligned} r^{\alpha} \frac{|u_{ij}(z) - u_{ij}(\bar z)|}{||\bar z^{-1} \circ z||^{\alpha}} &\le r^{\alpha} \frac{|v_{ij}(z) - v_{ij}(\bar z)|}{||\bar z^{-1} \circ z||^{\alpha}} + r^{\alpha} \frac{|w_{ij}(z) - w_{ij}(\bar z)|}{||\bar z^{-1} \circ z||^{\alpha}} \\ &\le c\,(|g|_{\alpha,d;B_r(\bar z)} + r^{-2} \sup_{B_r(\bar z)} |u|). \end{aligned}$$

This inequality proves (3.6).

(c) *Third step.* Let z_0, ζ_0 be any two distinct points of Ω, and assume that $d_{z_0} \le d_{\zeta_0}$. We write

$$L_{z_0} u(z) = Lu(z) + (L_{z_0} - L)u(z) = f(z) + \sum_{i,j=1}^{q} (a_{ij}(z_0) - a_{ij}(z)) u_{ij}(z) \equiv F(z)$$

and we consider $\bar\mu < 1$ to be a constant to be specified later but such that $B_r(z_0) \subset \Omega$ with $r = \bar\mu\, d_{z_0}$.

Suppose first that $\zeta_0 \in B_{\frac{r}{2a}}(z_0)$; then from (3.6)

$$(\bar\mu d_{z_0})^{2+\alpha} \, \frac{|u_{ij}(z_0) - u_{ij}(\zeta_0)|}{||\zeta_0^{-1} \circ z_0||^{\alpha}} \le c\,(\,|d^2 F|_{\alpha,d;B_r(z_0)} + \sup_{B_r(z_0)} |u|\,). \tag{3.18}$$

On the other hand, if $\zeta_0 \notin B_{\frac{r}{2a}}(z_0)$ we have

$$\begin{aligned} d_{z_0}^{2+\alpha} \, \frac{|u_{ij}(z_0) - u_{ij}(\zeta_0)|}{||\zeta_0^{-1} \circ z_0||^{\alpha}} &\le \big(\frac{2}{\bar\mu}\big)^{\alpha} \big(d_{z_0}^2 \, |u_{ij}(z_0)| + d_{\zeta_0}^2 \, |u_{ij}(\zeta_0)|\,\big) \\ &\le \frac{4}{\bar\mu^{\alpha}} \sup_{z \in \Omega} d_z^2 \, |u_{ij}(z)|, \end{aligned} \tag{3.19}$$

so that, from combining (3.18) and (3.19) we obtain

$$d_{z_0}^{2+\alpha} \frac{|u_{ij}(z_0) - u_{ij}(\zeta_0)|}{||\zeta_0^{-1} \circ z_0||^{\alpha}} \le \frac{c}{\bar\mu^{\alpha+2}} (|d^2 F|_{\alpha,d;B_r(z_0)} + \sup_{\Omega} |u|\,) + \frac{4}{\bar\mu^{\alpha}} \sup_{z \in \Omega} d_z^2 |u_{ij}(z)|. \tag{3.20}$$

We proceed by estimating $|d^2F|_{\alpha,d;B_r(z_0)}$ in terms of $|u_{ij}|_{\alpha,d;B_r(z_0)}$. We have

$$\begin{aligned}|d^2F|_{\alpha,d;B_r(z_0)} &\le \sum_{i,j=1}^{q} |d^2(a_{ij}(z_0)-a_{ij})u_{ij}|_{\alpha,d;B_r(z_0)} + |d^2 f|_{\alpha,d;B_r(z_0)} \\ &\le \sum_{i,j=1}^{q} |a_{ij}(z_0)-a_{ij}|_{\alpha,d;B_r(z_0)}\ |d^2u_{ij}|_{\alpha,d;B_r(z_0)} + |d^2 f|_{\alpha,d;B_r(z_0)},\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}|a_{ij}(z_0)-a_{ij}|_{\alpha,d;B_r(z_0)} &\le \sup_{B_r(z_0)} |a_{ij}(z_0)-a_{ij}| + r^{\alpha} \sup_{\substack{z,\bar z\in B_r(z_0)\\ z\ne \bar z}} \frac{|a_{ij}(z)-a_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}} \\ &\le 2^{1+\alpha}\,\bar\mu^{\alpha}\,|a_{ij}|_{\alpha,d;\Omega} \le c\,\mu^{\alpha}.\end{aligned} \tag{3.22}$$

It is then easy to prove that if c is the constant in the quasi-triangular inequality in Proposition 2.1 and $\bar\mu$ is a positive constant such that $\bar\mu \le \frac{1}{2c}$, then

$$|d^2u_{ij}|_{\alpha,d;B_r(z_0)} \le c\big(\bar\mu^2 \sup_{\Omega} d^2|u_{ij}| + \bar\mu^{\alpha+2} \sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}} d_{z,\bar z}^{\alpha+2}\,\frac{|u_{ij}(z)-u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}\big). \tag{3.23}$$

Hence, from (3.21)–(3.23) we arrive at the following estimate for the principal term in (3.20):

$$|d^2F|_{\alpha,d;B_r(z_0)} \le c\bar\mu^{\alpha+2}\big(\sup_{\Omega} d^2|u_{ij}| + \mu^{\alpha} \sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}} d_{z,\bar z}^{\alpha+2}\,\frac{|u_{ij}(z)-u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}\big) + |d^2 f|_{\alpha,d;\Omega}. \tag{3.24}$$

To proceed further, we need to remove the term $\frac{|u_{ij}(z)-u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}$ from (3.24) for which we need to prove the following interpolation inequalities.

Interpolation inequalities. *Let $u \in C_d^{2+\alpha}(\Omega;B)$; then for every $\epsilon > 0$ and some constant $c = c(\epsilon) > 0$ we have*

$$\sup_{\Omega} d^2|u_{ij}| \le c\ \sup_{\Omega}|u| + \epsilon \sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}} d_{z,\bar z}^{\alpha+2}\,\frac{|u_{ij}(z)-u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}} \tag{3.25}$$

and

$$\begin{aligned}&\sup_{\Omega}|u| + \sum_{i=1}^{q}\sup_{\Omega} d|u_i| + \sum_{i,j=1}^{q}\sup_{\Omega} d^2|u_{ij}| \\ &\qquad\le c\ \sup_{\Omega}|u| + \epsilon \sum_{i,j=1}^{q} \sup_{\substack{z,\bar z\in\Omega\\ z\ne\bar z}} d_{z,\bar z}^{\alpha+2}\,\frac{|u_{ij}(z)-u_{ij}(\bar z)|}{||\bar z^{-1}\circ z||^{\alpha}}.\end{aligned} \tag{3.26}$$

These inequalities will be proved at the end of the proof.

The conclusion of the proof then follows by applying (3.25) and (3.26). Indeed, inequalities (3.20), (3.24) and (3.25), with $\epsilon = \bar{\mu}^2$, now yield the estimate

$$\sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^{2+\alpha} \frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1}\circ z||^{\alpha}} \leq c\,(\sup_{\Omega}|u| + |d^2 f|_{\alpha,d;\Omega}).$$

By using (3.26) with ϵ small enough, we thus have

$$\begin{aligned} \sup_{\Omega}|u| + \sum_{i=1}^{q}\sup_{\Omega} d|u_i| + \sum_{i,j=1}^{q}\sup_{\Omega} d^2|u_{ij}| + \sup_{\substack{z,\bar{z}\in\Omega \\ z\neq\bar{z}}} d_{z,\bar{z}}^{2+\alpha} \frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1}\circ z||^{\alpha}} \\ \leq c\,(\sup_{\Omega}|u| + |d^2 f|_{\alpha,d;\Omega}). \end{aligned} \tag{3.27}$$

Finally, observing that $Yu = f - \sum_{i,j=1}^{q} a_{ij}u_{ij}$ we have

$$d_z^2|Yu(z)| \leq d_z^2|f(z)| + c\sum_{i,j=1}^{q} d_z^2|u_{ij}(z)|,$$

$$d_{z,\bar{z}}^{\alpha+2}\frac{|Yu(z) - Yu(\bar{z})|}{||\bar{z}^{-1}\circ z||^{\alpha}} \leq d_{z,\bar{z}}^{\alpha+2}\frac{|f(z) - f(\bar{z})|}{||\bar{z}^{-1}\circ z||^{\alpha}} + c\sum_{i,j=1}^{q} d_{z,\bar{z}}^{\alpha+2}\frac{|u_{ij}(z) - u_{ij}(\bar{z})|}{||\bar{z}^{-1}\circ z||^{\alpha}}.$$

Combining these estimates with (3.27) we arrive at the desired estimate (1.9).

Thus we are left with the proof of the interpolation inequalities (3.25) and (3.26). We prove (3.25) first establishing the interpolation inequality

$$\sup_{\Omega} d|u_i| \leq c\ \sup_{\Omega}|u| + \epsilon\ \sup_{\Omega} d^2\,|u_{ij}| \quad i,j = 1,\ldots,q, \quad c = c(\epsilon) > 0 \tag{3.28}$$

where $\epsilon > 0$ may be arbitrary. Let $z \in \Omega$ and $B_r(z) \subset\subset \Omega$ with $r = \mu\, d_z$ and μ a positive constant to be specified later.

Let $z', z'' \in B_r(z)$ be such that $z' = (x', t)$, $z'' = (x'', t)$, such that $||(z'')^{-1}\circ z'|| \cong c\,r$, and z', z'' are parallel to one of the x_i axes.

We note that $|z' - z''| = ||(z')^{-1}\circ z''||$, where $|\cdot|$ indicates the Euclidean norm in $\mathbf{R}^{N+1}$. Then, for some $\bar{z} \in B_r(z)$, we have

$$|u_i(\bar{z})| = \frac{|u(z') - u(z'')|}{|z' - z''|} \leq \frac{2\sup_{\Omega}|u|}{||(z')^{-1}\circ z''||}, \tag{3.29}$$

and

$$|u_i(z)| = \left|u_i(\bar{z}) + \int_{\bar{z}}^{z} u_{ij}(\zeta)\,d\zeta_j\right| \leq \frac{2\sup_{\Omega}|u|}{||(z')^{-1}\circ z''||} + |z - \bar{z}|\ \sup_{B_r(z)}|u_{ij}|$$

where $|z-\bar{z}| \le c\,||\bar{z}^{-1}\circ z||$. Therefore, from this estimate and (3.29) it follows that

$$\begin{aligned} |u_i(z)| &\le \frac{2\sup_\Omega |u|}{||(z')^{-1}\circ z''||} + c\,||\bar{z}^{-1}\circ z|| \sup_{\zeta\in B_r(z)} |u_{ij}(\zeta)| \\ &\le \frac{2}{cr}\sup_\Omega |u| + r\sup_{B_r(z)} |u_{ij}|. \end{aligned} \tag{3.30}$$

Since $d_\zeta \ge \frac{1}{2c} d_z$, we have by (3.30)

$$d_z|u_i(z)| \le \frac{2}{c\bar{\mu}}\sup_\Omega |u| + \frac{\mu}{4c^2}\sup_{\zeta\in\Omega} d_\zeta^2 |u_{ij}(\zeta)|$$

provided $0<\bar{\mu}\le \frac{1}{2c}$ (c is the positive constant in Proposition 2.1, (i)). Choosing the smaller value $\bar{\mu}$ corresponding to the two cases $\bar{\mu}<\frac{2}{c}$, $\bar{\mu}\le 4c^2\epsilon$, and taking the supremum over all $z\in\Omega$ we obtain inequality (3.28).

If we prove that for every $\epsilon>0$ there is a positive constant c such that

$$\sup_{z\in\Omega} d_z^2|u_{ij}(z)| \le c\sup_{z\in\Omega} d_z|u_i(z)| + \epsilon \sup_{\substack{z,\bar{z}\in\Omega \\ z\ne\bar{z}}} d_{z,\bar{z}}^{\alpha+2}\frac{|u_{ij}(z)-u_{ij}(\bar{z})|}{||\bar{z}^{-1}\circ z||^\alpha}, \tag{3.31}$$

then, from combining (3.28) and (3.31), we arrive at inequality (3.25).

The proof of (3.31) proceeds in much the same way after replacing u with u_i; the details of proof are therefore omitted.

Finally, interpolation inequality (3.26) follows from combining (3.25) and (3.28). Theorem 1.4 is thus completely proved.

Remark 3.1. If $(u_n)_n$ is a sequence of functions in $C^{2+\alpha}_{loc}(\Omega;B)$ such that $Lu_n=0$, $u_n\to u$ and $(u_n)_n$ is locally bounded, then from the estimates in Theorem 1.4 and by using a diagonal argument u must belong to $C^{2+\alpha}_{loc}(\Omega;B)$ and satisfy $Lu=0$ in Ω.

4. The Dirichlet problem for strongly regular sets. In this section we prove that the Dirichlet problem

$$\begin{cases} Lu=f \text{ in } \Omega, & f\in C^\alpha_d(\Omega;B) \\ u=\phi \text{ on } \partial\Omega, & \phi\in C(\partial\Omega) \end{cases} \tag{4.1}$$

has a unique solution $u\in C^{2+\alpha}_d(\Omega;B)\cap C(\bar{\Omega})$, when Ω belongs to a suitable class of open subsets of $\mathbf{R}^{N+1}$. This result is only a preliminary step in order to apply the Perron method to arbitrary open bounded subsets of $\mathbf{R}^{N+1}$.

Definition 4.1. We say that an open bounded subset Ω of $\mathbf{R}^{N+1}$ is strongly regular for L, in short it is L-regular, if for every $z_0\in\partial\Omega$ there is $\nu\in\mathbf{R}^{N+1}$ such that

$$B_{eucl}(z_0+\nu,|\nu|)\subset \mathbf{R}^{N+1}\setminus\Omega \quad\text{and}\quad \sum_{i,j=1}^q a_{ij}(z_0)\nu_i\nu_j>0;$$

here $B_{eucl}(z_0+\nu,|\nu|)$ indicates the Euclidean ball of center $z_0+\nu$ and radius $|\nu|$.

We can now assert the following existence result.

Proposition 4.2. *Let L be as in* (1.1) *satisfying hypothesis* (H) *and let $f \in C_d^\alpha(\Omega; B)$, $\phi \in C(\partial\Omega)$. If Ω is an L-regular open subset of $\mathbf{R}^{N+1}$, then there exists a unique solution u of the boundary value problem* (4.1) *and $u \in C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega})$.*

Proof. The proof of this proposition is split into three steps. We first prove the unique solvability of problem (4.1); next, using the continuity method, we prove the solvability of the homogeneous problem, ($\phi \equiv 0$); then we study the problem with $f \equiv 0$. From these steps the proof follows immediately.

Step 1. The following weak maximun principle can be proved in a rather standard manner: if u is a continuous function in $\bar{\Omega}$, Yu, $\partial^2_{x_i x_j} u$ for $i, j = 1, \dots, q$ are continuous in Ω and

$$\begin{cases} Lu \geq 0 & \text{in} \quad \Omega \\ u \leq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u \leq 0$ in Ω. Furthermore, if $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$ then there exists a positive constant c, independent of u, such that

$$\sup_{\Omega} |u| \leq c \sup_{\Omega} |f|.$$

Then, the uniqueness in problem (4.1) follows straightforwardly.

Step 2. We consider problem (4.1) with homogeneous boundary conditions. Let $z_0 \in \Omega$, $L_0 = L_{z_0}$ and for $\lambda \in [0, 1]$ let us now define the operator L_λ by

$$L_\lambda \equiv \lambda L + (1 - \lambda) L_0.$$

From now on, we shall indicate by $(P_{\lambda,f})$ the Dirichlet problem

$$\begin{cases} L_\lambda u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$P_{\lambda,f}$}$$

and by Λ the set

$$\Lambda = \big\{ \lambda \in [0,1] : \text{the problem } (P_{\lambda,f}) \text{ has a solution } u \in C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega}) \text{ for every } f \in C_d^\alpha(\Omega; B) \big\}.$$

We shall prove that
(i) Λ contains $\lambda = 0$;
(ii) Λ is an open set in the interval $[0, 1]$;
(iii) Λ is a closed set.
It will follow that $\Lambda = [0, 1]$, and, for $\lambda = 1$, we will get Step 2.

(i) Let $f \in C_d^\alpha(\Omega; B)$ and let Γ^0 be the fundamental solution of the operator L_0. If we set

$$v(z) = -\int_\Omega \Gamma^0(\zeta^{-1} \circ z) f(\zeta) \, d\zeta,$$

then $v \in C_d^{2+\alpha}(\Omega; B)$ and $L_0 v = f$ in Ω (see the proof of Theorem 1.4).

In order to prove that v belongs to $C(\bar{\Omega})$, let η be a bounded function in $C^2(\mathbf{R})$ such that $\eta(t) = 0$ if $|t| \geq 1$ and $\eta(t) = 1$ if $|t| \leq \frac{1}{2}$. We call $\eta_\epsilon(z;\zeta) = 1 - \eta(||D(\epsilon^{-1})(\zeta^{-1} \circ z)||)$ and

$$v_\epsilon(z) = -\int_\Omega \eta_\epsilon(z;\zeta)\Gamma^0(\zeta^{-1} \circ z) f(\zeta)\, d\zeta.$$

Now, $v_\epsilon \in C(\bar{\Omega})$ and from the translation and dilation invariance of Γ^0 we get

$$\begin{aligned} |v_\epsilon(z) - v(z)| &\leq c \sup_\Omega |f| \int_{||\zeta^{-1}\circ z|| \geq \epsilon} |\Gamma^0(\zeta^{-1} \circ z)|\, d\zeta \\ &= \epsilon^2 \sup_\Omega |f| \int_{||\zeta|| \geq 1} |\Gamma^0(\zeta)|\, d\zeta, \end{aligned}$$

so that v_ϵ uniformly converges to v in $\bar{\Omega}$.

Finally, from Theorem 5.2 in [1], there exists a unique solution $w \in C^\infty(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem

$$\begin{cases} L_0 w = 0 & \text{in} \quad \Omega \\ w = -v & \text{on} \quad \partial\Omega. \end{cases}$$

By virtue of Theorem 1.4 we have $w \in C_d^{2+\alpha}(\Omega; B)$; then $u = w + v \in C_d^{2+\alpha}(\Omega; B)$ $\cap C(\bar{\Omega})$ and it is the solution of $(P_{0,f})$.

(ii) Let $\lambda_0 \in \Lambda$ and let $f \in C_d^\alpha(\Omega; B)$. We write $L_\lambda u = f$ in the equivalent form

$$L_{\lambda_0} u = (L_{\lambda_0} - L_\lambda) u + f \equiv F(u).$$

We consider the linear transformation $\mathcal{A}$,

$$\mathcal{A} : C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega}) \to C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega}),$$

defined as follows: for every u, $\mathcal{A}u$ is the (unique) solution of $(P_{\lambda_0, F(u)})$.

From Theorem 1.4 and from Step 1 we obtain

$$|\mathcal{A}u|_{2+\alpha,d;\Omega} \leq c\,(\sup_\Omega |F| + |d^2 F|_{\alpha,d;\Omega}) \leq c\,(K|\lambda - \lambda_0|\, |u|_{2+\alpha,d;\Omega} + |f|_{\alpha,d;\Omega}), \quad (4.2)$$

where K is a positive constant independent of λ. Then, if

$$|u|_{2+\alpha,d;\Omega} \leq 2\,c\,|f|_{\alpha,d;\Omega} \quad \text{and} \quad |\lambda - \lambda_0| \leq \frac{1}{2\,c\,K}, \quad (4.3)$$

we have

$$|\mathcal{A}u|_{2+\alpha,d;\Omega} \leq 2\,c\,|f|_{\alpha,d;\Omega}.$$

If λ satisfies (4.3) then $\mathcal{A}$ is a contraction in the closed set

$$X = \{u \in C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega}) : |u|_{\alpha,d;\Omega} \leq 2\, c\, |f|_{\alpha,d;\Omega},\ u = 0 \text{ on } \partial\Omega\}.$$

Hence $\mathcal{A}$ has a unique fixed point $u \in X$ which is the solution of $(P_{\lambda,f})$. In other words, the set $\{\lambda \in [0,1] : |\lambda - \lambda_0| \leq \frac{1}{2\,c\,K}\}$ is contained in Λ.

(iii) Let $(\lambda_n)_n \subset \Lambda$ be such that $\lambda_n \to \sigma$ as $n \to \infty$. If $f \in C_d^\alpha(\Omega; B)$ and u_n is the solution of $(P_{\lambda_n,f})$, by using Theorem 1.4, Step 1 and standard arguments we can prove that u_n converges to a function $u \in C_d^{2+\alpha}(\Omega; B)$ such that $L_\sigma u = f$ in Ω and $|u|_{2+\alpha,d;\Omega} \leq c\,|f|_{\alpha,d;\Omega}$.

In order to prove that $u = 0$ on $\partial\Omega$ we fix $z_0 \in \partial\Omega$ and set

$$w_{z_0}(z) = \exp(-Mr^2) - \exp(-M|z - z_0 - r\nu|^2)$$

where $r > 0$ is chosen in such a way that $\overline{B_{eucl}(z_0 + r\nu, r)} \cap \Omega = \emptyset$. By construction, $w_{z_0}(z_0) = 0$ and if M is sufficiently large, w_{z_0} is a local barrier in z_0 for every L_λ; indeed, there is a neighborhood V of z_0 such that

$$w_{z_0} > 0 \text{ in } \overline{\Omega \cap V} \setminus \{z_0\}, \quad w_{z_0}(z_0) = 0 \text{ and } \quad L_\lambda w_{z_0}(z) \leq -1 \quad \forall z \in \Omega \cap V$$

for every $\lambda \in [0,1]$. Moreover, from Step 1, it follows that

$$|u(z)| \leq \sup_{\Omega \cap V} |f| \ \ w_{z_0}(z) \to 0, \quad \text{as } z \to z_0. \tag{4.4}$$

This proves that $\sigma \in \Lambda$, and Step 2 follows.

Step 3. We study problem (4.1) with $f \equiv 0$. Let $(\phi_n)_{n\in N} \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be uniformly convergent to ϕ in $\bar{\Omega}$, and let $u_n \in C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega})$ be solution of

$$\begin{cases} Lu = L\phi_n & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

(the existence of u_n is ensured by Step 2). If $v_n = u_n - \phi_n$, then

$$\begin{cases} Lv_n = 0 & \text{in} \quad \Omega \\ v_n = -\phi_n & \text{on} \quad \partial\Omega, \end{cases}$$

and from Step 1

$$\sup_{\bar{\Omega}} |v_n - v_m| \leq \sup_{\partial\Omega} |\phi_n - \phi_m| \quad \text{for all } m, n \in N.$$

From Remark 3.1 we deduce that v_n converges uniformly in $\bar{\Omega}$ to a function v belonging to $C_d^{2+\alpha}(\Omega; B) \cap C(\bar{\Omega})$ such that $Lv = 0$ in Ω and $v = \phi$ on $\partial\Omega$. Proposition 4.2 is thus proved.

5. The Dirichlet problem for arbitrary bounded open sets: generalized solution. In this section we study the problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = \phi & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is an arbitrary bounded open set, $f \in C^{\alpha}(\Omega; B)$ and $\phi \in C(\partial\Omega)$.

If $f \equiv 0$, a generalized Perron-Wiener solution can be determined by exploiting some results of potential theory on harmonic spaces. If f is not identically zero, we first need to define a function w of class $C^{2+\alpha}_{loc}(\Omega; B) \cap C(\bar{\Omega})$ such that

$$Lw = f \quad \text{in} \;\; \Omega$$

as follows: Let $\Gamma(z;\zeta)$ be the fundamental solution of the operator L; then we set

$$w(x,t) = -\int_0^t \int_{\mathbf{R}^N} \Gamma(x,t;y,s)\, \tilde{f}(y,s)\, dy\, ds,$$

where $\tilde{f} = f$ in Ω and $\tilde{f} = 0$ elsewhere.

We call the *generalized solution* of problem (5.1) the function $u = v + w$, where v is the generalized Perron-Wiener solution of the problem $Lv = 0$ in Ω, $v = \phi - w$ on $\partial\Omega$. We remark that

$$u \in C^{2+\alpha}_{loc}(\Omega; B), \quad Lu = f \quad \text{in} \;\; \Omega,$$

and u takes the boundary value ϕ in the L-regular points of $\partial\Omega$ (see below).

We define

$$u = \inf\{v : v \,\text{is}\; L\text{-superharmonic in}\; \Omega \;\text{and}\; \lim_{\zeta\to z} v(\zeta) \geq \phi(z),\; \forall z \in \partial\Omega\}. \tag{5.2}$$

A continuous function v is *L-superharmonic* in Ω if $v \geq H^V_{v|_{\partial V}}$ for every open L-regular subset $V \subset\subset \Omega$, when $H^V_{v|_{\partial V}}$ is the unique solution $C^{2+\alpha}_{loc}(V; B)$ of the problem

$$\begin{cases} Lw = 0 & \text{in} \quad V \\ w = v & \text{on} \quad \partial V. \end{cases}$$

Proposition 5.1. *Let Ω be an open bounded subset of $\mathbf{R}^{N+1}$. If ϕ belongs to $C(\partial\Omega)$, then the function u defined in* (5.2) *belongs to $C^{2+\alpha}_{loc}(\Omega; B)$, and it is the generalized Perron-Wiener solution of the problem*

$$\begin{cases} Lu = 0 & \textit{in} \quad \Omega \\ u = \phi & \textit{on} \quad \partial\Omega. \end{cases}$$

Proof. Let X be an open bounded set in $\mathbf{R}^{N+1}$. For every $V \subset X$ we set

$$\mathcal{H}^L(V) = \{u \in C^{2+\alpha}_{loc}(V;B) : Lu = 0 \quad \text{in } V\}.$$

The result follows by showing that $(X, \mathcal{H}^L)$ is a β-harmonic space, according to the classical definition (see for example the monograph of Constantinescu-Cornea, [3]).

The space $(X, \mathcal{H}^L)$ is a β-harmonic space if it satisfies the following properties:

(a) (Property of positivity). For every $z \in X$ there exist an open neighborhood V of z and $u \in \mathcal{H}^L(V)$ such that $u(\zeta) > 0$ for every $\zeta \in V$.

(b) (Bauer property of convergence). If Ω is an open subset of X and $(u_n)_n$ is a monotone increasing sequence in $\mathcal{H}^L(\Omega)$ such that $u = \sup_n u_n$ is locally bounded, then $u \in \mathcal{H}^L(\Omega)$.

(c) (Property of resolutivity). The Dirichlet problem $Lu = 0$ in Ω, $u = \phi$ in $\partial\Omega$, is solvable for a basis of open subsets of $\mathbf{R}^{N+1}$.

(d) (Property of separation). Let $z_1, z_2 \in X$, $z_1 \neq z_2$; then there exists a positive $\mathcal{H}^L$-superharmonic function v such that $v(z_1) \neq v(z_2)$.

The first property is satisfied since constant functions are $\mathcal{H}^L$-harmonic. The convergence property holds thanks to Remark 3.1. The third property follows from Proposition 4.2, since the family of the L-regular sets, according to Definition 4.1, is a basis for the topology on $\mathbf{R}^{N+1}$ (cf. [1, Corollario 5.2]).

In order to prove (d) we suppose that $X \subset \mathbf{R}^N \times [-T, T]$ and we fix $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2) \in X$. If $t_1 \neq t_2$, then we set $v_1(x,t) = t + T$. If $t_1 = t_2 = \bar{t}$ we choose $\gamma \in \mathbf{R}^N$ so that $\langle x_1 - x_2, E^T(-\bar{t})\gamma \rangle \neq 0$ and $M > 0$ so that

$$v_2(z) \equiv M - \langle x, E^T(-t)\gamma \rangle > 0, \quad \forall z \in X.$$

In both cases $v_{1,2}(z_1) \neq v_{1,2}(z_2)$, v_1 is a $\mathcal{H}^L$-superharmonic function ($Lv_1 \equiv -1$), while v_2 is a $\mathcal{H}^L$-harmonic function in X, since*

$$Lv_2(z) = \langle x, BDv_2(z) \rangle - \partial_t v_2(z) = -\langle x, BE^T(-t)\gamma \rangle + \langle x, BE^T(-t)\gamma \rangle = 0.$$

We next prove Theorem 1.5.

Proof of Theorem 1.5. In order to prove the main theorem it is sufficient to get a particular solution of $Lw = f$, continuous on $\bar{\Omega}$ and of class $C^{2+\alpha}_{loc}(\Omega; B)$. If $\Gamma(z;\zeta)$ is the fundamental solution of the operator L obtained from the parametrix method of Levi in [14], then for every continuous bounded function $g : \mathbf{R}^N \to \mathbf{R}$, it holds:

$$\lim_{t \to \tau^+} \int_{\mathbf{R}^N} \Gamma(x,t;\xi,\tau)\, g(\xi)\, d\xi = g(x), \qquad \text{for every } x \in \mathbf{R}^N, \tau \in \mathbf{R}.$$

We set

$$w(x,t) = -\int_0^t \int_{\mathbf{R}^N} \Gamma(x,t;y,s)\, \tilde{f}(y,s)\, dy\, ds, \tag{5.3}$$

*We thank A. Montanari for a suggestion on the construction of the function v_2.

where $\tilde{f} = f$ in Ω and $\tilde{f} = 0$ elsewhere, and we shall prove that $w \in C(\bar{\Omega}) \cap C^{2+\alpha}_{loc}(\Omega; B)$ and $Lw = f$ in Ω.

We first remark that $w \in C^{2+\alpha}_{loc}(\Omega; B)$. Indeed, if V is an open L-regular subset of Ω, $\bar{V} \subset \Omega$ and $u \in C^{2+\alpha}_{d}(V; B) \cap C(\bar{V})$ such that

$$\begin{cases} Lu = f & \text{in} \quad V \\ u = w & \text{on} \quad \partial V, \end{cases}$$

then from Proposition 4.2 we get $u = w$ in V so that $w \in C^{2+\alpha}_{loc}(\Omega; B)$.

We now prove that $w \in C(\mathbf{R}^{N+1})$. Recall that the fundamental solution $\Gamma(z;\zeta)$ can be expressed as $\Gamma(z;\zeta) = Z(z;\zeta) + J(z;\zeta)$, where $Z(z;\zeta)$ indicates the parametrix with pole at ζ, that is, the fundamental solution with pole at ζ of the operator L_ζ, and

$$J(x,t;\xi,\tau) = \int_\tau^t \Big(\int_{\mathbf{R}^N} Z(x,t;y,s)\, \Phi(y,s;\xi,\tau)\, dy \Big) ds,$$

for some unknown function Φ, which satisfies the integral equation

$$\Phi(z;\zeta) = (LZ)(z;\zeta) + \int_\tau^t \Big(\int_{\mathbf{R}^N} LZ(z;y,s)\, \Phi(y,s;\zeta)\, dy \Big) ds. \tag{5.4}$$

For the sake of brevity, in the sequel we suppose that $\Omega \subset \mathbf{R}^N \times [0,T]$ and we rewrite

$$\begin{aligned} w(x,t) = &- \int_0^t \int_{\mathbf{R}^N} Z(x,t;y,s)\, \tilde{f}(y,s)\, dy\, ds \\ &- \int_0^t \int_{\mathbf{R}^N} J(x,t;y,s)\, \tilde{f}(y,s)\, dy\, ds \equiv w_1(x,t) + w_2(x,t). \end{aligned} \tag{5.5}$$

We first prove that w_1 is continuous in $\mathbf{R}^{N+1}$. From Proposition 2.4 in [14] we get that there are a symmetric positive constant $q \times q$ matrix A_0^+ and a constant $c > 0$ such that

$$Z(z;\zeta) \le c\, \Gamma^+(z;\zeta), \quad \forall z, \zeta \in \mathbf{R}^{N+1}, z \neq \zeta \tag{5.6}$$

where $\Gamma^+(z;\zeta)$ is the fundamental solution of the operator $L^+ = \operatorname{div}(A^+ D) + Y$ and A^+ is the $N \times N$ constant matrix

$$A^+ = \begin{pmatrix} A_0^+ & 0 \\ 0 & 0 \end{pmatrix}.$$

We remark that

$$\int_{B(z,\epsilon)} |Z(z;\zeta)|\, d\zeta \to 0 \quad \text{as } \epsilon \to 0 \quad \text{uniformly in } \mathbf{R}^{N+1}.$$

Indeed, from (5.6) and from the dilation and translation invariance we have

$$\begin{aligned}\int_{B(z,\epsilon)} |Z(z;\zeta)|\, d\zeta &\le c \int_{||\zeta^{-1}\circ z||\le\epsilon} \Gamma^+(\zeta^{-1}\circ z)\, d\zeta \\ &= c\,\epsilon^{Q+2} \int_{||\zeta||\le\epsilon} \Gamma^+(D(\epsilon)\zeta)\, d\zeta = c\,\epsilon^2 \int_{||\zeta||\le 1} \Gamma^+(\zeta)\, d\zeta \to 0 \quad \text{as } \epsilon\to 0.\end{aligned}$$

Let $\eta \in C(\mathbf{R}^{N+1})$ be such that $\eta(z) = 1$ if $||z|| \ge 1$ and $\eta(z) = 0$ if $||z|| \le \frac{1}{2}$. We call $\eta_\epsilon(z) = \eta(D(\epsilon^{-1})z)$ and set

$$v_\epsilon(z) = -\int_0^t \int_{\mathbf{R}^N} \eta_\epsilon(\zeta^{-1}\circ z) Z(z;\zeta)\, \tilde{f}(\zeta)\, d\zeta.$$

Then $v_\epsilon \in C(\mathbf{R}^N \times [0,T])$ and

$$|v_\epsilon(z) - w_1(z)| \le c \sup_{\Omega} |f| \int_{B(z,\epsilon)} |Z(z;\zeta)|\, d\zeta,$$

so that v_ϵ uniformly converges to w_1 as $\epsilon \to 0$. We shall study the continuity of w_2 as defined in (5.5). We rewrite w_2 in the form

$$\begin{aligned} w_2(x,t) &= -\int_0^t \int_{\mathbf{R}^N} J(x,t;y,s)\, \tilde{f}(y,s)\, dy\, ds \\ &= -\int_0^t \int_{\mathbf{R}^N} \Big(\int_\tau^t \int_{\mathbf{R}^N} Z(x,t;y,s)\Phi(y,s;\xi,\tau)\, dy\, ds\Big)\, \tilde{f}(\xi,\tau)\, d\xi\, d\tau \\ &= -\int_0^t \int_{\mathbf{R}^N} Z(x,t;y,s)\, \hat{f}(y,s)\, dy\, ds, \end{aligned}$$

where

$$\hat{f}(y,s) = \int_0^s \int_{\mathbf{R}^N} \Phi(y,s;\xi,\tau)\, \tilde{f}(\xi,\tau)\, d\xi\, d\tau. \tag{5.7}$$

From Corollary 2.4 in [14] there exist $c, \epsilon > 0$ such that if $\tilde{\Gamma}$ is the fundamental solution of the operator $\tilde{L} = \operatorname{div}(\tilde{A}D) + Y$, $\tilde{A}$ is the $N \times N$ constant matrix

$$\tilde{A} = \begin{pmatrix} (\mu+\epsilon)I_q & 0 \\ 0 & 0 \end{pmatrix},$$

(μ is the constant in hypothesis (H) of Section 1) then for all $z, \zeta \in \Omega$, $z \ne \zeta$ it holds

$$|\Phi(z;\zeta)| \le c\, \frac{\tilde{\Gamma}(z;\zeta)}{(t-\tau)^{1-\frac{\alpha}{2}}} \qquad z = (y,t), \zeta = (\xi,\tau),$$

hence

$$\int_{B(z,\epsilon)} |\Phi(z;\zeta)|\, d\zeta \to 0 \quad \text{as } \epsilon \to 0, \quad \text{uniformly in } \mathbf{R}^{N+1},$$

and the continuity of the function $\hat{f}$ follows as before.

To complete the proof it remains to prove that $Lw = f$ in Ω. We shall show that

$$Lw_1(x,t) = \tilde{f}(x,t) - \int_0^t \int_{\mathbf{R}^N} LZ(x,t;y,s)\,\tilde{f}(y,s)\,dy\,ds \tag{5.8}$$

and

$$Lw_2(x,t) = \hat{f}(x,t) - \int_0^t \int_{\mathbf{R}^N} LZ(x,t;y,s)\,\hat{f}(y,s)\,dy\,ds, \tag{5.9}$$

therefore we immediately get

$$\begin{aligned} Lw(x,t) &= Lw_1(x,t) + Lw_2(x,t) \\ &= \tilde{f}(x,t) - \int_0^t \int_{\mathbf{R}^N} \tilde{f}(\xi,\tau)\big(-\Phi(x,t;\xi,\tau) + (LZ)(x,t;\xi,\tau) \\ &\quad + \int_\tau^t \int_{\mathbf{R}^N} LZ(x,t;\eta,\sigma)\,\Phi(\eta,\sigma;\xi,\tau)\big)\,d\eta\,d\sigma) d\xi\,d\tau = (5.4) = f(x,t), \end{aligned}$$

for all $(x,t) \in \Omega$.

The proof of (5.8) is split into the proof of the following statements:

(i) $\partial_{x_i} w_1$ exists and is a continuous function for all $i = 1, \dots, q$. Moreover

$$\partial_{x_i} w_1(z) = -\int_0^t \int_{\mathbf{R}^N} \partial_{x_i} Z(z;y,s)\,\tilde{f}(y,s)\,dy\,ds, \quad \forall z = (x,t) \in \mathbf{R}^{N+1}.$$

(ii) $\partial_{x_i x_j} w_1$ exists and is a continuous function for all $i,j = 1, \dots, q$. Moreover

$$\partial_{x_i x_j} w_1(z) = -\int_0^t \int_{\mathbf{R}^N} \partial^2_{x_i x_j} Z(z;y,s)\,\tilde{f}(y,s)\,dy\,ds, \quad \forall z(x,t) \in \mathbf{R}^{N+1}.$$

(iii) Yw_1 exists and is a continuous function. Moreover for all $z(x,t) \in \mathbf{R}^{N+1}$

$$Yw_1(z) = \tilde{f}(z) - \int_0^t \int_{\mathbf{R}^N} \sum_{i,j=1}^q a_{ij}(y,s)\partial^2_{x_i x_j} Z(z;y,s)\,\tilde{f}(y,s)\,dy\,ds.$$

We first prove that the integral in (i) converges. From Corollary 2.2 in [14] there exists a positive constant c such that

$$|\partial_{x_i} Z(z;\zeta)| \le \frac{c}{\sqrt{|t-\tau|}}\tilde{\Gamma}(z;\zeta) \quad \forall z, \zeta \in \mathbf{R}^{N+1},\ z \ne \zeta. \tag{5.10}$$

Let $\eta \in C^2(\mathbf{R})$ be bounded and such that $\eta(r) = 0$ if $|r| \ge 1$ and $\eta(r) = 1$ if $|r| \le \frac{1}{2}$. We call $\eta_\epsilon(z;\zeta) = 1 - \eta(||D(\epsilon^{-1})(\zeta^{-1} \circ z)||)$ and we set

$$v_\epsilon(z) = -\int_0^t \int_{\mathbf{R}^N} \eta_\epsilon(z;\zeta) Z(z;\zeta)\,\tilde{f}(\zeta)\,d\zeta.$$

Then $v_\epsilon \in C(\mathbf{R}^N \times [0,T])$ and v_ϵ uniformly converges to w_1 as $\epsilon \to 0$. We thus obtain

$$\partial_{x_i} v_\epsilon(z) = -\int_0^t \int_{\mathbf{R}^N} \partial_{x_i}(Z\,\eta_\epsilon)(z;\zeta)\,\tilde{f}(\zeta)\,d\zeta. \tag{5.11}$$

Indeed, from (5.10), it follows that

$$|\partial_{x_i}(Z\,\eta_\epsilon)(z;\zeta)| \le c_\epsilon \sup_{\zeta\in\mathbf{R}^N\setminus B(z,\frac{\epsilon}{2})} \frac{\tilde{\Gamma}(z;\zeta)}{\sqrt{|t-\tau|}} \equiv \tilde{c}_\epsilon < \infty.$$

(The proof of the last estimate is similar to the proof of (3.8) in [14]; the details of proof are therefore omitted.) Hence, the bound

$$|\partial_{x_i}(Z\,\eta_\epsilon)(z;\zeta)\,\tilde{f}(\zeta)| \le \tilde{c}_\epsilon|\tilde{f}(\zeta)|$$

holds uniformly with respect to z, the identity (5.11) follows by Lebesgue's theorem.

In order to prove that

$$\partial_{x_i} v_\epsilon(z) \to -\int_0^t \int_{\mathbf{R}^N} \partial_{x_i} Z(z;y,s)\,\tilde{f}(y,s)\,dy\,ds \quad \text{as} \quad \epsilon \to 0,$$

uniformly in $\mathbf{R}^N \times [0,T]$, we write $\partial_{x_i} v_\epsilon(z) - \int_0^t \int_{\mathbf{R}^N} \partial_{x_i} Z(z;\zeta)\,\tilde{f}(\zeta)\,d\zeta$ in the form

$$\begin{aligned}
\partial_{x_i} v_\epsilon(z) &- \int_0^t \int_{\mathbf{R}^N} \partial_{x_i} Z(z;\zeta)\,\tilde{f}(\zeta)\,d\zeta \\
&= \int_0^t \int_{\mathbf{R}^N} \partial_{x_i} Z(z;\zeta)(\eta_\epsilon(z;\zeta)-1)\,\tilde{f}(\zeta)\,d\zeta \\
&\quad + \int_0^t \int_{\mathbf{R}^N} Z(z;\zeta)\partial_{x_i}\eta_\epsilon(z;\zeta)\,\tilde{f}(\zeta)\,d\zeta \equiv I^1_\epsilon(z) + I^2_\epsilon(z),
\end{aligned} \tag{5.12}$$

and we estimate separately $I^1_\epsilon(z)$ and $I^2_\epsilon(z)$. From (5.10) it is readily seen that

$$|I^1_\epsilon(z)| \le \int_{B(z,\epsilon),s<t} \frac{c}{\sqrt{t-s}} \tilde{\Gamma}(z;\zeta)|\tilde{f}(\zeta)|\,d\zeta \le c\,\epsilon \sup_\Omega |f| \int_{B(0,1),s<0} \frac{\tilde{\Gamma}(0;\zeta)}{\sqrt{-s}}\,d\zeta,$$

hence

$$|I^1_\epsilon(z)| \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{5.13}$$

uniformly in $\mathbf{R}^N \times [0,T]$.

In order to evaluate $I^2_\epsilon(z)$ we first note that

$$\partial_{x_i}\eta_\epsilon(z;\zeta) = \frac{1}{\epsilon}\eta'(||D(\epsilon^{-1})(\zeta^{-1}\circ z)||),$$

and

$$|\partial_{x_i}\eta_\epsilon(z;\zeta)| \le \frac{1}{\epsilon}\sup_R |\eta'| = \frac{c}{\epsilon} \quad \forall \zeta \in \mathbf{R}^{N+1}, \tag{5.14}$$

besides $\partial_{x_i}\eta_\epsilon(z;\zeta) = 0$ for all $\zeta \in \mathbf{R}^{N+1} \setminus B(z,\epsilon)$.

Using (5.14) and (5.6) , we thus have

$$\begin{aligned} |I_\epsilon^2(z)| &\le \frac{c}{\epsilon}\int_{B(z,\epsilon),s<t} \Gamma^+(z;\zeta)\,d\zeta \\ &= c\,\epsilon \int_{B(0,1),s<0} \Gamma^+(0;\zeta)\,d\zeta \to 0 \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{5.15}$$

The proof of (i) is obtained from combining (5.12), (5.13) and (5.15).

Note that the integral in (ii) exists as a repeated integral. For every fixed $s \in (0,t)$ and for every $y' \in \mathbf{R}^N$ it turns out

$$\begin{aligned} \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z(z;y,s)\,\tilde f(y,s)\,dy &= \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z(z;y,s)\,(\tilde f(y,s) - \tilde f(y',s))\,dy \\ &\quad + \tilde f(y',s) \int_{\mathbf{R}^N} \partial^2_{x_ix_j}(Z_{(y,s)} - Z_{(y',s)})(z;y,s)\,dy \\ &\quad + \tilde f(y',s) \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z_{(y',s)}(z;y,s)\,dy \\ &\equiv I_1(z,s) + I_2(z,s) + I_3(z,s), \end{aligned} \tag{5.16}$$

where $Z_{\bar z}(z;\zeta)$ is the fundamental solution of the operator $L_{\bar z}$ with pole at ζ. We next give some estimates for I_1, I_2 and I_3. The condition $f \in C^\alpha(\Omega; B)$ implies that there also exists a constant $c > 0$ such that

$$|f(y,s) - f(y',s)| \le c\,|||y - y'|||^\alpha \quad \forall\,(y,s),(y',s) \in \Omega$$

($|||\cdot|||$ is introduced in Definition 1.1). Moreover, from Corollary 2.1 in [14], it follows that

$$|\partial^2_{x_ix_j} Z(z;\zeta)| \le \frac{c}{t-\tau}\,(1 + |D_0(\sqrt{t-\tau})(x - E(t-\tau)\xi)|^2)\,\Gamma^+(z;\zeta)$$

for every $z, \zeta \in \mathbf{R}^{N+1}$. Therefore

$$|I_1(z,s)| \le c \int_{\mathbf{R}^N} \frac{\Gamma^+(z;y,s)}{t-s}(1 + |D_0(\sqrt{t-s})(x - E(t-s)y)|^2)|||y - y'|||^\alpha\,dy,$$

and choosing $y' = E(s-t)x$ we obtain

$$\begin{aligned} |||y - y'|||^\alpha &= |||y - E(s-t)x|||^\alpha = (3.2) \\ &= \frac{1}{(t-s)^{\alpha/2}}\,|||D_0((t-s)^{1/2})(y - E(s-t)x)|||^\alpha, \end{aligned}$$

and

$$|I_1(z,s)| \le \frac{c}{(t-s)^{1-\alpha/2}}. \tag{5.17}$$

We evaluate I_2 in (5.16). From Lemma 3.2 in [14], it results

$$\left|\partial^2_{x_ix_j} Z_\zeta(z;w) - \partial^2_{x_ix_j} Z_{\zeta'}(z;w)\right| \le \frac{c}{t-s} ||\zeta^{-1}\circ\zeta'||^\alpha \tilde{\Gamma}(z;w), \quad \forall z,w,\zeta,\zeta' \in \mathbf{R}^{N+1} \tag{5.18}$$

for some positive constant c. Arguing as above and using (5.18)

$$|I_2(z,s)| \le \frac{c}{t-s}\,|f(y',s)| \int_{\mathbf{R}^N} \tilde{\Gamma}(z;y,s)|||y-y'|||^\alpha\, dy \le \frac{c}{(t-s)^{1-\alpha/2}}. \tag{5.19}$$

Noting that

$$\int_{\mathbf{R}^N} Z_{\bar{z}}(z;y,s)\, dy = 1 \quad \forall\ \bar{z} \in \mathbf{R}^{N+1},$$

by Lebesgue's theorem ($s<t$), we derive

$$0 = \partial^2_{x_ix_j} \int_{\mathbf{R}^N} Z_{\bar{z}}(z;y,s)\, dy = \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z_{\bar{z}}(z;y,s)\, dy, \tag{5.20}$$

then $I_3(z,s) \equiv 0$. Finally, the existence of the integral in (ii) follows from (5.17), (5.18) and (5.20).

Arguing as in the proof of (i), it can be shown that

$$\partial_{x_ix_j} v_\epsilon(z) = -\int_0^t \int_{\mathbf{R}^N} \partial^2_{x_ix_j}(Z\,\eta_\epsilon)(z;\zeta)\,\tilde{f}(\zeta)\, d\zeta, \tag{5.21}$$

and

$$\partial_{x_ix_j} v_\epsilon(z) \to -\int_0^t \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z(z;\zeta)\,\tilde{f}(\zeta)\, d\zeta \quad \text{as} \quad \epsilon \to 0, \tag{5.22}$$

uniformly in $\mathbf{R}^N \times [0,T]$.

In order to prove (5.22) we write, for a fixed $y' \in \mathbf{R}^N$,

$$\begin{aligned}
&\partial^2_{x_ix_j} v_\epsilon(z) - \int_0^t \int_{\mathbf{R}^N} \partial^2_{x_ix_j} Z(z;\zeta)\,\tilde{f}(\zeta)\, d\zeta \\
&\quad = \int_0^t \int_{\mathbf{R}^N} \partial^2_{x_ix_j}(\eta_\epsilon(\zeta^{-1}\circ z)-1) Z(z;y,s)\,(\tilde{f}(y,s)-\tilde{f}(y',s))\, dy\, ds \\
&\quad + \int_0^t \tilde{f}(y',s) \int_{\mathbf{R}^N} \partial^2_{x_ix_j}(\eta_\epsilon(\zeta^{-1}\circ z)-1)(Z_{(y,s)} - Z_{(y',s)})(z;y,s)\, dy\, ds \\
&\quad + \int_0^t \tilde{f}(y',s) \int_{\mathbf{R}^N} \partial^2_{x_ix_j}((\eta_\epsilon(\zeta^{-1}\circ z)-1) Z_{(y',s)})(z;y,s)\, dy\, ds \\
&\quad \equiv I_1^\epsilon(z,s) + I_2^\epsilon(z,s) + I_3^\epsilon(z,s).
\end{aligned}$$

We note that I_i^ϵ can be obtained by replacing Z with $(\eta_\epsilon(\zeta^{-1}\circ z)-1)Z$ in (5.16) for all $i=1,2,3$. Estimates analogous to (5.17) and (5.19) then guarantee that (5.21) and (5.22) hold.

To prove (iii) we can proceed as in the proof of Proposition 3.3 in [14] and with the same arguments as in (c).

Finally, from (i), (ii) and (iii) it follows that

$$\begin{aligned} Lw_1(x,t) &= \tilde{f}(x,t) \\ &\quad - \int_0^t \int_{\mathbf{R}^N} \sum_{i,j=1}^q (a_{ij}(x,t)-a_{ij}(y,s))\partial^2_{x_i x_j} Z(x,t;y,s)\tilde{f}(y,s)\,dy\,ds. \end{aligned}$$

Since $Z(z;\zeta)$ is the fundamental solution of L_ζ, we obtain (5.8).

The identity (5.9) follows as (5.8), observing that $\hat{f}$ satisfies the Hölder estimate

$$|\hat{f}(y,s)-\hat{f}(y',s)| \le c\,|||y-y'|||^{\gamma'}, \quad \forall y,y'\in\mathbf{R}^N, s\in[0,T] \tag{5.23}$$

for some constant $\gamma'\in(0,1)$.

Inequality (5.23) will then be a consequence of the following statement (see Lemma 3.1 in [14]). There exist three constants $c>0$, γ and $\gamma'\in(0,1)$ such that

$$|\Phi(x,t;\zeta)-\Phi(x',t;\zeta)| \le c\,\frac{|||x-x'|||^{\gamma'}}{(t-\tau)^{1-\frac{\gamma}{2}}}\left(\tilde{\Gamma}(x,t;\zeta)+\tilde{\Gamma}(x',t;\zeta)\right)$$

for every $\zeta=(\xi,\tau)\in\mathbf{R}^{N+1}$, $t,\tau\in[0,T], t>\tau$ and for every $x,x'\in\mathbf{R}^N$.

Thus, from the definition of $\hat{f}$ in (5.7), we get

$$\begin{aligned} &|\hat{f}(y,s)-\hat{f}(y',s)| \\ &\quad\le |||y-y'|||^{\gamma'}\int_0^s\int_{\mathbf{R}^N}\frac{1}{(s-\tau)^{1-\frac{\gamma}{2}}}\left(\tilde{\Gamma}(y,s;\xi,\tau)+\tilde{\Gamma}(y',s;\xi,\tau)\right)d\xi\,d\tau \end{aligned} \tag{5.24}$$

and

$$\int_0^s\int_{\mathbf{R}^N}\frac{1}{(s-\tau)^{1-\frac{\gamma}{2}}}\,\tilde{\Gamma}(y,s;\xi,\tau)\,d\xi\,d\tau = \int_0^s\frac{1}{(s-\tau)^{1-\frac{\gamma}{2}}}\,d\tau<\infty.$$

An analogous scheme can be adopted for the second integral in (5.24). Theorem 1.5 is thus completely proved.

6. Some geometric criteria for the regularity of the boundary points. We consider the Dirichlet problem

$$\begin{cases} Lu=0 & \text{in} \quad \Omega \\ u=\phi & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is an arbitrary bounded open subset of $\mathbf{R}^{N+1}$, and we indicate by H_ϕ^Ω its generalized solution.

We say that a point $z_0 \in \partial\Omega$ is *regular* with respect to the operator L for Ω (in short: *L-regular* for Ω), if

$$\lim_{z \to z_0} H_\phi^\Omega(z) = \phi(z_0) \quad \text{for all} \quad \phi \in C(\partial\Omega).$$

In the following Proposition 6.1 we prove that the boundary points of Ω for which there exists a noncharacteristic exterior normal at Ω are L-regular and we give a geometric condition which assures the regularity for the characteristic boundary points, when the Fichera function is positive definite.

Proposition 6.1 requires some preliminary notation. For every $z \in \mathbf{R}^{N+1}$ we write $z = (z', z'') \in \mathbf{R}^q \times \mathbf{R}^{N+1-q}$ and we set

$$d_\epsilon(z) \equiv \frac{|z'|^2}{\epsilon^2} + \frac{|z''|^2}{\epsilon^4},$$

where $|\cdot|$ indicates the Euclidean norm in $\mathbf{R}^q$ or in $\mathbf{R}^{N+1-q}$ and ϵ is a positive constant. Besides, we set

$$E(\zeta_0, \epsilon) = \{z \in \mathbf{R}^{N+1} : d_\epsilon(\zeta_0 - z) \le 1\}.$$

We can now state:

Proposition 6.1. *Let Ω be an open bounded subset of $\mathbf{R}^{N+1}$ and let $z_0 \in \partial\Omega$. If there exists an exterior normal $\nu(z_0) = (\nu_0(z_0), \nu_{N+1}(z_0)) \in \mathbf{R}^{N+1}$ at Ω in z_0 such that*

$$\langle A(z_0)\nu_0(z_0), \nu_0(z_0)\rangle \neq 0$$

($A(z_0)$ is the matrix as defined in (2.3)), then z_0 is an L-regular point for Ω.

Whereas, if

$$\langle A(z_0)\nu_0(z_0), \nu_0(z_0)\rangle = 0, \quad \langle x_0, B\nu_0(z_0)\rangle - \nu_{N+1}(z_0) > 0$$

and there exists a positive constant ϵ, $\epsilon \le \sqrt{\frac{\langle x_0, B\nu_0(z_0)\rangle - \nu_{N+1}(z_0)}{\operatorname{trace}(A_0(z_0))}}$ such that

$$E(\zeta_0, \epsilon) \subset \mathbf{R}^{N+1} \setminus \Omega,$$

where $\zeta_0 = z_0 + \epsilon^2 \nu(z_0)$, then z_0 is an L-regular point for Ω.

Proof. We first suppose that the first condition is fulfilled. Then there exists $r > 0$ such that

$$\overline{B_{eucl}(z_0 + r\nu(z_0), r)} \cap \overline{\Omega} = \emptyset \quad \text{and} \quad \langle A(z_0)\nu_0(z_0), \nu_0(z_0)\rangle > 0.$$

It is then easy to verify that the function w_{z_0} given by

$$w_{z_0}(z) = \exp(-Mr^2) - \exp(-M\,|z - z_0 - r\nu|^2), \quad z \in \Omega$$

is a barrier for Ω at z_0, provided M is a sufficiently large positive constant, so that z_0 is an L-regular point.

In the second case, the function

$$v_{z_0}(z) = \exp(-\epsilon^3) - \exp(-\epsilon^3 d_\epsilon(z-\zeta_0)), \quad z \in \Omega$$

is a barrier for Ω at z_0. Direct calculation shows that $v_{z_0}(z_0) = 0$, $v_{z_0}(z) > 0$ for every $z \in \Omega \setminus \{z_0\}$ and

$$Lv_{z_0}(z_0) = \frac{2}{\epsilon}\exp(-\epsilon^3)(\epsilon^2 \operatorname{trace} A_0(z_0) - \langle x_0, B\nu_0(z_0)\rangle + \nu_{N+1}(z_0)) < 0.$$

The following theorem furnishes a sufficient geometric criterion which extends the classical Zaremba criterion of exterior cone to our setting.

We now introduce the definition of L-cone.

Definition 6.2. We define the L-cone of vertex $(0,0)$, base K and height T, the subset of $\mathbf{R}^{N+1}$

$$C = \{D(r)\,(x,-T) : x \in K,\ 0 \le r \le 1\},$$

where $T > 0$ and K is a closed bounded set of $\mathbf{R}^N$ with finite and positive Lebesgue measure.

If $z_0 \in \mathbf{R}^{N+1}$ we define the L-cone of vertex z_0 to be the set

$$C_{z_0} = \{z_0 \circ z : z \in C\}.$$

Theorem 6.3. *Let Ω be an open set of $\mathbf{R}^{N+1}$ and let $z_0 \in \partial\Omega$. If there exists an L-cone C_{z_0} such that $C_{z_0} \subset \mathbf{R}^{N+1} \setminus \Omega$, then z_0 is L-regular.*

The proof of Theorem 6.3 requires some preliminary remarks.

If V is an L-regular open set according to Definition 4.1 and F is a compact subset of V, then, for all $\zeta \in F$, we set $G(z;\zeta) = \Gamma(z;\zeta) - h_\zeta(z)$, where $\Gamma(z;\zeta)$ is the fundamental solution of L with pole at ζ and h_ζ is the solution of the problem

$$\begin{cases} Lu = 0 & \text{in} \quad V \\ u = \Gamma(\,\cdot\,;\zeta) & \text{on} \quad \partial V. \end{cases}$$

We set

$$W_F^L(\zeta) = \inf\{u(\zeta) : u \text{ is } L\text{-superharmonic in } V,\ 0 \le u \le 1,\ u \equiv 1 \text{ in } F\}$$

and we indicate by V_F^L its semicontinuous regularization

$$V_F^L(z) = \liminf_{\zeta \to z} W_F^L(\zeta), \quad \forall z \in V.$$

Then (see [12, pages 97–98]) there exists a positive measure μ_F such that

$$V_F^L(z) = \int_V G(z;\zeta)\, d\mu_F(\zeta),$$

and ([12, Proposition 15])

$$\mu_F(F) = \sup\Big\{\mu(F) : \mu \in M^+(F),\ \int_V G(z;\zeta)\,d\mu(\zeta) \le 1,\ \ \forall z \in V\Big\}.$$

We call μ_F the *L-equilibrium measure of F* and we call *L-capacity* of F in V the total mass of μ_F; i.e., $\operatorname{cap}_L(F,V) \equiv \mu_F(F)$.

Lemma 6.4. *Let V be an open L-regular set and $z_0 \in V$. Then*

$$\lim_{r\to 0} cap_L\,(\overline{B_r(z_0)}, V) = 0.$$

In particular $cap_L(\{z_0\}, V) = 0$.

Proof. By the local estimate on Γ ([14, Theorem 2.1]) we infer that

$$\limsup_{z\to z_0} \Gamma(z; z_0) = +\infty.$$

Moreover, from the maximun principle (see the proof of Proposition 4.2), we obtain

$$\Gamma(z;\zeta) - \gamma \le G(z;\zeta) \le \Gamma(z;\zeta), \quad \text{for all}\, z \in V \text{ and for all}\, \zeta \in F \tag{6.1}$$

where $\gamma = \sup\{\Gamma(z;\zeta) : z \in \partial V, \zeta \in F\}$. Then, for every $k > 0$ there exists $z \in V$ such that $G(z; z_0) > k$. Since G is a smooth function in the complement of the diagonal of $V \times V$, there exists $\bar{r} > 0$ such that $G(z;\zeta) > k$ for all $\zeta \in \overline{B_r(z_0)}$ and for all $r \le \bar{r}$. Thus, if μ denotes the L-equilibrium measure of $\overline{B_r(z_0)}$, we get

$$1 \ge \int_{\overline{B_r(z_0)}} G(z;\zeta)\, d\mu(\zeta) \ge k\, \mu(\overline{B_r(z_0)}) = k\, \mathrm{cap}_L(\overline{B_r(z_0)}, V)$$

for every $r \le \bar{r}$. Consequently $\mathrm{cap}_L\,(\overline{B_r(z_0)}, V)$ tends to zero as $r \to 0$ as required.

Proof of Theorem 6.3. Let V be an open L-regular set such that $z_0 = (x_0, t_0) \in V$. For sufficiently small $r > 0$ the set

$$V_r(z_0) = \{z \in \mathbf{R}^{N+1} \setminus \Omega : ||z^{-1} \circ z_0|| \le r\}$$

is contained in V; then (see [12, Theorem 14]) it is enough to prove that

$$\liminf_{r\to 0} V^L_{V_r(z_0)} > 0. \tag{6.2}$$

Since there exists an L-cone C_{z_0} of vertex z_0, base K and height T, such that $C_{z_0} \subset \mathbf{R}^{N+1} \setminus \Omega$, we have for small r

$$V_r(z_0) \supset (D_0(r)K + E(-r^2T)x_0) \times \{t_0 - r^2T\} \equiv K_r(x_0) \times \{t_0 - r^2T\}.$$

If $\nu_r = m_N|_{K_r(z_0)} \otimes \delta_{t_0 - r^2T}$, then from (6.1), for every $z \in V$ it holds

$$\int_V G(z;\zeta)\, d\nu_r(\zeta) \le \int_{K_r(x_0)} \Gamma(x,t;\xi,t_0 - r^2T)\, d\xi \le 1;$$

then

$$\int_V G(z_0;\zeta)\, d\nu_r(\zeta) \le V^L_{V_r(z_0)}. \tag{6.3}$$

Moreover, from Theorem 2.1 in [14], there exists a constant $C > 0$ such that

$$\Gamma(z_0;\zeta) \geq \frac{1}{2}\Gamma^-(z_0;\zeta),$$

provided $\Gamma^-(z_0;\zeta) \geq C$. Γ^- is the fundamental solution of the operator $L^- = \mathrm{div}(A^- D) + Y$, where A^- is the $N \times N$ matrix

$$A^- = \begin{pmatrix} \frac{1}{\mu} I_q & 0 \\ 0 & 0 \end{pmatrix},$$

and μ is the positive constant in hypothesis (H) of Section 1.

It is easy to prove, by using $D(\lambda)$-homogeneity of Γ^- and identity (3.2), that

$$V_r(z_0) \subset \{\zeta \in \mathbf{R}^{N+1} : \Gamma^-(z_0;\zeta) \geq C\},$$

for r small enough. Hence, from (6.1) and (6.3) we obtain

$$\begin{aligned} V^L_{V_r(z_0)} &\geq c \int_{K_r(x_0)} \Gamma^-(z_0;\xi, t_0 - r^2T)\, d\xi - \gamma m_N(K_r(x_0)) \\ &\geq c \int_{K_r(x_0)} \Gamma^-(z_0;\xi, t_0 - r^2T)\, d\xi - \gamma\, r^Q\, m_N(K). \end{aligned} \tag{6.4}$$

Since $\Gamma^-(z_0;\zeta) = \Gamma^-(\zeta^{-1} \circ z_0)$ we have

$$\begin{aligned} V^L_{V_r(z_0)} &\geq c\, r^Q \int_K \Gamma^-(z_0; D_0(r)x + E(-r^2T)x_0, t_0 - r^2T)\, dx - \gamma\, r^Q\, m_N(K) \\ &= c\, r^Q \int_K \Gamma^-(z_0; z_0 \circ (D_0(r)x, -r^2T))\, dx - \gamma\, r^Q\, m_N(K) \\ &= c\, r^Q \int_K \Gamma^-((D_0(r)x, -r^2T)^{-1})\, dx - \gamma\, r^Q\, m_N(K) \\ &= c\, r^Q \int_K \Gamma^-(E(r^2T)D_0(r)x, r^2T)\, dx - \gamma\, r^Q\, m_N(K) \\ &= c\, T^{-Q/2} \int_K \Gamma^-(E(1)D_0(T^{-1/2})x, 1)\, dx - \gamma\, r^Q\, m_N(K). \end{aligned} \tag{6.5}$$

The last equality in (6.5) follows from (3.2), since

$$\begin{aligned} \Gamma^-(E(r^2T)D_0(r)x, r^2T) &= \Gamma^-(D_0(r\sqrt{T})E(1)D_0(\frac{1}{r\sqrt{T}})D_0(r)x, r^2T) \\ &= (\Gamma^- \text{ is } D(\lambda)\text{-homogeneous of degree } -Q) \\ &= r^{-Q}T^{-Q/2}\Gamma^-(E(1)D_0(T^{-1/2})x, 1). \end{aligned}$$

Finally, from (6.5) we infer that

$$\liminf_{r\to 0} V^L_{V_r(z_0)} \geq c\,T^{-Q/2} \int_K \Gamma^-(E(1)D_0(T^{-1/2})x, 1)\,dx > 0,$$

since c is a constant independent of r, therefore inequality (6.2) holds. This completes the proof of Theorem 6.3.

To conclude this section, we propose a simple example.

Example 6.5. Let L be the Kolmogorov operator in $\mathbf{R}^3$; that is,

$$L = \partial_x^2 + x\,\partial_y - \partial_t, \quad (x, y, t) \in \mathbf{R}^3$$

(see (1.3)). Then, with the notation of Section 1,

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

and $D(\lambda) = (\lambda, \lambda^3, \lambda^2)$ for every $\lambda > 0$.

Let Ω be the open set $\{(x, y, t) : |x|, |y|, |t| < 1\}$. In this case the points of the set $\{(1, y, t), (-1, y, t) : |y|, |t| \leq 1\}$ are L-regular since they are noncharacteristic boundary points; the points in the sets

$$\{(x, y, -1) : |x|, |y| \leq 1\}, \quad \{(x, 1, t) : x > 0, |t| \leq 1\},$$
$$\{(x, -1, t) : x < 0, |t| \leq 1\}$$

are L-regular since they satisfy the condition of Proposition 6.1. Furthermore, the points on the line

$$A = \{(0, 1, t),\ (0, -1, t) : |t| \leq 1\}$$

are regular points, since for every $z_0 \in A$ the exterior L-cone condition is satisfied (see Theorem 6.3). Indeed, if $z_0 = (0, 1, t_0) \in A$ and K is a compact subset of $\{(x, y) \in \mathbf{R}^2 : y > 1\}$, the set

$$C_{z_0} = \{(rx, r^3y + 1, -r^2T + t_0) : (x, y) \in K,\ 0 \leq r \leq 1\},$$

is an exterior L-cone of vertex z_0 for the set Ω.

Acknowledgments. The results of this paper are part of my Tesi di Dottorato di Ricerca ([10]), and were announced in [11]. We thank Professor E. Lanconelli for suggesting the problem and for several helpful conversations.

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